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DISTRIBUTION OF TRANSITION FREQUENCIES OF A MARKOV RENEWAL PROCESS, OVER AN ARBITRARY INTERVAL OF TIME

by

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ABSTRACT

The distribution of the transition count matrix of a Markov Renewal Process, over an arbitrary interval of time $(t_0, t_0 + t)$ is derived in this paper. Moments of the first and second order of the transition frequencies are also derived and asymptotic expressions for these moments are also obtained.

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1. INTRODUCTION

A Markov Renewal Process (M.R.P.) with $m(<\infty)$ states is one which records at each time t, the number of times a system visits each of the m states up to time t, when the system moves from state to state according to a Markov Chain with a transition probability matrix $P_0 = [p_{ij}]$ and the time required for each move is a random variable whose distribution function (d.f.) depends on the two states between which the move is made. Thus $F_{ij}(x)$ is the d.f. of x, the holding time in state i, if the next transition is to state j $(i, j = 1, 2, \cdots , m)$. We set $Q_{ij}(x) = p_{ij}F_{ij}(x)$ and denote by $Q(x)$, the $m \times m$ matrix of the elements $Q_{ij}(x)$. Let $N_{jk}(t)$ denote the number of direct transitions of such a process from state j to state k in the time interval $(0,t)$ and let $N_{jk}(t_0 , t)$ denote the same quantity in the interval $(t_0 , t_0 + t)$, for all j, k. The $m \times m$ matrices $N(t) = [N_{jk}(t)]$ and $N(t_0 , t) = [N_{jk}(t_0 , t)]$ are called the transition count matrices of the M.R.P. in the intervals $(0,t)$ and $(t_0 , t_0 + t)$, respectively. The authors have, in a previous paper (1969), derived the distribution, moments

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and asymptotic expressions for the moments of the elements of $N(t)$. In practical situations, however, it is not always possible to observe an M.R.P. from the beginning. In other words, a record of the different transitions and holding times of the M.R.P. is available only from some arbitrary instant of time t_0 (measured from the beginning of the process) and in such a case, it will be useful to study the distribution and moments of $N_{jk}(t_0\ ,\ t)$. This will be useful in statistical inference about the parameters of the M.R.P. and also for prediction of its future behaviour. These investigations have been carried out in this paper and are likely to be useful in inventory control of items, which are repairable and in counter problems (Pyke 1962, 1969), where the underlying process is an M.R.P.

Notation and Earlier Results. We define the following quantities

$$J(t) = \text{state of the process at time } t, \tag{1.1}$$

$$q_{jk}(s) = \text{Laplace-Stieltjes Transform (L.-S.-T.) of } Q_{jk}(t)$$
$$= \int_0^\infty e^{-st} d_t Q_{jk}(t) \ . \quad (j,\ k = 1,\ 2,\ \cdots\ ,\ m)\ . \tag{1.2}$$

In general, the L.-S.T. will be denoted by the corresponding small letter. We also set $q(s) = [q_{jk}(s)]$, the $m \times m$ matrix. Let, further,

$$Q_{jk}(t_0\ ,\ x|J(0) = i) = \text{Prob}\{J(t_0) = j\ , \text{ the first transition after } t_0 \text{ is to state } k, \text{ the holding time in state } j, \text{ measured from } t_0 \text{ being} \le x|J(0) = i\}\ ,\ \cdots \tag{1.3}$$

$$q_{jk}(t_0\ ,\ s|J(0) = i) = \int_0^\infty e^{-sx} d_x Q_{jk}(t_0\ ,\ x|J(0) = i)\ , \tag{1.4}$$

$$q^*_{jk}(s_0\ ,\ s|J(0) = i) = \int_0^\infty e^{-s_0 t_0} q_{jk}(t_0\ ,\ s|J(0) = i)\,dt_0$$
$$= \left\{(I - q(s_0))^{-1}\right\}_{ij} \left(q_{jk}(s) - q_{jk}(s_0)\right)/(s_0 - s) \cdots \tag{1.5}$$

Result (1.5) has been proved by Kshirsagar and Gupta (1970). We denote by $\psi_i^0(A,s)$, the L.-S.T. of the probability generating function (p.g.f.) of $N(t) = [N_{jk}(t)]$, conditional on $J(0) = i$. This is the i^{th} element $(i = 1, 2, \cdots, m)$ of

$$\underline{\psi}^0(A,s) = \left(I - q(s) \square A\right)^{-1}\left(I - q(s)\right)\underline{e} \tag{1.6}$$

where,

$$\begin{aligned} q(s) \square A &= \text{the Hadamard Product of the matrices } q(s) \text{ and } A = [a_{ij}] \\ &= [q_{jk}(s)a_{jk}(s)] , \end{aligned} \tag{1.7}$$

and

$$\underline{e}' = [1, 1, \cdots, 1] . \tag{1.8}$$

Result (1.6) has been proved by Kshirsagar and Wysocki (1970). We shall need the following results also, from the same paper:

$$N_j(t) = \text{number of visits to state } j, \text{ of the M.R.P. in } (0,t) \tag{1.9}$$

$$M_{ij}(t) = E[N_j(t) | J(0) = i] \tag{1.10}$$

$$m_{ij}(s) = \text{L.S.T. of } M_{ij}(s) \tag{1.11}$$

$$[m_{ij}(s)] = m(s) = \left(I - q(s)\right)^{-1} - I \tag{1.12}$$

$$\begin{aligned} \sigma_{\alpha\beta}(i,s) &= \text{L.S.T. of } E\{N_{\alpha\beta}(t) | J(0) = i\} = \text{L.S.T. of } M^i_{jk}(t) \\ &= q_{\alpha\beta}(s)\left\{\left(I - q(s)\right)^{-1}\right\}_{i\alpha} \end{aligned} \tag{1.13}$$

2. DISTRIBUTION OF THE TRANSITION COUNT MATRIX $N(t_0 , t)$

Let

$$V_i(N, t_0, t) = \text{Prob}\{N(t_0, t) = \text{a given matrix } N \text{ i.e.} [N_{jk}] | J(0) = i\} \quad (i = 1, 2, \cdots, m). \tag{2.1}$$

The p.g.f. of this distribution of $N(t_0, t)$ is

$$\phi_i(A, t_0, t) = \sum_{\text{all } N} \prod_{j,k=1}^{m} \left\{ a_{jk}^{N_{jk}} V_i(N, t_0, t) \right\} . \tag{2.2}$$

The L.-S.T. of this p.g.f. (with respect to t) is

$$\psi_i(A, t_0, s) = \int_0^\infty e^{-st} d_t \phi_i(A, t_0, t) , \tag{2.3}$$

and its Laplace transform again w.r.t. t_0 is

$$\psi_i^*(A, s_0, s) = \int_0^\infty e^{-s_0 t_0} \psi_i(A, t_0, s) dt_0 \quad (i = 1, 2, \cdots, m). \tag{2.4}$$

We shall denote by $\underline{\psi}^*(A, s_0, s)$, the column vector of the quantities in (2.4).

Let Γ denote the event that the M.R.P. is in state j at t_0 and that its holding time in this state, measured from t_0 is x and that it makes a transition to state k at the end of this holding time. Then it can be readily seen that, for all $\alpha, \beta = 1, 2, \cdots, m$

$$N_{\alpha\beta}(t_0, t)\left\{\text{given } J(0) = i\right\} = \begin{cases} \delta_{j\alpha}\delta_{k\beta} + N_{\alpha\beta}(t-x) \text{ given } J(0) = k, \text{ if } \Gamma \text{ occurs} \\ \qquad (j, k = 1, 2, \cdots, m) \\ 0, \text{ if there are no transitions in } (t_0, t_0 + t). \end{cases} \tag{2.5}$$

Using the results (1.3) to (1.7) in (2.5) and using (2.3), we obtain

$$\psi_i(A, t_0, s) = \sum_j \sum_k q_{jk}(t_0, s | J(0) = i) a_{jk} \psi_k^0(A, s) + \left\{ 1 - \sum_j \sum_k q_{jk}(t_0, s | J(0) = i) \right\} . \tag{2.6}$$

$$P_r = \left[p_{ij} \int_0^\infty x^r dF_{ij}(x)\right] = \left[p_{ij}\mu_{ij}^{(r)}\right] \quad , r = 0, 1, 2, \cdots ; \mu_{ij}^{(1)} = \mu_{ij}$$

$$a_r = \frac{\text{sum of principal minors of } P_1^{-1}(I - P_0), \text{ of order } m-r-1}{\text{sum of principal minors of } P_1^{-1}(I - P_0), \text{ of order } m-1} ,$$

$(r = 1, 2, \cdots , m-2)$

v_i = cofactor of any element in the i^{th} row of $I - P_0$,

$$H_0 = \text{adj.}(I - P_0) = \underline{e}\,\underline{v}' \;;\; \underline{v}' = [v_1 , \cdots , v_m]$$

$$\frac{1}{\alpha} = \underline{v}'P_1\underline{e} \quad , \quad k_2 = \underline{v}'P_2\underline{e} \quad , \quad k_3 = \underline{v}'P_3\underline{e}$$

$$H_1 = [h_{ij}] \quad , \quad L = [\ell_{ij}] ,$$

$$L = H_2 - a_1H_1 + (a_1^2 - a_2 - \alpha a_1k_2 - \frac{1}{6}\alpha k_3)H_0 + \frac{1}{2}\alpha H_0P_2H_1 + \frac{1}{2}\alpha H_1P_2H_0$$

It was proved by Kshirsagar and Wysocki (1970), using (4.1) that

$$\sigma_{jk}(i,s) = p_{jk}\left\{\frac{\alpha}{s} v_j + av_j + \alpha(h_{ij} - v_j\mu_{jk}) + s\left(\ell_{ij} - \mu_{jk}[av_j + \alpha h_{ij}] + \frac{1}{2}\alpha v_j\mu_{jk}^{(2)}\right) + o(s)\right\} . \quad (4.2)$$

Substituting this in (3.6) and inverting the L.-S.T. with respect to t, we get for large t,

$$m_{jk}^i(s_0 , t) = \frac{1}{s_0}\left[\alpha v_jp_{jk}t + av_jp_{jk} + \alpha p_{jk}(h_{ij} - v_j\mu_{jk}) - \sigma_{jk}(i, s_0) + \frac{\alpha v_jp_{jk}}{s_0} + o(1)\right] . \quad (4.3)$$

Inverting this with respect to s_0 , we get

$$M_{jk}^i(t_0 , t) = \alpha v_jp_{jk}t_0 + \alpha v_jp_{jk}t + av_jp_{jk} + \alpha p_{jk}(h_{ij} - v_j\mu_{jk}) - M_{jk}^i(t_0) + o(1) \quad (4.4)$$

In a similar manner, it can be further shown, for $j \neq \ell$ or $k \neq n$, that

$$
\begin{aligned}
c^{i}_{jk,\ell n}(t_0\ ,\ t) = {} & \alpha^2 p_{jk} p_{\ell n} v_j v_\ell t^2 + \Big[\alpha v_j p_{jk}\{a v_\ell p_{\ell n} + \alpha p_{\ell n}(h_{i\ell} - v_\ell \mu_{\ell n}) \\
& - M^{i}_{\ell n}(t_0) + \alpha v_\ell p_{\ell n} t_0\} + p_{jk}\{a v_j + \alpha(h_{nj} - v_j \mu_{jk})\}\alpha v_\ell p_{\ell n} \\
& + \alpha v_\ell p_{\ell n}\{a v_j p_{jk} + \alpha p_{jk}(h_{ij} - v_j \mu_{jk}) - M^{i}_{jk}(t_0) + \alpha v_j p_{jk} t_0\} \\
& + p_{\ell n}\{a v_\ell + \alpha(h_{k\ell} - v_\ell \mu_{\ell n})\}\alpha v_j p_{jk}\Big] t + o(t) \ . \qquad (4.5)
\end{aligned}
$$

We also can show that

$$
\begin{aligned}
\mathrm{Cov}\{N^{i}_{jk}(t_0\ ,t),\ N^{i}_{\ell n}(t_0\ ,t)\} = {} & 2a\alpha v_j v_\ell p_{jk} p_{\ell n} t + \alpha^2 v_\ell p_{jk} p_{\ell n} t(h_{nj} - v_j \mu_{jk}) \\
& + \alpha^2 v_j p_{jk} p_{\ell n} t(h_{k\ell} - v_\ell \mu_{\ell n}) - \alpha^2 v_j v_\ell p_{jk} p_{\ell n} t_0^2 \\
& - 2a\alpha v_j v_\ell p_{jk} p_\ell t_0 - \alpha^2 v_j p_{jk} p_{\ell n} t_0 (h_{i\ell} - v_\ell \mu_{\ell n}) \\
& - \alpha^2 v_\ell p_{jk} p_{\ell n} t_0 (h_{ij} - v_j \mu_{jk}) + v_j p_{jk} M^{i}_{\ell n}(t_0)(\alpha t_0 + a) \\
& + v_\ell p_{\ell n} M^{i}_{jk}(t_0)(\alpha t_0 + a) + \alpha p_{jk}(h_{ij} - v_j \mu_{jk}) M^{i}_{\ell n}(t_0) \\
& + \alpha p_{\ell n}(h_{i\ell} - v_\ell \mu_{\ell n}) M^{i}_{jk}(t_0) - M^{i}_{jk}(t_0) M^{i}_{\ell n}(t_0) + o(t)
\end{aligned}
$$

(4.6)

and

$$
\begin{aligned}
\mathrm{Var}\{N^{i}_{jk}(t_0\ ,\ t)\} = {} & \alpha v_j p_{jk} t + 2a\alpha v_j^2 p_{jk}^2 (t - t_0) - \alpha^2 v_j^2 p_{jk}^2 t_0^2 \\
& + 2\alpha^2 v_j p_{jk}^2 t(h_{kj} - v_j \mu_{jk}) - 2\alpha^2 v_j p_{jk}^2 t_0 (h_{ij} - v_j \mu_{jk}) \\
& + 2 v_j p_{jk} M^{i}_{jk}(t_0)(\alpha t_0 + a) + 2\alpha p_{jk}(h_{ij} - v_j \mu_{jk}) M^{i}_{jk}(t_0) \\
& - \left(M^{i}_{jk}(t_0)\right)^2 + o(t) \ . \qquad (4.7)
\end{aligned}
$$

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