THEMIS SIGNAL ANALYSIS STATISTICS RESEARCH PROGRAM

SIMPLIFIED EQUALLY-LIKELY REMOVALS IN SIMULATIONS

by

John E. Walsh

Technical Report No. 81
Department of Statistics THEMIS Contract

September 21, 1970

Research sponsored by the Office of Naval Research
Contract N00014-68-A-0515
Project NR 042-260

DEPARTMENT OF STATISTICS
Southern Methodist University

SIMPLIFIED EQUALLY-LIKELY REMOVALS IN SIMULATIONS

John E. Walsh*
Southern Methodist University**

ABSTRACT

Random simulation models often involve many equally-likely removals of determined numbers from identified groups of persons or objects. That is, all ways of selecting those actually removed are equally likely. The time and expense for such removals can be substantially reduced (compared to a separate equally-likely selection for each removal) by a simplified method that frequently is usable in at least part of a model. This simplified method can be applied to groups which consist of members of the same type and which, except for the initial group(s), are obtained from previous such groups by removals and/or subdivisions. For an application to members of a given type, a random ordering is assigned to the persons or objects that can be of that type (all possible orderings equally likely). This assignment is independent of all other aspects of the simulation and is used only for equally-likely removals involving the specified type. All such removals are made on the basis of this ordering. Attritions constitute an important class of removals.

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*Based on concepts developed in work performed at the Quality Evaluation Laboratory, U. S. Naval Torpedo Station, Keyport, Washington.

**Research partially supported by ONR Contract N00014-68-A-0515 and by Mobil Research and Development Corporation.

INTRODUCTION AND DISCUSSION

Consider stepwise simulation models wherein at least some of the events are based on random occurrences. A kind of occurrence that can happen many times is the equally-likely removal of a determined number from an identified group of persons or objects. All possible ways of selecting those actually removed are to be equally likely (regardless of how the group and the number to be removed are determined).

Nearly always, the members of such a group are, in some respect, of the same type (so that equally-likely removal is appropriate). Also, as the simulation continues, the groups with members of a given type often are obtained exclusively from previous such groups by removals and/or subdivisions. Here, members removed from a group do not occur later in groups with membership of this type and, of course, this way of obtaining successive groups does not apply to the initial group(s) with members of the specified type. An "initial" group, with members of the same type, is not obtained from a previous such group by removals and/or subdivision, but at least one group ordinarily is obtained in that manner from an initial group. To illustrate a situation of this nature, consider a large group of airplanes with the same appearance and capability that start a mission by flying together (the initial group). Further on in the flight they subdivide into moderate sized groups that go in different directions. Later, the moderate sized groups subdivide into small groups that go toward their ultimate destinations. Random attrition, of determined amounts, is experienced by these various groups. Within each group, the airplanes actually attrited are selected on an equally-likely basis.

A simplified method can be used for equally-likely removals from groups with membership of a given type when, except for the initial group(s), the groups arise exclusively from removals and/or subdivisions for previous such groups. The simplified method can substantially reduce the time and expense for performing the equally-likely removals (compared to a separate random selection for each removal). Incidentally, use of the simplified method for specific groups does not preclude the application of other removal methods to these groups. That is, other methods might result in some removals from groups with membership types for which the simplified method is used for equally-likely removals. For example, some of the airplanes previously discussed might be given orders to discontinue their participation in the mission and return. These orders could be for specific airplanes and their basis could be deterministic or random. In particular, they might have nothing to do with attritions that have been experienced.

As a simulation progresses, some groups with membership of a given type may be eligible for continued use of the simplified method while others are not. That is, some of the groups encountered may be ineligible because they are not required to be obtained from removals and/or subdivisions for previous such groups. Then, the simplified method continues to be used only for the eligible groups, and this use is discontinued when there are no more eligible groups. Again for illustration, consider the airplanes case. Suppose that later in the flight some of the small groups combine and this happens after attrition has occurred (within the small groups). The simplified method would not continue to be used for these

combined groups even though it was usable for all the small groups.

A simplified method could, of course, first be used after many steps of the simulation have occurred. In fact, an initial group for this use could have been obtained from persons or objects that previously were in one or more groups to which a simplified method was used for removals. When this can happen, extra care is needed in specifying what constitutes a "type" of persons or objects. That is, type can also indicate that some or all of the group members previously were involved in some other use(s) of the simplified method.

Applications of the simplified method sometimes can be appreciably increased by noting that groups which no longer are usable for continuations of a simplified method can represent initial groups for new uses of the simplified method. The airplanes case provides an illustration of such new uses. Consider groups that are obtained by combinations of some of the small groups later in the flight. Suppose that no other airplane join any of these combined groups and that these airplanes continue in this grouping for the remainder of the mission (except for the attrition experienced). Then, these combined groups could be used as initial groups for a new use of the simplified method (when equally-likely attrition is to occur within each combined group).

Next, consider some of the details in an application of the simplified method for a specified type of persons or objects. The random effects introduced are used exclusively for applying the simplified method to groups with this type of membership. These random effects are statistically independent of any other aspects of the simulation. In particular,

they are independent of any other random effects in the simulation (including those for applying the simplified method to groups with other types of membership).

To introduce the random effects, an ordering is assigned at the first simulation step where this use of the simplified method occurs. The ordering involves all persons or objects that are of the specified type at this step or could be of this type at further steps. The random assignment is done so that all possible orderings are equally likely (say, by an independent randomization). All removals for this use of the simplified method can be made on the basis of the resulting ordering (which remains fixed during a simulation).

If desired, some or all of the random effects for uses of the simplified method can be introduced prior to beginning a simulation. This approach can be desirable for uses where an initial group occurs early in the simulation, but can result in increased time and effort for uses where no initial group occurs until well into the simulation. To introduce random effects prior to the simulation, an ordering is given to the totality of persons or objects that (as viewed from the start of the simulation) could possibly be of the specified type at the first simulation step where this use of the simplified method occurs. In some cases, the possible totality as viewed from the start of a simulation is much larger than the possible totality when viewed from the first step where this use of the simplified method occurs. Also, identification of a person or object that could possibly be of the specified type at this step can be much more difficult at the start of the simulation.

An advantage of introducing the random effects prior to start of a simulation is that thereafter, for that use of the simplified method, the random removals can be made in a deterministic fashion. This can allow the simulation model to be expressed in a less complicated manner.

The big advantage of the simplified method is, of course, that a single random ordering can be used for many separate removals. This advantage decreases as the number of times that an ordering can be applied decreases. However, the simplified method has an advantage for almost all situations where its use extends over at least a few removals. The advantage is very great when a use extends over a large number of removals. Introduction of random effects prior to beginning a simulation can provide additional advantages for the simplified method.

Many of the kinds of removals are of an attrition nature. Frequently, attrition implies withdrawal of the persons or objects from further consideration in a simulation. However, this is not always the case. Sick or injured persons can recover and damaged objects can be repaired. Also, not all removals are attrition. For example, promotions of persons can change their type but are not ordinarily considered to represent attrition.

An additional motivation for the use of equally-likely removals is the ease with which an equally-likely removal can be performed. Thus an attempt is often made to construct the simulation model so that as many as possible of the random removals are of an equally-likely nature. The material of this paper provides additional motivation for constructing a simulation model in that manner.

The next section contains a statement of the simplified method. The final section furnishes verification that the simplified method has the equally-likely removal property for groups where it is applicable.

STATEMENT OF METHOD

As already mentioned, the simplified method can be applied in two ways. For the first way, an ordering is given at the first simulation step where the simplified method is to be used for the type of persons or objects considered. For the second way, the ordering is assigned prior to the beginning of the simulation. Both ways of application can be used in the same simulation.

The application approach is the same for the two ways. For a specified type, the first step at which the simplified method is to be used for this type is determined. Then, depending on the way, the totality of persons or objects that could possibly be of this type at that step is determined. An independent randomization is used to establish an ordering for the totality (all possible orderings equally likely). The value i is assigned to the person or object in the i-th position of the resulting ordering (say, starting from the left).

Now consider the removal procedure for any group that is eligible for this use of the simplified method. Suppose that n members are to be removed (from a group of size at least n). The numbers assigned for the specified type are examined for the group and the n members with the n smallest numbers are removed.

Now, consider in more detail what constitutes an eligible group for beginning, or for continuing, a use of the simplified method. First, for any eligible group, the group must be assured of occurring and of consisting only of members of the type considered. That is, the simulation model is such that this group is certain to exist and to have only members of this type. Second, except for an initial group, an eligible group is obtained from a previous eligible group by removals from this group and/or subdivision of this group. Here, one (and only one) removal is by application of the simplified method that is used for this type.

Third, an (eligible) initial group is obtained from the totality of persons or objects that could possibly be of the specified type at the step where this use of the simplified method first occurs. However, an initial group is not obtained from some previous eligible group by removals from this group or subdivision of this group. The requirements of existence and of membership entirely of the specified type are especially important for initial groups, since other groups are obtained in such a manner that these requirements are satisfied for them if they hold for every initial group. When there are two or more initial groups, they have no overlapping membership. Also, the removal from a given initial group by the simplified method does not necessarily happen at the first step where use of this simplified method occurs (one or more other initial groups could have this kind of removal at that step).

Some minor restrictions (not previously stated) are imposed on the nature of the simulation model. Namely, the simulation steps are defined so that a removal from a group occurs at an identifiable step. Also,

at most one removal is made from the members of a group at a step (including removals made other than by a use of the simplified method). However, removals could be made from two or more separate groups (non-overlapping membership) at a simulation step.

The only eligible groups considered are those from which a removal is made by the simplified method. Thus, for any eligible group, the first change that occurs is a removal by the simplified method. Other changes, if any, occur in following steps.

VERIFICATION

Consider use of the simplified method for an arbitrary but specified type of persons or objects. Determined first is the totality of persons or objects that could possibly be of the given type at the first step where this use of the simplified method occurs. The resulting totality is assigned an ordering according to the values of some unequal integers, where an integer is assigned to each member of the totality. This random assignment of the integers is done independently of all other aspects of the simulation and is such that all possible assignments are equally likely.

As the stepwise simulation continues, one or more initial groups are formed from the totality that can be of the specified type. Since determination of an initial group is independent of assignment of the integers, all possible orderings of the members of an initial group (on the basis of their integers) are equally likely. This is easily verified by taking

marginals, of the joint distribution of the integer assignment for the totality, with respect to the values received by the members of the totality that are not in this initial group.

Now consider any eligible group that is not an initial group. This group is obtained from a previous eligible group by a removal using the simplified method for this type and then subdivision and/or further removal of members, where any further removal is not by this use of the simplified method. The verification is completed by showing that all possible orderings are equally likely for the group considered if all possible orderings are equally likely for the previous group from which it is obtained. That is, the assignment of permissible integers within the group is such that all possible assignments that could occur are equally likely.

First, let us examine the (conditional) effect of the removal from the previous group by the simplified method for the type considered. This removal is such that the remaining group members are known to have assigned integers that are greater than all the integers for the members removed. Thus, the joint distribution for the values received by the remaining members is subject to this condition. The condition affects the class of integers that are permissible for assignment to the remaining group members. However, the members not removed are treated in a homogeneous fashion (and themselves are of a homogeneous nature with respect to the assignment of integers). Also, the members not removed were treated in a homogeneous fashion in any preceding removals by the use of the simplified method being considered. This implies that all possible assignments of

the permissible integers to the members not removed are equally likely. Of course, the permissible integers for the remaining members are a subclass of the permissible integers for the previous group being considered.

The eligible group considered is obtained from the remaining members of the previous group (after the removal through this use of the simplified method) by subdivision and/or further removal of members. Here any subdivision or further removal of members is made independently of the assignment of integers. Thus, since all orderings are equally likely for the remaining members of the previous group, all orderings are equally likely for the eligible group that results from this subdivision and/or further removal. This is readily verified in a manner like that used for initial groups. Here, the "totality" consists of the members of the previous group that are not removed by this use of the simplified method.

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)	2a. REPORT SECURITY CLASSIFICATION
SOUTHERN METHODIST UNIVERSITY	UNCLASSIFIED
	2b. GROUP UNCLASSIFIED

3. REPORT TITLE

"Simplified equally-likely removals in simulations"

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report

5. AUTHOR(S) (First name, middle initial, last name)

John E. Walsh

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
September 21, 1970	11	

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
N00014-68-A-0515	
b. PROJECT NO. NR 042-260	81
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

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11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Office of Naval Research

13. ABSTRACT

Random simulation models often involve many equally-likely removals of determined numbers from identified groups of persons or objects. That is, all ways of selecting those actually removed are equally likely. The time and expense for such removals can be substantially reduced (compared to a separate equally-likely selection for each removal) by a simplified method that frequently is usable in at least part of a model. This simplified method can be applied to groups which consist of members of the same type and which, except for the initial group(s), are obtained from previous such groups by removals and/or subdivisions. For an application to members of a given type, a random ordering is assigned to the persons or objects that can be of that type (all possible orderings equally likely). This assignment is independent of all other aspects of the simulation and is used only for equally-likely removals involving the specified type. All such removals are made on the basis of this ordering. Attritions constitute an important class of removals.