THEMIS SIGNAL ANALYSIS STATISTICS RESEARCH PROGRAM

# ESTIMATION AND CONFIDENCE INTERVALS FOR QUANTAL RESPONSE OR SENSITIVITY DATA

by

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Technical Report No. 66
Department of Statistics THEMIS Contract

February 25, 1970

Research sponsored by the Office of Naval Research
Contract N00014-68-A-0515
Project NR 042-260



DEPARTMENT OF STATISTICS
Southern Methodist University

# ESTIMATION AND CONFIDENCE INTERVALS FOR QUANTAL RESPONSE OR SENSITIVITY DATA

A Thesis Presented to the Faculty of the Graduate School

of

Southern Methodist University

in

Partial Fulfillment of the Requirements

for the degree of

Doctor of Philosophy

with a

Major in Statistics

by

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February 25, 1970

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Estimation and Confidence Intervals for Quantal Response or Sensitivity Data

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Doctor of Philosophy degree conferred May 24, 1970

Thesis completed April 5, 1970

The estimation of the quantal response curve parameters of a quantal bio-assay and the construction of confidence intervals for these parameters are the subjects considered in this paper. The method of minimum chi-square applied to quantal response data requires the minimization of the quantity

$$\sum_{i=1}^{t} \frac{n_i(p_i-P_i(\theta_1, \theta_2))^2}{P_i(\theta_1, \theta_2)Q_i(\theta_1, \theta_2)} \quad , \tag{1}$$

where $p_i$ and $P_i(\theta_1, \theta_2)$ are the observed and expected proportions respectively out of $n_i$ subjects at the $i^{th}$ level of stimulus. The expected response $P_i(\theta_1, \theta_2)$ is given by some distribution function

$$P_i(\theta_1, \theta_2) = F(\theta_1 + \theta_2 x_i) \quad , \tag{2}$$

where $x_i$ is the $i^{th}$ level of stimulus and $\theta_1$, $\theta_2$ are parameters to be estimated. An approximation for the chi-square quantity (1) is given by

$$\sum_{i=1}^{t} \frac{n_i(p_i-P_i(\theta_1, \theta_2))^2}{P_i(\theta_1, \theta_2)Q_i(\theta_1, \theta_2)} \doteq \sum_{i=1}^{t} \frac{n_i}{p_i q_i} [F'(y_i)]^2(Y_i(\theta_1, \theta_2) - y_i)^2 \quad , \tag{3}$$

where $Y_i(\theta_1, \theta_2) = \theta_1 + \theta_2 x_i$, $y_i = F^{-1}(p_i)$ and $F^{-1}(p_i)$ is the inverse of

the distribution function evaluated at $p_i$. The approximation (3) changes the estimation problem from a non-linear one to a weighted linear regression of $y_i$ on $Y_i(\theta_1, \theta_2)$. The particular choice of F(Y) used in this paper is the Burr distribution given by

$$F(X) = 1 - (1 + X^c)^{-k} \quad , \tag{4}$$

$$X > 0 \quad , \quad c, k \geq 1 \quad .$$

The minimization of the approximation (3) using the distribution (4) with c and k fixed is considered in detail. Burrit analysis is defined parallel to probit or logit analysis except the form of the underlying distribution is (4) instead of normal or logistic. Approximate and exact confidence regions are given for $(\theta_1, \theta_2)$ and for the $LD_\alpha$% point of the Burr distribution. The appendix contains tables for doing burrit analysis by desk calculator and a FORTRAN IV program is provided for doing the analysis by computer.

The more general estimation problem of estimating all four parameters of the Burr distribution by the method of maximum likelihood is also considered in this paper. This requires the solution of a set of four equations given by

$$\sum_{i=1}^{t} \frac{n_i\left(p_i - P_i(\Theta)\right)}{P_i(\Theta)Q_i(\Theta)} \frac{\partial P_i(\Theta)}{\partial \theta_j} = 0 \quad , \tag{5}$$

$$j = 1, 2, 3, 4$$

where $\Theta = (\theta_1, \theta_2, \theta_3 = c, \theta_4 = k)$. The solution to (5) is obtained by use of the Gauss-Newton method for solving a set of non-linear equations. Example problems are worked for both burrit analysis and the maximum likelihood solution given by (5). The Burr distribution when estimating all four parameters, offers a greater amount of flexibility in the shape of the quantal response curve than the normal or logistic which only have two parameters to be estimated.

## ACKNOWLEDGMENTS

I wish to express my deep appreciation to Professor Donald B. Owen for suggesting this dissertation topic and for his guidance and assistance throughout my research. I also wish to thank Associate Professor J. Wanzer Drane for his support and suggestions during the preparation of this paper. I also thank Associate Professor C. H. Kapadia for reading and commenting on the final draft. A special word of appreciation is due Professor Paul D. Minton for awarding to me a National Institutes of Health Fellowship which made my graduate training possible. Other faculty members of the Statistics Department who have contributed to my graduate training are Associate Professor Richard P. Bland and Associate Professor John T. Webster. I wish also to thank my wife, Mary Ann, and Mrs. Linda White whose combined efforts produced the typed manuscript.

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# INTRODUCTION

In some types of biological experiments a quantitative measurement of response under the action of a stimulus is not possible. In such situations it is observed whether or not the subject manifests some characteristic response. A typical type of quantal response often used is death. That is, one counts the number of subjects which die out of the total number which were subjected to a specified level of stimulus.

The general situation for quantal analysis is one in which $n_i$ subjects are exposed to some level of stimulus $x_i$ and $r_i$ out of the $n_i$ respond, where $i = 1, 2, \cdots, t$ and $t$ is the number of dose levels. The dose levels $x_i$ may be some transformation of the actual dose such as log dose. The reaction of each of the $n_i$ subjects is assumed to be independent of all the others. The probability that $r_i$ respond out of $n_i$ when subjected to the dose metameter $x_i$ is given by the binomial distribution,

$$e(r_i) = \binom{n_i}{r_i} P_i^{r_i} Q_i^{n_i - r_i} \quad , \quad r_i = 0, 1, \cdots, n_i$$

where

$$E(p_i) = P_i = \int_{-\infty}^{Y_i} f(u)\,du \quad , \tag{1}$$

and

$$p_i = r_i/n_i \quad .$$

The function $f(u)$ is a density function to be specified and $y_i = \theta_1 + \theta_2 x_i$ .

$P_i(\theta_1, \theta_2)$ then is the probability that a subject will respond when exposed to a dose level less than or equal to $x_i$ . This probability is given by specifying some absolutely continuous distribution function $F(Y_i)$, so that

$$P_i(\theta_1, \theta_2) = 1-Q_i(\theta_1, \theta_2) = F(Y_i) = F(\theta_1+\theta_2 x_i) \ .$$

Throughout the rest of this paper $P_i$ will be understood to be a function of $\theta_1$ and $\theta_2$ unless otherwise specified.

The log likelihood function for t doses is given by

$$L = \sum_{i=1}^{t} \left\{ \ln c_i + n_i(p_i \ln P_i + q_i \ln Q_i) \right\} \quad , \tag{2}$$

where $c_i = \binom{n_i}{r_i}$. Maximum likelihood estimates for the parameters $\theta_1, \theta_2$ are found as the solution to the equations

$$\frac{\partial L}{\partial \theta_j} = \sum_{i=1}^{t} \frac{n_i(p_i-P_i)}{P_i Q_i} \frac{\partial P_i}{\partial \theta_j} = 0 \quad . \tag{3}$$

These equations are nonlinear in the parameters $\theta_1, \theta_2$ and hence cannot be solved directly. Probit analysis provides for an iterative solution to (3) where F(Y) is taken as the normal distribution function. A detailed description of the probit solution is given by Finney [1952]. The solution makes use of a 'working deviate', $Y_i$ , defined as

$$y_i = Y_i + \frac{Q_i-q_i}{Z_i} \quad , \tag{4}$$

and weights

$$W_i = \frac{n_i Z_i^2}{P_i Q_i} \quad , \tag{5}$$

where $y_i = F^{-1}(p_i)$ and $z_i = \partial F(Y_i)/\partial Y_i$ . Each cycle consists of calculating $W_i$, $y_i$ from the previous cycle and then forming the weighted linear regression of $y_i$ on $x_i$ . The maximum likelihood solution (3) when $P_i$ is taken as the logistic distribution has been discussed by Berkson [1955 **A**]. A linear weighted regression solution has been given by Nelder [1968] which uses the Newton-Raphson iteration method and converges to the maximum likelihood solution.

Another method of estimating a quantal type bio-assay is that of minimum chi-square. Estimates found by this procedure minimize the statistic

$$\chi^2 = \sum_{i=1}^{t} \frac{(o_i - e_i)^2}{e_i} , \tag{6}$$

where $o_i$ and $e_i$ are the observed and expected frequencies of response at the $i^{th}$ dose level. At each dose level the observed and expected frequencies are given by the following table.

| | response | non-response | total |
|---|---|---|---|
| observed | pn | qn | n |
| expected | Pn | Qn | n |

The $\chi^2$ at each dose is given by

$$\chi^2 = \frac{(pn-Pn)^2}{Pn} + \frac{(qn-Qn)^2}{Qn} .$$

After combining terms and summing over all dose levels we have

$$\chi^2 = \sum_{i=1}^{t} \frac{n_i}{P_i Q_i} (p_i - P_i)^2 . \tag{7}$$

As in the case of the maximum likelihood solution (3) the normal equations for (7) are nonlinear in the parameters $\theta_1, \theta_2$ . An approximating $\chi^2$ for (7) has been given by Berkson ([1944], [1953], [1955 A]) where $P_i$ is taken to be the logistic distribution,

$$P_i = \left\{1+\exp[-(\theta_1+\theta_2 x_i)]\right\}^{-1} . \tag{8}$$

In equation (7), the quantity $(p_i-P_i)^2$ is replaced by

$$(p_i-P_i)^2 \doteq (P_i Q_i)(p_i q_i)(\ell_i-L_i)^2 , \tag{9}$$

where $\ell_i = \ln(p_i/q_i)$ and $L_i = \ln(P_i/Q_i) = \theta_1+\theta_2 x_i$ are the observed and estimated logits respectively. Using the approximation (9), equation (7) may be replaced by a 'minimum logit chi-square',

$$\chi^2 = \sum_{i=1}^{t} \frac{n_i}{P_i Q_i} (p_i-P_i)^2 \doteq \sum_{i=1}^{t} n_i p_i q_i (\ell_i-L_i)^2 . \tag{10}$$

The minimization of (10) may be viewed as a weighted linear regression of $\ell_i$ on $x_i$ with weights $w_i = n_i p_i q_i$ . The minimum $\chi^2$ method has been discussed by Berkson ([1955 B], [1957]) for normit analysis, where $P_i$ is taken to be the normal distribution function.

The estimation of $\theta_1, \theta_2$ may also be carried out by consideration of the following non-linear regression model,

$$p_i = P_i + \varepsilon_i , \tag{11}$$

with $E(\varepsilon_i) = 0$ $\mathrm{Var}(\varepsilon_i) = P_i Q_i/n_i$ . If the residual sum of squares $S(\Theta)$ is formed using weights $W_i = n_i/P_i Q_i$ we have

$$S(\Theta) = \sum_{i=1}^{t} \varepsilon_i^2 W_i = \sum_{i=1}^{t} W_i (p_i-P_i)^2 , \tag{12}$$

which is equivalent to (7). Further (Moore and Zeigler [1967]) if (12) is minimized without taking partials with respect to the weights, we obtain for the normal equations

$$\frac{\partial S(\Theta)}{\partial \theta_j} = \sum_{i=1}^{t} W_i (p_i - P_i) \frac{\partial P_i}{\partial \theta_j} = 0 \quad , \tag{13}$$

which is equivalent to the maximum likelihood normal equations (3).

In this paper we present a general family of distribution functions which will be used to estimate a quantal bio-assay. This distribution was originally proposed by Burr [1942] and is given by

$$F(x) = 1 - [(1+x^c)^k]^{-1} \quad x \geq 0 \quad , \quad c, k \geq 1 \tag{14}$$

with density function,

$$f(x) = \frac{ckx^{c-1}}{(1+x^c)^{k+1}} \tag{15}$$

which is unimodal at

$$x = \left(\frac{c-1}{ck+1}\right)^{1/c} \quad , \quad c > 1 \quad . \tag{16}$$

Burr and Cislak [1968] have shown that this system of distributions have curve shape characteristics which cover a large portion of the Pearson system types I, IV, and VI, including many transitional types, such as type III, or gamma distribution. The equation for the Pearson distributions is given in Kendall and Stuart [1963], page 148, as

$$\frac{df}{dx} = \frac{(x-a)f}{b_0 + b_1 x + b_2 x^2} \quad . \tag{17}$$

A generalization of (17) (K. and S., pg. 173) is

$$dF = F(1-F)\,g(x)\,dx \quad , \tag{18}$$

where $g(x)$ is some convenient function, which must be non-negative within the range of $x$ and for $0 \leq F \leq 1$ . The solution of (18) is

$$F(x) = [1 + \exp\{-G(x)\}]^{-1} \quad , \tag{19}$$

where

$$G(x) = \int_{-\infty}^{x} g(t)\,dt \quad .$$

If we choose $G(x) = \ln[(1+x^c)^k - 1]$ then one particular form of (19) is given by the Burr distribution (14).

Two distributions included within the range of approximation by the Burr are the normal and logistic. Thus, for appropriately chosen values of c and k both probit and logit analysis may be carried out using the Burr distribution. The analogous estimation procedure using the Burr distribution will be termed burrit analysis. The values of c and k which are used to approximate any given distribution are found by specifying the first four moments of the given distribution. In the case of the normal distribution these have been given by Burr [1967] as

$$c = 4.874 \quad , \quad k = 6.158 \quad . \tag{20}$$

# CHAPTER I

## APPROXIMATE MINIMUM CHI-SQUARE AND BURRIT ANALYSIS

In this chapter we present an approximation to the chi-square statistic (7) which will hold for any choice of absolutely continuous distribution function $F(Y) = P$ . This approximation will be used to estimate the parameters $\theta_1$ , $\theta_2$ and the particular solution will be given for the Burr distribution with c and k known.

We consider first the function defined by

$$G(Y) = \frac{n}{PQ} (p-P)^2 \quad . \tag{21}$$

The Taylors expansion of $G(Y)$ about the point $y = F^{-1}(p)$ to the first three terms is

$$G(Y) = G(y) + (Y-y)G'(y) + (Y-y)^2 G''(y)/2 \quad . \tag{22}$$

The first two terms of this expansion are zero since

$$G(y) = \frac{n\left(p-F(y)\right)^2}{F(y)[1-F(y)]} = 0$$

and

$$G'(y) = \left\{(p-P)\left[n(p-P)\ \frac{\partial(1/PQ)}{\partial Y} - \frac{2n}{PQ} \cdot \frac{\partial P}{\partial Y}\right]\right\}\Bigg|_{Y=y} = 0 \quad .$$

The third term of (22) is found from the expression for $G''(y)$,

$$G''(y) = \left\{\frac{\partial P}{\partial Y} \cdot \left[\frac{2n}{PQ}\frac{\partial P}{\partial Y} - 2n(p-P)\,\frac{\partial(1/PQ)}{\partial Y}\right]\right\}\Bigg|_{Y=y}$$

$$= 2n[F'(y)]^2/pq \quad .$$

The approximation for $G(Y)$ is now given by

$$G(Y) = \frac{n[F'(y)]^2}{pq}\,(Y-y)^2 \quad . \tag{23}$$

The desired approximation for the minimum chi-square (7) is

$$\sum_{i=1}^{t} \frac{n_i}{P_i Q_i}\,(p_i - P_i)^2 \doteq \sum_{i=1}^{t} \frac{n_i\left[F'(y_i)\right]^2}{p_i q_i}\,(Y-y)^2 \quad . \tag{24}$$

The approximate minimum chi-square (24) may be viewed as a weighted linear regression of y on x with weights

$$w = \frac{n[F'(y)]^2}{pq} \quad . \tag{25}$$

These estimates are the same as those resulting from the first iteration of the maximum likelihood solution if observed frequencies p, q are substituted for theoretical frequencies P, Q in equations (4) and (5). It is also of interest to note that if P is taken to be the logistic function (8), then we have

$$Y = \ln(P/Q) = L \qquad y = \ln(p/q) = \ell \quad ,$$

$$F'(y) = pq \qquad w = npq \quad , \tag{26}$$

so that (24) is equivalent to Berkson's minimum logit chi-square (10).

The approximate minimum $\chi^2$ (24) for burrit analysis becomes the minimum burrit $\chi^2$

$$\sum_{i=1}^{t} w_i (B_i - b_i)^2 \quad , \tag{32}$$

where $w_i$ is defined by (31). The estimates for $\theta_1$, $\theta_2$ are given by (28) with $y_i$, $w_i$ defined by (29) and (31) respectively. In the actual computing of (28) it is not necessary to carry along the quantity $c^2k^2$ in (31) since this is a constant and will cancel out of the expressions for $\hat{\theta}_1$, $\hat{\theta}_2$. A table is given in Appendix I which lists the weights and burrits for different values of $n_i$ and responses $r_i$. This table lists the weights without the quantity $c^2k^2$ and may be used along with a desk calculator to compute (28). A Fortran IV computer program which computes (28) and gives a residual analysis for the least square solution (32) is given in Appendix II. This program also evaluates the logit analysis solution for $\theta_1$, $\theta_2$ and a modified burrit analysis which will be discussed next.

Burrit Analysis Adjusted for Bias and Variance

As was noted above, (24) may be viewed as a weighted linear regression of y on x. However, in general it is not possible to define y and weights, w, such that

$$E(y) = Y$$

and

$$w = [\mathrm{var}(y)]^{-1} \quad .$$

For logit analysis this problem of bias and variance of the observed logit has been discussed by Gart and Zweifel [1967].

The mean and variance of y may be given in terms of the Taylor's

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## Burrit Analysis Adjusted for Bias and Variance

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and

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For logit analysis this problem of bias and variance of the observed logit has been discussed by Gart and Zweifel [1967].

The mean and variance of y may be given in terms of the Taylor's expansion of $y = F^{-1}(p)$ about the point P

$$y = y(p) = Y + \sum_{i=1}^{n} (p-P)^i y^{(i)}(P)/i! + R_{n+1} \quad , \tag{33}$$

where

$$R_{n+1} = \int_P^p \frac{(p-t)^n}{n!} \overset{(n+1)}{y(t)} \, dt \quad .$$

From the first five terms of (33) and the first four central moments of the binomial an approximation for bias(y) is

$$\xi = E(y)-Y \doteq PQy^{(2)}(P)/2n + PQ(Q-P)y^{(3)}(P)/6n^2$$
$$+ \{P^2Q^2/8n^2 + PQ(1-6PQ)/24n^3\}y^{(4)}(P) \quad . \tag{34}$$

Similarly the var(y) may be approximated from the first three terms of (33) as

$$\text{var}(y) \doteq PQy^{(1)}(P)^2/n + \{2P^2Q^2/n^2 + PQ(1-6PQ)/n^3\}$$
$$\cdot\, y^{(2)}(P)^2/4 + PQ(Q-P)y^{(1)}(P)y^{(2)}(P)/n^2 \quad . \tag{35}$$

The bias and variance of y may now be accounted for in the regression of y on x by regressing y* on x where

$$y^* = y - \xi \tag{36}$$

and the weights are given by

$$w^* = [\text{var}(y)]^{-1} \quad , \tag{37}$$

with var(y) defined by (35). To **estimate** $\theta_1$, $\theta_2$ the expected proportion P in (34) and (35) must be replaced by the observed proportion p. The where

$$\hat{P}_i = F(\hat{\theta}_1 + \hat{\theta}_2 x_i) \quad .$$

Data Set One

| | $\hat{\theta}_1$ | $\hat{\theta}_2$ | $\chi^2$ |
|---|---|---|---|
| Burrit | - 0.300915 | 0.519206 | 11.756911 |
| Burrit(Adj.) | - 0.296085 | 0.516622 | 11.773981 |
| Logit | -10.319148 | 5.655608 | 6.340549 |

Data Set Two

| | $\hat{\theta}_1$ | $\hat{\theta}_2$ | $\chi^2$ |
|---|---|---|---|
| Burrit | 0.518309 | 0.154945 | 2.766932 |
| Burrit(Adj.) | 0.525622 | 0.145185 | 2.997291 |
| Logit | - 1.363051 | 1.683767 | 1.735454 |

Data Set Three

| | $\hat{\theta}_1$ | $\hat{\theta}_2$ | $\chi^2$ |
|---|---|---|---|
| Burrit | 1.124552 | 0.209810 | 5.562210 |
| Burrit(Adj.) | 1.111075 | 0.203720 | 5.585712 |
| Logit | 4.841885 | 2.121552 | 5.588191 |

Data Set Four

| | $\hat{\theta}_1$ | $\hat{\theta}_2$ | $\chi^2$ |
|---|---|---|---|
| Burrit | 0.933090 | 0.151153 | 1.642129 |
| Burrit(Adj.) | 0.927188 | 0.147964 | 1.665029 |
| Logit | 2.889366 | 1.519151 | 1.643307 |

The above results all give a better fit for the simpler burrit analysis than for the adjusted burrit analysis. We feel that the additional amount of approximation required by the adjusted burrit analysis does more harm than good. For this reason it is recommended that burrit analysis be computed in the form given by (32). Also, unless one is sure of the choice for c and k, the values for $\hat{\theta}_1$, $\hat{\theta}_2$ will usually be used as starting values in a non-linear regression to estimate all four parameters of the Burr.

In this chapter we have discussed a general approximation for the $\chi^2$ statistic (7) and have given the specific approximation for the Burr. In Chapter II confidence intervals will be derived for $\theta_1$, $\theta_2$ and for $LD_\alpha$ point of the quantal response.

$$H_0: \theta_1 = \theta_{10} \quad \text{vs.} \quad H_1: \theta_1 \neq \theta_{10}$$
$$\theta_2 = \theta_{20} \qquad\qquad \theta_2 \neq \theta_{20} \tag{43}$$

A test of this hypothesis based on the likelihood ratio statistic is to reject $H_0$ at the $\alpha$ level of significance if

$$\lambda = \frac{L^*(\theta_{10}, \theta_{20})}{L^*(\hat{\theta}_1, \hat{\theta}_2)} < \lambda_0 \quad , \tag{44}$$

where

$$P[\lambda < \lambda_0] = \alpha \quad .$$

The function $L^*$ is the likelihood function for a quantal bio-assay with binomial response the log of which is given by (2). In the rest of this chapter the notation $\hat{\theta}_1$, $\hat{\theta}_2$ will refer specifically to the maximum likelihood estimates of $\theta_1$, $\theta_2$ respectively. The maximum likelihood estimates $\hat{\theta}_1$, $\hat{\theta}_2$ can be found by using the iterative solution given for probit analysis (4) and (5). The burrit analysis estimates can be used as initial values for the estimates $\hat{\theta}_1$, $\hat{\theta}_2$ .

The statistic $-2 \ln \lambda$ will be assymptotically distributed as $\chi^2$ with two degrees of freedom. Using this statistic $H_0$ is rejected at the $\alpha$ level of significance if

$$-2 \ln \lambda = -2\sum n_i p_i (\ln P_{i0} - \ln \hat{P}_i) - 2\sum n_i q_i (\ln Q_{i0} - \ln \hat{Q}_i) > \chi^2_\alpha \quad , \tag{45}$$

where

$$P\left[\chi^2_{(2)} > \chi^2_\alpha\right] = \alpha$$

and

$$P_{i0} = F(\theta_{10} + \theta_{20} x_i) \quad , \quad \hat{P}_i = F(\hat{\theta}_1 + \hat{\theta}_2 x_i) \quad .$$

An approximate $100(1-\alpha)\%$ confidence contour for $\theta_1, \theta_2$ is the set of all pairs of values $(\theta_{10}, \theta_{20})$ such that (45) is less than or equal to $\chi^2_\alpha$. The above procedure involves iterations to find $\hat{\theta}_1, \hat{\theta}_2$ and also some kind of search procedure for finding the set of $(\theta_{10}, \theta_{20})$ which make up the boundary of the confidence contour. Once the boundary is determined further searching would be necessary to obtain values of $(\theta_{10}, \theta_{20})$ which would give a confidence interval for (41).

## The Asymptotic Normal Distribution of $(\hat{\theta}_1, \hat{\theta}_2)$

The second confidence interval will be based on the assumption of a normal distribution for the maximum likelihood estimates $\hat{\theta}_1, \hat{\theta}_2$. Our assumption is that the vector $\hat{\underline{\theta}}' = (\hat{\theta}_1, \hat{\theta}_2)$ is distributed as

$$\hat{\underline{\theta}} \sim BVN(\underline{\theta}, V) \quad , \qquad (46)$$

where

$$\underline{\theta}' = (\theta_1, \theta_2) \quad , \quad V = R^{-1}$$

and

$$r_{ij} = -E\left[\frac{\partial^2}{\partial\theta_i \partial\theta_j} L(\theta_1, \theta_2)\right] \quad .$$

Now from (3) we have that

$$\frac{\partial L}{\partial \theta_i} = \sum \frac{n}{PQ} (p-P) \frac{\partial P}{\partial \theta_i} \quad ,$$

from which

$$\frac{\partial^2 L}{\partial\theta_i \partial\theta_j} = \sum \frac{n(p-P)}{PQ} \frac{\partial^2 P}{\partial\theta_i \partial\theta_j} + \sum \frac{n[p(2P-1)-P^2]}{P^2 Q^2} \frac{\partial P}{\partial\theta_i} \frac{\partial P}{\partial\theta_j} \quad ,$$

so that

$$r_{ij} = -E\left(\frac{\partial^2 L}{\partial\theta_i \partial\theta_j}\right) = \sum \frac{n}{PQ} \frac{\partial P}{\partial\theta_i} \frac{\partial P}{\partial\theta_j} \quad . \qquad (47)$$

$$P\left[(z-Y_\alpha)^2/\sigma^2 \leq Z_\alpha^2\right] = 1-\alpha \quad . \tag{52}$$

The upper and lower limits on $X_\alpha$ are given by the solution of

$$z^2 - 2zY_\alpha + Y_\alpha^2 = Z_\alpha^2(v_{11} + X_\alpha^2 v_{22} + 2X_\alpha v_{12}) \quad , \tag{53}$$

which can be written as a quadratic in $X_\alpha$

$$X_\alpha^2\left(\hat{\theta}_2^2 - Z_\alpha^2 v_{22}\right) + 2X_\alpha\left(\hat{\theta}_1\hat{\theta}_2 - Y_\alpha\hat{\theta}_2 - v_{12}Z_\alpha^2\right) + \left(\hat{\theta}_1^2 - 2Y_\alpha\hat{\theta}_1 + Y_\alpha^2 - v_{11}Z_\alpha^2\right) = 0 \quad . \tag{54}$$

An approximate $100(1-\alpha)\%$ confidence interval for $X_\alpha$ is

$$X_\alpha = \left(Y_\alpha\hat{\theta}_2 + v_{12}Z_\alpha^2 - \hat{\theta}_1\hat{\theta}_2\right)/\left(\hat{\theta}_2^2 - Z_\alpha^2 v_{22}\right)$$
$$\pm \left\{v_{12}Z_\alpha^2\left[v_{12}Z_\alpha^2 + 2\hat{\theta}_2(Y_\alpha - \hat{\theta}_1)\right] + v_{22}Z_\alpha^2\left[v_{11}Z_\alpha^2 - Y_\alpha^2 + \hat{\theta}_1(2Y_\alpha - \hat{\theta}_1)\right]\right.$$
$$\left. - \hat{\theta}_2^2 v_{11} Z_\alpha^2\right\}^{1/2}/\left(\hat{\theta}_2^2 - Z_\alpha^2 v_{22}\right) \quad . \tag{55}$$

The critical point $Z_\alpha$ in (55) may be approximately computed from a t-distribution with t-2 degrees of freedom since $\sigma^2$ is estimated as a function of $\hat{\theta}_1$ and $\hat{\theta}_2$ .

The above confidence interval involves finding $\hat{\theta}_1$, $\hat{\theta}_2$ by iteration as did the first interval. Additional computation and approximation is necessary since $v_{11}$, $v_{12}$, and $v_{22}$ must be estimated. However, a confidence interval for $X_\alpha$ can be found directly without searching the boundary of the $(\theta_1, \theta_2)$ confidence contour. The confidence region discussed next is exact and is based on a non-linear regression model with normal residuals. An approximate confidence region is also given which requires a minimal amount of computation.

## An Exact Confidence Region

The procedure which will be used is based on one given by Hartley [1964] for constructing exact confidence regions in non-linear estimation.

For a linear regression model of the form

$$Y = X\Theta + e \quad , \tag{56}$$

where $Y$, $e$ are $N \times 1$ matrices, $\Theta$ is a $m \times 1$ matrix of parameters, and $X$ is a $N \times m$ matrix of known input variables. An ellipsoidal confidence region for the $\Theta$ vector may be constructed by a decomposition of the error sum of squares $e'e$ into its regression and residual components

$$e'e = \text{reg}(e) + \text{res}(e) \quad . \tag{57}$$

The first component of (57) is defined as

$$\text{reg}(e) = (X'e)'(X'X)^{-1}(X'e) \quad , \tag{58}$$

and the second component by

$$\text{res}(e) = e'e - \text{reg}(e) \quad . \tag{59}$$

If the residual vector $e$ is distributed $N(0, \sigma^2)$ and the matrix $X$ is of rank $m$; then $\text{reg}(e)$ is distributed as $\sigma^2\chi^2$ with $m$ degrees of freedom and $\text{res}(e)$ is independently distributed as $\sigma^2\chi^2$ with $N-m$ degrees of freedom. An exact $100(1-\alpha)\%$ confidence region for $\Theta$ is given by

$$\text{reg}(Y-X\Theta)/\text{res}(Y-X\Theta) \le mF(\alpha;\, m, N-m)/(N-m) \quad , \tag{60}$$

where $F(\alpha;\, m, N-m)$ is the upper $100\alpha\%$ point of the F distribution with $m$ and $N-m$ degrees of freedom.

For the non-linear regression model given by

$$y_t = f(x_t, \Theta) + e_t \quad (t = 1, \cdots, N) \quad , \tag{61}$$

an exact $100(1-\alpha)\%$ confidence region for $\Theta$ is

$$\mathrm{reg}\big(Y-f(X,\Theta)\big)/\mathrm{res}\big(Y-f(X,\Theta)\big) \le mF(\alpha;\ m,\ N-m)/(N-m) \quad . \tag{62}$$

The confidence region (62) is exact for any decomposition of the sum of squares $e'e$ in which reg(e) is a quadratic form of rank m and res(e) is a quadratic form of rank N-m. That is, provided the decomposition is of the form

$$\begin{aligned} e'e &= e'[A + (I-A)]e \\ &= \mathrm{reg}(e) + \mathrm{res}(e) \quad , \end{aligned} \tag{63}$$

where A is an idempotent matrix of rank m.

A choice for the matrix A is made by considering a linear approximation of the form

$$f(x_t,\ \Theta) = \sum_{i=1}^{m} w_i(\Theta) u_{ti} \quad , \tag{64}$$

where the $w_i(\Theta)$ are a reparameterization of the $\Theta_i$ and the $u_{ti}$ form a $N \times m$ matrix U of rank m which does not depend on $\Theta$. The required quadratic forms are now given by

$$\begin{aligned} \mathrm{reg}(e) &= (U'e)'(U'U)^{-1}(U'e) \quad , \\ \mathrm{res}(e) &= e'e - \mathrm{reg}(e) \quad . \end{aligned} \tag{65}$$

The matrix A is now defined to be

$$A = U(U'U)^{-1}U' \quad , \tag{66}$$

which is idempotent and of rank m provided U is of rank m. The above decomposition of $e'e$ will give an exact confidence region for $\Theta$ for any choice of linear approximation (64). However, depending upon the choice for the approximation of $f(x_t,\ \Theta)$ the resulting confidence region may be useless. As a measure of the efficiency of the linear model in approximating the non-linear model we define

$$\frac{\mathrm{reg}(e^*)}{\mathrm{reg}(e)} = \frac{Y'Y - e^{*\prime}e^*_{min}}{Y'Y - e'e_{min}} \tag{67}$$

where e* is the residual vector associated with the linear model,

$$e^*_t = \left(y_t - \sum_{i=1}^{m} w_i(\Theta)u_{ti}\right) \quad . \tag{68}$$

It should be noted that situations can arise in which reg(e*) > reg(e), e.g., a poor non-linear model or a good non-linear model with high variance in the data. In such cases (67) may be greater than one, so that in general there is no upper bound.

The above technique for an exact confidence region will be used for the quantal response problem by considering the model

$$p_i = P_i + e_i \quad , \quad e_i \sim N(0,\ P_iQ_i/n_i) \quad . \tag{69}$$

A transformation, which gives a response variable with constant variance, yields the model

$$z_i = (n_iP_i/Q_i)^{1/2} + \delta_i \quad , \quad \delta_i \sim N(0,\ 1) \quad , \tag{70}$$

where

$$z_i = p_i/(P_iQ_i/n_i)^{1/2} \quad .$$

The non-linear function

$$H(Y) = (P/Q)^{1/2} \quad , \tag{71}$$

will be expanded to a first order approximation about the point $y = F^{-1}(p)$.

That is, a linear approximation for H(Y) is

$$H(Y) = H(y) + (Y-y)H'(y) \quad , \tag{72}$$

where

$$H(y) = (p/q)^{1/2} \quad , \quad H'(y) = \frac{q^{1/2}\ F'(y)}{2p^{1/2}q^2} \quad .$$

The approximation to (70) is now given by

$$z_i = (n_i p_i/q_i)^{1/2} + (Y_i - y_i)\left(\frac{n_i q_i}{p_i}\right)^{1/2} \frac{F'(y_i)}{2q_i^2} \quad , \tag{73}$$

which may be expressed as

$$z_i = u_{0i} + \theta_1 u_{1i} + \theta_2 u_{2i} + \delta_i^* \quad , \tag{74}$$

where

$$u_{0i} = (n_i p_i/q_i)^{1/2} - y_i u_{1i} \quad ,$$

$$u_{1i} = (n_i q_i/p_i)^{1/2}\ F'(y_i)/2q_i^2 \quad ,$$

$$u_{2i} = x_i u_{1i} \quad . \tag{75}$$

The constant term $u_{0i}$ may be removed by considering a model equivalent to the non-linear model (70)

$$z_i - u_{0i} = [(n_i P_i/Q_i)^{1/2} - u_{0i}] + \delta_i \quad , \quad \delta_i \sim N(0, 1) \quad . \tag{76}$$

The above model can now be approximated by

$$z_i - u_{0i} = \theta_1 u_{1i} + \theta_2 u_{2i} + \delta_i^* \quad .$$

The $t \times 2$ matrix U containing the transformed input variables $u_{1i}$, $u_{2i}$ is used to compute so that an exact $100(1-\alpha)\%$ confidence interval for $X_\alpha$ is

$$X_{min} \leq X_\alpha \leq X_{max} \quad . \tag{83}$$

The above confidence interval, though exact, requires a computer program which will search out enough values of $\Theta$ so that the contour boundary can be graphed and these values used to find $X_{max}$, $X_{min}$. An approximation to the above confidence interval and one which involves much less computing will now be given.

In Chapter I the approximation given by burrit analysis was

$$\sum_{i=1}^{t} \frac{n_i}{P_i Q_i} (p_i - P_i)^2 \doteq \sum_{i=1}^{t} w_i (b_i - B_i)^2 \quad , \tag{84}$$

where $w_i$, $b_i$, and $B_i$ are defined in (29), (30), and (31) respectively. From (84) we have the linear model

$$b_i = B_i + e_i \quad , \quad e_i \sim N(0,\ 1/w_i) \quad , \tag{85}$$

in place of the non-linear model (69). This model is equivalent to

$$z_i = B_i w_i^{1/2} + \delta_i \quad , \quad \delta_i \sim N(0,\ 1) \quad , \tag{86}$$

where $z_i = b_i w_i^{1/2}$ . The model (86) may also be expressed as

$$z_i = \theta_1 u_{1i} + \theta_2 u_{2i} + \delta_i \quad , \tag{87}$$

where

$$u_{1i} = w_i^{1/2} \quad , \quad u_{2i} = x_i w_i^{1/2} \quad .$$

The $t \times 2$ matrix U of input variables $u_{1i}$, $u_{2i}$ is used to form

$$\begin{aligned} reg(\delta) &= \delta' U (U'U)^{-1} U' \delta \\ &= \tilde{\Theta}' U' Z - 2\Theta' U' Z + \Theta' U' U \Theta \quad , \end{aligned} \tag{88}$$

and

$$\begin{aligned} \mathrm{res}(\delta) &= \delta'\delta - \mathrm{reg}(\delta) \\ &= Z'Z - \tilde{\Theta}'U'Z \quad , \end{aligned} \tag{89}$$

where $\tilde{\Theta} = (\tilde{\theta}_1, \tilde{\theta}_2)$ are computed from equation (28).

An approximate $100(1-\alpha)\%$ confidence region for $\Theta$ is

$$\frac{\mathrm{reg}(\delta)}{\mathrm{res}(\delta)} \leq 2F(\alpha;\ 2,\ t-2)/(t-2) = F_\alpha \quad , \tag{90}$$

which may also be written as

$$\Theta'U'U\Theta - 2\Theta'U'Z \leq F_\alpha Z'Z - \tilde{\Theta}'U'Z(F_\alpha+1) \ . \tag{91}$$

The terms needed for the region are

$$\begin{aligned} \tilde{\Theta}'U'Z &= \hat{\theta}_1\sum b_i w_i + \hat{\theta}_2\sum b_i x_i w_i \quad , \\ \Theta'U'Z &= \theta_1\sum b_i w_i + \theta_2\sum b_i x_i w_i \quad , \\ \Theta'U'U\Theta &= \theta_1^2\sum w_i + 2\theta_1\theta_2\sum x_i w_i + \theta_2^2\sum x_i^2 w_i \quad , \\ Z'Z &= \sum b_i^2 w_i \quad . \end{aligned} \tag{92}$$

The boundary of the confidence region is the set of $(\theta_1, \theta_2)$ which satisfy

$$s_1\theta_1^2 + s_2\theta_1\theta_2 + s_3\theta_2^2 + s_4\theta_1 + s_5\theta_2 = c \quad , \tag{93}$$

where

$$s_1 = \sum w_i \quad , \quad s_2 = 2\sum x_i w_i \quad , \quad s_3 = \sum x_i^2 w_i \quad ,$$

$$s_4 = -2\sum b_i w_i \quad , \quad s_5 = -2\sum b_i x_i w_i \quad ,$$

and

$$c = F_\alpha \sum b_i^2 w_i - (F_\alpha + 1)(\tilde{\theta}_1 \sum b_i w_i + \tilde{\theta}_2 \sum b_i x_i w_i) \quad . \tag{94}$$

The confidence contour (93) is the equation of an ellipse and may be plotted on the $\theta_1$, $\theta_2$ axis by the usual methods of analytical geometry. The following solution of (93) is for the purpose of constructing a confidence interval for $X_\alpha$. The contour equation (93) can be considered first as a quadratic in $\theta_1$

$$s_1\theta_1^2 + (s_2\theta_2 + s_4)\theta_1 + (s_3\theta_2^2 + s_5\theta_2 - c) = 0 \quad , \tag{95}$$

with solution

$$\theta_1 = -\frac{(s_2\theta_2+s_4)}{2s_1} \pm \frac{[(s_2\theta_2+s_4)^2 - 4s_1(s_3\theta_2^2+s_5\theta_2-c)]^{1/2}}{2s_1} \quad . \tag{96}$$

However, the above solution for $\theta_1$ will be real only if

$$\left(s_2^2 - 4s_1s_3\right)\theta_2^2 + (2s_2s_4 - 4s_1s_5)\theta_2 + \left(s_4^2 + 4s_1c\right) \geq 0 \quad . \tag{97}$$

The two values of $\theta_2$ for which equality holds are

$$\theta_2 = \frac{2s_1s_5-s_2s_4}{\left(s_2^2-4s_1s_3\right)} \pm \frac{\left[\left(2s_2s_4-4s_1s_5\right)^2 - 4\left(s_2^2-4s_1s_3\right)\left(s_4^2+4s_1c\right)\right]^{1/2}}{2\left(s_2^2-4s_1s_3\right)} \quad . \tag{98}$$

These conditions imply that values of $(\theta_1, \theta_2)$ which satisfy (93) may be found by first picking a $\theta_2$ from the interval

$$\frac{A-B}{D} \leq \theta_2 \leq \frac{A+B}{D} \quad , \tag{99}$$

where

$$A = 4s_1s_5 - 2s_2s_4 \quad ,$$

$$B = \left[(2s_2s_4 - 4s_1s_5)^2 - 4\left(s_2^2 - 4s_1s_3\right)\left(s_4^2 + 4s_1c\right)\right]^{1/2} \quad .$$

$$D = 2\left(s_2^2 - 4s_1s_3\right) \quad , \tag{100}$$

and then to determine $\theta_1$ from (96).

A confidence interval for $X_\alpha$ may be constructed from $(\theta_1, \theta_2)$ on the boundary defined by (93). The function $X_\alpha(\Theta)$ evaluated along the boundary is defined by

$$X_\alpha(\theta_2) = \frac{2s_1Y_\alpha + (s_2\theta_2+s_4) \pm [(s_2\theta_2+s_4)^2 - 4s_1(s_3\theta_2^2+s_3\theta_2-c)]^{1/2}}{2s_1\theta_2} \quad , \tag{101}$$

where $Y_\alpha$ is given in (50). An approximate $100(1-\alpha)\%$ confidence interval for $X_\alpha$ is

$$\inf_{\theta_2 \in \Phi} X_\alpha(\theta_2) \le X_\alpha \le \sup_{\theta_2 \in \Phi} X_\alpha(\theta_2) \quad , \tag{102}$$

where

$$\Phi = \left\{\theta_2 \;\middle|\; \frac{A-B}{D} \le \theta_2 \le \frac{A+B}{D}\right\} \quad . \tag{103}$$

In this chapter we have given several ways of constructing confidence contours for $\Theta$ and confidence intervals for $X_\alpha$. The exact confidence region given by (81) and (83) are felt to be optimum with regard to necessary assumptions and the exactness of the confidence level. However, these regions require a program which will search out the boundary values of $\Theta$. If ease of computation is the determining factor in one's choice of a region, then the approximate regions given by (93) and (102) are recommended.

# CHAPTER III

## COMPLETE ESTIMATION OF THE QUANTAL RESPONSE CURVE WHEN USING THE BURR DISTRIBUTION

In Chapter I, the quantal response curve using the Burr distribution was estimated by burrit analysis holding c and k fixed. In this chapter we give an iterative solution for the maximum likelihood estimates of all four parameters of the Burr distribution. That is, the vector $\Theta' = (\theta_1, \theta_2, \theta_3, \theta_4)$ will be estimated where the Burr distribution is given by

$$F(Y) = 1 - 1/(1 + Y^{\theta_3})^{\theta_4} \quad , \tag{104}$$

and

$$Y = \theta_1 + \theta_2 X \quad .$$

In estimating all four parameters the flexibility of the Burr is used which offers an advantage over specific choices of distributions such as logistic or normal. By doing this the observations themselves will select the correct distributional form, and it will be unnecessary to do so by assumption. Whenever a fixed choice for c and k is taken, then burrit analysis assumes a fixed distributional form. For the normal approximation of the Burr, burrit analysis is just another way of doing probit or normit analysis. However, even in this case there is an advantage in using the Burr in that the distribution function is a closed algebraic expression and hence also its inverse.

The normal equations for the maximum likelihood solution are

$$\frac{\partial L}{\partial \theta_i} = \sum \frac{n_j}{P_j Q_j} (p_j - P_j) \frac{\partial P_j}{\partial \theta_i}$$

$$= \sum w(p-P) \frac{\partial P}{\partial \theta_i} \quad , \quad i = 1, 2, 3, 4 \quad . \qquad (105)$$

These normal equations are equivalent to those for weighted non-linear least squares if partials are not taken with respect to the weights w. The normal equations are expanded in a first order Taylor's series about a vector of initial guesses $\Theta_0' = (\theta_{10}, \theta_{20}, \theta_{30}, \theta_{40})$. The initial guesses used for $\theta_1$, $\theta_2$ are those obtained from burrit analysis and $\theta_3$, $\theta_4$ are taken as values for the normal approximation. Expanding (105) about $\Theta_0$ yields

$$\frac{\partial L}{\partial \theta_i} + \delta_{\theta_1} \frac{\partial^2 L}{\partial \theta_i \partial \theta_1} + \delta_{\theta_2} \frac{\partial^2 L}{\partial \theta_i \partial \theta_2} + \delta_{\theta_3} \frac{\partial^2 L}{\partial \theta_i \partial \theta_3} + \delta_{\theta_4} \frac{\partial^2 L}{\partial \theta_i \partial \theta_4} = 0$$

$$i = 1, 2, 3, 4 \quad , \qquad (106)$$

where $\delta_{\theta_j} = (\theta_j - \theta_{j0})$ , $j = 1, 2, 3, 4$ and all partial derivatives are evaluated at $\Theta_0$ . This is a system of four equations which must be solved for the four $\delta_{\theta_j}$ . The normal equations may also be expressed as

$$G_{i1}\delta_{\theta_1} + G_{i2}\delta_{\theta_2} + G_{i3}\delta_{\theta_3} + G_{i4}\delta_{\theta_4} = -\frac{\partial L}{\partial \theta_i} \quad , \qquad (107)$$

where

$$G_{ij} = \frac{\partial^2 L}{\partial \theta_i \partial \theta_j} = \frac{\partial}{\partial \theta_j}\left[\sum w(p-P) \frac{\partial P}{\partial \theta_i}\right]$$

$$= \sum w(p-P)P_{ij} + \sum \frac{(p-P)n(2P-1)}{P^2Q^2} P_i P_j - \sum wP_i P_j \qquad (108)$$

$$= \sum w(p-P)P_{ij} + \sum \frac{w}{PQ} \{p(2P-1) - P^2\} P_i P_j \quad , \qquad (109)$$

and

$$P_{ij} = \frac{\partial^2 P}{\partial\theta_i \partial\theta_j} \quad , \quad P_i = \frac{\partial P}{\partial\theta_i} \quad .$$

The quantities $G_{ij}$ are found from (109) and are evaluated at $\Theta = \Theta_0$ . The solutions for $\delta_{\theta_j}$ after the $i^{th}$ iteration give corrections to $\theta_{j,i}$ by

$$\theta_{j,i} = \theta_{j,i-1} + \delta_{\theta_j} \qquad \begin{matrix} j = 1, 2, 3, 4 \\ i = 2, 3, \cdots \end{matrix} \quad . \qquad (110)$$

Iterations are continued until the process converges in terms of change in both error sum of squares and the $\delta_j$'s. A simplification of (109) may be used provided one is willing to substitute $E(G_{ij})$ for $G_{ij}$ and then solve the resulting normal equations. From (108) the $E(G_{ij})$ is

$$G^*_{ij} = E(G_{ij}) = - \sum wP_i P_j \quad , \qquad (111)$$

which eliminates the need for the second partials $P_{ij}$ .

The necessary first and second partials for the Burr are

$$P_1 = CK(V-1)V^{-(K+1)}/Y$$

$$P_2 = XP_1$$

$$P_3 = K(V-1)(\ln Y)V^{-(K+1)}$$

$$P_4 = (\ln V)V^{-K}$$

$$P_{11} = CKV^{-(K+2)}(V-1)\left\{C(K+1)-V(1+CK)\right\}/Y^2$$

$$P_{12} = XP_{11}$$

$$P_{13} = P_3 \cdot C[1-K(V-1)]/YV$$
$$P_{14} = P_1 \cdot (1-K \ln V)/K$$
$$P_{22} = XP_{12}$$
$$P_{21} = P_{12}$$
$$P_{23} = XP_{13}$$
$$P_{24} = XP_{14}$$
$$P_{31} = P_{13}+P_1/C$$
$$P_{32} = XP_{31}$$
$$P_{34} = P_3 \cdot (1-K \ln V)/K$$
$$P_{33} = P_3 \cdot (\ln Y)(1-K \cdot Y)/V$$
$$P_{41} = P_{14}$$
$$P_{42} = P_{24}$$
$$P_{43} = P_{34}$$
$$P_{44} = -P_4 \cdot \ln V \quad , \qquad (112)$$

where

$$Y = \theta_1 + \theta_2 X \quad , \quad \text{and} \quad V = (1 + Y^C) \quad .$$

The four sets of data given in Chapter I were used to estimate all four parameters of the Burr. The non-linear program used solved the normal equations (107) and computed the coefficient matrix by (111). The convergence of $\Theta$ was reached by use of a mixed algorithm which used both the Hartley and Marquardt algorithms. The results of the estimation are fiven in the following tables which also include the results of burrit analysis for comparison.

Data Set One

| | $\hat{\theta}_1$ | $\hat{\theta}_2$ | $\hat{\theta}_3$ | $\hat{\theta}_4$ | $\chi^2$ |
|---|---|---|---|---|---|
| Burrit | - 0.300915 | 0.519206 | 4.874000 | 6.158000 | 11.756911 |
| Logit | -10.319148 | 5.655608 | ---- | ---- | 6.340549 |
| Max.Lik. | - 0.126485 | 0.565127 | 7.763276 | 1.845379 | 6.808678 |

Data Set Two

| | $\hat{\theta}_1$ | $\hat{\theta}_2$ | $\hat{\theta}_3$ | $\hat{\theta}_4$ | $\chi^2$ |
|---|---|---|---|---|---|
| Burrit | 0.518309 | 0.154945 | 4.874000 | 6.158000 | 2.766932 |
| Logit | - 1.363051 | 1.683767 | ---- | ---- | 1.735454 |
| Max.Lik. | 0.586708 | 0.420008 | 3.672498 | 1.379782 | 0.778148 |

Data Set Three

| | $\hat{\theta}_1$ | $\hat{\theta}_2$ | $\hat{\theta}_3$ | $\hat{\theta}_4$ | $\chi^2$ |
|---|---|---|---|---|---|
| Burrit | 1.124552 | 0.209810 | 4.874000 | 6.158000 | 5.562210 |
| Logit | 4.841885 | 2.121552 | ---- | ---- | 5.588191 |
| Max.Lik. | 1.844753 | 0.310811 | 8.668435 | 0.540272 | 3.868488 |

Data Set Four

| | $\hat{\theta}_1$ | $\hat{\theta}_2$ | $\hat{\theta}_3$ | $\hat{\theta}_4$ | $\chi^2$ |
|---|---|---|---|---|---|
| Burrit | 0.933090 | 0.151153 | 4.874000 | 6.158000 | 1.642129 |
| Logit | 2.889366 | 1.519151 | ---- | ---- | 1.643307 |
| Max.Lik. | 1.167174 | 0.221299 | 4.047476 | 2.631725 | 1.536194 |

The above group of results, especially data set three, indicate the usefulness of the Burr in situations where the response curve is neither logistic nor normal. It should be pointed out that in all of these examples the chi-square for burrit analysis is non-significant, so that a further reduction by maximum likelihood does not represent a significantly better fit. We also note, as in data set one, that there are cases where the Burr does not offer any improvement over the logistic or normal. However, we feel that nothing is lost when using the Burr if all four parameters are estimated and a reduction in the lack of fit $\chi^2$ is likely.

# APPENDIX I

## TABLES OF BURRITS AND WEIGHTS

This appendix contains tables of burrits and weights to be used for burrit analysis when $c = 4.874$ and $k = 6.158$, the normal approximation. If $N$ is the number of subjects, then these tables contain burrits and weights for

$$N = 6, 7, 8, \cdots , 20$$

$$N = 25, 30, 35, \cdots , 95$$

$$N = 105, 125, 135, \cdots , 195 \quad .$$

For zero or all responding out of $N$ the response is taken to be $1/2N$ and $1 - 1/2N$ respectively.

| NUMBER | N= 6 | | N= 7 | | N= 8 | | N= 9 | | N= 10 | |
|---|---|---|---|---|---|---|---|---|---|---|
| RESPONDING | BURRIT | WEIGHT | BURRIT | WEIGHT | BURRIT | WEIGHT | BURRIT | WEIGHT | BURRIT | WEIGHT |
| 0 | .417920 | .074388 | .404294 | .079667 | .392923 | .084485 | .383208 | .088936 | .374752 | .093088 |
| 1 | .487190 | .107571 | .470477 | .115967 | .456661 | .123563 | .444936 | .130536 | .434787 | .137008 |
| 2 | .576161 | .146785 | .553885 | .161286 | .535926 | .174076 | .520951 | .185613 | .508158 | .196184 |
| 3 | .646300 | .162790 | .617141 | .184927 | .594535 | .203691 | .576161 | .220178 | .560742 | .235009 |
| 4 | .715281 | .156548 | .675252 | .190261 | .646300 | .217054 | .623688 | .239692 | .605199 | .259528 |
| 5 | .800337 | .119402 | .736762 | .174786 | .697423 | .213945 | .668787 | .245065 | .646300 | .271317 |
| 6 | .866399 | .080112 | .816204 | .128556 | .753968 | .190616 | .715281 | .234823 | .686993 | .270005 |
| 7 | | | .879576 | .084556 | .829337 | .136463 | .768253 | .204602 | .730168 | .253515 |
| 8 | | | | | .890652 | .088361 | .840508 | .143411 | .780425 | .217126 |
| 9 | | | | | | | .900186 | .091679 | .850205 | .149598 |
| 10 | | | | | | | | | .908543 | .094614 |

| NUMBER | N= 11 | | N= 12 | | N= 13 | | N= 14 | | N= 15 | |
|---|---|---|---|---|---|---|---|---|---|---|
| RESPONDING | BURRIT | WEIGHT | BURRIT | WEIGHT | BURRIT | WEIGHT | BURRIT | WEIGHT | BURRIT | WEIGHT |
| 0 | .367288 | .096989 | .360620 | .100676 | .354606 | .104179 | .349136 | .107520 | .344128 | .110718 |
| 1 | .425864 | .143067 | .417920 | .148776 | .410775 | .154185 | .404294 | .159334 | .398370 | .164253 |
| 2 | .497023 | .205981 | .487190 | .215142 | .478404 | .223767 | .470477 | .231935 | .463266 | .239705 |
| 3 | .547500 | .248568 | .535926 | .261114 | .525668 | .272829 | .516475 | .283849 | .508153 | .294275 |
| 4 | .589606 | .277328 | .576161 | .293571 | .564368 | .308577 | .553885 | .322571 | .544466 | .335721 |
| 5 | .627829 | .294276 | .612192 | .314840 | .598664 | .333576 | .586764 | .350862 | .576161 | .366964 |
| 6 | .664688 | .299668 | .646300 | .325580 | .630686 | .348761 | .617141 | .369854 | .605199 | .389292 |
| 7 | .702226 | .292531 | .680125 | .325436 | .661854 | .354165 | .646300 | .379844 | .632777 | .403188 |
| 8 | .742890 | .270447 | .715281 | .313097 | .693390 | .349095 | .675252 | .380522 | .659779 | .408599 |
| 9 | .791003 | .228459 | .753968 | .285924 | .726675 | .332028 | .704992 | .370987 | .686993 | .405007 |
| 10 | .858758 | .155167 | .800337 | .238805 | .763758 | .300178 | .736762 | .349573 | .715281 | .391371 |
| 11 | .915972 | .097240 | .866399 | .160225 | .808677 | .248315 | .772515 | .313386 | .745797 | .365923 |
| 12 | | | .922653 | .099610 | .873296 | .164852 | .816204 | .257111 | .780425 | .325689 |
| 13 | | | | | .928718 | .101768 | .879576 | .169112 | .823055 | .265289 |
| 14 | | | | | | | .934269 | .103746 | .885336 | .173055 |
| 15 | | | | | | | | | .939382 | .105569 |

| NUMBER | N= 16 | | N= 17 | | N= 18 | | N= 19 | | N= 20 | |
|---|---|---|---|---|---|---|---|---|---|---|
| RESPONDING | BURRIT | WEIGHT | BURRIT | WEIGHT | BURRIT | WEIGHT | BURRIT | WEIGHT | BURRIT | WEIGHT |
| 0 | .339515 | .113789 | .335243 | .116745 | .331268 | .119596 | .327555 | .122355 | .324074 | .125025 |
| 1 | .392923 | .168969 | .387887 | .173502 | .383208 | .177872 | .378842 | .182092 | .374752 | .186176 |
| 2 | .456661 | .247126 | .450574 | .254237 | .444936 | .261072 | .439689 | .267657 | .434787 | .274016 |
| 3 | .500575 | .304188 | .493615 | .313649 | .487190 | .322713 | .481229 | .331419 | .475674 | .339806 |
| 4 | .535926 | .348151 | .528124 | .359960 | .520951 | .371226 | .514319 | .382011 | .508158 | .392367 |
| 5 | .566612 | .382078 | .557939 | .396354 | .550001 | .409906 | .542693 | .422826 | .535926 | .435189 |
| 6 | .594535 | .407381 | .584914 | .424346 | .576161 | .440356 | .568139 | .455545 | .560742 | .470017 |
| 7 | .620830 | .424681 | .610144 | .444666 | .600487 | .463394 | .591688 | .481057 | .583614 | .497803 |
| 8 | .646300 | .434107 | .634373 | .457577 | .623688 | .479384 | .614020 | .499807 | .605199 | .519056 |
| 9 | .671612 | .435395 | .658192 | .462992 | .646300 | .488371 | .635632 | .511939 | .625968 | .533999 |
| 10 | .697423 | .427889 | .682139 | .460511 | .668787 | .490129 | .656940 | .517358 | .646300 | .542634 |
| 11 | .724510 | .410450 | .706790 | .449380 | .691607 | .484163 | .678327 | .515742 | .666531 | .544768 |
| 12 | .753968 | .381232 | .732866 | .428385 | .715281 | .469645 | .700198 | .506524 | .686993 | .540009 |
| 13 | .787630 | .337200 | .761416 | .395625 | .740491 | .445309 | .723037 | .488824 | .708053 | .527737 |
| 14 | .829337 | .272925 | .794239 | .348014 | .768253 | .409204 | .747495 | .461330 | .730168 | .507030 |
| 15 | .890652 | .176722 | .835132 | .280085 | .800337 | .358207 | .774566 | .422055 | .753968 | .476540 |
| 16 | .944120 | .107257 | .895586 | .180148 | .840508 | .286821 | .805995 | .367844 | .780425 | .434251 |
| 17 | | | .948533 | .108828 | .900186 | .183359 | .845517 | .293179 | .811269 | .376982 |
| 18 | | | | | .952661 | .110296 | .904494 | .186379 | .850205 | .299196 |
| 19 | | | | | | | .956537 | .111672 | .908543 | .189228 |
| 20 | | | | | | | | | .960190 | .112967 |

| NUMBER RESPONDING | N= 25 BURRIT | N= 25 WEIGHT | N= 30 BURRIT | N= 30 WEIGHT | N= 35 BURRIT | N= 35 WEIGHT | N= 40 BURRIT | N= 40 WEIGHT | N= 45 BURRIT | N= 45 WEIGHT |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | .309383 | .137297 | .297904 | .148165 | .288546 | .157995 | .280688 | .167014 | .273940 | .175384 |
| 1 | .357538 | .204899 | .344128 | .221437 | .333220 | .236366 | .324074 | .250051 | .316229 | .262734 |
| 2 | .414257 | .303030 | .398370 | .328506 | .385505 | .351414 | .374752 | .372351 | .365552 | .391718 |
| 3 | .452550 | .377849 | .434787 | .411025 | .420471 | .440724 | .408547 | .467785 | .398370 | .492759 |
| 4 | .482679 | .439032 | .463266 | .479409 | .447703 | .515375 | .434787 | .548033 | .423794 | .578098 |
| 5 | .508158 | .490459 | .487190 | .537854 | .470477 | .579837 | .456661 | .617814 | .444936 | .652680 |
| 6 | .530650 | .534132 | .508158 | .588551 | .490341 | .636456 | .475674 | .679611 | .463265 | .719114 |
| 7 | .551095 | .571215 | .527064 | .632802 | .508158 | .686642 | .492666 | .734924 | .479603 | .778976 |
| 8 | .570082 | .602425 | .544466 | .671442 | .524465 | .731313 | .508158 | .784734 | .494454 | .833305 |
| 9 | .588017 | .628206 | .560742 | .705025 | .539623 | .771095 | .522496 | .829723 | .508158 | .882826 |
| 10 | .605199 | .648820 | .576161 | .733927 | .553885 | .806428 | .535926 | .870378 | .520951 | .928064 |
| 11 | .621860 | .664391 | .590926 | .758395 | .567442 | .837630 | .548629 | .907060 | .533010 | .969411 |
| 12 | .638198 | .674931 | .605199 | .778584 | .580438 | .864922 | .560742 | .940034 | .544466 | 1.007163 |
| 13 | .654367 | .680356 | .619111 | .794573 | .592992 | .888462 | .572374 | .969500 | .555423 | 1.041550 |
| 14 | .670594 | .680479 | .632777 | .806376 | .605199 | .908348 | .583614 | .995605 | .565966 | 1.072748 |
| 15 | .686993 | .675011 | .646300 | .813951 | .617141 | .924636 | .594535 | 1.018453 | .576161 | 1.100891 |
| 16 | .703773 | .663540 | .659779 | .817198 | .628891 | .937338 | .605199 | 1.038112 | .586066 | 1.126081 |
| 17 | .721161 | .645500 | .673309 | .815959 | .640516 | .946432 | .615660 | 1.054623 | .595731 | 1.148393 |
| 18 | .739442 | .620128 | .686993 | .810014 | .652077 | .951860 | .625968 | 1.067999 | .605199 | 1.167876 |
| 19 | .759006 | .586384 | .700941 | .799068 | .663635 | .953528 | .636167 | 1.078225 | .614506 | 1.184562 |
| 20 | .780425 | .542814 | .715281 | .782742 | .675252 | .951304 | .646300 | 1.085268 | .623688 | 1.198460 |
| 21 | .804619 | .487312 | .730168 | .760545 | .686993 | .945016 | .656408 | 1.089066 | .632777 | 1.209564 |
| 22 | .833250 | .416622 | .745797 | .731846 | .698928 | .934443 | .666531 | 1.089535 | .641802 | 1.217851 |
| 23 | .869932 | .325176 | .762428 | .695822 | .711136 | .919312 | .676712 | 1.086565 | .650793 | 1.223281 |
| 24 | .925756 | .201428 | .780425 | .651377 | .723710 | .899283 | .686993 | 1.080018 | .659779 | 1.225796 |
| 25 | .975820 | .118465 | .800337 | .597011 | .736762 | .873932 | .697423 | 1.069724 | .668787 | 1.225324 |
| 26 | | | .823055 | .530577 | .750433 | .842733 | .708053 | 1.055474 | .677848 | 1.221770 |
| 27 | | | .850205 | .448794 | .764905 | .805021 | .718945 | 1.037020 | .686993 | 1.215021 |
| 28 | | | .885336 | .346109 | .780425 | .759940 | .730168 | 1.014060 | .696255 | 1.204941 |
| 29 | | | .939382 | .211137 | .797347 | .706367 | .741808 | .986229 | .705671 | 1.191369 |
| 30 | | | .988302 | .122791 | .816204 | .642778 | .753968 | .953081 | .715281 | 1.174113 |
| 31 | | | | | .837869 | .567007 | .766782 | .914067 | .725134 | 1.152948 |
| 32 | | | | | .863942 | .475771 | .780425 | .868503 | .735284 | 1.127606 |
| 33 | | | | | .897925 | .363557 | .795139 | .815515 | .745797 | 1.097769 |
| 34 | | | | | .950630 | .219149 | .811269 | .753964 | .756753 | 1.063057 |
| 35 | | | | | .998672 | .126327 | .829337 | .682313 | .768253 | 1.023010 |
| 36 | | | | | | | .850205 | .598392 | .780425 | .977066 |
| 37 | | | | | | | .875452 | .498929 | .793442 | .924527 |
| 38 | | | | | | | .908543 | .378456 | .807539 | .864509 |
| 39 | | | | | | | .960190 | .225933 | .823055 | .795866 |
| 40 | | | | | | | 1.007531 | .129298 | .840508 | .717054 |
| 41 | | | | | | | | | .860747 | .625905 |
| 42 | | | | | | | | | .885336 | .519164 |
| 43 | | | | | | | | | .917706 | .391417 |
| 44 | | | | | | | | | .968491 | .231792 |
| 45 | | | | | | | | | 1.015256 | .131847 |

| NUMBER | N= 50 | | N= 55 | | N= 60 | | N= 65 | | N= 70 | |
|---|---|---|---|---|---|---|---|---|---|---|
| RESPONDING | BURRIT | WEIGHT | BURRIT | WEIGHT | BURRIT | WEIGHT | BURRIT | WEIGHT | BURRIT | WEIGHT |
| 0 | .268046 | .183215 | .262827 | .190593 | .258153 | .197585 | .253928 | .204236 | .250080 | .210586 |
| 1 | .309383 | .274594 | .303325 | .285759 | .297904 | .296330 | .293006 | .306385 | .288546 | .315990 |
| 2 | .357538 | .409798 | .350458 | .426797 | .344128 | .442873 | .338417 | .458153 | .333220 | .472733 |
| 3 | .389524 | .516031 | .381719 | .537883 | .374752 | .558527 | .368472 | .578126 | .362764 | .596815 |
| 4 | .414257 | .606061 | .405858 | .632276 | .398370 | .657012 | .391628 | .680475 | .385505 | .702827 |
| 5 | .434787 | .685041 | .425864 | .715334 | .417920 | .743879 | .410775 | .770924 | .404294 | .796668 |
| 6 | .452550 | .755697 | .443146 | .789881 | .434787 | .822049 | .427277 | .852496 | .420471 | .881449 |
| 7 | .468349 | .819675 | .458492 | .857635 | .449743 | .893304 | .441893 | .927025 | .434787 | .959057 |
| 8 | .482679 | .878064 | .472387 | .919728 | .463266 | .958818 | .455094 | .995725 | .447703 | 1.030749 |
| 9 | .495872 | .931626 | .485155 | .976958 | .475674 | 1.019417 | .467191 | 1.059453 | .459528 | 1.097404 |
| 10 | .508158 | .980918 | .497023 | 1.029904 | .487190 | 1.075708 | .478404 | 1.118835 | .470477 | 1.159673 |
| 11 | .519708 | 1.026357 | .508158 | 1.079010 | .497977 | 1.128150 | .488894 | 1.174351 | .480708 | 1.218046 |
| 12 | .530650 | 1.068264 | .518684 | 1.124614 | .508158 | 1.177101 | .498780 | 1.226373 | .490341 | 1.272911 |
| 13 | .541086 | 1.106889 | .528701 | 1.166985 | .517828 | 1.222844 | .508158 | 1.275193 | .499466 | 1.324575 |
| 14 | .551095 | 1.142430 | .538285 | 1.206335 | .527064 | 1.265605 | .517100 | 1.321053 | .508158 | 1.373285 |
| 15 | .560742 | 1.175042 | .547500 | 1.242838 | .535926 | 1.305568 | .525668 | 1.364147 | .516475 | 1.419246 |
| 16 | .570082 | 1.204850 | .556398 | 1.276630 | .544466 | 1.342884 | .533911 | 1.404634 | .524465 | 1.462627 |
| 17 | .579161 | 1.231949 | .565022 | 1.307826 | .552726 | 1.377678 | .541870 | 1.442649 | .532170 | 1.503568 |
| 18 | .588017 | 1.256413 | .573411 | 1.336515 | .560742 | 1.410051 | .549580 | 1.478303 | .539623 | 1.542190 |
| 19 | .596686 | 1.278297 | .581596 | 1.362768 | .568545 | 1.440087 | .557071 | 1.511689 | .546853 | 1.578591 |
| 20 | .605199 | 1.297640 | .589606 | 1.386641 | .576161 | 1.467854 | .564368 | 1.542883 | .553885 | 1.612857 |
| 21 | .613582 | 1.314465 | .597466 | 1.408177 | .583614 | 1.493408 | .571495 | 1.571952 | .560742 | 1.645059 |
| 22 | .621860 | 1.328781 | .605199 | 1.427404 | .590926 | 1.516791 | .578470 | 1.598947 | .567442 | 1.675259 |
| 23 | .630058 | 1.340586 | .612824 | 1.444343 | .598115 | 1.538036 | .585314 | 1.623911 | .574002 | 1.703508 |
| 24 | .638198 | 1.349863 | .620362 | 1.459001 | .605199 | 1.557168 | .592040 | 1.646881 | .580438 | 1.729845 |
| 25 | .646300 | 1.356585 | .627829 | 1.471378 | .612192 | 1.574202 | .598664 | 1.667881 | .586764 | 1.754308 |
| 26 | .654387 | 1.360712 | .635243 | 1.481462 | .619111 | 1.589146 | .605199 | 1.686932 | .592992 | 1.776924 |
| 27 | .662477 | 1.362192 | .642621 | 1.489233 | .625968 | 1.601998 | .611657 | 1.704047 | .599133 | 1.797715 |
| 28 | .670594 | 1.360959 | .649977 | 1.494664 | .632777 | 1.612752 | .618051 | 1.719231 | .605199 | 1.816697 |
| 29 | .678758 | 1.356934 | .657327 | 1.497715 | .639550 | 1.621394 | .624390 | 1.732486 | .611198 | 1.833880 |
| 30 | .686993 | 1.350023 | .664688 | 1.498340 | .646300 | 1.627902 | .630686 | 1.743806 | .617141 | 1.849271 |
| 31 | .695322 | 1.340114 | .672074 | 1.496481 | .653039 | 1.632248 | .636949 | 1.753182 | .623036 | 1.862871 |
| 32 | .703773 | 1.327079 | .679504 | 1.492069 | .659779 | 1.634395 | .643187 | 1.760596 | .628891 | 1.874675 |
| 33 | .712375 | 1.310766 | .686993 | 1.485025 | .666531 | 1.634303 | .649411 | 1.766026 | .634715 | 1.884677 |
| 34 | .721161 | 1.290999 | .694560 | 1.475257 | .673309 | 1.631918 | .655631 | 1.769447 | .640516 | 1.892864 |
| 35 | .730168 | 1.267575 | .702226 | 1.462657 | .680125 | 1.627182 | .661854 | 1.770825 | .646300 | 1.899219 |
| 36 | .739442 | 1.240256 | .710012 | 1.447105 | .686993 | 1.620027 | .668093 | 1.770121 | .652077 | 1.903720 |
| 37 | .749033 | 1.208763 | .717942 | 1.428459 | .693926 | 1.610375 | .674355 | 1.767289 | .657852 | 1.906343 |
| 38 | .759006 | 1.172767 | .726044 | 1.406559 | .700941 | 1.598137 | .680651 | 1.762279 | .663635 | 1.907056 |
| 39 | .769437 | 1.131879 | .734348 | 1.381222 | .708053 | 1.583211 | .686993 | 1.755030 | .669432 | 1.905825 |
| 40 | .780425 | 1.085628 | .742890 | 1.352233 | .715281 | 1.565484 | .693390 | 1.745476 | .675252 | 1.902609 |
| 41 | .792096 | 1.033445 | .751712 | 1.319348 | .722646 | 1.544826 | .699856 | 1.733541 | .681103 | 1.897362 |
| 42 | .804619 | .974624 | .760866 | 1.282281 | .730168 | 1.521090 | .706402 | 1.719141 | .686993 | 1.890032 |
| 43 | .818225 | .908274 | .770412 | 1.240694 | .737876 | 1.494109 | .713044 | 1.702181 | .692931 | 1.880562 |
| 44 | .833250 | .833244 | .780425 | 1.194191 | .745797 | 1.463692 | .719796 | 1.682555 | .698928 | 1.868887 |
| 45 | .850205 | .747990 | .791003 | 1.142297 | .753968 | 1.429621 | .726675 | 1.660141 | .704992 | 1.854934 |
| 46 | .869932 | .650351 | .802269 | 1.084435 | .762428 | 1.391643 | .733701 | 1.634806 | .711136 | 1.838625 |
| 47 | .893980 | .537093 | .814390 | 1.019894 | .771227 | 1.349468 | .740895 | 1.606397 | .717371 | 1.819868 |
| 48 | .925756 | .402855 | .827595 | .947781 | .780425 | 1.302754 | .748283 | 1.574743 | .723710 | 1.798565 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 49 | .975820 | .236929 | .842216 | .866942 | .790097 | 1.251099 | .755893 | 1.539648 | .730168 | 1.774605 |
| 50 | 1.022099 | .134068 | .858758 | .775837 | .800337 | 1.194023 | .763758 | 1.500889 | .736762 | 1.747864 |
| 51 | | | .878058 | .672312 | .811269 | 1.130946 | .771920 | 1.458211 | .743510 | 1.718203 |
| 52 | | | .901653 | .553157 | .823055 | 1.061154 | .780425 | 1.411317 | .750433 | 1.685467 |
| 53 | | | .932925 | .413068 | .835924 | .983753 | .789334 | 1.359863 | .757555 | 1.649479 |
| 54 | | | .982376 | .241490 | .850205 | .897588 | .798720 | 1.303443 | .764905 | 1.610042 |
| 55 | | | 1.028239 | .136030 | .866399 | .801124 | .808677 | 1.241575 | .772515 | 1.566928 |
| 56 | | | | | .885336 | .692219 | .819325 | 1.173676 | .780425 | 1.519880 |
| 57 | | | | | .908543 | .567685 | .830829 | 1.099031 | .788683 | 1.468598 |
| 58 | | | | | .939382 | .422275 | .843412 | 1.016744 | .797347 | 1.412734 |
| 59 | | | | | .988302 | .245581 | .857401 | .925663 | .806490 | 1.351882 |
| 60 | | | | | 1.033803 | .137782 | .873296 | .824259 | .816204 | 1.285555 |
| 61 | | | | | | | .891919 | .710400 | .826610 | 1.213171 |
| 62 | | | | | | | .914790 | .580926 | .837869 | 1.134013 |
| 63 | | | | | | | .945252 | .430642 | .850205 | 1.047187 |
| 64 | | | | | | | .993707 | .249282 | .863942 | .951543 |
| 65 | | | | | | | 1.038890 | .139359 | .879576 | .845558 |
| 66 | | | | | | | | | .897925 | .727113 |
| 67 | | | | | | | | | .920500 | .593074 |
| 68 | | | | | | | | | .950630 | .438298 |
| 69 | | | | | | | | | .998672 | .252654 |
| 70 | | | | | | | | | 1.043572 | .140790 |

| NUMBER | N= 75 | | N= 80 | | N= 85 | | N= 90 | | N= 95 | |
|---|---|---|---|---|---|---|---|---|---|---|
| RESPONDING | BURRIT | WEIGHT | BURRIT | WEIGHT | BURRIT | WEIGHT | BURRIT | WEIGHT | BURRIT | WEIGHT |
| 0 | .246551 | .216678 | .243296 | .222527 | .240278 | .228167 | .237468 | .233616 | .234839 | .238887 |
| 1 | .284458 | .325188 | .280688 | .334029 | .277194 | .342547 | .273940 | .350768 | .270899 | .358725 |
| 2 | .328460 | .486692 | .324074 | .500101 | .320011 | .513008 | .316229 | .525468 | .312696 | .537515 |
| 3 | .357538 | .614696 | .352726 | .631859 | .348272 | .648378 | .344128 | .664310 | .340259 | .679711 |
| 4 | .379906 | .724201 | .374752 | .744702 | .369985 | .764423 | .365552 | .783436 | .361415 | .801810 |
| 5 | .398370 | .821265 | .392923 | .844845 | .387887 | .867512 | .383208 | .889360 | .378842 | .910458 |
| 6 | .414257 | .909091 | .408547 | .935570 | .403270 | .961011 | .398370 | .985518 | .393801 | 1.009177 |
| 7 | .428304 | .989616 | .422351 | 1.018869 | .416854 | 1.046960 | .411753 | 1.074004 | .406998 | 1.100099 |
| 8 | .440967 | 1.064133 | .434787 | 1.096066 | .429083 | 1.126710 | .423794 | 1.156195 | .418866 | 1.184634 |
| 9 | .452550 | 1.133546 | .446153 | 1.168091 | .440254 | 1.201217 | .434787 | 1.233074 | .429696 | 1.263784 |
| 10 | .463266 | 1.198523 | .456661 | 1.235628 | .450574 | 1.271185 | .444936 | 1.305359 | .439689 | 1.338286 |
| 11 | .473270 | 1.259573 | .466463 | 1.299202 | .460194 | 1.337151 | .454392 | 1.373600 | .448995 | 1.408701 |
| 12 | .482679 | 1.317096 | .475674 | 1.359222 | .469229 | 1.399535 | .463266 | 1.438227 | .457724 | 1.475467 |
| 13 | .491585 | 1.371406 | .484385 | 1.416015 | .477766 | 1.458669 | .471647 | 1.499584 | .465963 | 1.538938 |
| 14 | .500058 | 1.422763 | .492666 | 1.469849 | .485876 | 1.514832 | .479603 | 1.557952 | .473779 | 1.599401 |
| 15 | .508158 | 1.471377 | .500575 | 1.520938 | .493615 | 1.568247 | .487190 | 1.613563 | .481229 | 1.657097 |
| 16 | .515931 | 1.517426 | .508158 | 1.569469 | .501030 | 1.619104 | .494454 | 1.666610 | .488358 | 1.712221 |
| 17 | .523417 | 1.561057 | .515454 | 1.615594 | .508158 | 1.667560 | .501433 | 1.717261 | .495202 | 1.764944 |
| 18 | .530650 | 1.602396 | .522496 | 1.659445 | .515032 | 1.713755 | .508158 | 1.765652 | .501793 | 1.815411 |
| 19 | .537658 | 1.641548 | .529312 | 1.701135 | .521680 | 1.757802 | .514656 | 1.811908 | .508158 | 1.863744 |
| 20 | .544466 | 1.678605 | .535926 | 1.740757 | .528124 | 1.799801 | .520951 | 1.856128 | .514319 | 1.910053 |
| 21 | .551095 | 1.713645 | .542359 | 1.778394 | .534387 | 1.839841 | .527064 | 1.898407 | .520298 | 1.954432 |
| 22 | .557563 | 1.746734 | .548629 | 1.814119 | .540485 | 1.877997 | .533010 | 1.938821 | .526110 | 1.996960 |
| 23 | .563887 | 1.777927 | .554752 | 1.847993 | .546433 | 1.914333 | .538806 | 1.977441 | .531771 | 2.037713 |
| 24 | .570082 | 1.807276 | .560742 | 1.880068 | .552247 | 1.948908 | .544466 | 2.014327 | .537295 | 2.076751 |
| 25 | .576161 | 1.834818 | .566612 | 1.910391 | .557939 | 1.981770 | .550001 | 2.049531 | .542693 | 2.114132 |
| 26 | .582136 | 1.860590 | .572374 | 1.939001 | .563519 | 2.012963 | .555423 | 2.083100 | .547976 | 2.149907 |
| 27 | .588017 | 1.884619 | .578039 | 1.965932 | .568997 | 2.042525 | .560742 | 2.115076 | .553154 | 2.184115 |
| 28 | .593815 | 1.906930 | .583614 | 1.991210 | .574384 | 2.070489 | .565966 | 2.145495 | .558235 | 2.216800 |
| 29 | .599540 | 1.927539 | .589110 | 2.014862 | .579687 | 2.096883 | .571103 | 2.174388 | .563228 | 2.247993 |
| 30 | .605199 | 1.946461 | .594535 | 2.036906 | .584914 | 2.121728 | .576161 | 2.201782 | .568139 | 2.277725 |
| 31 | .610800 | 1.963704 | .599895 | 2.057356 | .590073 | 2.145048 | .581147 | 2.227700 | .572975 | 2.306023 |
| 32 | .616351 | 1.979274 | .605199 | 2.076225 | .595169 | 2.166856 | .586066 | 2.252162 | .577743 | 2.332909 |
| 33 | .621860 | 1.993172 | .610451 | 2.093521 | .600209 | 2.187167 | .590926 | 2.275186 | .582447 | 2.358404 |
| 34 | .627333 | 2.005396 | .615660 | 2.109247 | .605199 | 2.205989 | .595731 | 2.296786 | .587094 | 2.382524 |
| 35 | .632777 | 2.015940 | .620830 | 2.123407 | .610144 | 2.223331 | .600487 | 2.316972 | .591688 | 2.405285 |
| 36 | .638198 | 2.024794 | .625968 | 2.135998 | .615049 | 2.239196 | .605199 | 2.335753 | .596234 | 2.426698 |
| 37 | .643602 | 2.031945 | .631078 | 2.147015 | .619920 | 2.253586 | .609870 | 2.353136 | .600736 | 2.446772 |
| 38 | .648997 | 2.037377 | .636167 | 2.156451 | .624762 | 2.266499 | .614506 | 2.369123 | .605199 | 2.465517 |
| 39 | .654387 | 2.041068 | .641240 | 2.164296 | .629578 | 2.277935 | .619111 | 2.383719 | .609625 | 2.482937 |
| 40 | .659779 | 2.042994 | .646300 | 2.170536 | .634373 | 2.287883 | .623688 | 2.396920 | .614020 | 2.499034 |
| 41 | .665179 | 2.043128 | .651355 | 2.175154 | .639153 | 2.296338 | .628242 | 2.408725 | .618386 | 2.513813 |
| 42 | .670594 | 2.041438 | .656408 | 2.178132 | .643920 | 2.303288 | .632777 | 2.419128 | .622727 | 2.527271 |
| 43 | .676030 | 2.037887 | .661465 | 2.179446 | .648679 | 2.308719 | .637296 | 2.428124 | .627046 | 2.539406 |
| 44 | .681494 | 2.032434 | .666531 | 2.179070 | .653435 | 2.312616 | .641802 | 2.435703 | .631347 | 2.550217 |
| 45 | .686993 | 2.025034 | .671612 | 2.176976 | .658192 | 2.314961 | .646300 | 2.441853 | .635632 | 2.559695 |
| 46 | .692534 | 2.015637 | .676712 | 2.173131 | .662954 | 2.315731 | .650793 | 2.446562 | .639906 | 2.567834 |
| 47 | .698124 | 2.004186 | .681837 | 2.167498 | .667725 | 2.314904 | .655285 | 2.449814 | .644171 | 2.574625 |
| 48 | .703773 | 1.990619 | .686993 | 2.160036 | .672510 | 2.312453 | .659779 | 2.451593 | .648429 | 2.580056 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 49 | .709489 | 1.974867 | .692186 | 2.150702 | .677313 | 2.308347 | .664278 | 2.451878 | .652685 | 2.584116 |
| 50 | .715281 | 1.956855 | .697423 | 2.139447 | .682139 | 2.302554 | .668787 | 2.450647 | .656940 | 2.586789 |
| 51 | .721161 | 1.936499 | .702709 | 2.126215 | .686993 | 2.295039 | .673309 | 2.447877 | .661199 | 2.588059 |
| 52 | .727139 | 1.913705 | .708053 | 2.110948 | .691879 | 2.285761 | .677848 | 2.443539 | .665464 | 2.587908 |
| 53 | .733228 | 1.888372 | .713462 | 2.093581 | .696804 | 2.274677 | .682408 | 2.437605 | .669738 | 2.586315 |
| 54 | .739442 | 1.860384 | .718945 | 2.074040 | .701772 | 2.261740 | .686993 | 2.430041 | .674025 | 2.583258 |
| 55 | .745797 | 1.829615 | .724510 | 2.052249 | .706790 | 2.246898 | .691607 | 2.420813 | .678327 | 2.578712 |
| 56 | .752312 | 1.795924 | .730168 | 2.028120 | .711864 | 2.230093 | .696255 | 2.409882 | .682649 | 2.572650 |
| 57 | .759006 | 1.759151 | .735930 | 2.001558 | .717001 | 2.211264 | .700941 | 2.397205 | .686993 | 2.565043 |
| 58 | .765904 | 1.719119 | .741808 | 1.972458 | .722208 | 2.190344 | .705671 | 2.382738 | .691363 | 2.555860 |
| 59 | .773033 | 1.675626 | .747815 | 1.940702 | .727494 | 2.167258 | .710449 | 2.366429 | .695764 | 2.545065 |
| 60 | .780425 | 1.628443 | .753968 | 1.906161 | .732866 | 2.141926 | .715281 | 2.348226 | .700198 | 2.532620 |
| 61 | .788121 | 1.577309 | .760283 | 1.868692 | .738335 | 2.114257 | .720174 | 2.328070 | .704671 | 2.518487 |
| 62 | .796166 | 1.521923 | .766782 | 1.828134 | .743912 | 2.084156 | .725134 | 2.305896 | .709186 | 2.502619 |
| 63 | .804619 | 1.461936 | .773487 | 1.784306 | .749609 | 2.051515 | .730168 | 2.281635 | .713749 | 2.484971 |
| 64 | .813551 | 1.396938 | .780425 | 1.737005 | .755438 | 2.016215 | .735284 | 2.255212 | .718364 | 2.465490 |
| 65 | .823055 | 1.326443 | .787630 | 1.686001 | .761416 | 1.978125 | .740491 | 2.226543 | .723037 | 2.444121 |
| 66 | .833250 | 1.249866 | .795139 | 1.631029 | .767559 | 1.937098 | .745797 | 2.195538 | .727774 | 2.420803 |
| 67 | .844296 | 1.166492 | .803000 | 1.571789 | .773888 | 1.892974 | .751214 | 2.162098 | .732581 | 2.395471 |
| 68 | .856416 | 1.075427 | .811269 | 1.507928 | .780425 | 1.845568 | .756753 | 2.126114 | .737465 | 2.368054 |
| 69 | .869932 | .975527 | .820017 | 1.439036 | .787198 | 1.794677 | .762428 | 2.087465 | .742434 | 2.338475 |
| 70 | .885336 | .865273 | .829337 | 1.364626 | .794239 | 1.740069 | .768253 | 2.046019 | .747495 | 2.306649 |
| 71 | .903442 | .742563 | .839346 | 1.284114 | .801585 | 1.681478 | .774245 | 2.001629 | .752659 | 2.272485 |
| 72 | .925756 | .604283 | .850205 | 1.196785 | .809283 | 1.618602 | .780425 | 1.954131 | .757936 | 2.235881 |
| 73 | .955590 | .445346 | .862134 | 1.101746 | .817389 | 1.551088 | .786815 | 1.903341 | .763337 | 2.196728 |
| 74 | 1.003263 | .255746 | .875452 | .997858 | .825976 | 1.478525 | .793442 | 1.849053 | .768875 | 2.154904 |
| 75 | 1.047909 | .142098 | .890652 | .883611 | .835132 | 1.400426 | .800337 | 1.791034 | .774566 | 2.110276 |
| 76 | | | .908543 | .756913 | .844977 | 1.316206 | .807539 | 1.729019 | .780425 | 2.062694 |
| 77 | | | .930622 | .614678 | .855669 | 1.225151 | .815093 | 1.662702 | .786473 | 2.011994 |
| 78 | | | .960190 | .451866 | .867426 | 1.126374 | .823055 | 1.591732 | .792732 | 1.957991 |
| 79 | | | 1.007531 | .258596 | .880569 | 1.018737 | .831497 | 1.515696 | .799229 | 1.900479 |
| 80 | | | 1.051947 | .143300 | .895586 | .900738 | .840508 | 1.434107 | .805995 | 1.839222 |
| 81 | | | | | .913282 | .770300 | .850205 | 1.346383 | .813068 | 1.773956 |
| 82 | | | | | .935150 | .624359 | .860747 | 1.251810 | .820493 | 1.704374 |
| 83 | | | | | .964477 | .457927 | .872351 | 1.149502 | .828326 | 1.630123 |
| 84 | | | | | 1.011517 | .261237 | .885336 | 1.038328 | .836637 | 1.550793 |
| 85 | | | | | 1.055724 | .144409 | .900186 | .916794 | .845517 | 1.465895 |
| 86 | | | | | | | .917706 | .782834 | .855081 | 1.374848 |
| 87 | | | | | | | .939382 | .633412 | .865488 | 1.276942 |
| 88 | | | | | | | .968491 | .463583 | .876953 | 1.171292 |
| 89 | | | | | | | 1.015256 | .263694 | .889794 | 1.056771 |
| 90 | | | | | | | 1.059271 | .145438 | .904494 | .931895 |
| 91 | | | | | | | | | .921854 | .794612 |
| 92 | | | | | | | | | .943354 | .641906 |
| 93 | | | | | | | | | .972263 | .468881 |
| 94 | | | | | | | | | 1.018775 | .265987 |
| 95 | | | | | | | | | 1.062615 | .146395 |

| NUMBER | N=105 | | N=115 | | N=125 | | N=135 | | N=145 | |
|---|---|---|---|---|---|---|---|---|---|---|
| RESPONDING | BURRIT | WEIGHT | BURRIT | WEIGHT | BURRIT | WEIGHT | BURRIT | WEIGHT | BURRIT | WEIGHT |
| 0 | .230052 | .248950 | .225787 | .258460 | .221948 | .267493 | .218463 | .276116 | .215277 | .284362 |
| 1 | .265361 | .373915 | .260428 | .388265 | .255989 | .401898 | .251961 | .414891 | .248279 | .427324 |
| 2 | .306267 | .560512 | .300543 | .582231 | .295395 | .602838 | .290726 | .622482 | .286459 | .641269 |
| 3 | .333220 | .709099 | .326959 | .736832 | .321332 | .763138 | .316229 | .788202 | .311569 | .812169 |
| 4 | .353894 | .836843 | .347208 | .869883 | .341202 | .901214 | .335760 | .931051 | .330790 | .959572 |
| 5 | .370910 | .950671 | .363864 | .988576 | .357538 | 1.024494 | .351808 | 1.058688 | .346579 | 1.091364 |
| 6 | .385505 | 1.054241 | .378142 | 1.096690 | .371534 | 1.136898 | .365552 | 1.175154 | .360095 | 1.211702 |
| 7 | .398370 | 1.149771 | .390718 | 1.196535 | .383856 | 1.240802 | .377646 | 1.282903 | .371984 | 1.323109 |
| 8 | .409931 | 1.238737 | .402012 | 1.289630 | .394915 | 1.337783 | .388496 | 1.383562 | .382645 | 1.427257 |
| 9 | .420471 | 1.322173 | .412302 | 1.377060 | .404985 | 1.428961 | .398370 | 1.478277 | .392345 | 1.525335 |
| 10 | .430189 | 1.400847 | .421782 | 1.459613 | .414257 | 1.515152 | .407459 | 1.567897 | .401268 | 1.618204 |
| 11 | .439230 | 1.475345 | .430595 | 1.537899 | .422872 | 1.596981 | .415899 | 1.653067 | .409551 | 1.706535 |
| 12 | .447703 | 1.546124 | .438849 | 1.612395 | .430936 | 1.674947 | .423794 | 1.734293 | .417296 | 1.790848 |
| 13 | .455693 | 1.613553 | .446627 | 1.683480 | .438529 | 1.749436 | .431225 | 1.811985 | .424583 | 1.871563 |
| 14 | .463266 | 1.677932 | .453993 | 1.751466 | .445716 | 1.820780 | .438255 | 1.886476 | .431474 | 1.949019 |
| 15 | .470477 | 1.739510 | .461002 | 1.816611 | .452550 | 1.889243 | .444936 | 1.958039 | .438019 | 2.023508 |
| 16 | .477370 | 1.798492 | .467696 | 1.879136 | .459073 | 1.955048 | .451309 | 2.026913 | .444260 | 2.095266 |
| 17 | .483981 | 1.855062 | .474111 | 1.939223 | .465320 | 2.018389 | .457410 | 2.093287 | .450231 | 2.164497 |
| 18 | .490341 | 1.909367 | .480277 | 1.997031 | .471320 | 2.079429 | .463266 | 2.157341 | .455961 | 2.231375 |
| 19 | .496476 | 1.961539 | .486220 | 2.052696 | .477100 | 2.138311 | .468904 | 2.219216 | .461473 | 2.296049 |
| 20 | .502408 | 2.011693 | .491962 | 2.106339 | .482679 | 2.195161 | .474343 | 2.279039 | .466790 | 2.358658 |
| 21 | .508158 | 2.059927 | .497521 | 2.158062 | .488078 | 2.250083 | .479603 | 2.336928 | .471928 | 2.419316 |
| 22 | .513741 | 2.106331 | .502915 | 2.207960 | .493312 | 2.303177 | .484699 | 2.392978 | .476904 | 2.478125 |
| 23 | .519172 | 2.150979 | .508158 | 2.256111 | .498395 | 2.354527 | .489646 | 2.447281 | .481731 | 2.535177 |
| 24 | .524465 | 2.193940 | .513262 | 2.302591 | .503340 | 2.404212 | .494454 | 2.499915 | .486421 | 2.590555 |
| 25 | .529631 | 2.235277 | .518238 | 2.347464 | .508158 | 2.452295 | .499136 | 2.550951 | .490985 | 2.644332 |
| 26 | .534681 | 2.275042 | .523097 | 2.390786 | .512858 | 2.498840 | .503701 | 2.600454 | .495433 | 2.696574 |
| 27 | .539623 | 2.313285 | .527848 | 2.432610 | .517450 | 2.543902 | .508158 | 2.648478 | .499773 | 2.747341 |
| 28 | .544466 | 2.350048 | .532499 | 2.472984 | .521941 | 2.587529 | .512514 | 2.695080 | .504012 | 2.796687 |
| 29 | .549218 | 2.385369 | .537057 | 2.511949 | .526339 | 2.629769 | .516776 | 2.740304 | .508158 | 2.844662 |
| 30 | .553885 | 2.419285 | .541529 | 2.549543 | .530650 | 2.670659 | .520951 | 2.784193 | .512216 | 2.891312 |
| 31 | .558475 | 2.451826 | .545922 | 2.585802 | .534880 | 2.710239 | .525045 | 2.826787 | .516194 | 2.936676 |
| 32 | .562992 | 2.483019 | .550239 | 2.620753 | .539035 | 2.748541 | .529063 | 2.868122 | .520094 | 2.980791 |
| 33 | .567442 | 2.512889 | .554488 | 2.654427 | .543120 | 2.785596 | .533010 | 2.908232 | .523924 | 3.023694 |
| 34 | .571830 | 2.541458 | .558673 | 2.686850 | .547138 | 2.821431 | .536890 | 2.947145 | .527686 | 3.065417 |
| 35 | .576161 | 2.568745 | .562797 | 2.718042 | .551095 | 2.856076 | .540707 | 2.984890 | .531385 | 3.105989 |
| 36 | .580438 | 2.594767 | .566865 | 2.748026 | .554994 | 2.889547 | .544466 | 3.021490 | .535025 | 3.145433 |
| 37 | .584667 | 2.619540 | .570881 | 2.776819 | .558839 | 2.921871 | .548169 | 3.056970 | .538609 | 3.183778 |
| 38 | .588850 | 2.643076 | .574849 | 2.804440 | .562633 | 2.953062 | .551821 | 3.091351 | .542140 | 3.221047 |
| 39 | .592992 | 2.665386 | .578771 | 2.830900 | .566380 | 2.983142 | .555423 | 3.124650 | .545621 | 3.257258 |
| 40 | .597095 | 2.686480 | .582651 | 2.856215 | .570082 | 3.012126 | .558980 | 3.156889 | .549055 | 3.292432 |
| 41 | .601163 | 2.706364 | .586491 | 2.880393 | .573742 | 3.040025 | .562493 | 3.188082 | .552445 | 3.326587 |
| 42 | .605199 | 2.725045 | .590295 | 2.903449 | .577364 | 3.066855 | .565966 | 3.218243 | .555793 | 3.359741 |
| 43 | .609205 | 2.742527 | .594066 | 2.925388 | .580948 | 3.092626 | .569399 | 3.247388 | .559102 | 3.391907 |
| 44 | .613185 | 2.758814 | .597805 | 2.946218 | .584499 | 3.117349 | .572797 | 3.275527 | .562373 | 3.423099 |
| 45 | .617141 | 2.773907 | .601515 | 2.965944 | .588017 | 3.141032 | .576161 | 3.302672 | .565608 | 3.453331 |
| 46 | .621076 | 2.787805 | .605199 | 2.984573 | .591506 | 3.163684 | .579492 | 3.328834 | .568810 | 3.482615 |
| 47 | .624992 | 2.800508 | .608858 | 3.002108 | .594966 | 3.185312 | .582794 | 3.354023 | .571980 | 3.510962 |
| 48 | .628891 | 2.812013 | .612494 | 3.018550 | .598400 | 3.205922 | .586066 | 3.378243 | .575120 | 3.538380 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 49 | .632777 | 2.822316 | .616111 | 3.033902 | .601811 | 3.225517 | .589313 | 3.401505 | .578232 | 3.564879 |
| 50 | .636651 | 2.831413 | .619709 | 3.048164 | .605199 | 3.244101 | .592534 | 3.423815 | .581317 | 3.590469 |
| 51 | .640516 | 2.839296 | .623291 | 3.061335 | .608566 | 3.261680 | .595731 | 3.445179 | .584377 | 3.615157 |
| 52 | .644373 | 2.845958 | .626859 | 3.073414 | .611914 | 3.278253 | .598907 | 3.465600 | .587413 | 3.638946 |
| 53 | .648226 | 2.851390 | .630413 | 3.084396 | .615245 | 3.293823 | .602062 | 3.485082 | .590426 | 3.661848 |
| 54 | .652077 | 2.855581 | .633957 | 3.094281 | .618560 | 3.308390 | .605199 | 3.503629 | .593418 | 3.683863 |
| 55 | .655927 | 2.858519 | .637492 | 3.103061 | .621860 | 3.321953 | .608317 | 3.521244 | .596390 | 3.704998 |
| 56 | .659779 | 2.860192 | .641019 | 3.110733 | .625148 | 3.334513 | .611419 | 3.537929 | .599344 | 3.725257 |
| 57 | .663635 | 2.860584 | .644541 | 3.117288 | .628424 | 3.346066 | .614506 | 3.553685 | .602279 | 3.744643 |
| 58 | .667498 | 2.859681 | .648059 | 3.122720 | .631690 | 3.356611 | .617579 | 3.568513 | .605199 | 3.763157 |
| 59 | .671369 | 2.857464 | .651575 | 3.127019 | .634948 | 3.366143 | .620639 | 3.582411 | .608103 | 3.780805 |
| 60 | .675252 | 2.853913 | .655090 | 3.130177 | .638198 | 3.374657 | .623688 | 3.595380 | .610992 | 3.797587 |
| 61 | .679149 | 2.849010 | .658606 | 3.132182 | .641442 | 3.382151 | .626727 | 3.607418 | .613869 | 3.813502 |
| 62 | .683061 | 2.842729 | .662125 | 3.133021 | .644682 | 3.388616 | .629756 | 3.618523 | .616733 | 3.828554 |
| 63 | .686993 | 2.835048 | .665649 | 3.132683 | .647918 | 3.394049 | .632777 | 3.628692 | .619585 | 3.842739 |
| 64 | .690946 | 2.825941 | .669180 | 3.131154 | .651153 | 3.398439 | .635791 | 3.637924 | .622428 | 3.856062 |
| 65 | .694923 | 2.815378 | .672718 | 3.128417 | .654387 | 3.401779 | .638799 | 3.646212 | .625261 | 3.868517 |
| 66 | .698928 | 2.803330 | .676267 | 3.124457 | .657621 | 3.404062 | .641802 | 3.653554 | .628086 | 3.880105 |
| 67 | .702962 | 2.789765 | .679828 | 3.119256 | .660858 | 3.405276 | .644802 | 3.659943 | .630903 | 3.890823 |
| 68 | .707031 | 2.774646 | .683402 | 3.112795 | .664098 | 3.405411 | .647798 | 3.665375 | .633713 | 3.900670 |
| 69 | .711136 | 2.757937 | .686993 | 3.105052 | .667343 | 3.404455 | .650793 | 3.669843 | .636518 | 3.909641 |
| 70 | .715281 | 2.739597 | .690601 | 3.096008 | .670594 | 3.402397 | .653788 | 3.673340 | .639317 | 3.917734 |
| 71 | .719471 | 2.719583 | .694229 | 3.085637 | .673853 | 3.399222 | .656782 | 3.675858 | .642113 | 3.924944 |
| 72 | .723710 | 2.697848 | .697880 | 3.073915 | .677121 | 3.394916 | .659779 | 3.677389 | .644905 | 3.931269 |
| 73 | .728001 | 2.674342 | .701555 | 3.060815 | .680399 | 3.389464 | .662777 | 3.677926 | .647695 | 3.936700 |
| 74 | .732350 | 2.649011 | .705257 | 3.046309 | .683689 | 3.382851 | .665780 | 3.677456 | .650484 | 3.941234 |
| 75 | .736762 | 2.621796 | .708989 | 3.030366 | .686993 | 3.375057 | .668787 | 3.675971 | .653271 | 3.944866 |
| 76 | .741242 | 2.592636 | .712753 | 3.012953 | .690312 | 3.366065 | .671800 | 3.673460 | .656059 | 3.947588 |
| 77 | .745797 | 2.561462 | .716552 | 2.994036 | .693648 | 3.355855 | .674820 | 3.669910 | .658849 | 3.949393 |
| 78 | .750433 | 2.528200 | .720389 | 2.973578 | .697002 | 3.344406 | .677848 | 3.665309 | .661640 | 3.950273 |
| 79 | .755157 | 2.492772 | .724267 | 2.951539 | .700376 | 3.331694 | .680885 | 3.659643 | .664433 | 3.950221 |
| 80 | .759978 | 2.455091 | .728189 | 2.927877 | .703773 | 3.317698 | .683933 | 3.652900 | .667231 | 3.949228 |
| 81 | .764905 | 2.415062 | .732160 | 2.902546 | .707194 | 3.302391 | .686993 | 3.645062 | .670033 | 3.947283 |
| 82 | .769947 | 2.372584 | .736183 | 2.875498 | .710641 | 3.285747 | .690065 | 3.636115 | .672841 | 3.944380 |
| 83 | .775116 | 2.327545 | .740262 | 2.846681 | .714116 | 3.267737 | .693152 | 3.626041 | .675654 | 3.940505 |
| 84 | .780425 | 2.279820 | .744403 | 2.816040 | .717622 | 3.248332 | .696255 | 3.614823 | .678476 | 3.935648 |
| 85 | .785889 | 2.229274 | .748609 | 2.783514 | .721161 | 3.227498 | .699374 | 3.602441 | .681305 | 3.929798 |
| 86 | .791523 | 2.175756 | .752886 | 2.749038 | .724735 | 3.205202 | .702512 | 3.588876 | .684144 | 3.922942 |
| 87 | .797347 | 2.119101 | .757241 | 2.712544 | .728347 | 3.181409 | .705671 | 3.574106 | .686993 | 3.915066 |
| 88 | .803384 | 2.059123 | .761679 | 2.673957 | .732000 | 3.156078 | .708850 | 3.558109 | .689853 | 3.906158 |
| 89 | .809659 | 1.995612 | .766209 | 2.633195 | .735697 | 3.129170 | .712053 | 3.540862 | .692726 | 3.896202 |
| 90 | .816204 | 1.928333 | .770837 | 2.590171 | .739442 | 3.100640 | .715281 | 3.522339 | .695611 | 3.885182 |
| 91 | .823055 | 1.857020 | .775572 | 2.544789 | .743237 | 3.070442 | .718536 | 3.502515 | .698512 | 3.873083 |
| 92 | .830258 | 1.781366 | .780425 | 2.496945 | .747087 | 3.038526 | .721820 | 3.481359 | .701428 | 3.859887 |
| 93 | .837869 | 1.701020 | .785407 | 2.446526 | .750995 | 3.004839 | .725134 | 3.458844 | .704361 | 3.845578 |
| 94 | .845956 | 1.615568 | .790530 | 2.393407 | .754967 | 2.969323 | .728482 | 3.434938 | .707312 | 3.830134 |
| 95 | .854608 | 1.524526 | .795808 | 2.337450 | .759006 | 2.931919 | .731864 | 3.409608 | .710283 | 3.813537 |
| 96 | .863942 | 1.427314 | .801258 | 2.278505 | .763118 | 2.892559 | .735284 | 3.382818 | .713275 | 3.795765 |
| 97 | .874112 | 1.323225 | .806899 | 2.216405 | .767310 | 2.851173 | .738745 | 3.354532 | .716289 | 3.776797 |
| 98 | .885336 | 1.21[illegible]383 | .812753 | 2.150961 | .771587 | 2.8076[illegible] | .742248 | 3.324709 | .719326 | 3.756[illegible]7 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 99 | .897925 | 1.090670 | .818846 | 2.081965 | .775956 | 2.762015 | .745797 | 3.293308 | .722389 | 3.735173 |
| 100 | .912361 | .959617 | .825208 | 2.009181 | .780425 | 2.714071 | .749395 | 3.260283 | .725479 | 3.712468 |
| 101 | .929439 | .816199 | .831876 | 1.932342 | .785003 | 2.663758 | .753046 | 3.225588 | .728598 | 3.688466 |
| 102 | .950630 | .657447 | .838895 | 1.851142 | .789700 | 2.610970 | .756753 | 3.189171 | .731747 | 3.663135 |
| 103 | .979185 | .478549 | .846320 | 1.765228 | .794526 | 2.555594 | .760520 | 3.150978 | .734929 | 3.636448 |
| 104 | 1.025248 | .270157 | .854219 | 1.674190 | .799495 | 2.497503 | .764352 | 3.110952 | .738145 | 3.608370 |
| 105 | 1.068775 | .148129 | .862681 | 1.577545 | .804619 | 2.436560 | .768253 | 3.069029 | .741398 | 3.578869 |
| 106 | | | .871821 | 1.474716 | .809915 | 2.372613 | .772228 | 3.025145 | .744691 | 3.547908 |
| 107 | | | .881792 | 1.364999 | .815403 | 2.305493 | .776283 | 2.979226 | .748025 | 3.515449 |
| 108 | | | .892810 | 1.247527 | .821103 | 2.235013 | .780425 | 2.931197 | .751403 | 3.481451 |
| 109 | | | .905186 | 1.121192 | .827042 | 2.160965 | .784660 | 2.880974 | .754829 | 3.445872 |
| 110 | | | .919397 | .984539 | .833250 | 2.083110 | .788996 | 2.828467 | .758304 | 3.408666 |
| 111 | | | .936234 | .835571 | .839763 | 2.001183 | .793442 | 2.773580 | .761834 | 3.369785 |
| 112 | | | .957161 | .671361 | .846626 | 1.914878 | .798007 | 2.716206 | .765421 | 3.329178 |
| 113 | | | .985410 | .487178 | .853893 | 1.823843 | .802702 | 2.656231 | .769069 | 3.286788 |
| 114 | | | 1.031086 | .273860 | .861632 | 1.727671 | .807539 | 2.593528 | .772783 | 3.242558 |
| 115 | | | 1.074342 | .149661 | .869932 | 1.625878 | .812532 | 2.527959 | .776566 | 3.196425 |
| 116 | | | | | .878905 | 1.517892 | .817698 | 2.459372 | .780425 | 3.148322 |
| 117 | | | | | .888706 | 1.403015 | .823055 | 2.387597 | .784365 | 3.098177 |
| 118 | | | | | .899548 | 1.280385 | .828625 | 2.312448 | .788392 | 3.045911 |
| 119 | | | | | .911739 | 1.148906 | .834432 | 2.233715 | .792513 | 2.991443 |
| 120 | | | | | .925756 | 1.007138 | .840508 | 2.151161 | .796735 | 2.934680 |
| 121 | | | | | .942384 | .853109 | .846888 | 2.064520 | .801067 | 2.875526 |
| 122 | | | | | .963080 | .683932 | .853616 | 1.973488 | .805519 | 2.813874 |
| 123 | | | | | .991064 | .494952 | .860747 | 1.877714 | .810101 | 2.749608 |
| 124 | | | | | 1.036401 | .277180 | .868348 | 1.776791 | .814826 | 2.682603 |
| 125 | | | | | 1.079418 | .151029 | .876506 | 1.670241 | .819707 | 2.612719 |
| 126 | | | | | | | .885336 | 1.557492 | .824760 | 2.539805 |
| 127 | | | | | | | .894988 | 1.437853 | .830004 | 2.463690 |
| 128 | | | | | | | .905676 | 1.310468 | .835460 | 2.384187 |
| 129 | | | | | | | .917706 | 1.174252 | .841153 | 2.301088 |
| 130 | | | | | | | .931553 | 1.027780 | .847113 | 2.214157 |
| 131 | | | | | | | .947998 | .869104 | .853377 | 2.123127 |
| 132 | | | | | | | .968491 | .695375 | .859988 | 2.027694 |
| 133 | | | | | | | .996240 | .502011 | .867000 | 1.927511 |
| 134 | | | | | | | 1.041277 | .280183 | .874480 | 1.822171 |
| 135 | | | | | | | 1.084083 | .152260 | .882515 | 1.711199 |
| 136 | | | | | | | | | .891218 | 1.594028 |
| 137 | | | | | | | | | .900740 | 1.469970 |
| 138 | | | | | | | | | .911292 | 1.338177 |
| 139 | | | | | | | | | .923180 | 1.197575 |
| 140 | | | | | | | | | .936876 | 1.046753 |
| 141 | | | | | | | | | .953158 | .883785 |
| 142 | | | | | | | | | .973472 | .705861 |
| 143 | | | | | | | | | 1.001011 | .508465 |
| 144 | | | | | | | | | 1.045780 | .282917 |
| 145 | | | | | | | | | 1.088398 | .153377 |

| NUMBER | N=155 | | N=165 | | N=175 | | N=185 | | N=195 | |
|---|---|---|---|---|---|---|---|---|---|---|
| RESPONDING | BURRIT | WEIGHT | BURRIT | WEIGHT | BURRIT | WEIGHT | BURRIT | WEIGHT | BURRIT | WEIGHT |
| 0 | .212346 | .292282 | .209635 | .299903 | .207114 | .307253 | .204763 | .314357 | .202560 | .321235 |
| 1 | .244892 | .439261 | .241759 | .450741 | .238848 | .461833 | .236132 | .472544 | .233587 | .482923 |
| 2 | .282536 | .659306 | .278908 | .676652 | .275538 | .693390 | .272395 | .709556 | .269451 | .725220 |
| 3 | .307285 | .835162 | .303325 | .857278 | .299648 | .878602 | .296218 | .899213 | .293006 | .919155 |
| 4 | .326224 | .986926 | .322005 | 1.013232 | .318087 | 1.038588 | .314434 | 1.063083 | .311014 | 1.086793 |
| 5 | .341776 | 1.122687 | .337339 | 1.152808 | .333220 | 1.181832 | .329381 | 1.209868 | .325787 | 1.236998 |
| 6 | .355084 | 1.246725 | .350458 | 1.280390 | .346164 | 1.312824 | .342161 | 1.344155 | .338417 | 1.374458 |
| 7 | .366787 | 1.361628 | .361989 | 1.398639 | .357538 | 1.434292 | .353391 | 1.468711 | .349511 | 1.502011 |
| 8 | .377277 | 1.469109 | .372324 | 1.509311 | .367729 | 1.548030 | .363449 | 1.585395 | .359446 | 1.621542 |
| 9 | .386818 | 1.570387 | .381719 | 1.613649 | .376992 | 1.655306 | .372589 | 1.695506 | .368472 | 1.734377 |
| 10 | .395592 | 1.666352 | .390357 | 1.712574 | .385505 | 1.757069 | .380988 | 1.800000 | .376764 | 1.841498 |
| 11 | .403734 | 1.757687 | .398370 | 1.806783 | .393401 | 1.854029 | .388775 | 1.899604 | .384451 | 1.943655 |
| 12 | .411344 | 1.844935 | .405858 | 1.896828 | .400776 | 1.946754 | .396047 | 1.994897 | .391628 | 2.041425 |
| 13 | .418501 | 1.928519 | .412897 | 1.983147 | .407708 | 2.035688 | .402881 | 2.086349 | .398370 | 2.135289 |
| 14 | .425266 | 2.008796 | .419550 | 2.066105 | .414257 | 2.121213 | .409335 | 2.174326 | .404737 | 2.225632 |
| 15 | .431690 | 2.086052 | .425864 | 2.146001 | .420471 | 2.203622 | .415457 | 2.259146 | .410775 | 2.312772 |
| 16 | .437813 | 2.160541 | .431880 | 2.223083 | .426391 | 2.283181 | .421288 | 2.341077 | .416524 | 2.396985 |
| 17 | .443668 | 2.232466 | .437631 | 2.297568 | .432048 | 2.360108 | .426859 | 2.420341 | .422016 | 2.478490 |
| 18 | .449285 | 2.302011 | .443146 | 2.369644 | .437470 | 2.434593 | .432197 | 2.497134 | .427277 | 2.557489 |
| 19 | .454686 | 2.369330 | .448448 | 2.439465 | .442682 | 2.506801 | .437327 | 2.571618 | .432331 | 2.634154 |
| 20 | .459894 | 2.434560 | .453558 | 2.507177 | .447703 | 2.576874 | .442267 | 2.643945 | .437198 | 2.708639 |
| 21 | .464924 | 2.497822 | .458492 | 2.572904 | .452550 | 2.644940 | .447035 | 2.714242 | .441893 | 2.781074 |
| 22 | .469794 | 2.559221 | .463266 | 2.636750 | .457239 | 2.711107 | .451646 | 2.782620 | .446433 | 2.851570 |
| 23 | .474516 | 2.618853 | .467894 | 2.698815 | .461782 | 2.775478 | .456112 | 2.849192 | .450829 | 2.920242 |
| 24 | .479101 | 2.676800 | .472387 | 2.759185 | .466191 | 2.838142 | .460446 | 2.914038 | .455094 | 2.987174 |
| 25 | .483562 | 2.733141 | .476755 | 2.817937 | .470477 | 2.899183 | .464657 | 2.977247 | .459236 | 3.052453 |
| 26 | .487907 | 2.787940 | .481009 | 2.875145 | .474648 | 2.958663 | .468754 | 3.038894 | .463266 | 3.116160 |
| 27 | .492144 | 2.841267 | .485155 | 2.930873 | .478713 | 3.016659 | .472746 | 3.099038 | .467191 | 3.178358 |
| 28 | .496281 | 2.893170 | .489202 | 2.985174 | .482679 | 3.073225 | .476639 | 3.157748 | .471018 | 3.239109 |
| 29 | .500325 | 2.943704 | .493156 | 3.038106 | .486553 | 3.128413 | .480440 | 3.215077 | .474754 | 3.298473 |
| 30 | .504282 | 2.992915 | .497023 | 3.089714 | .490341 | 3.182278 | .484156 | 3.271080 | .478404 | 3.356506 |
| 31 | .508158 | 3.040845 | .500809 | 3.140044 | .494047 | 3.234866 | .487791 | 3.325800 | .481975 | 3.413249 |
| 32 | .511957 | 3.087537 | .504519 | 3.189135 | .497678 | 3.286215 | .491350 | 3.379280 | .485470 | 3.468753 |
| 33 | .515685 | 3.133023 | .508158 | 3.237029 | .501237 | 3.336366 | .494838 | 3.431561 | .488894 | 3.523053 |
| 34 | .519345 | 3.177338 | .511729 | 3.283756 | .504729 | 3.385353 | .498260 | 3.482681 | .492251 | 3.576192 |
| 35 | .522942 | 3.220513 | .515237 | 3.329352 | .508158 | 3.433212 | .501618 | 3.532671 | .495546 | 3.628202 |
| 36 | .526480 | 3.262573 | .518684 | 3.373842 | .511527 | 3.479972 | .504916 | 3.581568 | .498780 | 3.679118 |
| 37 | .529961 | 3.303551 | .522076 | 3.417256 | .514839 | 3.525662 | .508158 | 3.629396 | .501959 | 3.728965 |
| 38 | .533388 | 3.343464 | .525414 | 3.459619 | .518098 | 3.570310 | .511346 | 3.676188 | .505084 | 3.777780 |
| 39 | .536766 | 3.382339 | .528701 | 3.500954 | .521305 | 3.613935 | .514483 | 3.721962 | .508158 | 3.825580 |
| 40 | .540096 | 3.420195 | .531940 | 3.541282 | .524465 | 3.656567 | .517573 | 3.766747 | .511184 | 3.872397 |
| 41 | .543381 | 3.457050 | .535134 | 3.580626 | .527579 | 3.698223 | .520616 | 3.810566 | .514164 | 3.918248 |
| 42 | .546623 | 3.492926 | .538285 | 3.619005 | .530650 | 3.738923 | .523616 | 3.853435 | .517100 | 3.963160 |
| 43 | .549825 | 3.527836 | .541395 | 3.656435 | .533680 | 3.778686 | .526574 | 3.895380 | .519995 | 4.007151 |
| 44 | .552988 | 3.561796 | .544466 | 3.692932 | .536670 | 3.817533 | .529493 | 3.936413 | .522851 | 4.050237 |
| 45 | .556115 | 3.594821 | .547500 | 3.728513 | .539623 | 3.855475 | .532375 | 3.976554 | .525668 | 4.092439 |
| 46 | .559208 | 3.626923 | .550499 | 3.763190 | .542540 | 3.892528 | .535220 | 4.015819 | .528450 | 4.133774 |
| 47 | .562268 | 3.658115 | .553464 | 3.796978 | .545423 | 3.928711 | .538031 | 4.054224 | .531197 | 4.174258 |
| 48 | .565297 | 3.688408 | .556398 | 3.829891 | .548275 | 3.964031 | .540810 | 4.091779 | .533911 | 4.213903 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 49 | .568296 | 3.717814 | .559301 | 3.861938 | .551095 | 3.998506 | .543557 | 4.128502 | .536594 | 4.252724 |
| 50 | .571267 | 3.746342 | .562176 | 3.893130 | .553885 | 4.032142 | .546274 | 4.164402 | .539246 | 4.290735 |
| 51 | .574212 | 3.773999 | .565022 | 3.923478 | .556648 | 4.064955 | .548963 | 4.199495 | .541870 | 4.327948 |
| 52 | .577131 | 3.800794 | .567843 | 3.952991 | .559384 | 4.096952 | .551625 | 4.233785 | .544466 | 4.364374 |
| 53 | .580027 | 3.826735 | .570639 | 3.981677 | .562094 | 4.128145 | .554260 | 4.267289 | .547036 | 4.400023 |
| 54 | .582900 | 3.851829 | .573411 | 4.009544 | .564780 | 4.158540 | .556871 | 4.300013 | .549580 | 4.434910 |
| 55 | .585751 | 3.876080 | .576161 | 4.036600 | .567442 | 4.188149 | .559457 | 4.331968 | .552100 | 4.469039 |
| 56 | .588582 | 3.899497 | .578889 | 4.062850 | .570082 | 4.216976 | .562021 | 4.363158 | .554596 | 4.502420 |
| 57 | .591393 | 3.922083 | .581596 | 4.088303 | .572701 | 4.245029 | .564563 | 4.393596 | .557071 | 4.535066 |
| 58 | .594187 | 3.943843 | .584285 | 4.112963 | .575299 | 4.272315 | .567083 | 4.423290 | .559523 | 4.566981 |
| 59 | .596963 | 3.964779 | .586954 | 4.136834 | .577878 | 4.298842 | .569584 | 4.452244 | .561955 | 4.598173 |
| 60 | .599723 | 3.984896 | .589606 | 4.159923 | .580438 | 4.324612 | .572066 | 4.480463 | .564368 | 4.628649 |
| 61 | .602468 | 4.004198 | .592242 | 4.182232 | .582981 | 4.349635 | .574529 | 4.507957 | .566761 | 4.658417 |
| 62 | .605199 | 4.022685 | .594862 | 4.203767 | .585507 | 4.373910 | .576974 | 4.534729 | .569137 | 4.687484 |
| 63 | .607916 | 4.040361 | .597466 | 4.224530 | .588017 | 4.397445 | .579403 | 4.560786 | .571495 | 4.715855 |
| 64 | .610620 | 4.057227 | .600057 | 4.244523 | .590512 | 4.420244 | .581815 | 4.586130 | .573836 | 4.743533 |
| 65 | .613313 | 4.073282 | .602634 | 4.263751 | .592992 | 4.442310 | .584212 | 4.610766 | .576161 | 4.770527 |
| 66 | .615995 | 4.088528 | .605199 | 4.282213 | .595458 | 4.463646 | .586595 | 4.634703 | .578470 | 4.796840 |
| 67 | .618666 | 4.102966 | .607751 | 4.299914 | .597911 | 4.484256 | .588963 | 4.657940 | .580765 | 4.822475 |
| 68 | .621329 | 4.116595 | .610293 | 4.316853 | .600352 | 4.504140 | .591318 | 4.680480 | .583046 | 4.847439 |
| 69 | .623983 | 4.129412 | .612824 | 4.333030 | .602781 | 4.523300 | .593660 | 4.702330 | .585314 | 4.871734 |
| 70 | .626629 | 4.141421 | .615345 | 4.348448 | .605199 | 4.541741 | .595990 | 4.723488 | .587568 | 4.895365 |
| 71 | .629268 | 4.152617 | .617858 | 4.363106 | .607606 | 4.559463 | .598308 | 4.743960 | .589810 | 4.918334 |
| 72 | .631901 | 4.162998 | .620362 | 4.377004 | .610003 | 4.576468 | .600615 | 4.763745 | .592040 | 4.940642 |
| 73 | .634528 | 4.172563 | .622858 | 4.390143 | .612391 | 4.592755 | .602912 | 4.782848 | .594258 | 4.962297 |
| 74 | .637150 | 4.181307 | .625347 | 4.402520 | .614770 | 4.608323 | .605199 | 4.801269 | .596466 | 4.983297 |
| 75 | .639768 | 4.189230 | .627829 | 4.414134 | .617141 | 4.623178 | .607476 | 4.819011 | .598664 | 5.003644 |
| 76 | .642383 | 4.196326 | .630306 | 4.424984 | .619504 | 4.637314 | .609744 | 4.836074 | .600851 | 5.023344 |
| 77 | .644995 | 4.202592 | .632777 | 4.435068 | .621860 | 4.650735 | .612004 | 4.852457 | .603029 | 5.042395 |
| 78 | .647605 | 4.208024 | .635243 | 4.444386 | .624210 | 4.663438 | .614256 | 4.868162 | .605199 | 5.060797 |
| 79 | .650214 | 4.212617 | .637706 | 4.452932 | .626553 | 4.675423 | .616501 | 4.883190 | .607359 | 5.078557 |
| 80 | .652822 | 4.216365 | .640165 | 4.460703 | .628891 | 4.686689 | .618738 | 4.897540 | .609512 | 5.095671 |
| 81 | .655430 | 4.219264 | .642621 | 4.467699 | .631224 | 4.697233 | .620970 | 4.911213 | .611657 | 5.112141 |
| 82 | .658039 | 4.221306 | .645074 | 4.473916 | .633553 | 4.707054 | .623195 | 4.924207 | .613795 | 5.127969 |
| 83 | .660649 | 4.222485 | .647526 | 4.479347 | .635877 | 4.716151 | .625414 | 4.936522 | .615926 | 5.143152 |
| 84 | .663261 | 4.222795 | .649977 | 4.483991 | .638198 | 4.724520 | .627628 | 4.948157 | .618051 | 5.157694 |
| 85 | .665877 | 4.222228 | .652427 | 4.487842 | .640516 | 4.732160 | .629838 | 4.959110 | .620169 | 5.171593 |
| 86 | .668496 | 4.220776 | .654877 | 4.490896 | .642831 | 4.739067 | .632043 | 4.969381 | .622282 | 5.184847 |
| 87 | .671119 | 4.218431 | .657327 | 4.493146 | .645144 | 4.745239 | .634244 | 4.978969 | .624390 | 5.197459 |
| 88 | .673748 | 4.215183 | .659779 | 4.494587 | .647456 | 4.750670 | .636442 | 4.987868 | .626493 | 5.209425 |
| 89 | .676382 | 4.211026 | .662232 | 4.495215 | .649767 | 4.755359 | .638637 | 4.996080 | .628592 | 5.220745 |
| 90 | .679023 | 4.205946 | .664688 | 4.495020 | .652077 | 4.759301 | .640829 | 5.003602 | .630686 | 5.231419 |
| 91 | .681671 | 4.199934 | .667146 | 4.493999 | .654387 | 4.762491 | .643019 | 5.010428 | .632777 | 5.241444 |
| 92 | .684327 | 4.192979 | .669608 | 4.492142 | .656697 | 4.764925 | .645207 | 5.016561 | .634864 | 5.250821 |
| 93 | .686993 | 4.185071 | .672074 | 4.489442 | .659008 | 4.766600 | .647394 | 5.021993 | .636949 | 5.259545 |
| 94 | .689668 | 4.176196 | .674545 | 4.485893 | .661321 | 4.767506 | .649579 | 5.026722 | .639030 | 5.267614 |
| 95 | .692354 | 4.166342 | .677021 | 4.481484 | .663635 | 4.767641 | .651765 | 5.030744 | .641110 | 5.275029 |
| 96 | .695052 | 4.155495 | .679504 | 4.476207 | .665952 | 4.766999 | .653950 | 5.034057 | .643187 | 5.281787 |
| 97 | .697762 | 4.143641 | .681992 | 4.470054 | .668271 | 4.765572 | .656135 | 5.036655 | .645263 | 5.287882 |
| 98 | .700486 | 4.130766 | .684489 | 4.463013 | .670594 | 4.763355 | .658321 | 5.038535 | .647337 | 5.293313 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 99 | .703224 | 4.116854 | .686993 | 4.455075 | .672921 | 4.760341 | .660508 | 5.039690 | .649411 | 5.298079 |
| 100 | .705977 | 4.101888 | .689506 | 4.446230 | .675252 | 4.756522 | .662696 | 5.040119 | .651484 | 5.302174 |
| 101 | .708747 | 4.085852 | .692028 | 4.436466 | .677588 | 4.751891 | .664887 | 5.039813 | .653557 | 5.305597 |
| 102 | .711535 | 4.068727 | .694560 | 4.425770 | .679930 | 4.746440 | .667080 | 5.038769 | .655631 | 5.308342 |
| 103 | .714342 | 4.050495 | .697104 | 4.414131 | .682277 | 4.740160 | .669275 | 5.036980 | .657704 | 5.310406 |
| 104 | .717168 | 4.031135 | .699659 | 4.401537 | .684631 | 4.733044 | .671474 | 5.034441 | .659779 | 5.311785 |
| 105 | .720015 | 4.010627 | .702226 | 4.387972 | .686993 | 4.725080 | .673676 | 5.031145 | .661854 | 5.312475 |
| 106 | .722886 | 3.988949 | .704807 | 4.373423 | .689362 | 4.716261 | .675883 | 5.027085 | .663932 | 5.312472 |
| 107 | .725780 | 3.966079 | .707402 | 4.357875 | .691739 | 4.706576 | .678094 | 5.022256 | .666011 | 5.311769 |
| 108 | .728699 | 3.941991 | .710012 | 4.341314 | .694126 | 4.696013 | .680310 | 5.016648 | .668093 | 5.310362 |
| 109 | .731645 | 3.916661 | .712638 | 4.323722 | .696521 | 4.684564 | .682531 | 5.010256 | .670177 | 5.308247 |
| 110 | .734619 | 3.890062 | .715281 | 4.305082 | .698928 | 4.672217 | .684759 | 5.003071 | .672264 | 5.305418 |
| 111 | .737624 | 3.862166 | .717942 | 4.285376 | .701345 | 4.658958 | .686993 | 4.995084 | .674355 | 5.301868 |
| 112 | .740660 | 3.832943 | .720622 | 4.264587 | .703773 | 4.644777 | .689234 | 4.986289 | .676449 | 5.297592 |
| 113 | .743731 | 3.802362 | .723322 | 4.242695 | .706214 | 4.629659 | .691482 | 4.976673 | .678548 | 5.292584 |
| 114 | .746837 | 3.770391 | .726044 | 4.219678 | .708668 | 4.613593 | .693738 | 4.966232 | .680651 | 5.286836 |
| 115 | .749981 | 3.736994 | .728788 | 4.195516 | .711136 | 4.596562 | .696003 | 4.954951 | .682760 | 5.280343 |
| 116 | .753165 | 3.702135 | .731555 | 4.170186 | .713618 | 4.578553 | .698276 | 4.942823 | .684873 | 5.273096 |
| 117 | .756392 | 3.665775 | .734348 | 4.143665 | .716116 | 4.559549 | .700559 | 4.929837 | .686993 | 5.265089 |
| 118 | .759664 | 3.627874 | .737167 | 4.115928 | .718630 | 4.539537 | .702853 | 4.915981 | .689118 | 5.256314 |
| 119 | .762985 | 3.588386 | .740013 | 4.086948 | .721161 | 4.518497 | .705157 | 4.901245 | .691251 | 5.246763 |
| 120 | .766356 | 3.547267 | .742890 | 4.056700 | .723710 | 4.496413 | .707473 | 4.885616 | .693390 | 5.236427 |
| 121 | .769782 | 3.504468 | .745797 | 4.025154 | .726278 | 4.473267 | .709800 | 4.869082 | .695537 | 5.225298 |
| 122 | .773267 | 3.459935 | .748737 | 3.992279 | .728866 | 4.449040 | .712140 | 4.851630 | .697692 | 5.213366 |
| 123 | .776813 | 3.413615 | .751712 | 3.958045 | .731476 | 4.423709 | .714494 | 4.833247 | .699856 | 5.200623 |
| 124 | .780425 | 3.365448 | .754724 | 3.922418 | .734107 | 4.397257 | .716861 | 4.813918 | .702028 | 5.187058 |
| 125 | .784108 | 3.315370 | .757774 | 3.885363 | .736762 | 4.369660 | .719244 | 4.793629 | .704210 | 5.172663 |
| 126 | .787867 | 3.263314 | .760866 | 3.846842 | .739442 | 4.340896 | .721642 | 4.772366 | .706402 | 5.157424 |
| 127 | .791708 | 3.209207 | .764001 | 3.806816 | .742147 | 4.310941 | .724056 | 4.750111 | .708605 | 5.141332 |
| 128 | .795635 | 3.152971 | .767182 | 3.765244 | .744880 | 4.279768 | .726487 | 4.726849 | .710819 | 5.124377 |
| 129 | .799657 | 3.094522 | .770412 | 3.722082 | .747641 | 4.247353 | .728937 | 4.702562 | .713044 | 5.106544 |
| 130 | .803781 | 3.033770 | .773693 | 3.677283 | .750433 | 4.213667 | .731405 | 4.677232 | .715281 | 5.087824 |
| 131 | .808015 | 2.970616 | .777030 | 3.630797 | .753256 | 4.178681 | .733893 | 4.650841 | .717532 | 5.068202 |
| 132 | .812369 | 2.904953 | .780425 | 3.582574 | .756114 | 4.142364 | .736402 | 4.623369 | .719796 | 5.047664 |
| 133 | .816853 | 2.836667 | .783883 | 3.532555 | .759006 | 4.104686 | .738933 | 4.594796 | .722074 | 5.026198 |
| 134 | .821479 | 2.765629 | .787407 | 3.480682 | .761936 | 4.065611 | .741487 | 4.565100 | .724367 | 5.003790 |
| 135 | .826262 | 2.691703 | .791003 | 3.426891 | .764905 | 4.025104 | .744065 | 4.534259 | .726675 | 4.980423 |
| 136 | .831216 | 2.614733 | .794675 | 3.371113 | .767916 | 3.983128 | .746668 | 4.502250 | .729000 | 4.956083 |
| 137 | .836361 | 2.534553 | .798428 | 3.313277 | .770970 | 3.939642 | .749298 | 4.469048 | .731341 | 4.930754 |
| 138 | .841717 | 2.450975 | .802269 | 3.253304 | .774072 | 3.894606 | .751955 | 4.434628 | .733701 | 4.904418 |
| 139 | .847311 | 2.363788 | .806205 | 3.191109 | .777222 | 3.847974 | .754642 | 4.398963 | .736079 | 4.877058 |
| 140 | .853170 | 2.272760 | .810242 | 3.126601 | .780425 | 3.799699 | .757360 | 4.362025 | .738477 | 4.848656 |
| 141 | .859332 | 2.177623 | .814390 | 3.059682 | .783684 | 3.749733 | .760110 | 4.323785 | .740895 | 4.819192 |
| 142 | .865840 | 2.078076 | .818658 | 2.990245 | .787001 | 3.698021 | .762894 | 4.284211 | .743335 | 4.788647 |
| 143 | .872746 | 1.973772 | .823055 | 2.918174 | .790381 | 3.644509 | .765714 | 4.243272 | .745797 | 4.757000 |
| 144 | .880119 | 1.864306 | .827595 | 2.843343 | .793829 | 3.589135 | .768572 | 4.200932 | .748283 | 4.724230 |
| 145 | .888045 | 1.749207 | .832291 | 2.765613 | .797347 | 3.531836 | .771470 | 4.157157 | .750793 | 4.690313 |
| 146 | .896635 | 1.627909 | .837158 | 2.684830 | .800942 | 3.472543 | .774410 | 4.111907 | .753329 | 4.655225 |
| 147 | .906041 | 1.499732 | .842216 | 2.600827 | .804619 | 3.411184 | .777394 | 4.065144 | .755893 | 4.618944 |
| 148 | .916472 | 1.363835 | .847484 | 2.513416 | .808383 | 3.347679 | .780425 | 4.016825 | .758484 | 4.581442 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 149 | .928234 | 1.219151 | .852988 | 2.422388 | .812243 | 3.281945 | .783506 | 3.966905 | .761105 | 4.542693 |
| 150 | .941795 | 1.064286 | .858758 | 2.327511 | .816204 | 3.213889 | .786640 | 3.915338 | .763758 | 4.502668 |
| 151 | .957932 | .897335 | .864829 | 2.228518 | .820275 | 3.143414 | .789829 | 3.862072 | .766443 | 4.461336 |
| 152 | .978083 | .715523 | .871245 | 2.125108 | .824466 | 3.070412 | .793077 | 3.807055 | .769163 | 4.418669 |
| 153 | 1.005435 | .514399 | .878058 | 2.016936 | .828787 | 2.994768 | .796389 | 3.750231 | .771920 | 4.374633 |
| 154 | 1.049963 | .285422 | .885336 | 1.903602 | .833250 | 2.916354 | .799767 | 3.691539 | .774714 | 4.329191 |
| 155 | 1.092410 | .154397 | .893163 | 1.784633 | .837869 | 2.835033 | .803218 | 3.630915 | .777548 | 4.282310 |
| 156 | | | .901653 | 1.659471 | .842659 | 2.750651 | .806744 | 3.568290 | .780425 | 4.233951 |
| 157 | | | .910954 | 1.527438 | .847638 | 2.663040 | .810353 | 3.503591 | .783347 | 4.184073 |
| 158 | | | .921277 | 1.387702 | .852828 | 2.572014 | .814049 | 3.436740 | .786316 | 4.132634 |
| 159 | | | .932925 | 1.239204 | .858252 | 2.477364 | .817841 | 3.367650 | .789334 | 4.079589 |
| 160 | | | .946365 | 1.080566 | .863942 | 2.378857 | .821734 | 3.296233 | .792406 | 4.024891 |
| 161 | | | .962370 | .909901 | .869932 | 2.276229 | .825737 | 3.222388 | .795533 | 3.968489 |
| 162 | | | .982376 | .724471 | .876264 | 2.169181 | .829859 | 3.146009 | .798720 | 3.910330 |
| 163 | | | 1.009557 | .519883 | .882993 | 2.057368 | .834112 | 3.066979 | .801971 | 3.850358 |
| 164 | | | 1.053866 | .287730 | .890185 | 1.940392 | .838506 | 2.985173 | .805288 | 3.788511 |
| 165 | | | 1.096158 | .155332 | .897925 | 1.817784 | .843055 | 2.900451 | .808677 | 3.724726 |
| 166 | | | | | .906323 | 1.688988 | .847775 | 2.812661 | .812142 | 3.658934 |
| 167 | | | | | .915531 | 1.553334 | .852684 | 2.721636 | .815690 | 3.591061 |
| 168 | | | | | .925756 | 1.409994 | .857803 | 2.627189 | .819325 | 3.521029 |
| 169 | | | | | .937300 | 1.257919 | .863157 | 2.529112 | .823055 | 3.448752 |
| 170 | | | | | .950630 | 1.095745 | .868774 | 2.427173 | .826887 | 3.374139 |
| 171 | | | | | .966516 | .921606 | .874690 | 2.321110 | .830829 | 3.297093 |
| 172 | | | | | .986389 | .732793 | .880948 | 2.210623 | .834889 | 3.217504 |
| 173 | | | | | 1.013416 | .524974 | .887601 | 2.095371 | .839080 | 3.135260 |
| 174 | | | | | 1.057525 | .289865 | .894715 | 1.974956 | .843412 | 3.050231 |
| 175 | | | | | 1.099676 | .156195 | .902374 | 1.848914 | .847899 | 2.962280 |
| 176 | | | | | | | .910691 | 1.716692 | .852556 | 2.871256 |
| 177 | | | | | | | .919813 | 1.577625 | .857401 | 2.776990 |
| 178 | | | | | | | .929948 | 1.430890 | .862456 | 2.679297 |
| 179 | | | | | | | .941399 | 1.275450 | .867745 | 2.577969 |
| 180 | | | | | | | .954628 | 1.109952 | .873296 | 2.472776 |
| 181 | | | | | | | .970406 | .932549 | .879145 | 2.363456 |
| 182 | | | | | | | .990157 | .740564 | .885336 | 2.249711 |
| 183 | | | | | | | 1.017042 | .529719 | .891919 | 2.131201 |
| 184 | | | | | | | 1.060967 | .291850 | .898962 | 2.007531 |
| 185 | | | | | | | 1.102989 | .156994 | .906549 | 1.878239 |
| 186 | | | | | | | | | .914790 | 1.742777 |
| 187 | | | | | | | | | .923834 | 1.600483 |
| 188 | | | | | | | | | .933888 | 1.450542 |
| 189 | | | | | | | | | .945252 | 1.291925 |
| 190 | | | | | | | | | .958390 | 1.123292 |
| 191 | | | | | | | | | .974067 | .942815 |
| 192 | | | | | | | | | .993707 | .747846 |
| 193 | | | | | | | | | 1.020461 | .534158 |
| 194 | | | | | | | | | 1.064217 | .293700 |
| 195 | | | | | | | | | 1.106119 | .157738 |

# APPENDIX II

## COMPUTER PROGRAM FOR BURRIT AND LOGIT ANALYSIS

The program given in this appendix may be used to find estimates for the parameters of a quantal bio-assay, where the quantal response is assumed to be either the Burr or logistic distribution. Input which is read by the program is defined as follows:

C - the parameter c in the Burr distribution.
FK - the parameter k in the Burr distribution.
K - the number of dose levels.
IT - the number of iterations, IT=1.
SS(I) - the number of subjects at the $I^{th}$ dose.
OP(I) - the proportion of subjects responding at the $I^{th}$ dose level.
X(I) - the value of the $I^{th}$ dose level.

The user must specify values for all of the above variables. In addition to these will be needed a set of control cards which direct the computer to compile, link edit, and execute the program. The control cards will depend on the users particular machine installation. The following breakdown will illustrate the order and format for submitting a job deck:

a) control cards (machine installation dependent).

b) Parameter cards

1st parameter card, FORMAT (8F10.6)

| column # | variable | |
|---|---|---|
| 1-10 | C | 〉 punch with a decimal |
| 11-20 | FK | |

2nd parameter card, FORMAT (8I10)

| column # | variable | |
|---|---|---|
| 1-10 | K | punch right justified |
| 11-20 | IT=1 | |

c) Data cards, FORMAT (F18.6, 2F10.6)

Ith data card, K data cards are needed.

| column # | variable | |
|---|---|---|
| 1-18 | SS(I) | punch with decimal . |
| 19-28 | OP(I) | |
| 29-38 | X(I) | |

The parameter and data cards would take the following form for analysis of Data Set Two given in Chapter I.

1st parameter card

| column # | variable |
|---|---|
| 1-10 | 4.874 |
| 11-20 | 6.158 |

2nd parameter card

| column # | variable |
|---|---|
| 10 | 5 |
| 20 | 1 |

Data Cards

| Col. 1-18 | Col. 19-28 | Col. 29-38 |
|---|---|---|
| 40. | .175 | 0.0 |
| 40. | .450 | 0.693 |
| 40. | .800 | 1.386 |
| 40. | .875 | 2.079 |
| 40. | .950 | 2.772 |

FORTRAN IV G LEVEL 1, MOD 3     MAIN

```
      DIMENSION X(100), OP(100), OQ(100), EP(100), EQ(100
     1), U(100), GAMA(100), SS(100), OB(100), EB(100)
    1 FORMAT(8I10)
    2 FORMAT(8F10.6)
    3 FORMAT(1HO)
    4 FORMAT(1H1)
    5 FORMAT(60X, 'ITERATION NO', 2X, I2)
   12 FORMAT(65X, 'C=', 2X, F10.6, 2X, 'K=', 2X, F10.6)
   13 FORMAT(F18.6, 2F10.6)
   14 FORMAT(////57X, 'BURRIT ANALYSIS')
   15 FORMAT(////58X, 'LOGIT ANALYSIS')
   16 FORMAT(6E20.6)
      READ 2,C,FK
   21 READ 1,K,IT
      READ 13,(SS(I), OP(I), X(I), I=1, K)
C
C        IN STATEMENT 20 SET X(I) EQUAL TO THE DESIRED TRANSFORMATION
C        OF DOSE AND REMOVE THE 'C' FROM COLUMN 6 OF STATEMENTS 18, 19 AND 20
C
C  18 DO 20 I=1,K
C  19 TMPS=X(I)
C  20 X(I)=ALOG(TMPS)
      C1=1./C
      C2=1./FK
      IF(OP(1).EQ.0) OP(1)=1./2.*SS(1)
      IF(OP(K).EQ.1) OP(K)=1.-1./2.*SS(K)
      DO 30 I=1,K
      OQ(I)=1.-OP(I)
   30 EP(I)=OP(I)
   40 L=1
C
C        BURRIT ANALYSIS ADJUSTED FOR BIAS AND VARIANCE
C
   50 CALL ADJST(K,C,FK,C1,C2,SS,U,GAMA,EP,EQ)
      DO 100 I=1,K
  100 OB(I)=(1./(OQ(I)**C2)-1.)**C1
      CALL PARAM(X,OB,U,GAMA,A,B,K)
      PRINT 4
      PRINT 14
      PRINT 3
      PRINT 5,L
      PRINT 3
      PRINT 12,C,FK
      DO 101 I=1,K
  101 EP(I)=1.-1./((1.+(A+B*X(I))**C)**FK)
      CALL RESID(X,OB,U,GAMA,EB,K,A,B,EP,OP,SS)
      L=L+1
      IF(L-IT)50,50,102
  102 L=1
```

```
C
C        BURRIT ANALYSIS UNADJUSTED FOR BIAS WITH
C        WEIGHT=C*C*K*K*Q(1.-Q**C2)**2*N/B*B*P
C
      DO 103 I=1,K
      EP(I)=OP(I)
      EQ(I)=1.-OP(I)
  103 U(I)=0.0
  104 DO 105 I=1,K
  105 GAMA(I)=(OB(I)*OB(I)*EP(I))/(C*C*FK*FK*OQ(I)*OQ(I)*((1.-(OQ(
     1I)**C2))**2)*SS(I))
      CALL PARAM(X,OB,U,GAMA,A,B,K)
      PRINT 4
      PRINT 14
      PRINT 3
      PRINT 5,L
      PRINT 3
      PRINT 12,C,FK
      DO 106 I=1,K
  106 EP(I)=1.-1./((1.+(A+B*X(I))**C)**FK)
      CALL RESID(X,OB,U,GAMA,EB,K,A,B,EP,OP,SS)
      L=L+1
      IF(L-IT)104,104,107
  107 L=1
C
C        LOGIT ANALYSIS WITH WEIGHT=NPQ
C
      DO 108 I=1,K
      EP(I)=OP(I)
      TMPL=OP(I)/OQ(I)
      OB(I)=ALOG(TMPL)
  108 EQ(I)=1.-OP(I)
  109 DO 110 I=1,K
  110 GAMA(I)=1./(SS(I)*EP(I)*EQ(I))
      CALL PARAM(X,OB,U,GAMA,A,B,K)
      PRINT 4
      PRINT 15
      PRINT 3
      PRINT 5,L
      DO 111 I=1,K
  111 EP(I)=1./(1.+EXP(-(A+B*X(I))))
      CALL RESID(X,OB,U,GAMA,EB,K,A,B,EP,OP,SS)
      L=L+1
      IF(L-IT)109,109,112
  112 GO TO 21
      END
```

FORTRAN IV G LEVEL 1, MOD 3     ADJST

```
      SUBROUTINE ADJST(K,C,FK,C1,C2,SS,U,GAMA,EP,EQ)
      DIMENSION SS(1),U(1),GAMA(1),EP(1),EQ(1)
```

```
   16 FORMAT(6E20.6)
      A=-C2
      A1=A-1.
      A2=A-2.
      A3=A-3.
      A4=A-4.
      B=C1
      B1=B-1.
      B2=B-2.
      B3=B-3.
      A11=A*B
      A21=A*B1
      A22=A*A1*B
      A31=A*B2
      A32=(2.*B-1.)*A*A1
      A33=B*A*A1*A2
      A41=A*B3
      A42=(3.*B1*A*A1*A2
      A44=B*A*A1*A2*A3
      DO 105 I=1,K
      EQ(I)=1.-EP(I)
      Q=EQ(I)
      QA1=(Q**A)-1.
      F0=QA1**B
      F1=-(A11*Q**A1*F0)QA1
      F2=(-A21*Q**A1*F1+A22*Q**A2*F0)/QA1
      F3=(-A31*Q**A1*F2+A32*Q**A2*F1-A33*Q**A3*F0)/QA1
      F4=(-A41*Q**A1*F3+A42*Q**A2*F2-A43*Q**A3*F1+A44*Q**A4*F0)/QA1
      U(I)=(EP(I)*EQ(I)*F2)/(2.*SS(I))+(EP(I)*EQ(I)*(EQ(I)-EP(I))*F3)/(6
     1.*SS(I)*SS(I))+((EP(I)*EP(I)*EQ(I))/(8.*SS(I)*SS*I))+(EP(I)*
     2EQ(I)*(1.-6.*EP(I)*EQ(I)))/(24.*SS(I)**3.))*F4
      GAMA(I)=(EP(I)*EQ(I)*F1*F1)/SS(I)+(F2*F2*(2.*SS(I)*EP(I)*EP(I)*EQ(
     1I)*EQ(I)*(EQ(I)*(1.-6.*EP(I)*EQ(I))))/(4.*(SS(I)**3.))+(F1*F2
     2*EP(I)*EQ(I)*(EQ(I)-EP(I)))/(SS(I)**2.)
  105 CONTINUE
      RETURN
      END
```

FORTRAN IV G LEVEL 1, MOD 3     PARAM

```
      SUBROUTINE PARAM(X,OB,U,GAMA,A,B,K)
      DIMENSION X(1),OB(1),U(1),GAMA(1)
   16 FORMAT(6E20.6)
      SX2G=0.
      SBUG=0.
      SXG=0.
      SBUXG=0.
      SG=0.
      DO 51 I=1,K
```

```
      SX2G=SX2G+(X(I)*X(I))/GAMA(I)
      SBUG=SBUG+(OB(I)-U(I))/GAMA(I)
      SXG=SXG+X(I)/GAMA(I)
      SBUXG=SBUXG+((OB(I)-U(I))*X(I))/GAMA(I)
   51 SG=SG+1./GAMA(I)
      DIV=SG*SX2G-(SXG*SXG)
      A=(SX2G*SBUG-SXG*SBUXG)/DIV
      B=(SG*SBUXG-SXG*SBUG)/DIV
      RETURN
      END
```

FORTRAN IV G LEVEL 1, MOD 3     RESID

```
      SUBROUTINE RESID(X,OB,U,GAMA,EB,K,A,B,EP,OP,SS)
      DIMENSION X(1),OB(1),U(1),GAMA(1),EB(1),EP(1),OP(1),SS(1)
    3 FORMAT(1H0
    6 FORMAT(11X,'OBSERVED',12X,'BIAS',14X,'WEIGHT',11X,'ESTIMATED',1
     11X,'RESIDUAL',11X,'SSE/CELL')
    7 FORMAT(6F20.6)
    8 FORMAT(13X,'ALPHA',15X,'BETA',16X,'TTS',17X,'SSR',17X,'SSE',14X,'R
     1-SQUARED')
    9 FORMAT(////13X,'DOSE',13X,'NO SUBJECTS',9X,'OBSERVED ρ',10X,'ESTIM
     1ATED P',11X,'RESIDUAL',9X,'CHI-SQUARE/CELL')
   10 FORMAT(60X,'CHI-SQUARE')
   11 FORMAT(45X,F20.6)
      PRINT 3
      PRINT 6
      SSET=0.
      TSS=0.
      DO 52 I=1,K
      EB(I)=A+B*X(I)
      RB=OB(I)-EB(I)
      SSE=((RB-U(I)*(RB-U(I)))/GAMA(I)
      SSET=SSET+SSE
      TSS=TSS+((OB(I)=U(I))*(OB(I)-U(I)))/GAMA(I)
      GAMA(I)=1./GAMA(I)
   52 PRINT 7, OB(I),U(I),GAMA(I),EB(I),RB,SSE
      SSR=TSS-SSET
      R2=SSR/TSS
      PRINT 3
      PRINT 8
      PRINT 7, A,B,TSS,SSR,SSET,R2
      PRINT 9
      CHI2T=0.
      DO 55 I=1,K
   54 RP=OP(I)-EP(I)
      CHI2=(RP*RP*SS(I))/(EP(I)*(1.-EP(I)))
      CHI2T=CHI2T+CHI2
   55 PRINT 7, X(I),SS(I),OP(I),EP(I),RP,CHI2
      PRINT 3
      PRINT 10
      PRINT 11, CHI2T
      RETURN
      END
```

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UNCLASSIFIED
Security Classification

## DOCUMENT CONTROL DATA - R & D

*Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)*

1. ORIGINATING ACTIVITY *(Corporate author)*
SOUTHERN METHODIST UNIVERSITY

2a. REPORT SECURITY CLASSIFICATION
UNCLASSIFIED

2b. GROUP
UNCLASSIFIED

3. REPORT TITLE
"Estimation and Confidence Intervals for Quantal Response or Sensitivity Data"

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)* Technical Report

5. AUTHOR(S) *(First name, middle initial, last name)*
Guy B. Seibert, Jr.

| 6. REPORT DATE | 7a. TOTAL NO. OF PAGES | 7b. NO. OF REFS |
|---|---|---|
| February 25, 1970 | 66 | 19 |

8a. CONTRACT OR GRANT NO.
N00014-68-A-0515
b. PROJECT NO.
NR 042-260
c.
d.

9a. ORIGINATOR'S REPORT NUMBER(S)
66

9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

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11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY
Office of Naval Research

13. ABSTRACT

If different levels of stimuli are applied and then counts are made of the number reacting and not-reacting to the stimuli, a sensitivity analysis is usually the best analysis to be applied. Sensitivity analysis may be used to determine the probability of destroying a target with different size bombs, without the necessity of actually having data on all the different sizes. It may also be used in determining the effect of a drug on a patient or the effect of polution on a body of water. This paper is concerned with a generalized model for sensitivity analysis based on the Burr distribution. This generalization gives a model that will fit quite well data that comes from probit or logit models, but also covers many additional situations that the probit or logit models would not fit. A FORTRAN IV program is included for doing the analysis on a computer. Confidence intervals for any quantile are also given.


DD FORM 1 NOV 65 1473 (PAGE 1)
S/N 0101-807-6811

UNCLASSIFIED
Security Classification

A-31408