THEMIS SIGNAL ANALYSIS STATISTICS RESEARCH PROGRAM

SOME USEFUL GENERALIZATIONS IN MARKOV RENEWAL PROCESSES

by

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Technical Report No. 47
Department of Statistics THEMIS Contract

November 4, 1969

Research sponsored by the Office of Naval Research
Contract N00014-68-A-0515
Project NR 042-260

DEPARTMENT OF STATISTICS
Southern Methodist University

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A Dissertation Presented to the Faculty of the Graduate School

of

Southern Methodist University

in

Partial Fulfillment of the Requirements

for the degree of

Doctor of Philosophy

with a

Major in Statistics

by

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September 22, 1969

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Some Useful Generalizations in Markov Renewal Processes

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Doctor of Philosophy degree conferred December 22, 1969

Dissertation completed September 22, 1969

Consider a particular system which at time t may be in one of a finite number of distinguishable states, labeled for convenience by $1, 2, \cdots, m$. Once the system enters a particular state, say i, it instantaneously selects the next state to be visited, say j, with probability p_{ij}. However, transition to state j occurs after holding in state i for a random time (sojourn time) whose distribution function is $F_{ij}(\cdot)$. These processes are known as Semi-Markov Processes and the associated renewal process is called a Markov Renewal Process. In this paper we introduce a new counting process which at time t counts the number of times the system has made a one-step transition from state i to state j, $i,j = 1, 2, \cdots, m$. The matrix N(t) denotes these counts. The distribution and moments of N(t) are derived and cumulative processes associated with N(t) are discussed. We also extend these results to arbitrary intervals of the form $(t_0, t_0 + t]$. Certain limiting results are given and some important special cases discussed. Several open problems are given in the summary chapter.

ACKNOWLEDGMENTS

I am deeply indebted to my adviser Dr. A. M. Kshirsagar for his guidance and genuine interest during the writing of this paper and for his comments as to future research problems in Markov Renewal Processes. I would like to thank the faculty for their guidance during my studies and especially Dr. Paul Minton for his generous financial assistance through an NDEA Fellowship and Dr. Donald B. Owen for the THEMIS Contract which financed my dissertation research.

A special thanks to Mrs. Linda White for her perseverence and excellent typing of a difficult manuscript. Without the many sacrifices and encouragements of my wife Nancy and my family, none of this would have been possible.

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CHAPTER I

PRELIMINARIES

1. Introduction

The theory of Semi-Markov Processes (S.M.P.) was introduced independently by Lévy (1954) and Smith (1955). The first thorough study of S.M.P.'s as well as the introduction of Markov Renewal Processes (M.R.P.) is due to Pyke (1961a, 1961b) and Pyke and Schaufele (1964). The associated problem of estimation is treated at some length in a recent paper by Pyke and Moore (1968).

Examples of M.R.P.'s abound in the physical and social sciences. In general, consider a particular system which at time t may be in one of a finite number of distinguishable states, labeled for convenience by $1, 2, \cdots, m$. Once the system enters a particular state, say i , it instantaneously selects the next state to be visited, say j , with probability p_{ij} . However, transition to state j occurs after holding in state i for a random time (sojourn time) whose distribution function is $F_{ij}(\cdot)$. Note that, in general, the sojourn time distribution depends on both the current state of the system as well as the next state to be visited. A special class of M.R.P.'s in which the imbedded Markov chains are of zero-order arises when $F_{ij}(\cdot)$ depends either on i or j . Pyke (1962) and Kshirsagar and Gupta (1968) have studied these processes called zero-order Markov Renewal Processes.

Note that a S.M.P. is Markovian only at transition instants and thus one may loosely describe a S.M.P. as a Markov chain in which the time axis has been randomly transformed. If the $F_{ij}(\cdot)$ are exponential distributions, then the S.M.P. reduces to a Markov process. If the $F_{ij}(\cdot)$ are degenerate, then the S.M.P. reduces to an ordinary Markov chain with finite state space. Also note that M.R.P. theory embodies ordinary renewal theory on the real line simply by letting $m = 1$.

In the following chapters our main interest centers on distributions which may arise in the study of S.M.P.'s. Our point of departure is the M.R.P. with finite state space as defined by Pyke (1961b). In this chapter we present the known results in M.R.P.'s that will be of use in later chapters.

2. Definitions and Notation

Let $\{\underline{J}_n \; ; \; n \geq 0\}$ be a Markov chain with finite state space denoted by $\{\underline{e}_i \; ; \; i = 1, 2, \cdots, m\}$, where $\underline{e}_j$ is the $m \times 1$ vector of zeroes except for the j^{th} entry which is one. Thus $\underline{J}_n = \underline{e}_i$ means that at the n^{th} transition the system is in state i . Denote the $m \times 1$ vector of initial probabilities, $(a_1 , a_2 , \cdots , a_m)'$, by

$$\Pr\{\underline{J}_0 = \underline{e}_i\} = a_i \quad , \quad i = 1, 2, \cdots , m \quad . \tag{1.2.1}$$

Let X_n be a sequence of non-negative random variables such that

$$\Pr\{\underline{J}_n = \underline{e}_j \; , \; X_n \leq x | \underline{J}_0 \; , \; \underline{J}_1 \; , \cdots , \; \underline{J}_{n-1} \; ; \; X_1 \; , \; X_2 \; , \cdots , \; X_{n-1}\}$$

$$= \Pr\{\underline{J}_n = \underline{e}_j \; , \; X_n \leq x | \underline{J}_{n-1}\} = Q_{\underline{J}_{n-1} , \underline{e}_j}(x) \quad . \tag{1.2.2}$$

Let $\underline{J}_{n-1} = \underline{e}_i$ for some i , then (1.2.2) will be written $Q_{ij}(x)$ for notational convenience. Clearly, $Q_{ij}(x) = p_{ij}F_{ij}(x)$, where $F_{ij}(x)$ is the holding time distribution in state i before transition to state j . We can also write that the probability of transition to any state in the interval $(0,x]$ is given by

$$H_i(x) = \sum_{j=1}^{m} Q_{ij}(x) . \tag{1.2.3}$$

The probability of ultimate transition into state j is given by $Q_{ij}(+\infty) \leq 1$. If the imbedded Markov chain is an irreducible recurrent aperiodic class, which will be assumed throughout this paper, then $H_i(+\infty) = 1$, $i = 1, 2, \cdots, m$. Further, let $S_n = X_0 + X_1 + \cdots + X_n$, $n \geq 0$, $X_0 = 0$, be the time of the n^{th} transition. Then the process $\{\underline{J}_n ; X_n ; n \geq 0\}$ is equivalent to the S.M.P. $\{\underline{J}(t);t \geq 0\} = \underline{J}_n$ for $S_n \leq t < S_{n+1}$. X_n is called the n^{th} sojourn time of the process $\{\underline{J}(t);t \geq 0\}$. Let

$$N(t) = \max\{n ; S_n \leq t\} . \tag{1.2.4}$$

Then $N(t)$ is the number of transitions that the system makes in the interval $(0,t]$. Define the vector-valued random variable

$$\underline{N}(t) = \big(N_1(t), N_2(t), \cdots, N_m(t)\big)' = \begin{cases} \underline{J}_1 + \cdots + \underline{J}_{N(t)} & , \ N(t) > 0 \\ \underline{0} & , \ N(t) = 0 . \end{cases} \tag{1.2.5}$$

Then the process $\{\underline{N}(t);t \geq 0\}$ is a M.R.P. whose j^{th} component represents the number of times the system enters state j in the interval $(0,t]$. We shall say that the M.R.P. $\{\underline{N}(t);t \geq 0\}$ is determined by the triplet $\{m, \underline{a}, Q\}$, where Q is the $m \times m$ matrix of the $Q_{ij}(\cdot)$.

The following notation will be used throughout the paper. Random variables and their distribution functions will be denoted by upper case letters and their corresponding Laplace-Stieltjes Transforms (L.S.T) by lower case letters; i.e., the distribution functions $Q_{ij}(t)$ have L.S.T.'s given by

$$q_{ij}(s) = \int_0^\infty e^{-st} d_t Q_{ij}(t) \quad , \tag{1.2.6}$$

whenever they exist. Also q(s) will denote the $m \times m$ matrix of the $q_{ij}(s)$. Let Q and R be matrix-valued functions and define a convolution operation $Q * R$ by

$$Q * R = \left(\sum_{k=1}^{m} Q_{ik} * R_{kj} \right) \quad ,$$

$$Q_{ik} * R_{kj} = \int_0^t Q_{ik}(t - u) d_u R_{kj}(u) \quad , \tag{1.2.7}$$

which is the same as matrix multiplication except the product is replaced with convolution. Let E, $\underline{e}$ e_{ij} , and $\underline{e}_j$ denote respectively an $m \times m$ matrix of ones, an $m \times 1$ vector of ones, an $m \times m$ null matrix except for the $(i,j)^{th}$ element which equals one, and an $m \times 1$ vector of zeroes whose j^{th} component equals one. If B is an $m \times m$ matrix, then ${}_dB$ and ${}_oB$ are, respectively,

$${}_dB = (\delta_{ij} b_{ij}) \quad , \quad {}_oB = B - {}_dB \quad . \tag{1.2.8}$$

Pyke (1961b) derives the distribution and first moment of $N_j(t)$ in terms of generating functions. We extend his results and derive the distribution and first two moments of $\underline{N}(t)$ in terms of generating functions. Using results from Kshirsagar and Gupta (1967) we find asymptotic expressions for the moments of $\underline{N}(t)$ from series expansions of the L.S.T.'s of the moments. The resulting expressions are in terms of the basic quantities specified by $(m, \underline{a}, Q)$ rather than in terms of mean recurrence distributions as

Pyke (1961b) has done. We also discuss a generalization of $\underline{N}(t)$; namely, let $N_{ij}(t)$ be the number of times the system makes a one step transition from state i to state j in the interval (0,t] and let N(t) be the m × m matrix of the $N_{ij}(t)$. The distribution and moments of N(t) are derived in terms of L.S.T.'s and asymptotic means and variances derived. Cumulative processes associated with $\underline{N}(t)$ and N(t) are discussed. Lastly, the results for the interval (0,t] are extended to the interval $(t_0, t_0 + t]$, for t_0 arbitrary. Certain limiting results are given and some important special cases are discussed.

3. Renewal Theory

A close analogy exists between M.R.P.'s and Renewal Processes on the real line. So that these similarities may be pointed out we present a brief account of Renewal Theory in one dimension. A more detailed account of the following discussion is given by Cox (1962).

Let F(t) be the distribution function of the time between renewals in a one unit system. For an ordinary renewal process we assume that the times between renewals are independent and identically distributed with distribution function F(t). If N(t) denotes the number of renewals in (0,t] and S_r denotes the time of the r^{th} renewal, then

$$\Pr\{N(t) < r\} = \Pr\{S_r > t\} = 1 - F^{(r)}(t) \quad , \qquad (1.3.1)$$

where $F^{(r)}(t)$ is the r-fold convolution of F(t). From (1.3.1) we obtain

$$\begin{aligned} \Pr\{N(t) = r\} &= \Pr\{N(t) < r + 1\} - \Pr\{N(t) < r\} \\ &= 1 - F^{(r+1)}(t) - \{1 - F^{(r)}(t)\} \\ &= F^{(r)}(t) - F^{(r+1)}(t) \quad . \qquad (1.3.2) \end{aligned}$$

The statistical properties of $N(t)$ are often more easily discussed from the probability generating function (p.g.f.) of $N(t)$ and so we define

$$\begin{aligned} G(t,z) &= \sum_{r=0}^{\infty} z^r \Pr\{N(t) = r\} \\ &= \sum_{r=0}^{\infty} z^r F^{(r)}(t) - \sum_{r=0}^{\infty} z^r F^{(r+1)}(t) \\ &= 1 + \sum_{r=1}^{\infty} z^r F^{(r)}(t) - \sum_{r=1}^{\infty} z^{r-1} F^{(r)}(t) \\ &= 1 + \sum_{r=1}^{\infty} z^{r-1}(z-1) F^{(r)}(t) \quad . \end{aligned} \tag{1.3.3}$$

Define

$$f(s) = \int_0^{\infty} e^{-st} d_t F(t) \tag{1.3.4}$$

as the L.S.T. of $F(t)$. The reader will note that Cox (1962) defines

$$f^*(s) = \int_0^{\infty} e^{-st} f(t)\, dt \tag{1.3.5}$$

as the L.S.T. of $f(t)$. However in Cox's notation we have that the L.S.T. of $F(t)$ is

$$\int_0^{\infty} e^{-st} F(t)\, dt = \frac{1}{s} \int_0^{\infty} e^{-st} f(t)\, dt = \frac{1}{s} f^*(s) \tag{1.3.6}$$

and so we make the substitution

$$f(s) = \frac{1}{s} f^*(s) \tag{1.3.7}$$

in order that our notation agrees with that of Cox.

Taking the L.S.T. of (1.3.3) we have in Cox's notation

$$g_0(s,z) = \frac{1}{s} + \frac{1}{s}\sum_{r=1}^{\infty} z^{r-1}(z-1)\big(f^*(s)\big)^r \tag{1.3.8}$$

$$= \frac{1}{s}\big(1 - f^*(s)\big)\Big[1 + zf^*(s) + \big(zf^*(s)\big)^2 + \cdots\Big]$$

$$= \frac{1 - f^*(s)}{s\{1 - zf^*(s)\}} \tag{1.3.9}$$

for the ordinary renewal process. For the modified renewal process we obtain

$$g_m(s,z) = \frac{1 - zf^*(s) + zf_1^*(s) - f_1^*(s)}{s\{1 - zf^*(s)\}} \quad . \tag{1.3.10}$$

In our notation

$$g_0(s,z) = [1 - f(s)][1 - zf(s)]^{-1} \tag{1.3.11}$$

and

$$g_m(s,z) = 1 - (1 - z)f_1(s)[1 - zf(s)]^{-1} \quad . \tag{1.3.12}$$

The transform of the renewal function, $H(t) = E\big(N(t)\big)$, is in our notation

$$h_0(s) = \frac{\partial g_0(s,z)}{\partial z}\bigg|_{z=1} = f(s)[1 - f(s)]^{-1} \quad , \tag{1.3.13}$$

$$h_m(s) = \frac{\partial g_m(s,z)}{\partial z}\bigg|_{z=1} = f_1(s)[1 - f(s)]^{-1} \quad . \tag{1.3.14}$$

In all but the simplest cases $h_0(s)$ cannot be inverted exactly and one is forced to consider the asymptotic behavior of the transform of the renewal function. Now for an arbitrary function $k(x)$,

$$k(s) = \int_0^\infty e^{-sx} d_x k(x) \quad .$$

If x is large, k(s) is negligible unless s is small. Hence the form of the function $H_0(t)$ for large t can be studied by examining the behavior of $h_0(s)$ for small s . Letting μ and σ^2 be the mean and variance of F(t) we have for large t; i.e., small s,

$$f(s) = 1 - s\mu + \frac{1}{2} s^2(\mu^2 + \sigma^2) + 0(s^2) \quad . \tag{1.3.15}$$

Substituting into (1.3.13) and simplifying gives

$$h_0(s) = \frac{1}{s\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} + 0(1) \quad , \tag{1.3.16}$$

whose inverse is, for large t,

$$H_0(t) = \frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} + 0(1) \quad . \tag{1.3.17}$$

At this point we require no further results from ordinary renewal theory. As M.R.P. theory is developed we shall point out some of the analogies with ordinary renewal theory.

4. Basic Results in M.R.P.'s

We turn now to a discussion of the main results in M.R.P.'s that have appeared in the literature since the publication of Pyke's papers in 1961.

Define

$$V_{ij}(k;t) = \Pr\{N_j(t) = k \mid \underline{J}(0) = \underline{e}_i\} = \Pr\{N_j^i(t) = k\} \quad . \tag{1.4.1}$$

Then we immediately have that

$$V_{ij}(0,t) = 1 - H_i(t) + \sum_{r \neq j} Q_{ir}(t) * V_{rj}(0;t) \quad ,$$

$$V_{ij}(k;t) = Q_{ij}(t) * V_{jj}(k-1;t) + \sum_{r \neq j} Q_{ir}(t) * V_{rj}(k;t) \quad . \qquad (1.4.2)$$

Let, for $|z| \leq 1$, $i,j = 1, 2, \cdots , m$,

$$\phi_{ij}(z;t) = \sum_{k=0}^{\infty} z^k V_{ij}(k;t) \qquad (1.4.3)$$

be the probability generating function (p.g.f.) of the $V_{ij}(k;t)$ and define

$$\psi_{ij}(z;s) = \int_0^{\infty} e^{-st} d_t \phi_{ij}(z;t) \qquad (1.4.4)$$

as the L.S.T. of $\phi_{ij}(z;t)$. Applying (1.4.3) to the system of equations (1.4.2) gives

$$\phi_{ij}(z;t) = 1 - H_i(t) + \sum_{r \neq j} Q_{ir}(t) * V_{rj}(0;t) + z \sum_{k=1}^{\infty} z^{k-1} V_{ij}(k;t)$$

$$= 1 - H_i(t) + \sum_{r \neq j} Q_{ir}(t) * V_{rj}(0;t)$$

$$+ z \sum_{k=1}^{\infty} z^{k-1} Q_{ij}(t) * V_{jj}(k-1;t)$$

$$+ z \sum_{k=1}^{\infty} z^{k-1} \sum_{r \neq j} Q_{ir}(t) * V_{rj}(k;t)$$

$$= 1 - H_i(t) + zQ_{ij}(t) * \phi_{jj}(z;t) + \sum_{k=0}^{\infty} z^k \sum_{r \neq j} Q_{ir}(t) * V_{rj}(k;t)$$

$$= 1 - H_i(t) + (z-1)Q_{ij}(t) * \phi_{jj}(z;t) + \sum_{r=1}^{m} Q_{ir}(t) * \phi_{rj}(z;t)$$

whose L.S.T. is

$$\psi_{ij}(z;s) = 1 - h_i(s) + (z - 1)q_{ij}(s)\phi_{jj}(z;s) + \sum_{r=1}^{m} q_{ir}(s)\phi_{rj}(z;s) \ , \tag{1.4.5}$$

or in matrix notation

$$\psi_z = \big(I - q(s)\big)E + (z - 1)q(s)\,{}_d\psi_z + q(s)\psi_z \ . \tag{1.4.6}$$

$\big(I - q(s)\big)$ is nonsingular hence

$$\psi_z + (1 - z)\big(I - q(s)\big)^{-1}q(s)\,{}_d\psi_z = E \ . \tag{1.4.7}$$

Taking diagonal elements on both sides of (1.4.7) gives

$${}_d\psi_z + (1 - z)\,{}_d\Big\{\big(I - q(s)\big)^{-1}q(s)\Big\}\,{}_d\psi_z = I \ ,$$

so that

$${}_d\psi_z = \Big[I + (1 - z)\,{}_d\Big\{\big(I - q(s)\big)^{-1}q(s)\Big\}\Big]^{-1} \ . \tag{1.4.8}$$

Substituting into (1.4.7) gives

$$\psi_z = E - (1 - z)\big(I - q(s)\big)^{-1}q(s)\Big[I + (1 - z)\,{}_d\Big\{\big(I - q(s)\big)^{-1}q(s)\Big\}\Big]^{-1} \ . \tag{1.4.9}$$

<u>Lemma</u>: $(I - A)^{-1}A = A(I - A)^{-1} = (I - A)^{-1} - I$.

PROOF: $(I - A)^{-1}A = (I + A + A^2 + \cdots)A$

$$= A + A^2 + \cdots$$

$$= A(I + A + A^2 + \cdots) = A(I - A)^{-1} \ .$$

Also

$$A + A^2 + \cdots = (I + A + A^2 + \cdots) - I$$

$$= (I - A)^{-1} - I \quad .$$

Using the Lemma in (1.4.9) gives

$$\psi_z = E - (1 - z)\big(I - q(s)\big)^{-1}q(s)\Big[I + (1 - z)_d\big\{\big(I - q(s)\big)^{-1} - I\big\}\Big]^{-1}$$

$$= E - (1 - z)\big(I - q(s)\big)^{-1}q(s)\Big[zI + (1 - z)_d\big\{\big(I - q(s)\big)^{-1}\big\}\Big]^{-1} \quad , \qquad (1.4.10)$$

a result obtained by Pyke (1961b). The L.S.T.'s of the moments of the $V_{ij}(k;t)$ are easily obtained by differentiation of (1.4.10). In particular define

$$M_{ij}(t) = E\{N_j^i(t)\} \qquad (1.4.11)$$

$$R_{ij}(t) = E\{N_j^i(t)\,[N_j^i(t) - 1]\} \qquad (1.4.12)$$

and let $m_{ij}(s)$ and $r_{ij}(s)$ denote their L.S.T.'s respectively. Setting $m(s) = \big(m_{ij}(s)\big)$ we have

$$m(s) = \frac{d}{dz}\psi_z \Big|_{z=1} \quad .$$

Differentiating (1.4.10) gives

$$\frac{d}{dz}\psi_z = \big(I - q(s)\big)^{-1}q(s)\Big[zI + (1 - z)_d\big\{\big(I - q(s)\big)^{-1}\big\}\Big]^{-1}$$

$$+ (1 - z)\big(I - q(s)\big)^{-1}q(s)\Big[zI + (1 - z)_d\big\{\big(I - q(s)\big)^{-1}\big\}\Big]^{-1} \times$$

$$\Big[I - {}_d\big\{\big(I - q(s)\big)^{-1}\big\}\Big]\Big[zI + (1 - z)_d\big\{\big(I - q(s)\big)^{-1}\big\}\Big]^{-1} \quad .$$

Setting $z = 1$ gives

$$m(s) = \left(I - q(s)\right)^{-1} q(s) \tag{1.4.13}$$

as first derived by Pyke (1961b). In ordinary renewal theory we have shown that the L.S.T. of $E\{N(t)\}$ is given by

$$h_0(s) = f(s)[1 - f(s)]^{-1}$$

which is the same as $m(s)$ with the matrix $q(s)$ in place of the scalar $f(s)$.

Note from (1.4.13) that $\left(m, \underline{a}, M(t)\right)$ completely determines the M.R.P. Differentiating a second time and setting $z = 1$ gives the matrix $r(s) = \left(r_{ij}(s)\right)$; i.e.,

$$r(s) = 2\left[\left(I - q(s)\right)^{-1} q(s)\right]_d \left\{\left(I - q(s)\right)^{-1} - I\right\}$$

$$= 2m(s)_d\{m(s)\} \quad . \tag{1.4.14}$$

From (1.4.13) and (1.4.14) the second moments and variance are obtainable.

5. Probability Distribution of $\underline{N}(t)$

Çinlar (1968) and Kshirsagar and Gupta (1969a) generalized the results of I.4 and derived the joint distribution of the vector $\underline{N}(t)$ and $\underline{J}(t)$. For the purposes of this paper we derive, in a manner analogous to Çinlar's, the joint distribution of the vector $\underline{N}(t)$. Letting $\underline{k}$ be a vector of non-negative integers define

$$V_i(\underline{k};t) = \Pr\{\underline{N}(t) = \underline{k} \mid \underline{J}(0) = \underline{e}_i\} = \Pr\{\underline{N}^i(t) = \underline{k}\} \tag{1.5.1}$$

Let

$$\phi_i(z_1, z_2, \cdots, z_m; t) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} z_1^{k_1} \cdots z_m^{k_m} V_i(\underline{k};t)$$

and

$$\psi_i(z_1, \cdots, z_m; s) = \int_0^\infty e^{-st} d_t \phi_i(z_1, \cdots, z_m; t) ,$$

where $|z_i| \le 1$, $i = 1, 2, \cdots, m$. Let $Z = \text{diag}(z_1, \cdots, z_m)$ and denote $\underline{\psi}_Z = \left(\psi_i(z_1, \cdots, z_m; s)\right)'$. Then

$$V_i(\underline{0};t) = 1 - H_i(t) ,$$

$$V_i(\underline{k};t) = \sum_{r=1}^{m} Q_{ir}(t) * V_r(\underline{k} - \underline{e}_r ; t) , \tag{1.5.2}$$

where it is understood that summation in (1.5.2) extends over those states $\underline{e}_r$ having $k_r > 0$. Taking the p.g.f. in (1.5.2) one obtains

$$\phi_i(z_1, \cdots, z_m; t) = 1 - H_i(t) + \sum_{r=1}^{m} Q_{ir}(t) z_r \phi_r(z_1, \cdots, z_m; t)$$

whose L.S.T. is

$$\psi_i(z_1, \cdots, z_m; s) = 1 - h_i(s) + \sum_{r=1}^{m} q_{ir}(s) z_r \psi_r(z_1, \cdots, z_m; s) \tag{1.5.3}$$

or in matrix notation

$$\underline{\psi}_Z = q(s) Z \underline{\psi}_Z + \left(I - q(s)\right)\underline{e} . \tag{1.5.4}$$

Note that $|z_i| \le 1$, $i = 1, 2, \cdots, m$ and $\left(I - q(s)\right)$ nonsingular imply that $\left(I - q(s)Z\right)$ is also nonsingular so that we can solve (1.5.4) for $\underline{\psi}_Z$ obtaining

$$\underline{\psi}_Z = \left(I - q(s)Z\right)^{-1}\left(I - q(s)\right)\underline{e} . \tag{1.5.5}$$

Note that $\underline{\psi}_0 = (I - q(s))\underline{e}$ and $\underline{\psi}_E = \underline{e}$ as required. The analogous result for ordinary renewal theory is $(1 - f(s)z)^{-1}(1 - f(s))$ which is remarkably similar to $\underline{\psi}_Z$ above. Let

$$\underline{M}(t) = E\{\underline{N}^i(t)\} \tag{1.5.6}$$

and

$$c^i_{jk}(t) = E\{N_j(t)N_k(t) \,|\, \underline{J}(0) = \underline{e}_i\} = E\{N^i_j(t)N^i_k(t)\} \quad . \tag{1.5.7}$$

Their L.S.T.'s will be denoted by $\underline{m}(s)$ and $c^i_{jk}(s)$ respectively. Differentiating (1.5.5) and setting $Z = I$ we obtain, omitting details,

$$\underline{m}(s) = (I - q(s))^{-1}q(s)\underline{e}$$

$$= m(s)\underline{e} \quad . \tag{1.5.8}$$

To obtain the product moments (1.5.7) we take the mixed partial $\partial/\partial z_j \partial z_k$, and evaluate the resulting expression at $Z = I$. In particular,

$$\frac{\partial \underline{\psi}_Z}{\partial z_j} = (I - q(s)Z)^{-1}q(s)e_{jj}(I - q(s)Z)^{-1}(I - q(s))\underline{e}$$

and

$$\frac{\partial^2 \underline{\psi}_Z}{\partial z_j \partial z_k} = (I - q(s)Z)^{-1}q(s)e_{kk}(I - q(s)Z)^{-1}q(s)e_{jj}(I - q(s)Z)^{-1}(I - q(s))\underline{e}$$

$$+ (I - q(s)Z)^{-1}q(s)e_{jj}(I - q(s)Z)^{-1}q(s)e_{kk}(I - q(s)Z)^{-1}(I - q(s))\underline{e} \ .$$

For $Z = I$ we obtain the vector $\underline{c}_{jk}(s) = (c^i_{jk}(s))'$; i.e.,

$$\underline{c}_{jk}(s) = m(s)e_{kk}m(s)e_{jj}\underline{e} + m(s)e_{jj}m(s)e_{kk}\underline{e} \quad , \tag{1.5.9}$$

whose i^{th} element is

$$c^i_{jk}(s) = m_{ik}(s)m_{kj}(s) + m_{ij}(s)m_{jk}(s) \quad . \qquad (1.5.10)$$

For $j = k$,

$$c^i_{jj}(s) = 2m_{ij}(s)m_{jj}(s) + m_{ij}(s) \quad . \qquad (1.5.11)$$

So that for any j and k

$$c^i_{jk}(s) = \delta_{jk}m_{ij}(s) + m_{ik}(s)m_{kj}(s) + m_{ij}(s)m_{jk}(s) \quad . \qquad (1.5.12)$$

Çinlar (1968) also defines a new vector-valued process which at time t gives the amounts of times spent in the various states in the interval (0,t]. He also considers the vector-valued process which gives the amounts of times spent in the various states at the instant of the n^{th} transition. Various distributions associated with M.R.P.'s in space are introduced and briefly discussed. We do not present details of his results here, since they do not pertain to results in later chapters. It suffices to note that Çinlar's results in m dimensional space are the exact analogue for a simple renewal process on the real line.

From (1.5.3) or (1.5.5) various marginal distributions are easily obtained. For example, in (1.4.3) let $Z^* = e + (z - 1)e_{jj}$ to obtain an expression for the marginal distribution of $N_j(t)$. Formally we obtain, deleting arguments of the ψ_i ,

$$\psi^0_i = 1 - h_i(s) + \sum_{r=1}^{m} q_{ir}(s)\psi^0_r + (z - 1)q_{ij}(s)\psi^0_j \quad . \qquad (1.5.13)$$

Let $q_{(i)}(s)$ be the i^{th} column of the matrix $q(s)$. Then (1.5.13) may be written in matrix notation in the form

$$\underline{\psi}^0 = (z - 1)\psi_j^0\left(I - q(s)\right)^{-1}q_{(j)}(s) + \underline{e} \quad . \tag{1.5.14}$$

The j^{th} element of $\underline{\psi}^0$ is then

$$\psi_j^0 = (z - 1)\psi_j^0\left\{\left(I - q(s)\right)^{-1}q(s)\right\}_{jj} + 1 \quad .$$

Let $a_{ij}(s) = \left\{\left(I - q(s)\right)^{-1}q(s)\right\}_{ij}$. Then

$$\psi_j^0 = [1 - (z - 1)a_{jj}(s)]^{-1} \quad .$$

Substituting into (1.5.14) gives

$$\underline{\psi}^0 = \frac{(z - 1)\left(I - q(s)\right)^{-1}q_{(j)}(s)}{1 - (z - 1)a_{jj}(s)} + \underline{e} \quad .$$

Whose i^{th} element is

$$\psi_{ij} = \psi_i^0 = \frac{(z - 1)a_{ij}(s)}{1 - (z - 1)a_{jj}(s)} + 1 \quad , \tag{1.5.15}$$

or matrix notation

$$\psi_z = (z - 1)\left(I - q(s)\right)^{-1}q(s)\left[I - (z - 1)_d\left\{\left(I - q(s)\right)^{-1}q(s)\right\}\right]^{-1} + E$$

$$= E - (1 - z)\left(I - q(s)\right)^{-1}q(s)\left[zI + (1 - z)_d\left\{\left(I - q(s)\right)^{-1}\right\}\right]^{-1} \quad ,$$

as first derived by Pyke (1961b). Similarly, the joint marginal distribution of $\left(N_j(t), N_k(t)\right)$ is found by setting $Z^* = E + (z_j - 1)e_{jj} + (z_k - 1)e_{kk}$ in (1.5.3). Then it is easily shown that

$$\underline{\psi}^0 = \alpha\psi_j^0\left(I - q(s)\right)^{-1}q_{(j)}(s) + \beta\psi_k^0\left(I - q(s)\right)^{-1}q_{(k)}(s) + \underline{e} \quad , \tag{1.5.16}$$

where $\alpha = z_j - 1$, $\beta = z_k - 1$. As in the previous case we take the j^{th} and k^{th} elements on both sides of (1.5.16) giving the following system of equations

$$\psi_j^0 = \alpha\psi_j^0 a_{jj}(s) + \beta\psi_k^0 a_{jk}(s) + 1 \quad ,$$

$$\psi_k^0 = \alpha\psi_j^0 a_{kj}(s) + \beta\psi_k^0 a_{kk}(s) + 1 \quad . \qquad (1.5.17)$$

The solution is easily shown to be

$$\psi_j^0 = \frac{1 - \beta\left(a_{kk}(s) - a_{jk}(s)\right)}{\left(1 - \alpha a_{jj}(s)\right)\left(1 - \beta a_{kk}(s)\right) - \alpha\beta a_{kj}(s) a_{jk}(s)} \quad ,$$

$$\psi_k^0 = \frac{1 - \alpha\left(a_{jj}(s) - a_{kj}(s)\right)}{\left(1 - \alpha a_{jj}(s)\right)\left(1 - \beta a_{kk}(s)\right) - \alpha\beta a_{kj}(s) a_{jk}(s)} \quad .$$

Since

$$\psi_i^0 = \alpha\psi_j^0 a_{ij}(s) + \beta\psi_k^0 a_{kj}(s) + 1 \quad ,$$

we may write

$$\psi_i^0 = \frac{\alpha a_{ij}(s)\left(1 - \beta a_{kk}(s) + \beta a_{jk}(s)\right) + \beta a_{ik}(s)\left(1 - \alpha a_{jj}(s) + \alpha a_{kj}(s)\right)}{\left(1 - \alpha a_{jj}(s)\right)\left(1 - \beta a_{kk}(s)\right) - \alpha\beta a_{jk}(s) a_{kj}(s)} + 1 \ . \qquad (1.5.18)$$

Note that ψ_i^0 at $z_j = z_k = 1$ equals one as required and that

$$\left.\frac{\partial\psi_i^0}{\partial\alpha}\right|_{\alpha=\beta=0} = a_{ij}(s) \quad , \quad \left.\frac{\partial\psi_i^0}{\partial\beta}\right|_{\alpha=\beta=0} = a_{ik}(s) \quad ,$$

in agreement with (1.4.13). From (1.5.18) or (1.5.5) one can determine the $c_{jk}^i(s)$, the L.S.T.'s of $E\{N_j^i(t)N_k^i(t)\}$.

6. Extension to Arbitrary Intervals

In this section we show how the results of Sections 4 and 5 may be extended to arbitrary intervals $(t_0 , t_0 + t]$. One often encounters this situation in physical applications since seldom do we observe systems from the start of operation. One is more likely to begin observing a system after it has been in operation for a time t_0 , where t_0 may be large. Following the derivation given in Kshirsagar and Gupta (1969b) the distribution of the vector $\underline{N}(t_0 , t)$, whose j^{th} component is the number of times the system visits the j^{th} state in $(t_0 , t_0 + t]$, is derived. Expressions for the L.S.T.'s of the first two moments are also given. As in Sections 4 and 5 denote the moments for the interval $(t_0 , t_0 + t]$ by $M_{ij}(t_0 , t)$, $R_{ij}(t_0 , t)$, and $C^i_{jk}(t_0 , t)$. Their L.S.T.'s, with respect to t, will be denoted by $m_{ij}(t_0 , s)$, $r_{ij}(t_0 , s)$, and $c^i_{jk}(t_0 , s)$ respectively. Also we define their L.S.T.'s, with respect to t_0 , by

$$m_{ij}(s_0 , s) = \int_0^\infty e^{-s_0 t_0} m_{ij}(t_0 , s)\,dt_0 \quad ,$$

$$r_{ij}(s_0 , s) = \int_0^\infty e^{-s_0 t_0} r_{ij}(t_0 , s)\,dt_0 \quad ,$$

$$c^i_{jk}(s_0 , s) = \int_0^\infty e^{-s_0 t_0} c^i_{jk}(t_0 , s)\,dt_0 \quad ,$$

respectively. From Cox (1962, pg. 26) we have that the renewal density, which specifies the mean number of renewals in small intervals near t , is defined by

$$h_{ij}(t) = \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} E\left(N^i_j(t, t + \Delta t)\right)$$

$$= \frac{d}{dt} M_{ij}(t) \quad .$$

This represents the probability of a transition from i to j in the interval $(t, t + \Delta t)$. Also its L.S.T. is given by

$$h_{ij}(s) = \int_0^\infty e^{-st} h_{ij}(t)\,dt = m_{ij}(s)$$

$$= \left\{ \left(I - q(s)\right)^{-1} \right\}_{ij} - \delta_{ij} \quad , \tag{1.6.1}$$

as noted by Kshirsagar and Gupta (1969b). We consider now a generalization to S.M.P.'s of the distribution of forward recurrence-time. Note that the distribution of the first transition after t_0 is different than all subsequent transitions in $(t_0 \, , \, t_0 + t]$. Let $Q_{jk}(t_0 \, , \, x^0 | \underline{J}(0) = \underline{e}_i)$ denote the probability that the first transition after time t_0 is from state j to k with sojourn time, measured from t_0 , less than or equal to x^0 . For transition to occur in the interval $(x^0 \, , \, x^0 + \Delta x^0)$, we have from Cox (1962, pg. 62) that

a) the first transition occurs in the interval $(t_0 + x^0 \, , \, t_0 + x^0 + \Delta x^0)$. This occurs with probability $\delta_{ij} \dfrac{d}{dx^0} Q_{jk}(t_0 + x^0)$, or

b) for some u transition to state j occurs in the interval $(t_0 - u, \, t_0 - u + \delta u)$ and then transition to state k occurs in the interval $(u + x^0 \, , \, u + x^0 + \Delta x^0)$. This occurs with probability

$$\int_0^{t_0} h_{ij}(t_0 - u) \frac{d}{dx^0} Q_{jk}(u + x^0)\,du \quad .$$

Summing the contributions from (a) and (b) gives the p.d.f. of the first transition after t_0 . Thus we have shown

$$\frac{d}{dx^0} Q_{jk}\left(t_0 \, , \, x^0 | \underline{J}(0) = \underline{e}_i\right) = \delta_{ij} \frac{d}{dx^0} Q_{jk}(t_0 + x^0)$$

$$+ \int_0^{t_0} h_{ij}(t_0 - u) \frac{d}{dx^0} Q_{jk}(u + x_0)\,du \quad , \tag{1.6.2}$$

which is analogous to the p.d.f. of forward recurrence time in ordinary renewal theory given by Cox (1962, pg. 63). Let $q^*_{jk}(s_0\ ,\ s|\underline{J}(0) = \underline{e}_i)$ be the L.S.T. of (1.6.2), i.e.

$$q^*_{jk}(s_0\ ,\ s|\underline{J}(0) = \underline{e}_i) = \int_0^\infty e^{-s_0t_0} \int_0^\infty e^{-sx}\left[\frac{d}{dx} Q_{jk}(t_0\ ,\ x|\underline{J}(0) = \underline{e}_i)\right]dt_0 \ .$$

$$= \delta_{ij} \int_0^\infty e^{-s_0t_0} \int_0^\infty e^{-sx} \frac{d}{dx} Q_{jk}(t_0 + x)\,dt_0$$

$$+ \int_0^\infty e^{-s_0t_0} \int_0^\infty e^{-sx} \int_0^{t_0} h_{ij}(t_0-u)\frac{d}{dx} Q_{jk}(u+x)\,du\,dt_0 \ . \tag{1.6.3}$$

In the first integral let $v = t_0 + x$ giving

$$\delta_{ij} \int_0^\infty e^{-s_0t_0} \int_{t_0}^\infty e^{-s(v-t_0)} \frac{d}{dv} Q_{jk}(v)\,dt_0 = \delta_{ij} \int_0^\infty e^{-sv} \int_0^v e^{-t_0(s_0-s)} dt_0 \frac{d}{dv} Q_{jk}(v)$$

$$= \frac{-\delta_{ij}}{s_0-s} \int_0^\infty e^{-sv}\left[e^{-v(s_0-s)} - 1\right]\frac{d}{dv} Q_{jk}(v)$$

$$= \frac{\delta_{ij}}{s_0-s} \left[q_{jk}(s) - q_{jk}(s_0)\right] . \tag{1.6.4}$$

The second integral can be written in the form

$$\int_0^\infty e^{-sx} \int_0^\infty \frac{d}{dx} Q_{jk}(u + x) \int_u^\infty e^{-s_0t_0} h_{ij}(t_0 - u)\,dt_0\,du \ .$$

Let $v = t_0 - u$ giving

$$\int_0^\infty e^{-sx} \int_0^\infty \frac{d}{dx} Q_{jk}(u + x) \int_0^\infty e^{-s_0(u+v)} h_{ij}(v)\,dv\,du$$

$$= h_{ij}(s_0) \int_0^\infty e^{-s_0u} \int_0^\infty e^{-sx} \frac{d}{dx} Q_{jk}(u + x)\,du \ .$$

Let $v = u + x$ giving

$$= h_{ij}(s_0) \int_0^\infty e^{-sv} \frac{d}{dv} Q_{jk}(v) \int_0^v e^{-s_0 u + su}\, du$$

$$= \frac{-h_{ij}(s_0)}{s_0 - s} \int_0^\infty e^{-sv}\left[e^{-v(s_0-s)} - 1\right] \frac{d}{dv} Q_{jk}(v)$$

$$= \frac{h_{ij}(s_0)}{s_0 - s}\left[q_{jk}(s) - q_{jk}(s_0)\right] \quad . \tag{1.6.5}$$

Using (1.6.4) and (1.6.5) in (1.6.3) gives

$$q^*_{jk}(s_0 \,,\, s \,|\, \underline{J}(0) = \underline{e}_i) = \frac{\left(\delta_{ij} + h_{ij}(s_0)\right)}{s_0 - s}\left(q_{jk}(s) - q_{jk}(s_0)\right)$$

$$= \frac{\left\{\left(I - q(s_0)\right)^{-1}\right\}_{ij}}{s_0 - s}\left(q_{jk}(s) - q_{jk}(s_0)\right) \,, \tag{1.6.6}$$

where we have used (1.6.1). Define $v_i(Z \,,\, t_0 \,,\, s)$ as the L.S.T. with respect to t of the p.g.f. of $\underline{N}(t_0 \,,\, t)$ given the initial state is i . Then

$$v_i(Z,t_0,s) = \sum_j \sum_k q^*_{jk}(t_0 \,,\, s \,|\, \underline{J}(0) = \underline{e}_i) z_k v_k(Z,s) + 1 - \sum_j \sum_k q^*_{jk}(t_0, s \,|\, \underline{J}(0) = \underline{e}_i) \quad . \tag{1.6.7}$$

Taking L.S.T.'s with respect to t_0 gives

$$v_i(Z;s_0 \,,\, s) = \sum_{jk} \left\{\left(I - q(s_0)\right)^{-1}\right\}_{ij}\left(q_{jk}(s) - q_{jk}(s_0)\right) z_k v_k(Z,s)/(s_0 - s)$$

$$+ \frac{1}{s_0} - \sum_{jk} \left\{\left(I - q(s_0)\right)^{-1}\right\}_{ij}\left(q_{jk}(s) - q_{jk}(s_0)\right)/(s_0 - s) \quad , \tag{1.6.8}$$

or in matrix notation

$$\underline{v}(Z\ ,\ s_0\ ,\ s) = \frac{1}{s_0}\underline{e} + \frac{1}{s_0-s}\Big(I - q(s_0)\Big)^{-1}\Big(q(s) - q(s_0)\Big)\Big(Z\underline{v}(Z;s) - \underline{e}\Big)\ . \tag{1.6.9}$$

Kshirsagar and Gupta (1969b) find the L.S.T.'s of the first and second order moments of the $N_j(t_0\ ,\ t)$ by differentiating (1.6.9). Omitting details one obtains

$$m(s_0\ ,\ s) = \frac{1}{s_0-s}\Big(m(s) - m(s_0)\Big)\ , \tag{1.6.10}$$

$$r(s_0\ ,\ s) = \frac{2}{s_0-s}\{m(s) - m(s_0)\}_d m(s)\ , \tag{1.6.11}$$

and

$$c^i_{uv}(s_0\ ,\ s) = \sum_{jk}\left\{\Big(I - q(s_0)\Big)^{-1}\right\}_{ij}\Big(q_{jk}(s) - q_{jk}(s_0)\Big)\Big(c^k_{uv}(s) + \beta m_{uv}(s)\Big)\ , \tag{1.6.12}$$

where $\beta = 1$ if $u = k$ or if $v = k$.

7. Asymptotic Results

The results of the previous sections are all in terms of the L.S.T.'s of various distributions and their first two moments. Inversion of these L.S.T.'s is possible only in the simplest cases. A feasible approach to this problem involves the inversion of power series expansions in s of the quantities of interest. Kshirsagar and Gupta (1967) take this approach and show how one obtains asymptotic results in terms of the basic quantities $(m, \underline{a}, Q)$ rather than in terms of recurrence-time distributions as done by Pyke (1961b). The details of the following argument can be found in Kshirsagar and Gupta (1967). Referring to the results of the previous sections, the reader will observe that the quantity $\Big(I - q(s)\Big)^{-1}$ appears often in the L.S.T.'s of the moments and distributions derived. In the following argument, due to Kshirsagar and Gupta (1967), we derive an asymptotic expression for $\left(I - q(s)\right)^{-1}$ and show how the resulting expression is used to obtain limiting results for several cases discussed in the previous section. Recall that $Q_{ij}(t) = p_{ij}F_{ij}(t)$ and so $q_{ij}(s) = p_{ij}f_{ij}(s)$. Let $\mu_{ij}^{(k)}$ denote the k^{th} order moment of $F_{ij}(t)$ and write

$$q_{ij}(s) = p_{ij}f_{ij}(s) = p_{ij}\{1 - s\mu_{ij} + \frac{s^2}{2!}\mu_{ij}^{(2)} - \frac{s^3}{3!}\mu_{ij}^{(3)} + \cdots\} \quad . \quad (1.7.1)$$

In matrix notation the first few terms provide a power series in s of $\left(I - q(s)\right)$; that is,

$$\left(I - q(s)\right) = I - P_0 + sP_1 - \frac{1}{2}s^2P_2 + \frac{1}{6}s^3P_3 - \frac{1}{24}s^4P_4 + o(s^4)$$

$$= (I - P_0 + sP_1)[I - \frac{1}{2}s^2(I - P_0 + sP_1)^{-1}$$

$$\times \{P_2 - \frac{1}{3}sP_3 + \frac{1}{12}s^2P_4 + o(s^2)\}] \quad ,$$

where the matrix $P_k = (p_{ij}\mu_{ij}^{(k)})$. Hence from the above we immediately have that

$$\left(I - q(s)\right)^{-1} = [I - \frac{1}{2}s^2(I - P_0 + sP_1)^{-1}\{P_2 - \frac{1}{3}sP_3 + \frac{1}{12}s^2P_4 + o(s^2)\}]^{-1}$$

$$\times (I - P_0 + sP_1)^{-1} \quad . \quad (1.7.2)$$

In order to evaluate this expression we first need an expression for $(I - P_0 + sP_1)^{-1}$. Now

$$\frac{1}{|I - P_0 + sP_1|} = \frac{1}{|P_1|\cdot|P_1^{-1}(I - P_0) + sI|}$$

$$= \frac{\alpha}{s}\prod_{i=1}^{m-1}(1 + s/\lambda_i)^{-1} \quad , \quad (1.7.3)$$

where $\lambda_1, \lambda_2, \cdots, \lambda_{m-1}, 0$ are the eigenvalues of $P_1^{-1}(I - P_0)$. The root zero is of multiplicity one, since $\sum_{j=1}^{m} p_{ij} = 1,\ i = 1, 2, \cdots, m$ implies that $(I - P_0)\underline{e} = \underline{0}$. Also $1/\alpha = |P_1|\lambda_1\lambda_2 \cdots \lambda_{m-1}$. The first few terms in (1.7.3) are

$$\frac{\alpha}{s}\left\{1 - s\sum_{i=1}^{m-1}\frac{1}{\lambda_i} + s^2\left(\sum_{i=1}^{m-1}\frac{1}{\lambda_i^2} + \sum_{i<j}\frac{1}{\lambda_i\lambda_j}\right)\right.$$

$$\left. - s^3\left(\sum_{i=1}^{m-1}\frac{1}{\lambda_i^3} + \sum_{i<j}\frac{1}{\lambda_i^2\lambda_j} + \sum_{i<j}\frac{1}{\lambda_i\lambda_j^2} + \sum_{i<j<k}\frac{1}{\lambda_i\lambda_j\lambda_k}\right) + o(s^3)\right\}$$

$$= \frac{\alpha}{s}\left\{1 - a_1 s + a_2 s^2 - a_3 s^3 + o(s^3)\right\} . \qquad (1.7.4)$$

Rewriting $\mathrm{adj}(I - P_0 + sP_1)$ in powers of s as

$$\mathrm{adj}(I - P_0 + sP_1) = \sum_{r=0}^{m-1} s^r H_r , \qquad (1.7.5)$$

where H_r is an $m \times m$ matrix. Kshirsagar and Gupta (1967) have shown that

$$H_0 P_1 H_0 = \underline{e}\underline{v}' P_1 \underline{e}\underline{v}' = k_1 H_0 , \quad k_1 = \underline{v}' P_1 \underline{e} , \qquad (1.7.6)$$

where v_i is the cofactor of any element in the i^{th} row of $(I - P_0)$ and

$$H_0 = \underline{e}\underline{v}' = (\underline{v}', \underline{v}', \cdots, \underline{v}') . \qquad (1.7.7)$$

From (1.7.6) and (1.7.7) one easily shows that

$$H_0 P_i H_0 = k_i H_0 , \quad k_i = \underline{v}' P_i \underline{e} , \quad i = 2, 3, \cdots .$$

Combining (1.7.4) and (1.7.5) gives $(I - P_0 + sP_1)^{-1}$ which when substituted into (1.7.2) gives, after a little algebra,

$$\left(I - q(s)\right)^{-1} = \frac{\alpha}{s} H_0 + \alpha H_1 + \alpha(\frac{1}{2}\alpha k_2 - a_1)H_0$$
$$+ \alpha s\{H_2 - a_1 H_1 + a_2 H_0 + \frac{1}{2}\alpha(H_0 P_2 H_1 + H_1 P_2 H_0)$$
$$- \frac{1}{2}\alpha^2(a_1 k_2 + \frac{1}{3} k_3)H_0\} + o(s) \qquad (1.7.8)$$

$$= \frac{\alpha}{s} H_0 + \alpha H_1 + aH_0 + sL + o(s) \quad , \qquad (1.7.9)$$

where $a = \alpha(\frac{1}{2}\alpha k_2 - a_1)$ and L is an $m \times m$ matrix. The $(i,j)^{th}$ element of $\left(I - q(s)\right)^{-1}$ is given by

$$\left\{\left(I - q(s)\right)^{-1}\right\}_{ij} = \frac{\alpha}{s} v_j + av_j + \alpha h_{ij} + s\ell_{ij} + o(s) \quad . \qquad (1.7.10)$$

Recall from Section 1.4 that

$$m(s) = \left(I - q(s)\right)^{-1} q(s) = \left(I - q(s)\right)^{-1} - I \quad , \qquad (1.7.11)$$

so that we may substitute (1.7.8) into the above and invert to obtain, for large t ,

$$M(t) = [M_{ij}(t)] = \alpha t H_0 + \alpha H_1 + \alpha(\frac{1}{2}\alpha k_2 - a_1)H_0 - I + o(1) \quad . \qquad (1.7.12)$$

(1.7.8) can also be used in (1.4.14) and inverting the resulting expression to obtain an asymptotic expression for $E[N_j^i(t)\{N_j^i(t) - 1\}]$. Kshirsagar and Gupta (1967) give this result, which is not reproduced here. At this point no further examples using (1.7.8) or (1.7.10) are given; in later chapters, however, we shall make extensive use of these equations. From (1.7.12) we have that

$$\lim_{t \to \infty} \frac{1}{t} M(t) = \alpha H_0 \quad . \qquad (1.7.13)$$

Note that, in the limit, the mean number of visits is proportional to the corresponding cofactor and independent of the initial state.

CHAPTER II

TRANSITION FREQUENCIES IN M.R.P.'S

P. Whittle (1955) derived the probability distribution of the matrix $[n_{jk}]$ whose $(j,k)^{th}$ element is the number of one-step transitions from state j to state k , for finite Markov Chains. Here we consider time continuous and derive the analogous distribution for M.R.P.'s. Thus we provide a generalization of both Whittle's result and Pyke's results.

1. Derivation

Let $N_{ij}(t)$ be the number of one-step transitions from state i to state j in the interval (0,t] for the M.R.P. specified by $(m, \underline{a}, Q)$ and let N(t) be the $m \times m$ matrix of the $N_{ij}(t)$. We shall call N(t) the matrix of transition counts. Further, let k be an $m \times m$ matrix of non-negative integers and define

$$W_{ij}(k;t) = \Pr\{N(t) = k \ , \ \underline{J}(t) = \underline{e}_j \mid \underline{J}(0) = \underline{e}_i\}$$

$$= \Pr\{N^i(t) = k \ , \ \underline{J}(t) = \underline{e}_j\} \ . \qquad (2.1.1)$$

Observe that every entry into state i must be followed by an exit from state i except for the initial and final states. Hence the matrix N(t) is constrained by

$$N_{k\cdot}(t) - N_{\cdot k}(t) = \delta_{ik} - \delta_{jk} \ , \quad k = 1, 2, \cdots , m \ , \qquad (2.1.2)$$

$N_{k\cdot}(t)$ and $N_{\cdot k}(t)$ being the row and column sums of N(t) and where i and j are the initial and final states respectively. For a matrix k satisfying (2.1.2) we have that

$$W_{ij}(k;t) = \sum_{r=1}^{m} Q_{ir}(t) * W_{rj}(k - e_{ir}\ ;\ t) \quad ,$$

$$W_{ij}(0;t) = \delta_{ij}\Big(1 - H_i(t)\Big) \quad . \tag{2.1.3}$$

Define

$$w_{ij}(Z;s) = \sum_{k\geq 0} \int_0^\infty e^{-st} d_t W_{ij}(k;t) \prod_{i,j} z_{ij}^{k_{ij}} \quad , \tag{2.1.4}$$

where $Z = (z_{ij})$, $|z_{ij}| \leq 1$, $i,j = 1, 2, \cdots , m$; i.e., $w_{ij}(Z;s)$ is the L.S.T. of the p.g.f. of $W_{ij}(k;t)$. Using (2.1.4) in (2.1.3) gives

$$w_{ij}(Z;s) = \delta_{ij}\Big(1 - h_i(s)\Big) + \sum_{r=1}^{m} q_{ir}(s) z_{ir} w_{rj}(z;s) \quad ,$$

which in matrix notation becomes

$$w(Z;s) = I - h(s) + \Big(q(s) \,\square\, Z\Big) w(Z;s) \quad ,$$

where $A \,\square\, B$ is a matrix with $(i,j)^{th}$ element $a_{ij}b_{ij}$. Solving for $w(Z;s)$ gives

$$w(Z;s) = \Big(I - q(s) \,\square\, Z\Big)^{-1}\Big(I - h(s)\Big) \quad . \tag{2.1.5}$$

Summing the rows of $w(Z;s)$ gives

$$\underline{w}(Z;s) = \Big(I - q(s) \,\square\, Z\Big)^{-1}\Big(I - q(s)\Big)\underline{e} \tag{2.1.6}$$

which is the L.S.T. of the p.g.f. of

$$W_i(k;t) = \Pr\{N^i(t) = k\} \tag{2.1.7}$$

<u>Lemma</u>. [Whittle (1955)]. The coefficient of

$$\prod_{k,\ell} a_{k\ell}^{n_{k\ell}}$$

in the $(i,j)^{th}$ element of the $m \times m$ matrix $(I - A)^{-1}$, where $A = (a_{ij})$, is

$$N^*_{ji} \frac{\prod_k n_{k\cdot}!}{\prod_{k,\ell} n_{k\ell}!} , \tag{2.1.8}$$

if $n_{k\ell}$ are non-negative integers such that

$$n_{k\cdot} - n_{\cdot k} = \delta_{ik} - \delta_{jk} \quad , \quad k = 1, 2, \cdots , m \quad ,$$

and is zero otherwise. Here N^*_{ji} is the cofactor of the $(j,i)^{th}$ element of the matrix $N^* = (n^*_{ij})$, defined by

$$n^*_{ij} = \begin{cases} \delta_{ij} - n_{ij}/n_{i\cdot} & ; \ n_{i\cdot} > 0 \\ \delta_{ij} & ; \ n_{i\cdot} = 0 \ . \end{cases} \tag{2.1.9}$$

With the aid of the lemma the coefficient of $\prod_{k,\ell} z_{k\ell}^{n_{k\ell}}$ in $w_{ij}(z;s)$ is

$$N^*_{ji} \frac{\prod_k N_{k\cdot}!}{\prod_{k,\ell} N_{k\ell}!} \prod_{k,\ell} \Big(q_{k\ell}(s)\Big)^{N_{k\ell}} \Big(1 - h_k(s)\Big) \tag{2.1.10}$$

if $N(t)$ satisfies (2.1.2) and zero otherwise. The analogous result for $w_i(Z;s)$ is obtained by summing (2.1.10) with respect to the state j . Martin (1967, pg. 119) gives a simple proof showing that the solution for k in the system of equations (2.1.2) is unique given the matrix $N(t)$ and the initial state i . Hence the only contribution from summing (2.1.10) with respect to the state j arises when $j = k$ where k is the solution of

(2.1.2). Hence the coefficient of $\prod_{k,\ell} z_{k\ell}^{n_{k\ell}}$ in $w_i(Z;s)$ is

$$N^*_{ki} \frac{\prod_{\alpha} N_{\alpha\cdot}!}{\prod_{\alpha,\beta} N_{\alpha\beta}!} \prod_{\alpha,\beta} \left(q_{\alpha\beta}(s)\right)^{N_{\alpha\beta}} \left(1 - h_k(s)\right) \quad , \tag{2.1.11}$$

where k is the unique solution of (2.1.2).

2. Moments of $N^i(t)$

Define

$$M^i_{jk}(t) = E\left\{N^i_{jk}(t)\right\}$$

$$C^i_{jk,ln}(t) = E\left\{N^i_{jk}(t) N^i_{\ell n}(t)\right\}$$

and let $\underline{M}_{jk}(t)$ and $\underline{C}_{jk,\ell n}(t)$ be the $m \times 1$ vectors of these moments. Denote the L.S.T.'s of these moments by $\underline{m}_{jk}(s)$ and $\underline{c}_{jk,\ell n}(s)$ respectively. We give two independent derivations of the moments. The first is an extension of a method developed by Martin (1967, pg. 122ff) for Markov Chains and may also be found in a recent paper by Kshirsagar and Wysocki (1969a). The second method involves differentiation of (2.1.6) and is perhaps the more direct approach. For the first method note that

$$N^i_{\alpha\beta}(t) = \begin{cases} \delta_{i\alpha}\delta_{k\beta} + N^k_{\alpha\beta}(t - x_1) \ , \ \underline{J}_1 = \underline{e}_k \ , \\ \qquad\qquad k = 1, 2, \cdots, m \\ 0 \ , \ \text{if there is no transition in } (0,t]. \end{cases} \tag{2.2.1}$$

Taking L.S.T.'s of the expectation on both sides,

$$m^i_{\alpha\beta}(s) = \sum_{k=1}^{m} q_{ik}(s)\delta_{i\alpha}\delta_{k\beta} + \sum_{k=1}^{m} q_{ik}(s) m^k_{\alpha\beta}(s)$$

$$= q_{i\beta}(s)\delta_{i\alpha} + \sum_{k=1}^{m} q_{ik}(s) m^k_{\alpha\beta}(s) \quad , \tag{2.2.2}$$

or in matrix notation

$$\underline{m}_{\alpha\beta}(s) = q_{\alpha\beta}(s)\underline{e}_{\alpha} + q(s)\underline{m}_{\alpha\beta}(s) \quad . \qquad (2.2.3)$$

Solving for $\underline{m}_{\alpha\beta}(s)$ gives

$$\underline{m}_{\alpha\beta}(s) = q_{\alpha\beta}(s)\Big(I - q(s)\Big)^{-1}\underline{e}_{\alpha} \qquad (2.2.4)$$

whose i^{th} element is

$$m^{i}_{\alpha\beta}(s) = q_{\alpha\beta}(s)\Big\{\Big(I - q(s)\Big)^{-1}\Big\}_{i\alpha} \quad . \qquad (2.2.5)$$

Just as in (2.2.1) we can write

$$N^{i}_{\alpha\beta}(t)N^{i}_{\gamma\delta}(t) = \delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{ai}\delta_{\beta k} + \delta_{i\gamma}\delta_{k\delta}N^{k}_{\alpha\beta}(t - x_1)$$

$$+ \delta_{i\alpha}\delta_{k\beta}N^{k}_{\gamma\delta}(t - x_1) + N^{k}_{\alpha\beta}(t - x_1)N^{k}_{\gamma\delta}(t - x_1) \qquad (2.2.6)$$

if $\underline{J}_1 = \underline{e}_k$ for $k = 1, 2, \cdots , m$ and is zero if there is no transition in $(0,t]$. Taking L.S.T.'s of the expectations on both sides of (2.2.6) one can write in matrix notation that

$$\underline{c}_{\alpha\beta,\gamma\delta}(s) = \delta_{\alpha\gamma}\delta_{\beta\delta}\underline{m}_{\alpha\beta}(s) + m^{\delta}_{\alpha\beta}(s)\underline{m}_{\gamma\delta}(s) + m^{\beta}_{\gamma\delta}(s)\underline{m}_{\alpha\beta}(s) \quad , \qquad (2.2.7)$$

whose i^{th} element is

$$c^{i}_{\alpha\beta,\gamma\delta}(s) = \delta_{\alpha\gamma}\delta_{\beta\delta}m^{i}_{\alpha\beta}(s) + m^{\delta}_{\alpha\beta}(s)m^{i}_{\gamma\delta}(s) + m^{\beta}_{\gamma\delta}(s)m^{i}_{\alpha\beta}(s) \quad . \qquad (2.2.8)$$

The second method follows from the appropriate differentiation of

$$\underline{w}(Z;s) = \Big(I - q(s)\,\Box\, Z\Big)^{-1}\Big(I - q(s)\Big)\underline{e} \qquad (2.2.9)$$

and evaluation at $Z = e$. Rather than proceeding from (2.2.9) directly, it is more convenient and algebraically simpler to find an expression for the L.S.T. of the marginal distribution of $\Big(N^{i}_{\alpha\beta}(t)\ ,\ N^{i}_{\gamma\delta}(t)\Big)$ and then find moments from the resulting expression. To that end set

$$Z = Z^* = E + (z_1 - 1)e_{\alpha\beta} + (z_2 - 1)e_{\gamma\delta} \quad . \tag{2.2.10}$$

Substituting (2.2.10) into (2.2.9) one easily obtains

$$\underline{w}(Z^*,s) = \Big[I - (z_1 - 1)\big(I - q(s)\big)^{-1}q_{\alpha\beta}(s)e_{\alpha\beta} - (z_2 - 1)\big(I - q(s)\big)^{-1}q_{\gamma\delta}(s)e_{\gamma\delta}\Big]^{-1}\underline{e} \tag{2.2.11}$$

Then the L.S.T. of the vector of conditional means $\underline{m}_{\alpha\beta}(s)$ is given by

$$\underline{m}_{\alpha\beta}(s) = \left.\frac{\partial \underline{w}(Z^*,s)}{\partial z_1}\right|_{z_1=z_2=1} = q_{\alpha\beta}(s)\big(I - q(s)\big)^{-1}e_{\alpha\beta}\underline{e}$$

$$= q_{\alpha\beta}(s)\big(I - q(s)\big)^{-1}\underline{e}_\alpha \quad , \tag{2.2.12}$$

in agreement with (2.2.4). Note that

$$\big(m_{i\beta}(s)\big) = \left(\sum_{\alpha=1}^{m} m^i_{\alpha\beta}(s)\right) = \big(I - q(s)\big)^{-1}q(s) \quad , \tag{2.2.13}$$

which is the L.S.T. of the matrix of means with $(i,\beta)^{th}$ element the L.S.T. of $E\{N^i_\beta(t)\}$ derived by Pyke (1961b). For $\alpha \neq \gamma$ or $\beta \neq \delta$ the vector of product moments $\underline{C}_{\alpha\beta,\gamma\delta}(t)$ has L.S.T. given by

$$\underline{c}_{\alpha\beta,\gamma\delta}(s) = \left.\frac{\partial^2 \underline{w}(Z^*,s)}{\partial z_1 \partial z_2}\right|_{z_1=z_2=1}$$

$$= q_{\alpha\beta}(s)\big(I - q(s)\big)^{-1}e_{\alpha\beta}\underline{m}_{\gamma\delta}(s) + q_{\gamma\delta}(s)\big(I - q(s)\big)^{-1}e_{\gamma\delta}\underline{m}_{\alpha\beta}(s) \quad , \tag{2.2.14}$$

whose i^{th} element is

$$c^i_{\alpha\beta,\gamma\delta}(s) = m^i_{\gamma\delta}(s)m^\delta_{\alpha\beta}(s) + m^i_{\alpha\beta}(s)m^\beta_{\gamma\delta}(s) \quad . \tag{2.2.15}$$

For $\alpha = \gamma$ and $\beta = \delta$ we obtain the L.S.T. of $E\{N^i_{\alpha\beta}(t)[N^i_{\alpha\beta}(t) - 1]\}$ which is $2m^i_{\alpha\beta}m^\beta_{\alpha\beta}$ so that the L.S.T. of $E\left\{\left(N^i_{\alpha\beta}(t)\right)^2\right\}$ is

$$c^i_{\alpha\beta,\alpha\beta} = m^i_{\alpha\beta}(zm^\beta_{\alpha\beta} + 1) \quad . \tag{2.2.16}$$

Combining (2.2.15) and (2.2.16) we have that for any i, α, β, γ, δ that

$$c^i_{\alpha\beta,\gamma\delta}(s) = \delta_{\alpha\gamma}\delta_{\beta\delta}m^i_{\alpha\beta}(s) + m^\delta_{\alpha\beta}(s)m^i_{\gamma\delta}(s) + m^\beta_{\gamma\delta}(s)m^i_{\alpha\beta}(s) \quad , \tag{2.2.17}$$

in agreement with (2.2.8).

3. Cumulative Processes

In this chapter we have associated the holding time X_n with the transition from state $\underline{J}_{n-1}$ to state $\underline{J}_n$. We can generalize this concept by considering, in place of X_n , the real-valued function $g(\underline{J}_{n-1}, \underline{J}_n, X_n)$. For n = 1, 2, $\cdots$ we assume that the g-functions are independent and identically distributed. Then the random variable defined at time t by

$$W_g(t) = \begin{cases} \sum_{n=1}^{N(t)} g(\underline{J}_{n-1}, \underline{J}_n, X_n) & , \quad N(t) > 0 \\ 0 & , \quad N(t) = 0 \quad , \end{cases} \tag{2.3.1}$$

where $N(t) = \sum_j N_j(t)$, gives rise to the stochastic process $\{W_g(t) \ ; \ t \geq 0\}$ which is known as a cumulative process. The reader is referred to Cox (1962, Chapter 8) for examples of how cumulative processes arise in applications.

Let

$$\xi_{ij}(p) = \int_0^\infty e^{-pg(i,j,x)} d_x Q_{ij}(x) \tag{2.3.2}$$

be the moment generating function of g(i,j,x). Then, since the g-functions are independent, we can write

$$E\left\{e^{-pW_g(t)}\,\middle|\,N^i(t)\right\} = \prod_{j,k}\left[\xi_{jk}(p)\right]^{N^i_{jk}(t)} \quad . \tag{2.3.3}$$

So that

$$E\left\{e^{-pW_g(t)}\,\middle|\,\underline{J}(0) = \underline{e}_i\right\} = E\left\{\prod_{j,k}\left[\xi_{jk}(p)\right]^{N^i_{jk}(t)}\right\} \quad . \tag{2.3.4}$$

Comparing (2.3.4) with (2.1.4) we see that this is nothing but the p.g.f. of $N^i_{jk}(t)$ with argument $\xi_{jk}(p)$ hence the L.S.T. of $(W_g(t)|\underline{J}(0) = \underline{e}_i)$ is the i^{th} element of the vector

$$[I - q(s) \square \xi(p)]^{-1}\left(I - q(s)\right)\underline{e} \quad , \tag{2.3.5}$$

where $\xi(p) = \left(\xi_{jk}(p)\right)$. To find the moments of $W_g(t)$ one may proceed by differentiating (2.3.5), but a more direct approach is available. To find the mean note that

$$E\left\{W_g(t)\,|\,\underline{J}(0) = \underline{e}_i\right\} = E\,\underset{N}{E}\left\{W_g(t)\,|\,N(t)\ ,\ \underline{J}(0) = \underline{e}_i\right\}$$

$$= E\left\{\sum_{jk} N^i_{jk}(t)\int_0^\infty g(j,k,x)\,dF_{jk}(x)\right\} \quad . \tag{2.3.6}$$

Following the notation in Pyke and Moore (1968) we set

$$A_{jk} = \int_0^\infty g(j,k,x)\,dQ_{jk}(x) \quad , \quad \sum_k A_{jk} = A_j \quad , \tag{2.3.7}$$

$$B_{jk} = \int_0^\infty g^2(j,k,x)\,dQ_{jk}(x) \quad , \quad \sum_k B_{jk} = B_j \quad . \tag{2.3.8}$$

Then

$$E\left\{W_g(t)\,|\,\underline{J}(0) = \underline{e}_i\right\} = E\left\{\sum_{jk}\frac{A_{jk}}{P_{jk}}N^i_{jk}(t)\right\}$$

$$= \sum_{jk}\frac{A_{jk}}{P_{jk}}M^i_{jk}(t) \quad . \tag{2.3.9}$$

From the identity

$$\mathrm{Var}\left\{W_g(t)\,|\,\underline{J}(0) = \underline{e}_i\right\} = \underset{N}{E}\ \mathrm{Var}\left\{W_g(t)\,|\,N^i(t)\right\} + \underset{N}{\mathrm{Var}}\ E\left\{W_g(t)\,|\,N^i(t)\right\}$$

we obtain

$$\mathrm{Var}\left\{W_g(t)\,|\,\underline{J}(0) = \underline{e}_i\right\} = \underset{N}{E}\left\{\sum_{jk}\left[\frac{B_{jk}}{P_{jk}} - \frac{A_{jk}^2}{P_{jk}^2}\right]N_{jk}^i(t)\right\} + \underset{N}{\mathrm{Var}}\left\{\sum_{jk}\frac{A_{jk}}{P_{jk}}N_{jk}^i(t)\right\}$$

$$= \sum_{j,k}\left[\frac{B_{jk}}{P_{jk}} - \frac{A_{jk}^2}{P_{jk}^2}\right]M_{jk}^i(t) + \sum_{j,k}\ \sum_{\ell,n}\frac{A_{jk}A_{\ell n}}{P_{jk}P_{\ell n}}\mathrm{Cov}\left\{N_{jk}^i(t), N_{\ell n}^i(t)\right\}. \tag{2.3.10}$$

4. Asymptotic Results

The results of Section 1.7 can be used to determine the asymptotic values of the first few moments of the distribution of $N(t)$, the matrix of transition counts. Recall that in Section 1.7 we had shown that

$$\left\{\left(I - q(s)\right)^{-1}\right\}_{ij} = \frac{\alpha}{s}v_j + av_j + \alpha h_{ij} + s\ell_{ij} + o(s) \tag{2.4.1}$$

and $q_{jk}(s)$ could be written in the form

$$q_{jk}(s) = p_{jk}\left\{1 - s\mu_{jk} + \frac{1}{2}s^2\mu_{jk}^{(2)} + o(s^2)\right\} \quad . \tag{2.4.2}$$

Using (2.4.1) and (2.4.2) in (2.2.5) one obtains

$$\begin{aligned} m_{jk}^i(s) = p_{jk}\Big\{&\frac{\alpha}{s}v_j + av_j + \alpha(h_{ij} - v_j\mu_{jk}) \\ &+ s\left(\ell_{ij} - \mu_{jk}\left\{av_j + \alpha h_{ij}\right\} + \frac{1}{2}\alpha v_j\mu_{jk}^{(2)}\right) \\ &+ o(s)\Big\} \ , \end{aligned} \tag{2.4.3}$$

whose inverse transform is, for large t,

$$E\left\{N_{jk}^i(t)\right\} = p_{jk}\left\{\alpha v_j t + av_j + \alpha(h_{ij} - v_j\mu_{jk}) + o(1)\right\} \ . \tag{2.4.4}$$

Hence

$$\lim_{t \to \infty} \frac{1}{t} E\left\{N^i_{jk}(t)\right\} = \alpha v_j p_{jk} \quad , \tag{2.4.5}$$

which is independent of the initial state of the system as one would expect. Using (2.4.3) in (2.2.17) gives

$$\begin{aligned} c^i_{jk,\ell n}(s) = {} & \frac{2}{s^2} \alpha^2 v_j v_\ell p_{jk} p_{\ell n} + \frac{1}{s} \delta_{j\ell}\delta_{kn} p_{jk} \\ & + \frac{1}{s} p_{jk} p_{\ell n} [4a\alpha v_j v_\ell + \alpha^2 (v_j h_{k\ell} + v_\ell h_{ij}) \\ & - 2\alpha^2 v_j v_\ell (\mu_{jk} + \mu_{\ell n}) + \alpha^2 (v_j h_{i\ell} + v_\ell h_{nj})] \\ & + o(1/s) \quad . \end{aligned} \tag{2.4.6}$$

From the inverses of (2.4.6) and (2.4.3) we get, for large t,

$$\begin{aligned} \mathrm{Cov}\left\{N^i_{jk}(t) \ , \ N^i_{\ell n}(t)\right\} = {} & \delta_{j\ell}\delta_{kn} \alpha v_j p_{jk} t \\ & + p_{jk} p_{\ell n} t [2a\alpha v_j v_\ell - \alpha^2 v_j v_\ell (\mu_{jk} + \mu_{\ell n}) \\ & + \alpha^2 (v_j h_{k\ell} + v_\ell h_{nj})] + o(t) \end{aligned} \tag{2.4.7}$$

and

$$\mathrm{Var}\left\{N^i_{jk}(t)\right\} = \alpha v_j p_{jk} t + \alpha v_j p^2_{jk} t (2av_j - 2\alpha v_j \mu_{jk} + 2\alpha h_{k\ell}) + o(t) \quad . \tag{2.4.8}$$

Taking the appropriate limits in (2.4.7) and (2.4.8) gives

$$\begin{aligned} \lim_{t \to \infty} \frac{1}{t} \mathrm{Cov}\left\{N^i_{jk}(t) \ , \ N^i_{\ell n}(t)\right\} = {} & p_{jk} p_{\ell n} [2a\alpha v_j v_\ell - \alpha^2 v_j v_\ell (\mu_{jk} + \mu_{\ell n}) \\ & + \alpha^2 (v_j h_{k\ell} + v_\ell h_{nj})] \end{aligned} \tag{2.4.9}$$

and

$$\lim_{t \to \infty} \frac{1}{t} \mathrm{Var}\left\{N^i_{jk}(t)\right\} = \alpha v_j p_{jk} + 2\alpha v_j p^2_{jk} (av_j - \alpha v_j \mu_{jk} + \alpha h_{kj}) \quad . \tag{2.4.10}$$

The reader will note that the above expressions are independent of the initial state of the system. From (2.4.4) we also have

$$\lim_{t \to \infty} \frac{1}{t} E\{W_g(t) \mid \underline{J}(0) = \underline{e}_i\} = \sum_{j,k} \frac{A_{jk}}{P_{jk}} \alpha p_{jk} v_j = \sum_{j,k} \alpha v_j A_{jk} \quad . \tag{2.4.11}$$

The asymptotic variance is formed as follows:

$$\lim_{t \to \infty} \frac{1}{t} \mathrm{Var}\{W_g(t) \mid \underline{J}(0) = \underline{e}_i\}$$

$$= \sum_{j,k} \alpha v_j B_{jk} - \sum_{j,k} \frac{\alpha v_j A_{jk}^2}{P_{jk}} + \sum_{j,k} \sum_{\ell,n} \frac{A_{jk} A_{\ell n}}{P_{jk} P_{\ell n}} \Big\{ \delta_{j\ell} \delta_{kn} \alpha v_j p_{jk}$$

$$+ p_{jk} p_{\ell n} [2a\alpha v_j v_\ell - \alpha^2 v_j v_\ell (\mu_{jk} + \mu_{\ell n}) + \alpha^2 (v_j h_{k\ell} + v_\ell h_{nj})] \Big\}$$

$$= \alpha \sum_j v_j B_j - \alpha \sum_{j,k} \frac{v_j}{P_{jk}} A_{jk}^2 + \alpha \sum_j \frac{v_j}{P_{jj}} A_{jj}^2 + \alpha \sum_{j,k} \sum_{\ell,n} A_{jk} A_{\ell n} \{ 2a v_j v_\ell$$

$$- \alpha v_j v_\ell (\mu_{jk} + \mu_{\ell n}) + \alpha (v_j h_{k\ell} + v_\ell h_{nj}) \}$$

$$= \alpha \sum_j v_j B_j - \alpha \sum_{j,k} \frac{v_j}{P_{jk}} A_{jk}^2 + \alpha \sum_j \frac{v_j}{P_{jj}} A_{jj}^2 + 2a\alpha \sum_{j,k} A_j A_k v_j v_k$$

$$- 2\alpha^2 \sum_{j,k} \sum_\ell A_{jk} A_\ell v_j v_\ell \mu_{jk} + 2\alpha^2 \sum_{j,k} \sum_\ell A_{jk} A_\ell v_j h_{k\ell}$$

$$= \alpha \sum_j v_j B_j + \alpha \left[\sum_j \frac{v_j}{P_{jj}} A_{jj}^2 - \sum_{j,k} \frac{v_j}{P_{jk}} A_{jk}^2 \right] + 2a\alpha \sum_{j,k} A_j A_k v_j v_k$$

$$- 2\alpha^2 \sum_{j,k} \sum_\ell A_{jk} A_\ell v_j (v_\ell \mu_{jk} - h_{k\ell}) \quad . \tag{2.4.12}$$

CHAPTER III

DISTRIBUTION OF TRANSITION FREQUENCIES FOR ARBITRARY INTERVALS

1. Derivation

In this chapter the results of Section 1.6 and Chapter II are combined and the distribution of the matrix of transition counts in the interval $(t_0 , t_0 + t]$ is derived. Specifically, let $N_{jk}(t_0 , t)$ denote the number of direct transitions from state i to state j in the interval $(t_0 , t_0 + t]$. Further we define

$$M^i_{jk}(t_0 , t) = E\{N^i_{jk}(t_0 , t)\} \quad , \tag{3.1.1}$$

$$R^i_{jk}(t_0 , t) = E\{N^i_{jk}(t_0 , t)[N^i_{jk}(t_0 , t) - 1]\} \quad , \tag{3.1.2}$$

and

$$C^i_{jk,\ell n}(t_0 , t) = E\{N^i_{jk}(t_0 , t)N^i_{\ell n}(t_0 , t)\} \quad . \tag{3.1.3}$$

The following L.S.T.'s will also be used:

$$m^i_{jk}(t_0 , s) = \int_0^\infty e^{-st} M^i_{jk}(t_0 , t)\,dt \quad , \tag{3.1.4}$$

$$m^i_{jk}(s_0 , s) = \int_0^\infty e^{-s_0 t_0} m^i_{jk}(t_0 , t)\,dt_0 \quad . \tag{3.1.5}$$

Similar notation will be used for the L.S.T.'s of $R^i_{jk}(t_0 , t)$ and $C^i_{jk,\ell n}(t_0 , t)$. Letting

$$W_i(k;t) = Pr\{N^i(t) = k\} \tag{3.1.6}$$

it has been previously shown that the L.S.T. of the p.g.f. of the vector $\underline{W}(k;t)$ is

$$\underline{w}(Z;s) = \left(I - q(s) \square Z\right)^{-1}\left(I - q(s)\right)\underline{e} \quad . \tag{3.1.7}$$

Recall from Section 1.6 that the p.d.f. of the first transition after t_0 is

$$\frac{d}{dx^0} Q_{jk}(t_0 , x^0 | \underline{J}(0) = \underline{e}_i) = \delta_{ij} \frac{d}{dx_0} Q_{jk}(t_0 + x^0) + \int_0^{t_0} h_{ij}(t_0 - u) \frac{d}{dx^0} Q_{jk}(u + x_0)du \tag{3.1.8}$$

and that its L.S.T. is given by

$$q^*_{jk}(s_0 , s | \underline{J}(0) = \underline{e}_i) = \frac{\left\{\left(I - q(s_0)\right)^{-1}\right\}_{ij}}{s_0 - s} \left(q_{jk}(s) - q_{jk}(s_0)\right) \quad . \tag{3.1.9}$$

Define $\underline{w}(Z;\, t_0 , s)$ as the L.S.T. with respect to t of the p.g.f. of $N(t_0 , t)$ conditional on the initial state. Then, using (3.1.7) we can write

$$w_i(Z ; t_0 , s) = \sum_j \sum_k q_{jk}(t_0 , s | \underline{J}(0) = \underline{e}_i) Z_{jk} w_k(Z;s) + 1 - \sum_j \sum_k q_{jk}(t_0 , s | \underline{J}(0) = \underline{e}_i) \quad , \tag{3.1.10}$$

whose L.S.T. with respect to t_0 , using (3.1.9), is

$$w_i(Z ; s_0 , s) = \int_0^\infty e^{-s_0 t_0} w_i(Z ; t_0 , s)dt_0$$

$$= \sum_{jk} \left\{\left(I - q(s_0)\right)^{-1}\right\}_{ij} \left(q_{jk}(s) - q_{jk}(s_0)\right) z_{jk} w_k(Z,s)/(s_0 - s) + \frac{1}{s_0} - \sum_{jk} \left\{\left(I - q(s_0)\right)^{-1}\right\}_{ij} \left(q_{jk}(s) - q_{jk}(s_0)\right)/(s_0 - s) \quad . \tag{3.1.11}$$

In matrix notation the vector $\underline{w}(Z\ ;\ s_0\ ,\ s)$ is

$$\underline{w}(Z\ ;\ s_0\ ,\ s) = \frac{1}{s_0}\underline{e} + \frac{1}{s_0-s}\Big[\big(I - q(s_0)\big)^{-1}\Big]\Big[\Big\{\big(q(s) - q(s_0)\big)\,\square\, Z\Big\}\underline{w}(Z;s) - \big(q(s) - q(s_0)\big)\underline{e}\Big] \quad . \qquad (3.1.12)$$

2. Moments

From (3.1.12) we can obtain the L.S.T.'s of the moments of $N(t_0\ ,\ t)$ by appropriate differentiation and evaluating these expressions at $z = e$. One method involves substituting

$$Z^* = Z = E + (z_1 - 1)e_{jk} + (z_2 - 1)e_{\ell n} \qquad (3.2.1)$$

into (3.1.12) and proceeding as in Section 2.2. However, it is less cumbersome to proceed directly from (3.1.12). To that end

$$\frac{\partial \underline{w}(Z\ ;\ s_0\ ,\ s)}{\partial z_{jk}} = \frac{1}{s_0-s}\big(I - q(s_0)\big)^{-1}\big\{q(s) - q(s_0)\big\}\,\square\, e_{jk}\underline{w}(Z;s) + \frac{1}{s_0-s}\big(I - q(s_0)\big)^{-1}\big\{q(s) - q(s_0)\big\}\,\square\, Z\,\frac{\partial \underline{w}(Z;s)}{\partial z_{jk}} \quad . \qquad (3.2.2)$$

At Z = E we obtain the L.S.T. of the vector of conditional means

$$\begin{aligned}\underline{m}_{jk}(s_0\ ,\ s) &= \frac{1}{s_0-s}\big(I - q(s_0)\big)^{-1}\big(q_{jk}(s) - q_{jk}(s_0)\big)e_{jk}\underline{e} \\ &\quad + \frac{1}{s_0-s}\big(I - q(s_0)\big)^{-1}\big(q(s) - q(s_0)\big)\underline{m}_{jk}(s) \\ &= \frac{1}{s_0-s}\Big[\big(I - q(s_0)\big)^{-1}q_{jk}(s)e_{jk}\underline{e} - \underline{m}_{jk}(s_0) \\ &\quad + \big(I - q(s_0)\big)^{-1}q(s)\big(I - q(s)\big)^{-1}q_{jk}(s)e_{jk}\underline{e} \\ &\quad - \big(I - q(s_0)\big)^{-1}\big(I - q(s)\big)^{-1}q_{jk}(s)e_{jk}\underline{e} + \underline{m}_{jk}(s)\Big] \\ &= \frac{1}{s_0-s}\big(\underline{m}_{jk}(s) - \underline{m}_{jk}(s_0)\big) \quad . \end{aligned} \qquad (3.2.3)$$

The product moments may be obtained from a second differentiation. In particular,

$$\frac{\partial^2 \underline{w}(Z;s_0\,,\,s)}{\partial z_{jk}\partial z_{\ell n}} = \frac{1}{s_0-s}\Big(I - q(s_0)\Big)^{-1}\Big\{\Big(q(s) - q(s_0)\Big)\square\, e_{jk}\Big\}\frac{\partial \underline{w}(Z;s)}{\partial z_{\ell n}}$$

$$+ \frac{1}{s_0-s}\Big(I - q(s_0)\Big)^{-1}\Big\{\Big(q(s) - q(s_0)\Big)\square\, e_{\ell n}\Big\}\frac{\partial \underline{w}(Z;s)}{\partial z_{jk}}$$

$$+ \frac{1}{s_0-s}\Big(I - q(s_0)\Big)^{-1}\Big\{\Big(q(s) - q(s_0)\Big)\square\, Z\Big\}\frac{\partial^2 \underline{w}(Z;s)}{\partial z_{jk}\partial z_{\ell n}}$$

at $Z = E$,

$$\underline{c}_{jk,\ell n}(s_0\,,\,s) = \frac{1}{s_0-s}\Big(I - q(s_0)\Big)^{-1}\Big(q_{jk}(s) - q_{jk}(s_0)\Big)e_{jk}\underline{m}_{\ell n}(s)$$

$$+ \frac{1}{s_0-s}\Big(I - q(s_0)\Big)^{-1}\Big(q_{\ell n}(s) - q_{\ell n}(s_0)\Big)e_{\ell n}\underline{m}_{jk}(s)$$

$$+ \frac{1}{s_0-s}\Big(I - q(s_0)\Big)^{-1}\Big(q(s) - q(s_0)\Big)\underline{c}_{jk,\ell n}(s)$$

$$= \frac{1}{s_0-s}\Big\{\underline{c}_{jk,\ell n}(s) - q_{jk}(s_0)\Big(I - q(s_0)\Big)^{-1}e_{jk}\underline{m}_{\ell n}(s) - q_{\ell n}(s_0)\Big(I - q(s_0)\Big)^{-1}e_{\ell n}\underline{m}_{jk}(s)\Big\} \qquad (3.2.4)$$

whose i^{th} element is, after some simplification,

$$c^i_{jk,\ell n}(s_0\,,\,s) = \frac{1}{s_0-s}\, m^n_{jk}(s)\,[m^i_{\ell n}(s) - m^i_{\ell n}(s_0)] + \frac{1}{s_0-s}\, m^k_{\ell n}(s)\,[m^i_{jk}(s) - m^i_{jk}(s_0)] \qquad (3.2.5)$$

$$= m^n_{jk}(s)m^i_{\ell n}(s_0\,,\,s) + m^k_{\ell n}(s)m^i_{jk}(s_0\,,\,s) \quad , \qquad (3.2.6)$$

for $j \neq \ell$ or $k \neq n$. If $j = \ell$ and $k = n$, then

$$r^i_{jk}(s_0\,,\,s) = 2m^k_{jk}(s)m^i_{jk}(s_0\,,\,s) \quad . \qquad (3.2.7)$$

In general, for any i, j, k, ℓ, and n

$$c^i_{jk,\ell n}(s_0\,,\,s) = \delta_{j\ell}\delta_{kn}m^i_{jk}(s_0\,,\,s) + m^n_{jk}(s)m^i_{\ell n}(s_0\,,\,s) + m^k_{\ell n}(s)m^i_{jk}(s_0\,,\,s) \quad . \qquad (3.2.8)$$

3. Cumulative Processes in $(t_0 , t_0 + t]$

In order to study cumulative processes in an arbitrary interval we define $W_g(t_0 , t)$ similar to $W_g(t)$ as given by (2.3.1). Specifically let

$$W_g(t_0 , t) = \begin{cases} \sum_{n=1}^{N(t_0,t)} g(\underline{J}_{n-1} , \underline{J}_n , X_n) , & N(t_0 , t) > 0 \\ 0 , & N(t_0 , t) = 0 , \end{cases} \tag{3.3.1}$$

where $N(t_0 , t) = \sum_{j,k} N_{jk}(t_0 , t)$. Then the stochastic process $\{W_g(t_0 , t) ; t \geq 0\}$ gives rise to the cumulative process in the arbitrary interval $(t_0 , t_0 + t]$ for t_0 arbitrary. Define

$$\xi_{jk}(t_0 , p) = \int_0^\infty e^{-pg(j,k,x)} \frac{d}{dx} Q_{jk}(t_0 , x | \underline{J}(0) = \underline{e}_i) dx \tag{3.3.2}$$

and

$$\xi_{jk}(s_0 , p) = \int_0^\infty e^{-s_0 t_0} \xi_{jk}(t_0 , p) dt_0 \tag{3.3.3}$$

as the moment generating function and L.S.T. of the moment generating function, respectively, of the first transition after t_0 from state j to state k with sojourn time, measured from t_0 , less than or equal to x^0 . Subsequent transitions in $(t_0 , t_0 + t]$ are the same as the transitions in the interval $(0, t-x^0]$ so that the results of Section 2.3 can be employed. Substituting (3.1.8) into (3.3.2) and applying (3.3.3) gives

$$\xi_{jk}(s_0 , p) = \int_0^\infty e^{-s_0 t_0} \int_0^\infty e^{-pg(j,k,x)} \delta_{ij} \frac{d}{dx} Q_{jk}(t_0 + x) dx dt_0$$
$$+ \int_0^\infty e^{-s_0 t_0} \int_0^\infty e^{-pg(j,k,x)} \int_0^{t_0} h_{ij}(t_0 - u) \frac{d}{dx} Q_{jk}(u + x) du dx dt_0 . \tag{3.3.4}$$

The first integral can be written in the form

$$\delta_{ij} \int_0^\infty e^{-s_0 t_0} \theta_{jk}(t_0 , p)\, dt_0 , \tag{3.3.5}$$

where $\theta_{jk}(t_0 , p)$ is the moment generating function of $g(j,k,x)$, when it exists, in the distribution of $Q_{jk}(x)$ truncated on the left at t_0 . The L.S.T. with respect to t_0 of (3.3.5) will be denoted by

$$\delta_{ij}\theta_{jk}(s_0 , p) \quad . \tag{3.3.6}$$

The second integral in (3.3.4) is

$$\int_0^\infty e^{-pg(j,k,x)} \int_0^\infty \frac{d}{dx} Q_{jk}(u + x) \int_u^\infty e^{-s_0 t_0} h_{ij}(t_0 - u)\, dt_0 du dx \quad .$$

Letting $v = t_0 - u$ gives

$$\int_0^\infty e^{-pg(j,k,x)} \int_0^\infty \frac{d}{dx} Q_{jk}(u + x) e^{-s_0 u} \int_0^\infty e^{-s_0 v} h_{ij}(v)\, dv du dx$$

$$= h_{ij}(s_0) \int_0^\infty e^{-pg(j,k,x)} \int_0^\infty e^{-s_0 u} \frac{d}{dx} Q_{jk}(u + x)\, du dx$$

$$= h_{ij}(s_0)\theta_{jk}(s_0 , p) \quad . \tag{3.3.7}$$

So that

$$\xi_{jk}(s_0 , p) = [\delta_{ij} + h_{ij}(s_0)]\theta_{jk}(s_0 , p)$$

$$= \left\{\left(I - q(s_0)\right)^{-1}\right\}_{ij}\theta_{jk}(s_0 , p) \quad . \tag{3.3.8}$$

Since the g-functions are independent we have

$$E\{W_g(t_0 , t)|N^i(t_0 , t)\} = \xi_{jk}(s_0 , p) \prod_{\ell,m} \xi_{\ell m}(p)^{N_{\ell m}(t_0,t)-\delta_{j\ell}\delta_{km}} , \tag{3.3.9}$$

if the first transition after t_0 is from state j to state k . Therefore

$$E\{W_g(t_0, t) \mid \underline{J}(0) = \underline{e}_i\} = E\left\{\xi_{jk}(s_0, p) \prod_{\ell m} \xi_{\ell m}(p)^{N_{\ell m}(t_0,t) - \delta_{j\ell}\delta_{km}} \mid \underline{J}(0) = \underline{e}_i\right\}. \tag{3.3.10}$$

Unfortunately we are unable to relate this expectation to the p.g.f. of $N_{jk}(t_0, t)$ similar to the method used for the interval (0,t]. The difficulty arises because the time to the first transition after t_0 , say x_0 , has a distribution different than subsequent transition times. Although we cannot find the L.S.T. of the distribution $\{W_g(t_0, t) \mid \underline{J}(0) = \underline{e}_i\}$ we can still determine its mean and variance.

Given that the first transition after t_0 is from $j \to k$ with sojourn time, measured from t_0 , less than x_0 and given the transition count matrix $N^k(t - x_0)$ we write

$$\{W_g(t_0, t) \mid j \to k \leq x^0, N^k(t - x_0)\}$$

$$= g(j,k,x^0) + \sum_{\alpha\beta} N^k_{\alpha\beta}(t - x_0) g(\alpha,\beta,x) . \tag{3.3.11}$$

The expected value of the first term, conditional only on $\underline{J}(0) = \underline{e}_i$, is

$$\sum_{jk} \int_0^\infty g(j,k,x) \frac{d}{dx} Q_{jk}(t_0, x \mid \underline{J}(0) = \underline{e}_i) dx$$

$$= \sum_{jk} \delta_{ij} \int_0^\infty g(j,k,x) \frac{d}{dx} Q_{jk}(t_0 + x) dx$$

$$+ \sum_{jk} \int_0^\infty g(j,k,x) \int_0^{t_0} h_{ij}(t_0 - u) \frac{d}{dx} Q_{jk}(u + x) du dx$$

$$= \sum_k \frac{A_{ik}(t_0)}{P_{ik}} + \sum_{jk} \int_0^{t_0} h_{ij}(t_0 - u) \frac{A_{jk}(u)}{P_{jk}} du . \tag{3.3.12}$$

Taking expectation with respect to the distribution of X in the second term of (3.3.11) gives

$$\sum_{\alpha\beta} N^k_{\alpha\beta}(t - x_0) \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \quad .$$

Taking expectation with respect to the distribution of the transition count matrix gives

$$\sum_{\alpha\beta} M^k_{\alpha\beta}(t - x_0) \frac{A_{\alpha\beta}}{P_{\alpha\beta}}$$

whose expectation with respect to the first transitions $j \to k$ is

$$\sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_{jk} \int_0^{\infty} M^k_{\alpha\beta}(t - x) \frac{d}{dx} Q_{jk}(t_0 \ , \ x | \underline{J}(0) = \underline{e}_i) dx$$

$$= \sum_{\alpha,\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \left[\sum_{jk} \delta_{ij} \int_0^t M^k_{\alpha\beta}(t - x) \frac{d}{dx} Q_{jk}(t_0 + x | \underline{J}(0) = \underline{e}_i) dx \right.$$

$$\left. + \sum_{jk} \int_0^t M^k_{\alpha\beta}(t - x) \int_0^{t_0} h_{ij}(t_0 - u) \frac{d}{dx} Q_{jk}(u + x) du dx \right]$$

$$= \sum_{\alpha,\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \left[\sum_{k} \int_0^t M^k_{\alpha\beta}(t - x) \frac{d}{dx} Q_{ik}(t_0 + x | \underline{J}(0) = \underline{e}_i) dx \right.$$

$$\left. + \sum_{jk} \int_0^t M^k_{\alpha\beta}(t - x) \int_0^{t_0} h_{ij}(t_0 - u) \frac{d}{dx} Q_{jk}(u + x) du dx \right] . \qquad (3.3.12)$$

Combining (3.3.11) and (3.3.12) gives

$$E\{W_g(t_0 \ , \ t) | \underline{J}(0) = \underline{e}_i\} = \sum_k \frac{A_{ik}(t_0)}{P_{ik}} + \sum_{j,k} \int_0^{t_0} h_{ij}(t_0 - u) \frac{A_{jk}(u)}{P_{jk}} du$$

$$+ \sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \left\{ \sum_k \int_0^t M^k_{\alpha\beta}(t - x) \frac{d}{dx} Q_{ik}(t_0 + x | \underline{J}(0) = \underline{e}_i) dx \right.$$

$$+ \sum_{jk} \int_0^t M_{\alpha\beta}^k (t - x) \int_0^{t_0} h_{ij}(t_0 - u) \frac{d}{dx} Q_{jk}(u + x)\, du dx \Bigg\} . \tag{3.3.13}$$

Note that as $t_0 \to 0$

$$E\{W_g(0,t | \underline{J}(0) = \underline{e}_i)\} = \sum_k \frac{A_{ik}}{P_{ik}} + \sum_{\alpha,\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_k \int_0^\infty M_{\alpha\beta}^k (t-x) \frac{d}{dx} Q_{ik}(x)\, dx \quad . \tag{3.3.14}$$

Recall from Section 2.2 that

$$N_{\alpha\beta}^i (t) = \delta_{i\alpha}\delta_{k\beta} + N_{\alpha\beta}^k (t - x) \quad , \tag{3.3.15}$$

if the first transition is from state i to state k . Taking expectations in (3.3.15) gives

$$M_{\alpha\beta}^i (t) = \sum_k \delta_{i\alpha}\delta_{k\beta} + \sum_k \int_0^\infty M_{\alpha\beta}^k (t - x) \frac{d}{dx} Q_{ik}(x)\, dx \quad ;$$

that is,

$$\sum_k \int_0^\infty M_{\alpha\beta}^k (t - x) \frac{d}{dx} Q_{ik}(x)\, dx = M_{\alpha\beta}^i (t) - \sum_k \delta_{i\alpha}\delta_{k\beta} \quad . \tag{3.3.16}$$

Substituting into (3.3.14) gives

$$\begin{aligned} E\{W_g(0,t) | \underline{J}(0) = \underline{e}_i\} &= \sum_k \frac{A_{ik}}{P_{ik}} + \sum_{\alpha,\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \left[M_{\alpha\beta}^i (t) - \sum_k \delta_{i\alpha}\delta_{k\beta} \right] \\ &= \sum_k \frac{A_{ik}}{P_{ik}} + \sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} M_{\alpha\beta}^i (t) - \sum_{\alpha\beta} \delta_{i\alpha} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \\ &= \sum_{\alpha,\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} M_{\alpha\beta}^i (t) \\ &= E\{W_g(t) | \underline{J}(0) = \underline{e}_i\} \quad , \end{aligned} \tag{3.3.17}$$

as it should be. Although we have an expression for the mean of $W_g(t_0\ , t)$ the reader will note that it involves quantities for which we do not have expressions; namely, $M^k_{\alpha\beta}(t - x)$ and $h_{ij}(t_0 - u)$. However we do have expressions for the L.S.T.'s of these quantities suggesting that we may be able to obtain an expression for the L.S.T. of $W_g(t_0\ , t)$ in terms of known L.S.T.'s.

To find the L.S.T. of $E\{W_g(t_0\ , t)|\underline{J}(0) = \underline{e}_i\}$, which we denote by $w_i(s_0\ , s)$, first write (3.3.13) in the form

$$\sum_k \frac{A_{ik}(t_0)}{P_{ik}} + \sum_{j,k} \int_0^{t_0} h_{ij}(t_0 - u)\, \frac{A_{jk}(u)}{P_{jk}}\, du$$
$$+ \sum_{\alpha,\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_{j,k} \int_0^\infty M^k_{\alpha\beta}(t - x)\, \frac{d}{dx} Q_{jk}(t_0\ , x|\underline{J}(0) = \underline{e}_i)\,dx \quad . \tag{3.3.18}$$

Let

$$\alpha_{ik}(s_0) = \int_0^\infty e^{-s_0t_0} A_{ik}(t_0)\,dt_0 \tag{3.3.19}$$

be the L.S.T. of $A_{ik}(t_0)$. Then the L.S.T. of (3.3.18) with respect to t_0 is

$$\sum_k \frac{\alpha_{ik}(s_0)}{P_{ik}} + \sum_{jk} \frac{1}{P_{jk}}\, m_{ij}(s_0)\alpha_{jk}(s_0)$$
$$+ \sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_{jk} \int_0^\infty e^{-s_0t_0} \int_0^\infty M^k_{\alpha\beta}(t - x)\, \frac{d}{dx} Q_{jk}(t_0, x|\underline{J}(0) = \underline{e}_i)\,dx\,dt_0$$

and taking the L.S.T. with respect to t gives

$$\frac{1}{s} \sum_{jk} \frac{\alpha_{jk}(s_0)}{P_{jk}}\Big(\delta_{ij}+m_{ij}(s_0)\Big) + \sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_{jk}\int_0^\infty e^{-s_0t_0}\int_0^\infty e^{-st}\int_0^\infty M^k_{\alpha\beta}(t-x)\,\frac{d}{dx}Q_{jk}(t_0,x|\underline{J}(0)=\underline{e}_i)\,dx\,dt_0\,dt \quad . \tag{3.3.20}$$

$$\delta_{ij}\int_0^\infty e^{-s_0t_0}\int_0^\infty e^{-st}\int_0^t M^k_{\alpha\beta}(t-x)\,\frac{d}{dx}Q_{jk}(t_0+x)\,dxdtdt_0$$

$$= \delta_{ij}\int_0^\infty e^{-s_0t_0}\int_0^\infty \frac{d}{dx}Q_{jk}(t_0+x)\int_x^\infty e^{-st}M^k_{\alpha\beta}(t-x)\,dtdxdt_0 \quad .$$

Let $v = t - x$

$$\delta_{ij}\int_0^\infty e^{-s_0t_0}\int_0^\infty \frac{d}{dx}Q_{jk}(t_0+x)\int_0^\infty e^{-s(v+x)}M^k_{\alpha\beta}(v)\,dvdxdt_0$$

$$= \delta_{ij}m^k_{\alpha\beta}(s)\int_0^\infty e^{-s_0t_0}\int_0^\infty e^{-sx}\frac{d}{dx}Q_{jk}(t_0+x)\,dxdt_0 \quad .$$

Let $v = t_0 + x$

$$= \delta_{ij}m^k_{\alpha\beta}(s)\int_0^\infty e^{-s_0t_0}\int_{t_0}^\infty e^{-s(v-t_0)}\frac{d}{dv}Q_{jk}(v)\,dvdt_0$$

$$= \delta_{ij}m^k_{\alpha\beta}(s)\int_0^\infty e^{-sv}\frac{d}{dv}Q_{jk}(v)\int_0^v e^{-t_0(s_0-s)}\,dt_0dv$$

$$= \frac{-\delta_{ij}m^k_{\alpha\beta}(s)}{s_0-s}\int_0^\infty e^{-sv}\frac{d}{dv}Q_{jk}(v)\left[e^{-v(s_0-s)}-1\right]dv$$

$$= \frac{\delta_{ij}m^k_{\alpha\beta}(s)}{s_0-s}\left[q_{jk}(s)-q_{jk}(s_0)\right] . \qquad (3.3.21)$$

$$\int_0^\infty e^{-s_0t_0}\int_0^\infty e^{-st}\int_0^\infty M^k_{\alpha\beta}(t-x)\int_0^{t_0} h_{ij}(t_0-u)\,\frac{d}{dx}Q_{jk}(u+x)\,dudxdtdt_0$$

$$= \int_0^\infty e^{-st}\int_0^\infty M^k_{\alpha\beta}(t-x)\int_0^\infty \frac{d}{dx}Q_{jk}(u+x)\int_u^\infty e^{-s_0t_0}h_{ij}(t_0-u)\,dt_0dudxdt \, .$$

Let $v = t_0 - u$

$$= \int_0^\infty e^{-st}\int_0^\infty M^k_{\alpha\beta}(t-x)\int_0^\infty e^{-s_0u}\frac{d}{dx}Q_{jk}(u+x)\int_0^\infty e^{-s_0v}h_{ij}(v)\,dvdudxdt$$

$$= h_{ij}(s_0) \int_0^\infty e^{-st} \int_0^\infty M^k_{\alpha\beta}(t - x) \int_0^\infty e^{-s_0 u} \frac{d}{dx} Q_{jk}(u + x)\, dudxdt$$

$$= h_{ij}(s_0) \int_0^\infty \int_x^\infty e^{-st} M^k_{\alpha\beta}(t - x) \int_0^\infty e^{-s_0 u} \frac{d}{dx} Q_{jk}(u + x)\, dudtdx$$

$$= h_{ij}(s_0) \int_0^\infty \int_0^\infty e^{-s_0 u} \frac{d}{dx} Q_{jk}(u + x)\, du \int_x^\infty e^{-st} M^k_{\alpha\beta}(t - x)\, dtdx\ .$$

Let $v = t - x$

$$= h_{ij}(s_0) \int_0^\infty \int_0^\infty e^{-s_0 u} \frac{d}{dx} Q_{jk}(u + x)\, du \int_0^\infty e^{-s(v+x)} M^k_{\alpha\beta}(v)\, dvdx$$

$$= h_{ij}(s_0) m^k_{\alpha\beta}(s) \int_0^\infty \int_0^\infty e^{-s_0 u} e^{-sx} \frac{d}{dx} Q_{jk}(u + x)\, dudx\ .$$

Let $v = u + x$

$$= h_{ij}(s_0) m^k_{\alpha\beta}(s) \int_0^\infty e^{-sv} \frac{d}{dv} Q_{jk}(v) \int_0^v e^{-u(s_0 - s)}\, dudv$$

$$= \frac{-h_{ij}(s_0) m^k_{\alpha\beta}(s)}{s_0 - s} \int_0^\infty e^{-sv} \left[e^{-v(s_0 - s)} - 1\right] \frac{d}{dv} Q_{jk}(v)\, dv$$

$$= \frac{h_{ij}(s_0) m^k_{\alpha\beta}(s)}{s_0 - s} [q_{jk}(s) - q_{jk}(s_0)]\ . \qquad (3.3.22)$$

Combining (3.3.20) to (3.3.22) gives

$$w_i(s_0, s) = \frac{1}{s} \sum_{jk} \frac{\alpha_{jk}(s_0)}{P_{jk}} \Big(\delta_{ij} + m_{ij}(s_0)\Big) + \sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_{jk} \left\{ \frac{\delta_{ij} m^k_{\alpha\beta}(s)}{s_0 - s} [q_{jk}(s) - q_{jk}(s_0)] \right.$$

$$\left. + \frac{h_{ij}(s_0) m^k_{\alpha\beta}(s)}{s_0 - s} [q_{jk}(s) - q_{jk}(s_0)] \right\}$$

$$= \frac{1}{s} \sum_{jk} \frac{\alpha_{jk}(s_0)}{P_{jk}} \left(\delta_{ij} + m_{ij}(s_0)\right)$$

$$+ \sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_{jk} m^k_{\alpha\beta}(s) \left\{ \frac{\delta_{ij} + h_{ij}(s_0)}{s_0 - s} \left(q_{jk}(s) - q_{jk}(s_0)\right) \right\}$$

$$= \frac{1}{s} \sum_{jk} \frac{\alpha_{jk}(s_0)}{P_{jk}} \left(\delta_{ij} + m_{ij}(s_0)\right) + \sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_{jk} m^k_{\alpha\beta}(s) q^*_{jk}(s_0 \, , s) \quad . \tag{3.3.23}$$

So that we have expressed the L.S.T. of $E\{W_g(t_0 \, , t) | \underline{J}(0) = \underline{e}_i\}$ in terms of known quantities. From previous results we can express (3.3.23) in another form. For easier reference we note here those results which are used to obtain the alternate expression.

$$m_{ij}(s_0) = h_{ij}(s_0)$$

and

$$m^k_{\alpha\beta}(s) = q_{\alpha\beta}(s) \left\{ \left(I - q(s)\right)^{-1} \right\}_{k\alpha} \quad .$$

So we have

$$w_i(s_0 \, , s) = \frac{1}{s} \sum_{jk} \frac{\alpha_{jk}(s_0)}{P_{jk}} \left\{ \left(I - q(s_0)\right)^{-1} \right\}_{ij}$$

$$+ \frac{1}{s_0 - s} \sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_{jk} q_{\alpha\beta}(s) \left\{ \left(I - q(s)\right)^{-1} \right\}_{k\alpha}$$

$$\times \left\{ \left(I - q(s_0)\right)^{-1} \right\}_{ij} \left(q_{jk}(s) - q_{jk}(s_0)\right)$$

$$= \sum_{jk} \left\{ \left(I - q(s_0)\right)^{-1} \right\}_{ij} \left[\frac{1}{s} \frac{\alpha_{jk}(s_0)}{P_{jk}} \right.$$

$$\left. + \frac{1}{s_0 - s} \sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} q_{\alpha\beta}(s) \left\{ \left(I - q(s)\right)^{-1} \right\}_{k\alpha} \left(q_{jk}(s) - q_{jk}(s_0)\right) \right] \, , \tag{3.3.24}$$

which is in terms of the L.S.T.'s of the basic quantities (m, $\underline{a}$, Q) and the g-functions.

To derive the variance of $W_g(t_0\,,\, t|\underline{J}(0) = \underline{e}_i)$ we begin with the well-known identity

$$\mathrm{Var}\{W_g(t_0\,, t)|\underline{J}(0) = \underline{e}_i\} = \underset{\Lambda}{E}\ \mathrm{Var}\{W_g(t_0\,, t)|\Lambda\,,\, \underline{J}(0) = \underline{e}_i\} + \underset{\Lambda}{\mathrm{Var}}\ E\{W_g(t_0\,, t)|\Lambda\,,\, J(0) = \underline{e}_i\}\ , \tag{3.3.25}$$

where Λ is the event that the first transition after t_0 is from state j to state k with sojourn time, measured from t_0 , less than or equal to x . Using the same identity for the first term on the right gives

$$\begin{aligned}\mathrm{Var}\{W_g(t_0\,, t)|\underline{J}(0) = \underline{e}_i\} &= \underset{\Lambda}{E}\ \underset{N}{E}\ \mathrm{Var}\left\{g(j,k,x^0) + \sum_{\alpha\beta} N^k_{\alpha\beta}(t - x^0)\,g(\alpha,\beta,x)\right\} \\ &+ \underset{\Lambda}{E}\ \underset{N}{\mathrm{Var}}\ E\left\{g(j,k,x^0) + \sum_{\alpha\beta} N^k_{\alpha\beta}(t - x^0)\,g(\alpha,\beta,x)\right\} \\ &+ \underset{\Lambda}{\mathrm{Var}}\left\{g(j,k,x^0) + \sum_{\alpha\beta} M^k_{\alpha\beta}(t - x^0)\,\frac{A_{\alpha\beta}}{P_{\alpha\beta}}\right\}\ .\end{aligned} \tag{3.3.26}$$

For the first term we get

$$\underset{\Lambda}{E}\ \underset{N}{E}\ \sum_{\alpha\beta}\left[N^k_{\alpha\beta}(t - x^0)\right]\left[\frac{B_{\alpha\beta}}{P_{\alpha\beta}} - \frac{A^2_{\alpha\beta}}{P^2_{\alpha\beta}}\right] = \underset{\Lambda}{E}\ \sum_{\alpha\beta}\left[\frac{B_{\alpha\beta}}{P_{\alpha\beta}} - \frac{A^2_{\alpha\beta}}{P^2_{\alpha\beta}}\right] M^k_{\alpha\beta}(t - x^0)$$

and using (3.3.12) gives for the first term

$$\sum_{\alpha,\beta}\left[\frac{B_{\alpha\beta}}{P_{\alpha\beta}} - \frac{A^2_{\alpha\beta}}{P^2_{\alpha\beta}}\right]\sum_{jk}\int_0^\infty M^k_{\alpha\beta}(t - x)\ \frac{d}{dx}\,Q_{jk}(t_0\ ,\ x|\underline{J}(0) = \underline{e}_i)\,dx\ . \tag{3.3.27}$$

The second term is

$$\begin{aligned}&\underset{\Lambda\ N|\Lambda}{E\ \mathrm{Var}}\left\{g(j,k,s^0) + \sum_{\alpha\beta}\frac{A_{\alpha\beta}}{P_{\alpha\beta}}\,N^k_{\alpha\beta}(t - x^0)\right\} \\ &= \sum_{\alpha,\beta}\ \sum_{\gamma,\delta}\ \frac{A_{\alpha\beta}A_{\gamma\delta}}{P_{\alpha\beta}P_{\gamma\delta}}\ \sum_{j,k}\int_0^\infty \mathrm{Cov}\left\{N^k_{\alpha\beta}(t - x)\,,\ N^k_{\gamma\delta}(t - x)\right\}\frac{d}{dx}\,Q_{jk}(t_0,x|\underline{J}(0) = \underline{e}_i)\,dx\ .\end{aligned} \tag{3.3.28}$$

The third term is

$$\operatorname*{Var}_{\Lambda}\left\{g(j,k,x^0) + \sum_{\alpha,\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} M^k_{\alpha\beta}(t - x^0)\right\}$$

$$= \sum_{j,k} \int_0^\infty \left[g(j,k,x) + \sum_{\alpha,\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} M^k_{\alpha\beta}(t - x)\right]^2 \frac{d}{dx} Q_{jk}(t_0, x|\underline{J}(0) = \underline{e}_i)dx$$

$$- \left[\sum_k \frac{A_{ik}(t_0)}{P_{ik}} + \sum_{j,k} \int_0^{t_0} h_{ij}(t_0 - u) \frac{A_{jk}(u)}{P_{jk}} du\right.$$

$$\left. + \sum_{\alpha,\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_{j,k} \int_0^\infty M^k_{\alpha\beta}(t - x) \frac{d}{dx} Q_{jk}(t_0, x|\underline{J}(0) = \underline{e}_i)dx\right]^2 . \tag{3.3.29}$$

Few simplifications of the last three equations are possible. Since we will ultimately find the L.S.T. of $\mathrm{Var}\{W_g(t_0, t|\underline{J}(0) = \underline{e}_i\}$, we gain little by simplifying the expression for the variance. For easier reference we combine our results giving

$$\mathrm{Var}\{W_g(t_0, t)|\underline{J}(0) = \underline{e}_i\}$$

$$= \sum_{\alpha,\beta} \left[\frac{B_{\alpha\beta}}{P_{\alpha\beta}} - \frac{A^2_{\alpha\beta}}{P^2_{\alpha\beta}}\right] \sum_{jk} \int_0^\infty M^k_{\alpha\beta}(t - x) \frac{d}{dx} Q_{jk}(t_0, x|\underline{J}(0) = \underline{e}_i)dx$$

$$+ \sum_{\alpha,\beta} \sum_{\gamma\delta} \frac{A_{\alpha\beta}A_{\gamma\delta}}{P_{\alpha\beta}P_{\gamma\delta}} \sum_{jk} \int_0^\infty \mathrm{Cov}\left\{N^k_{\alpha\beta}(t - x), N^k_{\gamma\delta}(t - x)\right\} \frac{d}{dx} Q_{jk}(t_0, x|\underline{J}(0) = \underline{e}_i)dx$$

$$+ \sum_{jk} \int_0^\infty \left[g(j,k,x) + \sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} M^k_{\alpha\beta}(t - x)\right]^2 \frac{d}{dx} Q_{jk}(t_0, x|\underline{J}(0) = \underline{e}_i)dx$$

$$- \left[\sum_k \frac{A_{ik}(t_0)}{P_{ik}} + \sum_{jk} \int_0^{t_0} h_{ij}(t_0 - u) \frac{A_{jk}(u)}{P_{jk}} du\right.$$

$$\left. + \sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_{jk} \int_0^\infty M^k_{\alpha\beta}(t - x) \frac{d}{dx} Q_{jk}(t_0, x|\underline{J}(0) = \underline{e}_i)dx\right]^2 . \tag{3.3.30}$$

For $t_0 = 0$,

$\mathrm{Var}\{W_g(0,t) \mid \underline{J}(0) = \underline{e}_i\}$

$$= \sum_{\alpha,\beta} \left[\frac{B_{\alpha\beta}}{P_{\alpha\beta}} - \frac{A^2_{\alpha\beta}}{P^2_{\alpha\beta}} \right] \sum_k \int_0^\infty M^k_{\alpha\beta}(t - x) \frac{d}{dx} Q_{ik}(x)\,dx$$

$$+ \sum_{\alpha,\beta} \sum_{\gamma,\delta} \frac{A_{\alpha\beta}A_{\gamma\delta}}{P_{\alpha\beta}P_{\gamma\delta}} \sum_k \int_0^\infty \mathrm{Cov}\left\{N^k_{\alpha\beta}(t - x),\ N^k_{\gamma\delta}(t - x)\right\} \frac{d}{dx} Q_{ik}(x)\,dx$$

$$+ \sum_k \int_0^\infty \left[g(i,k,x) + \sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} M^k_{\alpha\beta}(t - x) \right]^2 \frac{d}{dx} Q_{ik}(x)\,dx$$

$$- \sum_{\alpha\beta} \sum_{\gamma\delta} \frac{A_{\alpha\beta}A_{\gamma\delta}}{P_{\alpha\beta}P_{\gamma\delta}} M^i_{\alpha\beta}(t) M^i_{\gamma\delta}(t) \quad . \tag{3.3.31}$$

$$= \sum_{\alpha\beta} \left[\frac{B_{\alpha\beta}}{P_{\alpha\beta}} - \frac{A^2_{\alpha\beta}}{P^2_{\alpha\beta}} \right] \left[M^i_{\alpha\beta}(t) - \sum_k \delta_{i\alpha}\delta_{k\beta} \right]$$

$$+ \sum_{\alpha,\beta} \sum_{\gamma,\delta} \frac{A_{\alpha\beta}A_{\gamma\delta}}{P_{\alpha\beta}P_{\gamma\delta}} \sum_k \int_0^\infty c^k_{\alpha\beta,\gamma\delta}(t - x) \frac{d}{dx} Q_{ik}(x)\,dx$$

$$- \sum_{\alpha\beta} \sum_{\gamma\delta} \frac{A_{\alpha\beta}A_{\gamma\delta}}{P_{\alpha\beta}P_{\gamma\delta}} \sum_k \int_0^\infty M^k_{\alpha\beta}(t - x) M^k_{\gamma\delta}(t - x) \frac{d}{dx} Q_{ik}(x)\,dx$$

$$+ \sum_k \frac{B_{ik}}{P_{ik}} + 2 \sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P\alpha\beta} \sum_k \int_0^\infty g(i,k,x) \left[M^i_{\alpha\beta}(t) - \sum_k \delta_{i\alpha}\delta_{k\beta} \right] \frac{d}{dx} Q_{ik}(x)\,dx$$

$$+ \sum_{\alpha\beta} \sum_{\gamma\delta} \frac{A_{\alpha\beta}A_{\gamma\delta}}{P_{\alpha\beta}P_{\gamma\delta}} \sum_k \int_0^\infty M^k_{\alpha\beta}(t - x) M^k_{\gamma\delta}(t - x) \frac{d}{dx} Q_{ik}(x)\,dx$$

$$- \sum_{\alpha\beta} \sum_{\gamma\delta} \frac{A_{\alpha\beta}A_{\gamma\delta}}{P_{\alpha\beta}P_{\gamma\delta}} M^i_{\alpha\beta}(t) M^i_{\gamma\delta}(t)$$

$$= \sum_{\alpha\beta} \left[\frac{B_{\alpha\beta}}{P_{\alpha\beta}} - \frac{A^2_{\alpha\beta}}{P^2_{\alpha\beta}} \right] M^i_{\alpha\beta}(t) - \sum_\beta \left[\frac{B_{i\beta}}{P_{i\beta}} - \frac{A^2_{i\beta}}{P^2_{i\beta}} \right]$$

$$+ \sum_{\alpha\beta} \sum_{\gamma\delta} \frac{A_{\alpha\beta}A_{\gamma\delta}}{P_{\alpha\beta}P_{\gamma\delta}} \sum_k \int_0^\infty c^k_{\alpha\beta,\gamma\delta}(t - x) \frac{d}{dx} Q_{ik}(x)\,dx$$

$$+ \sum_k \frac{B_{ik}}{P_{ik}} + 2 \sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} M^i_{\alpha\beta}(t) \sum_k \frac{A_{ik}}{P_{ik}} - 2 \sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_k \delta_{i\alpha} \frac{A_{ik}}{P_{ik}}$$

$$- \sum_{\alpha\beta} \sum_{\gamma\delta} \frac{A_{\alpha\beta} A_{\gamma\delta}}{P_{\alpha\beta} P_{\gamma\delta}} M^i_{\alpha\beta}(t) M^i_{\gamma\delta}(t) \quad . \qquad (3.3.32)$$

Recall that

$$c^k_{\alpha\beta,\gamma\delta}(t - x) = E\left\{N^k_{\alpha\beta}(t - x) N^k_{\gamma\delta}(t - x)\right\} \quad .$$

Now

$$N^i_{\alpha\beta}(t) N^i_{\gamma\delta}(t) = \delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{i\alpha}\delta_{k\beta} + \delta_{i\gamma}\delta_{k\delta} N^k_{\alpha\beta}(t - x)$$
$$+ \delta_{i\alpha}\delta_{k\beta} N^k_{\gamma\delta}(t - x) + N^k_{\alpha\beta}(t - x) N^k_{\gamma\delta}(t - x) \quad .$$

Taking expectations gives

$$c^i_{\alpha\beta,\gamma\delta}(t) = \delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{i\alpha} + \delta_{i\gamma} \sum_k \delta_{k\delta} \int_0^\infty M^k_{\alpha\beta}(t - x) \frac{d}{dx} Q_{ik}(x)\,dx$$
$$+ \delta_{i\alpha} \sum_k \delta_{k\beta} \int_0^\infty M^k_{\gamma\delta}(t - x) \frac{d}{dx} Q_{ik}(x)\,dx$$
$$+ \sum_k \int_0^\infty c^k_{\alpha\beta,\gamma\delta}(t - x) \frac{d}{dx} Q_{ik}(x)\,dx \quad .$$

Hence

$$\sum_k \int_0^\infty c^k_{\alpha\beta,\gamma\delta}(t - x) \frac{d}{dx} Q_{ik}(x)\,dx = c^i_{\alpha\beta,\gamma\delta}(t) - \delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{i\alpha}$$
$$- \delta_{i\gamma} \int_0^\infty M^\delta_{\alpha\beta}(t - x) \frac{d}{dx} Q_{i\delta}(x)\,dx$$
$$- \delta_{i\alpha} \int_0^\infty M^\beta_{\gamma\delta}(t - x) \frac{d}{dx} Q_{i\beta}(x)\,dx \quad ,$$

but

$$N^{i}_{\alpha\beta}(t) = \delta_{i\alpha}\delta_{k\beta} + N^{k}_{\alpha\beta}(t - x)$$

whose expectation, conditional on the first transition, is

$$M^{i}_{\alpha\beta}(t) = \delta_{i\alpha}\delta_{k\beta} + \int_0^{\infty} M^{k}_{\alpha\beta}(t - x)\,\frac{d}{dx}\,Q_{ik}(x)\,dx \quad .$$

Using this in the above gives

$$\sum_k \int_0^{\infty} c^{k}_{\alpha\beta\,,\gamma\delta}(t - x)\,\frac{d}{dx}\,Q_{ik}(x)\,dx = c^{i}_{\alpha\beta,\gamma\delta}(t) - \delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{i\alpha}$$
$$- \delta_{i\gamma}[M^{i}_{\alpha\beta}(t) - \delta_{i\alpha}\delta_{\delta\beta}]$$
$$- \delta_{i\alpha}[M^{i}_{\gamma\delta}(t) - \delta_{i\gamma}\delta_{\beta\delta}] \quad .$$

Substituting into (3.3.32) we finally arrive at

$$\mathrm{Var}\{W_g(0,t)\,|\,\underline{J}(0) = \underline{e}_i\} = \sum_{\alpha\beta}\left[\frac{B_{\alpha\beta}}{P_{\alpha\beta}} - \frac{A^2_{\alpha\beta}}{P^2_{\alpha\beta}}\right] M^{i}_{\alpha\beta}(t) + \sum_k \frac{A^2_{ik}}{P^2_{ik}}$$
$$+ \sum_{\alpha\beta}\sum_{\gamma\delta} \frac{A_{\alpha\beta}A_{\gamma\delta}}{P_{\alpha\beta}P_{\gamma\delta}} \Big\{ c^{i}_{\alpha\beta,\gamma\delta}(t) - \delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{i\alpha}$$
$$- \delta_{i\gamma}M^{i}_{\alpha\beta}(t) + \delta_{i\alpha}\delta_{\delta\beta}\delta_{i\gamma} - \delta_{i\alpha}M^{i}_{\gamma\delta}(t) + \delta_{i\alpha}\delta_{i\gamma}\delta_{\beta\delta}\Big\}$$
$$+ 2\sum_{\alpha\beta}\sum_k \frac{A_{\alpha\beta}A_{ik}}{P_{\alpha\beta}P_{ik}} M^{i}_{\alpha\beta}(t) - 2\sum_{\beta}\frac{A^2_{i\beta}}{P^2_{i\beta}}$$
$$- \sum_{\alpha\beta}\sum_{\gamma\delta} \frac{A_{\alpha\beta}A_{\gamma\delta}}{P_{\alpha\beta}P_{\gamma\delta}} M^{i}_{\alpha\beta}(t)M^{i}_{\gamma\delta}(t)$$
$$= \sum_{\alpha\beta}\left[\frac{B_{\alpha\beta}}{P_{\alpha\beta}} - \frac{A^2_{\alpha\beta}}{P^2_{\alpha\beta}}\right] M^{i}_{\alpha\beta}(t) + \sum_{\alpha,\beta}\sum_{\gamma,\delta} \frac{A_{\alpha\beta}A_{\gamma\delta}}{P_{\alpha\beta}P_{\gamma\delta}}$$

$$\times \operatorname{Cov}\left\{n^i_{\alpha\beta}(t) \;,\; n^i_{\gamma\delta}(t)\right\} + \sum_k \frac{A^2_{ik}}{P^2_{ik}}$$

$$- \sum_{\alpha\beta} \sum_{\gamma\delta} \frac{A_{\alpha\beta}A_{\gamma\delta}}{P_{\alpha\beta}P_{\gamma\delta}} M^i_{\alpha\beta}(t) + \sum_{\beta} \sum_{\gamma\delta} \frac{A_{i\beta}A_{\gamma\delta}}{P_{i\beta}P_{\gamma\delta}} \delta_{\delta\beta}\delta_{i\gamma}$$

$$- \sum_{\beta} \sum_{\gamma,\delta} \frac{A_{i\beta}A_{\gamma\delta}}{P_{i\beta}P_{\gamma\delta}} M^i_{\gamma\delta}(t) + 2 \sum_{\alpha\beta} \sum_{k} \frac{A_{\alpha\beta}A_{ik}}{P_{\alpha\beta}P_{ik}} M^i_{\alpha\beta}(t)$$

$$- 2 \sum_{\beta} \frac{A^2_{i\beta}}{P^2_{i\beta}}$$

$$= \sum_{\alpha\beta} \left[\frac{B_{\alpha\beta}}{P_{\alpha\beta}} - \frac{A^2_{\alpha\beta}}{P^2_{\alpha\beta}}\right] M^i_{\alpha\beta}(t) + \sum_{\alpha\beta} \sum_{\gamma\delta} \frac{A_{\alpha\beta}A_{\gamma\delta}}{P_{\alpha\beta}P_{\gamma\delta}} \operatorname{Cov}\left\{n^i_{\alpha\beta}(t), n^i_{\gamma\delta}(t)\right\}$$

$$= \operatorname{Var}\{W_g(t) \,|\, \underline{J}(0) = \underline{e}_i\} \quad ,$$

as required.

We now find the L.S.T. of $\operatorname{Var}\{W_g(t_0 \;,\; t) \,|\, \underline{J}(0) = \underline{e}_i\}$. Define

$$\sigma^2_i(s_0 \;,\; s) = \int_0^\infty e^{-s_0 t_0} \int_0^\infty e^{-st} \operatorname{Var}\{W_g(t_0 \,,\, t) \,|\, \underline{J}(0) = \underline{e}_i\} dt dt_0 \; . \quad (3.3.33)$$

Taking L.S.T.'s term by term in (3.3.30) we have that the first term, from previous results, is

$$\int_0^\infty e^{-s_0 t_0} \int_0^\infty e^{-st} \sum_{\alpha\beta} \left[\frac{B_{\alpha\beta}}{P_{\alpha\beta}} - \frac{A^2_{\alpha\beta}}{P^2_{\alpha\beta}}\right] \sum_{jk} \int_0^\infty M^k_{\alpha\beta}(t-x) \frac{d}{dx} Q_{jk}(t_0 \,,\, x \,|\, \underline{J}(0) = \underline{e}_i) dx dt dt_0$$

$$= \sum_{\alpha\beta} \left[\frac{B_{\alpha\beta}}{P_{\alpha\beta}} - \frac{A^2_{\alpha\beta}}{P^2_{\alpha\beta}}\right] \sum_{jk} m^k_{\alpha\beta}(s) q^*_{jk}(s_0 \;,\; s) \quad . \quad (3.3.34)$$

For the second term

$$\sum_{\alpha,\beta} \sum_{\gamma,\delta} \frac{A_{\alpha\beta}A_{\gamma\delta}}{P_{\alpha\beta}P_{\gamma\delta}} \sum_{jk} \int_0^\infty e^{-s_0 t_0} \int_0^\infty e^{-st} \int_0^\infty \left[c^k_{\alpha\beta,\gamma\delta}(t-x) - M^k_{\alpha\beta}(t-x) M^k_{\gamma\delta}(t-x)\right]$$

$$+ \sum_{\alpha\beta} \sum_{\gamma\delta} \frac{A_{\alpha\beta} A_{\gamma\delta}}{P_{\alpha\beta} P_{\gamma\delta}} \sum_{jk} \int_0^\infty e^{-s_0 t_0} \int_0^\infty e^{-st} \int_0^\infty M^k_{\alpha\beta}(t-x) M^k_{\gamma\delta}(t-x) \frac{d}{dx} Q_{jk}(t_0, x | \underline{J}(0) = \underline{e}_i) \, dx dt dt_0 \, . \tag{3.3.36}$$

Consider the first term only which we expand giving

$$\sum_{j,k} \int_0^\infty e^{-s_0 t_0} \int_0^\infty e^{-st} \int_0^\infty \delta_{ij} g^2(j,k,x) \frac{d}{dx} Q_{jk}(t_0 + x) \, dx dt dt_0$$

$$+ \sum_{jk} \int_0^\infty e^{-s_0 t_0} \int_0^\infty e^{-st} \int_0^\infty g^2(j,k,x) \int_0^{t_0} h_{ij}(t_0 - u) \frac{d}{dx} Q_{jk}(u+x) \, du dx dt dt_0$$

$$= \frac{1}{s} \sum_{jk} \delta_{ij} \int_0^\infty e^{-s_0 t_0} \int_0^\infty g^2(j,k,x) \frac{d}{dx} Q_{jk}(t_0 + x) \, dx dt_0$$

$$+ \frac{1}{s} \sum_{jk} \int_0^\infty g^2(j,k,x) \int_0^\infty \frac{d}{dx} Q_{jk}(u+x) \int_u^\infty e^{-s_0 t_0} h_{ij}(t_0 - u) \, dt_0 du dx$$

in the second integral let $v = t_0 - u$

$$= \frac{1}{s} \sum_{jk} \delta_{ij} \int_0^\infty e^{-s_0 t_0} \frac{B_{jk}(t_0)}{P_{jk}} dt_0 + \frac{1}{s} \sum_{jk} m_{ij}(s_0) \int_0^\infty g^2(j,k,x) \int_0^\infty e^{-s_0 u} \frac{d}{dx} \times Q_{jk}(u+x) \, du dx \, .$$

Define

$$\beta_{jk}(s_0) = \int_0^\infty e^{-s_0 t_0} B_{jk}(t_0) \, dt_0$$

and interchange the order of integration in the second term

$$= \frac{1}{s} \sum_{jk} \delta_{ij} \frac{\beta_{jk}(s_0)}{P_{jk}} + \frac{1}{s} \sum_{jk} m_{ij}(s_0) \int_0^\infty e^{-s_0 u} \frac{B_{jk}(u)}{P_{jk}} du$$

$$= \frac{1}{s} \sum_{jk} \frac{\delta_{ij} + m_{ij}(s_0)}{P_{jk}} \beta_{jk}(s_0) \quad .$$

$$2\sum_{\alpha\beta}\frac{A_{\alpha\beta}}{P_{\alpha\beta}}\sum_{jk}\delta_{ij}\int_0^\infty e^{-s_0t_0}\int_0^\infty e^{-st}\int_0^\infty g(j,k,x)M^k_{\alpha\beta}(t-x)\frac{d}{dx}Q_{jk}(t_0+x)\,dx\,dt\,dt_0$$

$$+\,2\sum_{\alpha\beta}\frac{A_{\alpha\beta}}{P_{\alpha\beta}}\sum_{jk}\int_0^\infty e^{-s_0t_0}\int_0^\infty e^{-st}\int_0^\infty g(j,k,x)M^k_{\alpha\beta}(t-x)\int_0^{t_0}h_{ij}(t_0-u)\frac{d}{dx}Q_{jk}(u+x)\,du\,dx\,dt\,dt_0\ . \qquad (3.3.37)$$

For convenience consider the terms individually. The first term becomes

$$2\sum_{\alpha\beta}\frac{A_{\alpha\beta}}{P_{\alpha\beta}}\sum_{jk}\delta_{ij}\int_0^\infty e^{-s_0t_0}\int_0^\infty g(j,k,x)\frac{d}{dx}Q_{jk}(t_0+x)\int_x^\infty e^{-st}M^k_{\alpha\beta}(t-x)\,dt\,dx\,dt_0$$

let $v = t - x$

$$2\sum_{\alpha\beta}\frac{A_{\alpha\beta}}{P_{\alpha\beta}}\sum_{jk}\delta_{ij}\int_0^\infty e^{-s_0t_0}\int_0^\infty e^{-sx}g(j,k,x)\frac{d}{dx}Q_{jk}(t_0+x)\int_0^\infty e^{-sv}M^k_{\alpha\beta}(v)\,dv\,dx\,dt_0$$

$$= 2\sum_{\alpha\beta}\frac{A_{\alpha\beta}}{P_{\alpha\beta}}\sum_{jk}\delta_{ij}m^k_{\alpha\beta}(s)\int_0^\infty e^{-s_0t_0}\int_0^\infty e^{-sx}g(j,k,x)\frac{d}{dx}Q_{jk}(t_0+x)\,dx\,dt_0\quad .$$

Define

$$\alpha_{jk}(t_0\ ,\ s) = \int_0^\infty e^{-sx}g(j,k,x)\frac{d}{dx}Q_{jk}(t_0+x)\,dx$$

and

$$\alpha_{jk}(s_0\ ,\ s) = \int_0^\infty e^{-s_0t_0}\alpha_{jk}(t_0\ ,\ s)\,dt_0\quad .$$

Then

$$= 2\sum_{\alpha\beta}\frac{A_{\alpha\beta}}{P_{\alpha\beta}}\sum_{jk}\delta_{ij}m^k_{\alpha\beta}(s)\alpha_{jk}(s_0\ ,\ s)\,. \qquad (3.3.38)$$

For the second term in write

$$2\sum_{\alpha\beta}\frac{A_{\alpha\beta}}{P_{\alpha\beta}}\sum_{j,k}\int_0^\infty e^{-st}\int_0^\infty g(j,k,x)M^k_{\alpha\beta}(t-x)\int_0^\infty\frac{d}{dx}Q_{jk}(u+x)\int_u^\infty e^{-s_0t_0}h_{ij}(t_0-u)\,dt_0\,du\,dx\,dt$$

let $v = t_0 - u$

$$2 \sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_{jk} m_{ij}(s_0) \int_0^\infty e^{-st} \int_0^\infty g(j,k,x) M^k_{\alpha\beta}(t-x) \int_0^\infty e^{-s_0 u} \frac{d}{dx} Q_{jk}(u+x)\, du dx dt$$

$$= 2 \sum_{\alpha,\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_{jk} m_{ij}(s_0) \int_0^\infty g(j,k,x) \int_0^\infty e^{-s_0 u} \frac{d}{dx} Q_{jk}(u+x) \int_x^\infty e^{-st} M^k_{\alpha\beta}(t-x)\, dt du dx$$

let $v = t - x$

$$= 2 \sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_{jk} m_{ij}(s_0) m^k_{\alpha\beta}(s) \int_0^\infty e^{-s_0 u} \int_0^\infty e^{-sx} g(j,k,x) \frac{d}{dx} Q_{jk}(u+x)\, dx du$$

$$= 2 \sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_{jk} m_{ij}(s_0) m^k_{\alpha\beta}(s) \quad {}_{jk}(s_0 , s) \quad .$$

We have now shown that the second term in (3.3.36) is

$$2 \sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_{jk} [\delta_{ij} + m_{ij}(s_0)] m^k_{\alpha\beta}(s) \alpha_{jk}(s_0 , s) \quad . \tag{3.3.39}$$

The third term in (3.3.36) is

$$\sum_{\alpha,\beta} \sum_{\gamma\delta} \frac{A_{\alpha\beta} A_{\gamma\delta}}{P_{\alpha\beta} P_{\gamma\delta}} \sum_{jk} \delta_{ij} \int_0^\infty e^{-s_0 t_0} \int_0^\infty e^{-st} \int_0^\infty M^k_{\alpha\beta}(t-x) M^k_{\gamma\delta}(t-x) \frac{d}{dx} Q_{jk}(t_0, x | \underline{J}(0) = \underline{e}_i)\, dx dt dt_0 \ ,$$

which from previous results becomes

$$\sum_{\alpha\beta} \sum_{\gamma\delta} \frac{A_{\alpha\beta} A_{\gamma\delta}}{P_{\alpha\beta} P_{\gamma\delta}} \sum_{jk} [m^k_{\alpha\beta}(s) * m^k_{\gamma\delta}(s)] q^*_{jk}(s_0 , s) \quad . \tag{3.3.40}$$

So that the third term in (3.3.30) is

$$\frac{1}{s} \sum_{jk} \frac{\delta_{ij} + m_{ij}(s_0)}{P_{jk}} \beta_{jk}(s_0) + 2 \sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_{jk} [\delta_{ij} + m_{ij}(s_0)] m^k_{\alpha\beta}(s) \delta_{jk}(s_0 , s)$$

$$+ \sum_{\alpha\beta} \sum_{\gamma\delta} \frac{A_{\alpha\beta}A_{\gamma\delta}}{P_{\alpha\beta}P_{\gamma\delta}} \sum_{jk} [m^k_{\alpha\beta}(s) * m^k_{\gamma\delta}(s)]q^*_{jk}(s_0\ ,\ s) \quad . \tag{3.3.41}$$

Squaring the last term in (3.3.30) and using (3.3.23) and (3.3.35) we can immediately write that the L.S.T. of the fourth term is

$$\frac{1}{s}\sum_h \sum_k \frac{\alpha_{ih}(s_0) * \alpha_{ik}(s_0)}{P_{ih}P_{ik}} + \frac{2}{s}\sum_h \sum_{jk} \frac{\alpha_{ih}(s_0) * [\alpha_{jk}(s_0)m_{ij}(s_0)]}{P_{ih}P_{jk}}$$

$$+ \frac{2}{s}\sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_h \sum_{jk} \frac{1}{P_{ih}} \alpha_{ij}(s_0) * [m^k_{\alpha\beta}(s)q^*_{jk}(s_0\ ,\ s)]$$

$$+ \frac{1}{s}\sum_{jk} \sum_{\ell m} \frac{1}{P_{jk}P_{\ell m}} [\alpha_{jk}(s_0)m_{ij}(s_0)] * [\alpha_{\ell m}(s_0)m_{i\ell}(s_0)]$$

$$+ \frac{2}{s}\sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_{jk} \sum_{\ell m} \frac{1}{P_{jk}} [\alpha_{jk}(s_0)m_{ij}(s_0)] * [m^m_{\alpha\beta}(s)q^*_{\ell m}(s_0\ ,\ s)]$$

$$+ \sum_{\alpha\beta} \sum_{\gamma\delta} \frac{A_{\alpha\beta}A_{\gamma\delta}}{P_{\alpha\beta}P_{\gamma\delta}} \sum_{j,k} \sum_{\ell,m} [m^k_{\alpha\beta}(s)q^*_{jk}(s_0, s)] * [m^m_{\gamma\delta}(s) * q^*_{\ell m}(s_0, s)] \quad . \tag{3.3.42}$$

Combining the appropriate terms we have shown that the L.S.T. of $\mathrm{Var}\{W_g(t_0\ ,\ t)|\underline{J}(0) = \underline{e}_i\}$ is

$$\sigma^2_i(s_0,s) = \sum_{\alpha\beta} \left[\frac{B_{\alpha\beta}}{P_{\alpha\beta}} - \frac{A^2_{\alpha\beta}}{P^2_{\alpha\beta}}\right] \sum_{jk} m^k_{\alpha\beta}(s)q^*_{jk}(s_0\ ,\ s) + \sum_{\alpha\beta} \sum_{\gamma\delta} \frac{A_{\alpha\beta}A_{\gamma\delta}}{P_{\alpha\beta}P_{\gamma\delta}}$$

$$\times \sum_{jk} \left\{c^k_{\alpha\beta,\gamma\delta}(s)q^*_{jk}(s_0, s) - [m^k_{\alpha\beta}(s) * m^k_{\gamma\delta}(s)]q^*_{jk}(s_0\ ,\ s)\right\}$$

$$+ \frac{1}{s}\sum_{jk} \frac{\delta_{ij} + m_{ij}(s_0)}{P_{jk}} \beta_{jk}(s_0) + 2\sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_{jk} [\delta_{ij} + m_{ij}(s_0)]m^k_{\alpha\beta}(s)\alpha_{jk}(s_0,s)$$

$$+ \sum_{\alpha\beta} \sum_{\gamma\delta} \frac{A_{\alpha\beta}A_{\gamma\delta}}{P_{\alpha\beta}P_{\gamma\delta}} \sum_{jk} [m^k_{\alpha\beta}(s) * m^k_{\gamma\delta}(s)]q^*_{jk}(s_0\ ,\ s)$$

$$- \frac{1}{s}\sum_h \sum_k \frac{1}{P_{ih}P_{ik}} \alpha_{ih}(s_0) * \alpha_{ik}(s_0)$$

$$- \frac{2}{s} \sum_h \sum_{jk} \frac{1}{P_{ih} P_{jk}} \alpha_{ih}(s_0) * [\alpha_{jk}(s_0) m_{ij}(s_0)]$$

$$- \frac{2}{s} \sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_h \sum_{jk} \frac{1}{P_{ih}} \alpha_{ih}(s_0) * [m^k_{\alpha\beta}(s) q^*_{jk}(s_0 \ , \ s)]$$

$$- \frac{1}{s} \sum_{jk} \sum_{\ell m} \frac{1}{P_{jk} P_{\ell m}} [\alpha_{jk}(s_0) m_{ij}(s_0)] * [\alpha_{\ell m}(s_0) m_{i\ell}(s_0)]$$

$$- \frac{2}{s} \sum_{\alpha\beta} \frac{A_{\alpha\beta}}{P_{\alpha\beta}} \sum_{jk} \sum_{\ell m} \frac{1}{P_{jk}} [\alpha_{jk}(s_0) m_{ij}(s_0)] * [m^m_{\alpha\beta}(s) q^*_{\ell m}(s_0 \ , \ s)]$$

$$- \sum_{\alpha\beta} \sum_{\gamma\delta} \frac{A_{\alpha\beta} A_{\gamma\delta}}{P_{\alpha\beta} P_{\gamma\delta}} \sum_{jk} \sum_{\ell m} [m^k_{\alpha\beta}(s) q^*_{jk}(s_0, s)] * [m^m_{\gamma\delta}(s) q^*_{\ell m}(s_0 \ , \ s)] \ . \tag{3.3.43}$$

4. Asymptotic Results

Recall that the L.S.T. of $E\{N^i_{jk}(t_0 \ , \ t)\}$ is the i^{th} element of the vector $\underline{m}_{jk}(s_0 \ , \ s)$; namely

$$m^i_{jk}(s_0 \ , \ s) = \frac{1}{s_0 - s} \left(m^i_{jk}(s) - m^i_{jk}(s_0) \right) \ . \tag{3.4.1}$$

For small s ; i.e., large t , replace $m^i_{jk}(s)$ with (2.4.3). Then

$$\begin{aligned} m^i_{jk}(s_0, s) &= \frac{1}{s_0 - s} \Bigg[p_{jk} \Big\{ \frac{\alpha}{s} v_j + av_j + \alpha(h_{ij} - v_j \mu_{jk}) \\ &\quad + s\Big(\ell_{ij} - \mu_{jk}\{av_j + \alpha h_{ij}\} + \frac{1}{2} \alpha v_j \mu^{(2)}_{jk}\Big) + o(s) \Big\} - m^i_{jk}(s_0) \Bigg] \\ &= \frac{1}{s_0} \Bigg[\frac{\alpha v_j p_{jk}}{s} + av_j p_{jk} + \alpha p_{jk}(h_{ij} - v_j \mu_{jk}) - m^i_{jk}(s_0) \\ &\quad + \frac{\alpha v_j p_{jk}}{s_0} + s \Big\{ \frac{\alpha v_j p_{jk}}{s_0^2} + \frac{av_j p_{jk}}{s_0} + \frac{\alpha p_{jk}(h_{ij} - v_j \mu_{jk})}{s_0} \\ &\quad - \frac{m^i_{jk}(s_0)}{s_0} + p_{jk}\Big(\ell_{ij} - \mu_{jk}\{av_j + \alpha h_{ij}\} \\ &\quad + \frac{1}{2} \alpha v_j \mu^{(2)}_{jk}\Big) \Big\} + o(s) \Bigg] \ , \end{aligned} \tag{3.4.2}$$

whose inverse with respect to s is, for large t ,

$$m^i_{jk}(s_0\ ,\ t) = \frac{1}{s_0}\Big[\alpha v_j p_{jk} t + a v_j p_{jk} + \alpha p_{jk}(h_{ij} - v_j \mu_{jk}) - m^i_{jk}(s_0) + \frac{\alpha v_j p_{jk}}{s_0} + o(1)\Big]$$

and inverting with respect to s_0 gives

$$M^i_{jk}(t_0\ , t) = \alpha v_j p_{jk} t_0 + \alpha v_j p_{jk} t + a v_j p_{jk} + \alpha p_{jk}(h_{ij} - v_j \mu_{jk}) - M^i_{jk}(t_0) + o(1)\ . \tag{3.4.3}$$

Note that for $t_0 = 0$, $M^i_{jk}(0\ , t) = M^i_{jk}(t)$ for large t . Also

$$\lim_{t \to \infty} \frac{1}{t} M^i_{jk}(t_0\ , t) = \alpha v_j p_{jk} \tag{3.4.4}$$

which is independent of both t_0 and the initial state. It can be further shown, for $j \neq \ell$ or $k \neq n$, that

$$\begin{aligned} c^i_{jk,\ell n}(t_0\ , t) = \alpha^2 p_{jk} p_{\ell n} v_j v_\ell t^2 + \Big[&\alpha v_j p_{jk}\{a v_\ell p_{\ell n} + \alpha p_{\ell n}(h_{i\ell} - v_\ell \mu_{\ell n}) \\ &- M^i_{\ell n}(t_0) + \alpha v_\ell p_{\ell n} t_0\} + p_{jk}\{a v_j + \alpha(h_{nj} - v_j \mu_{jk})\}\alpha v_\ell p_{\ell n} \\ &+ \alpha v_\ell p_{\ell n}\{a v_j p_{jk} + \alpha p_{jk}(h_{ij} - v_j \mu_{jk}) - M^i_{jk}(t_0) + \alpha v_j p_{jk} t_0\} \\ &+ p_{\ell n}\{a v_\ell + \alpha(h_{k\ell} - v_\ell \mu_{\ell n})\}\alpha v_j p_{jk}\Big] t + O(t)\ . \end{aligned} \tag{3.4.5}$$

For $t_0 = 0$ this reduces to $c^i_{jk,\ell n}(t)$ as required. We also can show that

$$\begin{aligned} \mathrm{Cov}\{N^i_{jk}(t_0,t), N^i_{\ell n}(t_0,t)\} = {} & 2a\alpha v_j v_\ell p_{jk} p_{\ell n} t + \alpha^2 v_\ell p_{jk} p_{\ell n} t(h_{nj} - v_j \mu_{jk}) \\ & + \alpha^2 v_j p_{jk} p_{\ell n} t(h_{k\ell} - v_\ell \mu_{\ell n}) - \alpha^2 v_j v_\ell p_{jk} p_{\ell n} t_0^2 \\ & - 2a\alpha v_j v_\ell p_{jk} p_\ell t_0 - \alpha^2 v_j p_{jk} p_{\ell n} t_0(h_{i\ell} - v_\ell \mu_{\ell n}) \\ & - \alpha^2 v_\ell p_{jk} p_{\ell n} t_0(h_{ij} - v_j \mu_{jk}) + v_j p_{jk} M^i_{\ell n}(t_0)(\alpha t_0 + a) \end{aligned}$$

$$+ v_\ell p_{\ell n} M^i_{jk}(t_0)(\alpha t_0 + a) + \alpha p_{jk}(h_{ij} - v_j \mu_{jk}) M^i_{\ell n}(t_0)$$

$$+ \alpha p_{\ell n}(h_{i\ell} - v_\ell \mu_{\ell n}) M^i_{jk}(t_0) - M^i_{jk}(t_0) M^i_{\ell n}(t_0) + o(t) \tag{3.4.6}$$

and

$$\mathrm{Var}\{N^i_{jk}(t_0,t)\} = \alpha v_j p_{jk} t + 2a\alpha v_j^2 p_{jk}^2 (t - t_0) - \alpha^2 v_j^2 p_{jk}^2 t_0^2$$

$$+ 2\alpha^2 v_j p_{jk}^2 t (h_{kj} - v_j \mu_{jk}) - 2\alpha^2 v_j p_{jk}^2 t_0 (h_{ij} - v_j \mu_{jk})$$

$$+ 2 v_j p_{jk} M^i_{jk}(t_0)(\alpha t_0 + a) + 2\alpha p_{jk}(h_{ij} - v_j \mu_{jk}) M^i_{jk}(t_0)$$

$$- \left(M^i_{jk}(t_0)\right)^2 + o(t) \quad . \tag{3.4.7}$$

Furthermore

$$\lim_{t \to \infty} \frac{1}{t} \mathrm{Cov}\left\{N^i_{jk}(t_0\,, t)\,,\ N^i_{\ell n}(t_0\,, t)\right\}$$

$$= 2a\alpha v_j v_\ell p_{jk} p_{\ell n} + \alpha^2 v_\ell p_{jk} p_{\ell n}(h_{nj} - v_j \mu_{jk}) + \alpha^2 v_j p_{jk} p_{\ell n}(h_{k\ell} - v_\ell \mu_{\ell n}) \tag{3.4.8}$$

and

$$\lim_{t \to \infty} \frac{1}{t} \mathrm{Var}\{N^i_{jk}(t_0\,, t)\} = \alpha v_j p_{jk} + 2a\alpha v_j^2 p_{jk}^2 + 2\alpha^2 v_j p_{jk}^2 (h_{kj} - v_j \mu_{jk}) \tag{3.4.9}$$

CHAPTER IV

SPECIAL CASES

In this chapter we specialize the results of the preceding chapters and derive the results for two special cases. First we consider Zero-Order M.R.P.'s. Recall from (1.1) that a Zero-Order M.R.P. arises when $F_{ij}(x)$ is a function of either i or j but not both. For the second special case we treat the two-state M.R.P. and derive results analogous to the case of m states.

1. Zero-Order M.R.P.'s

For this special case the matrix P_0 of transition probabilities is of the form

$$P_0 = \begin{bmatrix} p_1 & p_2 & \cdots & p_m \\ p_1 & p_2 & \cdots & p_m \\ \vdots & \vdots & \ddots & \vdots \\ p_1 & p_2 & \cdots & p_m \end{bmatrix} = \underline{e}\,\underline{p}' \quad , \qquad (4.1.1)$$

where $\underline{p}'$ is the row vector of the p_i . Also the matrix $Q(t)$ has $(i,j)^{th}$ element $Q_{ij}(t) = p_j F_i(t)$, hence $q(s) = \underline{p}\,\underline{f}'(s)$, where $\underline{f}(s)$ is the column vector of L.S.T.'s of the $F_i(t)$. Then

$$\left(I - q(s)\right)^{-1} = \left(I - \underline{p}\,\underline{f}'(s)\right)^{-1} \quad . \qquad (4.1.2)$$

<u>Lemma:</u> $(I + PQ)^{-1} = I - P(I + QP)^{-1}Q$.

Proof: $(I + PQ)^{-1} = I - PQ + (PQ)^2 - (PQ)^3 + \cdots$

$$= I - P[I - QP + (QP)^2 - \cdots]Q$$

$$= I - P(I + QP)^{-1}Q \quad .$$

Using the Lemma (4.1.2) becomes

$$\left(I - \underline{p}\underline{f}'(s)\right)^{-1} = I + \underline{p}\left(I - \underline{f}'(s)\underline{p}\right)^{-1}\underline{f}'(s)$$

$$= I + \frac{\underline{p}\underline{f}'(s)}{1 - \underline{f}'(s)\underline{p}} \quad . \tag{4.1.3}$$

Hence for the Zero-Order M.R.P. the basic quantity $\left(I - q(s)\right)^{-1}$ can be exactly determined. From (4.1.3) we immediately have

$$\left\{\left(I - \underline{p}\underline{f}'(s)\right)^{-1}\right\}_{ij} = \delta_{ij} + p_j g_i(s)\left[1 - \sum_k p_k f_k(s)\right]^{-1} \quad . \tag{4.1.4}$$

Let

$$\sum_k p_k f_k(s) = 1 - s\mu_{\cdot} + \frac{1}{2} s^2 \mu_{\cdot}^{(2)} - \frac{1}{6} s^3 \mu_{\cdot}^{(3)} + o(s^3) \quad , \tag{4.1.5}$$

where

$$\mu_{\cdot}^{(i)} = \sum_{k=1}^{m} p_k \mu_k^{(i)}$$

and the $\mu_k^{(i)}$ are the i^{th} order raw moments of the $F_k(x)$, $k = 1, 2, \cdots , m$. In powers of s then (4.1.4) is

$$\left\{\left(I - \underline{p}\underline{f}'(s)\right)^{-1}\right\}_{ij} = \delta_{ij} + p_j\left[\frac{1}{s\mu_{\cdot}} + \frac{1}{\mu_{\cdot}}\left(\frac{1}{2}\gamma_2 - \mu_i\right) + \frac{s}{2\mu_{\cdot}}\left\{\gamma_2\left(\frac{1}{2}\gamma_2 - \mu_i\right) - \frac{1}{3}\gamma_3 + \mu_i^{(2)}\right\} + o(s)\right] \quad , \tag{4.1.6}$$

where

$$\gamma_2 = \frac{\mu_{\bullet}^{(2)}}{\mu_{\bullet}} \quad , \quad \gamma_3 = \frac{\mu_{\bullet}^{(3)}}{\mu_{\bullet}} \tag{4.1.7}$$

Using (4.1.6) in (2.2.5) we get

$$m_{jk}^{i}(s) = p_k f_j(s)\left\{\left(I - \underline{\underline{p}}\underline{f}'(s)\right)^{-1}\right\}_{ij}$$

$$= \frac{1}{s}\frac{p_j p_k}{\mu_{\bullet}} + p_k \delta_{ij} + \frac{p_j p_k}{\mu_{\bullet}}\left(\frac{1}{2}\gamma_2 - \mu_i - \mu_j\right)$$

$$+ s\left\{\frac{p_j p_k}{\mu_{\bullet}}\left[\frac{1}{2}\mu_j^{(2)} + \mu_i^{(2)} + \left(\frac{1}{2}\gamma_2 - \mu_i\right)\left(\frac{1}{2}\gamma_2 - \mu_j\right) - \frac{1}{3}\gamma_3\right]\right.$$

$$\left. - \delta_{ij} p_k \mu_j\right\} + o(s) \quad . \tag{4.1.8}$$

Formal inversion gives, for large t ,

$$M_{jk}^{i}(t) = \frac{p_j p_k}{\mu_{\bullet}} t + p_k \delta_{ij} + \frac{p_j p_k}{\mu_{\bullet}}\left(\frac{1}{2}\gamma_2 - \mu_i - \mu_j\right) + o(1) \quad . \tag{4.1.9}$$

Using (4.1.8) and the results of Section 2.2 we obtain, after a little algebra,

$$\mathrm{Cov}\left\{N_{jk}^{i}(t)\, ,\, N_{\ell n}^{i}(t)\right\} = \delta_{j\ell}\delta_{kn}\frac{p_j p_k}{\mu_{\bullet}} t + \frac{p_k p_n}{\mu_{\bullet}} t(\delta_{k\ell} p_j + \delta_{nj} p_\ell)$$

$$+ \frac{p_j p_k p_\ell p_n}{\mu_{\bullet}^2} t(\gamma_2 - \mu_j - \mu_k - \mu_\ell - \mu_n)$$

$$+ o(t) \tag{4.1.10}$$

and in particular

$$\mathrm{Var}\left\{N_{jk}^{i}(t)\right\} = \frac{p_j p_k t}{\mu_{\bullet}} + \frac{2p_j p_k^2}{\mu_{\bullet}}\delta_{jk} t + \left(\frac{p_j p_k}{\mu_{\bullet}}\right)^2 t(\gamma_2 - 2\mu_j - 2\mu_k) + o(t) \tag{4.1.11}$$

For large t we have shown for Zero-Order M.R.P.'s that

$$\lim_{t \to \infty} \frac{1}{t} M^{i}_{jk}(t) = \frac{p_j p_k}{\mu_{\cdot}} , \tag{4.1.12}$$

$$\lim_{t \to \infty} \frac{1}{t} \operatorname{Cov}\left\{N^{i}_{jk}(t),\ N^{i}_{\ell n}(t)\right\} = \delta_{j\ell}\delta_{kn} \frac{p_j p_k}{\mu_{\cdot}} + \frac{p_k p_n}{\mu_{\cdot}} (\delta_{k\ell} p_j + \delta_{nj} p_\ell)$$
$$+ \frac{p_j p_k p_\ell p_n}{\mu_{\cdot}^2} (\gamma_2 - \mu_j - \mu_k - \mu_\ell - \mu_n) , \tag{4.1.13}$$

and

$$\lim_{t \to \infty} \frac{1}{t} \operatorname{Var}\left\{N^{i}_{jk}(t)\right\} = \frac{p_j p_k}{\mu_{\cdot}} + \frac{2 p_j p_k^2}{\mu_{\cdot}} \delta_{jk} + \left(\frac{p_j p_k}{\mu_{\cdot}}\right)^2 (\gamma_2 - 2\mu_j - 2\mu_k) . \tag{4.1.14}$$

Note that these asymptotic moments are independent of the initial state of the process, as expected.

2. Two-State M.R.P.'s

For the special case m = 2 we write

$$\left(I-q(s)\right) = \begin{bmatrix} 1-a+sc_{11}-\frac{1}{2}s^2 d_{11}+\frac{1}{6}s^3 e_{11} & a-1+sc_{12}-\frac{1}{2}s^2 d_{12}+\frac{1}{6}s^3 e_{12} \\ -b+sc_{21}-\frac{1}{2}s^2 d_{21}+\frac{1}{6}s^3 e_{21} & b+sc_{22}-\frac{1}{2}s^2 d_{22}+\frac{1}{6}s^3 e_{22} \end{bmatrix},$$

where

$$p_{11} = 1 - p_{12} = a , \quad p_{21} = 1 - p_{22} = b . \tag{4.2.1}$$

Then

$$\det\left(I - q(s)\right) = \alpha s + \beta s^2 + \gamma s^3 + o(s^3) , \tag{4.2.2}$$

where

$$\alpha = (1 - a)(c_{21} + c_{22}) + b(c_{11} + c_{12}) ,$$

$$\beta = \det P_1 - \frac{1}{2}\Big\{(1-a)(d_{21}+d_{22}) + b(d_{11}+d_{12})\Big\} ,$$

$$\gamma = \frac{1}{6}\Big\{(1-a)(e_{21}+e_{22}) + b(e_{11}+e_{12})\Big\} - \frac{1}{2}\Big\{c_{11}d_{22}+c_{22}d_{11}-c_{12}d_{21}-c_{21}d_{12}\Big\} ,$$

$$P_1 = \begin{bmatrix} p_{11}\mu_{11} & p_{12}\mu_{12} \\ p_{21}\mu_{21} & p_{22}\mu_{22} \end{bmatrix} .$$

So that

$$\frac{1}{\det\big(I - q(s)\big)} = \frac{1}{\alpha s} - \frac{\beta}{\alpha^2} + \frac{s}{\alpha^3}(\beta^2 - \alpha\gamma) + o(s) . \qquad (4.2.3)$$

Letting $A(s) = \big(I - q(s)\big)^{-1}$ we get, omitting the algebra,

$$a_{11}(s) = \frac{1}{s}\left(\frac{1-a}{\alpha}\right) + \frac{1}{\alpha^2}[\alpha c_{11} - \beta(1-a)] + \frac{s}{\alpha^3}[\theta_1(1-a) - \alpha\beta c_{11} - \frac{1}{2}\alpha^2 d_{11}] + o(s) ,$$

$$a_{12}(s) = \frac{1}{s}\left(\frac{b}{\alpha}\right) - \frac{1}{\alpha^2}[\alpha c_{21} + b\beta] + \frac{s}{\alpha^3}[\theta_1 b + \alpha\beta c_{21} + \frac{1}{2}\alpha^2 d_{21}] + o(s) ,$$

$$a_{21}(s) = \frac{1}{s}\left(\frac{1-a}{\alpha}\right) - \frac{1}{\alpha^2}[\alpha c_{12} + \beta(1-a)] + \frac{s}{\alpha^3}[\theta_1(1-a) + \alpha\beta c_{12} + \frac{1}{2}\alpha^2 d_{12}] + o(s) ,$$

and

$$a_{22}(s) = \frac{1}{s}\left(\frac{b}{\alpha}\right) + \frac{1}{\alpha^2}[\alpha c_{22} - b\beta] + \frac{s}{\alpha^3}[\theta_1 b - \alpha\beta c_{22} - \frac{1}{2}\alpha^2 d_{22}] + o(s) ,$$

where

$$\theta_1 = \beta^2 - \alpha\gamma .$$

From the previous result for m-state processes

$$m^i_{jk}(s) = q_{jk}(s)\Big\{\big(I - q(s)\big)^{-1}\Big\}_{ij}$$

$$= q_{jk}(s)\,a_{ij}(s) \qquad (4.2.4)$$

we may obtain, by direct evaluation, the means for the two-state process.

We omit the algebraic details, but one can easily show that

$$m_{11}^{1}(s) = \frac{p_{11}}{\alpha^{3}}\left\{\frac{1}{s}\,\alpha^{2}(1 - a) + \alpha^{2}c_{11} - \alpha(1 - a)(\beta + \alpha\mu_{11}) + s[\theta_1(1-a) - \alpha\beta c_{11} - \frac{1}{2}\alpha^{2}d_{11} - \mu_{11}(\alpha^{2}c_{11} - \alpha\beta(1-a)) + \frac{1}{2}\mu_{11}^{(2)}\alpha^{2}(1 - a)] + o(s)\right\} . \quad (4.2.5)$$

To compute $m_{12}^{1}(s)$ replace $\mu_{11}^{(k)}$ by $\mu_{12}^{(k)}$ and p_{11} by p_{12} in (4.2.5).

$$m_{22}^{1}(s) = \frac{p_{22}}{\alpha^{3}}\left\{\frac{1}{s}\,\alpha^{2}b - \alpha^{2}c_{21} - \alpha b(\beta + \alpha\mu_{22}) + s[\theta_1 b + \alpha\beta c_{21} + \frac{1}{2}\alpha^{2}d_{21} + \mu_{22}(\alpha^{2}c_{21} + \alpha b(\beta + \alpha\mu_{22})) + \frac{1}{2}\mu_{22}^{(2)}\alpha^{2}b] + o(s)\right\} . \quad (4.2.6)$$

To compute $m_{21}^{1}(s)$ replace $\mu_{22}^{(k)}$ by $\mu_{21}^{(k)}$ and p_{22} by p_{21} in (4.2.6).

$$m_{11}^{2}(s) = \frac{p_{11}}{\alpha^{3}}\left\{\frac{1}{s}\,\alpha^{2}(1 - a) - \alpha^{2}c_{12} - \alpha(1 - a)(\beta + \alpha\mu_{11}) + s[\theta_1(1-a) + \alpha\beta c_{12} + \frac{1}{2}\alpha^{2}d_{12} + \mu_{11}(\alpha^{2}c_{12} + \alpha\beta(1-a)) + \frac{1}{2}\mu_{11}^{(2)}\alpha^{2}(1 - a)] + o(s)\right\} . \quad (4.2.7)$$

To compute $m_{12}^{2}(s)$ replace $\mu_{11}^{(k)}$ by $\mu_{12}^{(k)}$ and p_{11} by p_{12} in (4.2.7).

$$m_{22}^{2}(s) = \frac{p_{22}}{\alpha^{3}}\left\{\frac{1}{s}\,\alpha^{2}b + \alpha^{2}c_{22} - \alpha b(\beta + \alpha\mu_{22}) + s[b\theta_1 - \alpha\beta c_{22} - \frac{1}{2}\alpha^{2}d_{22} - \mu_{22}(\alpha^{2}c_{22} - \alpha\beta b) + \frac{1}{2}\mu_{22}^{(2)}\alpha^{2}b] + o(s)\right\} . \quad (4.2.8)$$

To compute $m_{21}^{2}(s)$ replace $\mu_{22}^{(k)}$ by $\mu_{21}^{(k)}$ and p_{22} by p_{21} in (4.2.8). From the above expressions we obtain

$$\lim_{t \to \infty} \frac{1}{t} M_{11}^{1}(t) = \frac{p_{11}}{\alpha}(1 - a) = \frac{1}{\alpha}\,p_{11}(1 - p_{11}) , \quad (4.2.9)$$

$$\lim_{t \to \infty} \frac{1}{t} M^1_{22}(t) = \frac{p_{22}}{\alpha} b = \frac{1}{\alpha} p_{22}(1 - p_{22}) , \qquad (4.2.10)$$

$$\lim_{t \to \infty} \frac{1}{t} M^2_{11}(t) = \frac{p_{11}}{\alpha} (1 - a) = \frac{1}{\alpha} p_{11}(1 - p_{11}) , \qquad (4.2.11)$$

$$\lim_{t \to \infty} \frac{1}{t} M^2_{22}(t) = \frac{p_{22}}{\alpha} b = \frac{1}{\alpha} p_{22}(1 - p_{22}) , \qquad (4.2.12)$$

recall that

$$\alpha = (1 - a)(c_{21} + c_{22}) + b(c_{11} + c_{12})$$

$$= (1 - p_{11})(p_{21}\mu_{21} + p_{22}\mu_{22}) + (1 - p_{22})(p_{11}\mu_{11} + p_{12}\mu_{12}) . \qquad (4.2.13)$$

We may summarize and write in matrix notation

$$\lim_{t \to \infty} \frac{1}{t} M^1(t) = \lim_{t \to \infty} \frac{1}{t} M^2(t) = \frac{1}{\alpha} \begin{bmatrix} p_{11}p_{12} & p_{11}p_{12} \\ p_{22}p_{21} & p_{22}p_{21} \end{bmatrix} . \qquad (4.2.14)$$

CHAPTER V

LIMITING DISTRIBUTIONS

In this chapter certain limiting distributions for previous results are derived. Results are given in terms of the basic quantities $Q(t)$ rather than in terms of the mean recurrence times as in Pyke (1961b).

First recall from Section 1.7 that

$$\alpha = \left(\det(P_1) \prod_{i=1}^{m-1} \lambda_i\right)^{-1} , \tag{5.1}$$

where the λ_i are the nonzero latent roots of $P_1^{-1}(I - P_0)$. We shall show that

$$\frac{1}{\alpha} = \underline{v}'\underline{\eta} , \tag{5.2}$$

where

$$\eta_i = \sum_j p_{ij}\mu_{ij} . \tag{5.3}$$

Now, from (1.7.3)

$$\frac{1}{\alpha} = \det(P_1) \prod_{i=1}^{m-1} \lambda_i = \lim_{s \to 0} \frac{1}{s} |I - P_0 + sP_1| .$$

But $\lim_{s \to 0} |I - P_0 + sP_1| = |I - P_0| = 0$, hence using L'Hôpital's Rule

$$\lim_{s \to 0} \frac{1}{s} |I - P_0 + sP_1| = \lim_{s \to 0} \frac{d}{ds} |I - P_0 + sP_1| .$$

From Aitken (1951, pg. 131) we have that

$$\frac{d}{ds}|A + sB| = \sum_{i,j} b_{ij}|A + sB|_{ij} ,$$

where $B = (b_{ij})$ and $|A + sB|_{ij}$ is the cofactor of the $(i,j)^{th}$ element of $|A + sB|$. Hence

$$\lim_{s \to 0} \frac{d}{ds}|I - P_0 + sP_1| = \lim_{s \to 0} \sum_{i,j} p_{ij}\mu_{ij}|I - P_0 + sP_1|_{ij}$$

$$= \sum_{i,j} p_{ij}\mu_{ij}|I - P_0|_{ij} ,$$

but $(I - P_0)$ is singular and therefore the cofactors of the elements in any row of $(I - P_0)$ are the same. Let v_i be these cofactors. Then

$$\lim_{s \to 0} \frac{d}{ds}|I - P_0 + sP_1| = \sum_{i,j} v_i p_{ij}\mu_{ij} = \underline{v}'P_1\underline{e} = \underline{v}'\underline{\eta} .$$

Hence we have shown that $\frac{1}{\alpha} = \underline{v}'\underline{\eta}$. Note also that $\underline{v}'$ is the left eigenvector for $\lambda = 1$ in $|P_0 - \lambda I| = 0$ and therefore are the steady-state probabilities for the transition probability matrix P_0 . With the above results in hand we can write

$$\left(I - q(s)\right)^{-1} = \frac{1}{s}\frac{\underline{e}\underline{v}'}{\underline{v}'\underline{\eta}} + \text{constant} + o(1) . \tag{5.4}$$

Before discussing the limiting results for the distributions derived in this paper we will use (5.4) to give alternate limiting expressions for results due to Pyke (1961b). First, define

$$P_{ij}(t) = Pr\{J(t) = \underline{e}_j | \underline{J}(0) = \underline{e}_i\} . \tag{5.5}$$

Pyke has shown that the matrix of L.S.T.'s of the $P_{ij}(t)$ is given by

$$\pi(s) = \big(I - q(s)\big)^{-1}\big(I - h(s)\big) \quad . \tag{5.6}$$

Using (5.4) we obtain

$$\lim_{t \to \infty} P(t) = \lim_{s \to 0} \pi(s) = \lim_{s \to 0} \left\{ \frac{1}{s} \frac{\underline{e}\underline{v}'}{\underline{v}'\underline{\eta}} + \text{const.} + o(1) \right\}$$

$$\times \left\{ I - [I - s\,\text{diag}(\eta_1 , \cdots , \eta_m) + o(s)] \right\}$$

$$= \frac{\underline{e}\underline{v}'}{\underline{v}'\underline{\eta}} \,\text{diag}(\eta_1 , \eta_2 , \cdots , \eta_m)$$

$$= \frac{1}{\underline{v}'\underline{\eta}} \begin{bmatrix} v_1\eta_1 & v_2\eta_2 & \cdots & v_m\eta_m \\ \vdots & \vdots & \ddots & \vdots \\ v_1\eta_1 & v_2\eta_2 & \cdots & v_m\eta_m \end{bmatrix} ,$$

whose $(i,j)^{th}$ element is

$$\lim_{t \to \infty} P_{ij}(t) = \frac{v_j \eta_j}{\underline{v}'\underline{\eta}} \quad . \tag{5.7}$$

Thus we obtain an expression for the stationary distribution of $P_{ij}(t)$ in terms of the basic quantities specified by the matrix $Q(t)$. Pyke obtains the stationary distribution of $P_{ij}(t)$ as follows: define

$$G_{ij}(t) = \Pr\{N_j^i(t) > 0\} \quad . \tag{5.8}$$

Pyke then proves that

$$\lim_{t \to \infty} P_{ij}(t) = G_{ij}(\infty) \frac{\eta_j}{\gamma_{jj}} \quad , \tag{5.9}$$

where γ_{jj} is the mean of G_{jj} . (5.7) seems more useful to the practitioner since it is in terms of basic quantities rather than in terms of the mean recurrence times.

From (5.4) we can obtain the limiting forms for the moments of several distributions discussed in previous chapters. In Section 1.4 we obtained

$$\int_0^\infty e^{-st} dM(t)\,dt = m(s) = \left(I - q(s)\right)^{-1} - I \quad . \tag{5.10}$$

So that

$$\lim_{t \to \infty} \frac{1}{t} M(t) = \lim_{s \to 0} sm(s)$$

$$= \lim_{s \to 0} s\left\{\frac{1}{s}\frac{\underline{\underline{e}}\underline{v}'}{\underline{v}'\underline{\eta}} + \text{const.} + o(1)\right\}$$

$$= \frac{\underline{\underline{e}}\underline{v}'}{\underline{v}'\underline{\eta}} \quad , \tag{5.11}$$

whose $(i,j)^{th}$ element is

$$\lim_{t \to 0} \frac{1}{t} M_{ij}(t) = \frac{v_j}{\underline{v}'\underline{\eta}} \quad . \tag{5.12}$$

From the definition of $G_{ij}(t)$ given in (5.8) one has recursively that

$$G_{ij}(t) = [1 - G_{jj}(t)] * Q_{ij}(t) + \sum_{k=1}^{m} Q_{ik}(t) * G_{kj}(t) \tag{5.13}$$

and

$$g_{ij}(s) = [1 - g_{jj}(s)]q_{ij}(s) + \sum_{k=1}^{m} q_{ik}(s)g_{kj}(s) \quad . \tag{5.14}$$

Pyke shows that the solution to (5.14) is

$$g(s) = q(s)\left(I - q(s)\right)^{-1}\left\{{}_d\left[\left(I - q(s)\right)^{-1}\right]\right\}^{-1} \quad , \tag{5.15}$$

whose $(i,j)^{th}$ element is

$$g_{ij}(s) = \frac{m_{ij}(s)}{\left(1 + m_{jj}(s)\right)} \quad . \tag{5.16}$$

So from (5.16) we have

$$s\left(1 + m_{jj}(s)\right) = \frac{s}{\left(1 - g_{jj}(s)\right)}$$

$$= \frac{s}{\left(s\gamma_{jj} + o(s)\right)} \quad .$$

Taking limits on both sides gives

$$\frac{1}{\gamma_{jj}} = \lim_{s \to 0} s\left(1 + m_{jj}(s)\right)$$

$$= \frac{v_j}{\underline{v}'\underline{\eta}} \quad . \tag{5.17}$$

Note that if the state j is recurrent then from (5.9)

$$\lim_{t \to \infty} P_{ij}(t) = \frac{\eta_j}{\gamma_{jj}} = \frac{v_j \eta_j}{\underline{v}'\underline{\eta}} \quad ,$$

which shows the equivalence of (5.7) and (5.9). One is often interested in ratios of means and their limits. To this end we have

$$\lim_{t \to \infty} \frac{M_{ij}(t)}{M_{jj}(t)} = \lim_{s \to 0} \frac{m_{ij}(s)}{m_{jj}(s)}$$

$$= \lim_{s \to 0} \frac{\frac{1}{s}\frac{v_j}{\underline{v}'\underline{\eta}} + \text{const.} + o(1)}{\frac{1}{s}\frac{v_i}{\underline{v}'\underline{\eta}} + \text{const.} + o(1)}$$

$$= \frac{v_j}{v_i} \quad . \tag{5.18}$$

Using (5.4) we can also obtain the asymptotic moments of the distributions

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studied in Chapter II. For example,

$$\lim_{t \to \infty} \frac{1}{t} \underline{M}_{\alpha\beta}(t) = \lim_{s \to 0} s\underline{m}_{\alpha\beta}(s)$$

$$= \lim_{s \to 0} sq_{\alpha\beta}(s)\left[\frac{1}{s}\frac{\underline{e}\underline{v}'}{\underline{v}'\underline{\eta}} + \text{const.} + 0(1)\right]\underline{e}_{\alpha}$$

$$= \lim_{s \to 0} q_{\alpha\beta}(s) \frac{\underline{e}\underline{v}'}{\underline{v}'\underline{\eta}} \underline{e}_{\alpha}$$

$$= p_{\alpha\beta} \frac{\underline{e}\underline{v}'\underline{e}_{\alpha}}{\underline{v}'\underline{\eta}}$$

$$= \frac{p_{\alpha\beta} v_{\alpha} \underline{e}}{\underline{v}'\underline{\eta}} \quad , \tag{5.19}$$

whose i^{th} element is

$$\lim_{t \to \infty} \frac{1}{t} M^{i}_{\alpha\beta}(t) = \frac{p_{\alpha\beta} v_{\alpha}}{\underline{v}'\underline{\eta}} \tag{5.20}$$

in agreement with the asymptotic expression (2.4.5) as required. We can also use (5.4) to determine limiting moments for the arbitrary interval $(t_0 \, , \, t_0 + t]$ as for example

$$\lim_{t \to \infty} \frac{1}{t} M^{i}_{jk}(t_0 \, , \, t) = \lim_{s \to 0} \frac{s}{s_0 - s} [m^{i}_{jk}(s) - m^{i}_{jk}(s_0)]$$

$$= \frac{1}{s_0} \lim_{s \to 0} s\left[1 + \frac{s}{s_0} + \left(\frac{s}{s_0}\right)^2 + \cdots\right]$$

$$\times \left\{\frac{1}{s}\frac{v_j p_{jk}}{\underline{v}'\underline{\eta}} + \text{const.} + o(1) - \underline{m}_{jk}(s_0)\right\}$$

$$= \frac{1}{s_0} \frac{v_j p_{jk}}{\underline{v}'\underline{\eta}} \quad . \tag{5.21}$$

Inverting with respect to s_0 gives

$$\lim_{t \to \infty} \frac{1}{t} M^{i}_{jk}(t_0 \, , \, t) = \frac{v_j p_{jk}}{\underline{v}'\underline{\eta}} \tag{5.22}$$

which is independent of t_0 and the initial state.

CHAPTER VI

SUMMARY AND FURTHER RESEARCH

Those who possess a cursory knowledge of S.M.P.'s and M.R.P.'s will agree that the two papers published in 1961 by Ronald Pyke were the catalyst for the several papers that have appeared in the literature since then. The researcher soon realizes that the theoretical and applied papers in S.M.P.'s and M.R.P.'s number well above 100. In this paper we have made no attempt to present a complete historical account of the theory, but rather to include in the first chapter those results of direct use in later chapters. In this sense we have tried to make the paper self-contained.

After deriving the distribution and moments of $N_j^i(t)$ and the vector $\underline{N}^i(t)$ we generalize these results in Chapter II where the distribution and moments of the transition count matrix $[N_{jk}^i(t)]$ are derived. Having derived this distribution we consider a generalization of cumulative process in which the g-functions depend on the states between which the transitions are made and the associated sojourn times. The distribution and moments of this generalized cumulative process are then derived.

Chapter III further generalizes the results of Chapter II. In Chapter III we treat the case of arbitrary intervals $(t_0 , t_0 + t]$ whereas in Chapter II we were concerned with intervals $(0,t]$. This material is to be published in a separate paper by Wysocki and Kshirsagar (1969).

In Chapter IV we consider the results of Chapter II for Zero-Order M.R.P.'s and Two-State M.R.P.'s.

Chapter V utilizes a power series expansion of $\left(I - q(s)\right)^{-1}$ to derive limiting distributions and their moments in terms of the basic quantities $Q_{ij}(t)$.

Although several researchers have been attracted to M.R.P. theory they have severely neglected the estimation problems that arise in M.R.P.'s. The reader may consult recent bibliographies by Neuts (1968) and Fox (1968) to see that the theoretical papers discuss distribution theory almost exclusively. A recent paper by Pyke and Moore (1968) is the only paper, to this author's knowledge, that discusses the estimation problem for M.R.P.'s. Those anticipating research in this area would do well to study the results of Bartlett (1951) concerning goodness of fit tests for Markov chains and a book by Billingsley (1961) on inference in Markov Processes. The bridge between theory and application is yet to be built.

One further generalization of this paper has indications of being a fruitful avenue of research. We motivate the problem by means of a physical example from inventory models. Consider an inventory of n spare parts which supply a machine and assume that these parts have been previously used and therefore, at machine time $t = 0$, have ages specified by the $n \times 1$ vector $\underline{X}(0)$. Assume that the distribution of failure times for the spare parts, given that the spares are $\underline{X}(0)$ old, is not the same as the unconditional distribution; i.e., failure time distributions are not exponential. If one considers the M.R.P. associated with this spare parts problem, call it a Modified M.R.P. (M.M.R.P.), then several questions come to mind.

(i) Find the distribution and moments of

$$\Pr\{N(t) = k \mid \underline{X}(0) \;,\; \underline{J}(0) = \underline{e}_i\} \;,$$

$$\Pr\{N(t_0, t) = k \mid \underline{X}(0) \;,\; \underline{J}(0) = \underline{e}_i\} \;.$$

(ii) If $\underline{X}(0)$ is given but we do not know how to assign an element of $\underline{X}(0)$ to a spare part, then what can be said of the distributions in (i)?

(iii) We may have prior probabilities for $\underline{X}(0)$, in which case find the distribution and moments of

$$\Pr\{N^i(t) = k\} \quad ,$$
$$\Pr\{N^i(t_0,t) = k\} \quad .$$

(iv) Assuming that we have a second machine of a different kind that breaks down according to failure time distribution G(x) find the distribution of N^i_{jk} in the time interval between successive failures of the second machine in the interval $(t_0 \, , \, t_0 + t]$. We call this distribution the G-counts of the M.R.P. For $t_0 = 0$ and G a Poisson random variable the reader is referred to Kshirsagar (1969).

(v) Instead of one machine and one spare parts inventory assume we have several machines each having their own spare parts inventory. What can be said about the superposition of these M.M.R.P.'s? What is the distribution of the interval between successive failures in the superposed process? If we begin observing the superposed process at t_0 , then what is the distribution of time to the next renewal?

This list of research problems is presented to give the reader a flavor for the avenues of further study. Indeed one could extend the list to include a generalization of Markov Chain theory to S.M.P.'s.

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UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)

SOUTHERN METHODIST UNIVERSITY

2a. REPORT SECURITY CLASSIFICATION

UNCLASSIFIED

2b. GROUP

UNCLASSIFIED

3. REPORT TITLE

Some Useful Generalizations in Markov Renewal Processes

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report

5. AUTHOR(S) (First name, middle initial, last name)

Robert K. Wysocki

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
November 4, 1969	83	24

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
N00014-68-A-0515	47
b. PROJECT NO. NR 042-260	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Office of Naval Research

13. ABSTRACT

Consider a particular system which at time t may be in one of a finite number of distinguishable states, labeled for convenience by 1, 2, ···, m. Once the system enters a particular state, say i, it instantaneously selects the next state to be visited, say j, with probability P_{ij}. However, transition to state j occurs after holding in state i for a random time (sojourn time) whose distribution function is $F_{ij}(\cdot)$. These processes are known as Semi-Markov Processes and the associated renewal process is called a Markov Renewal Process. In this paper we introduce a new counting process which at time t counts the number of times the system has made a one-step transition from state i to state j, i,j = 1, 2, ···, m. The matrix N(t) denotes these counts. The distribution and moments of N(t) are derived and cumulative processes associated with N(t) are discussed. We also extend these results to arbitrary intervals of the form $(t_0, t_0 + t]$. Certain limiting results are given and some important special cases discussed. Several open problems are given in the summary chapter.

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