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BIVARIATE PROBIT, LOGIT, AND BURRIT ANALYSIS

by

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The problem of a mixture of two stimulants in a biological quantal assay is investigated from a mathematical standpoint. The basic assumption is made that the response region does not depend on biological considerations - i.e., given a specified mixture of stimulants $\underline{z}$, the response region is defined by the point $\underline{z}'$ in the p-variate space where there are p stimulants under consideration; instead, the probability functions, themselves, may take on different forms. A general form is proposed and investigated. Three analytic models (one utilizing the bivariate normal distribution, one a bivariate logistic distribution developed by Gumbel (1961), and one a bivariate Burr distribution developed by this author) are employed in this investigation. The investigation includes the analysis of data, under the three analytic models, which had been classified by previous investigators as examples of synergistic action, simple similar action, independent action, and additive action. The residual analyses are included as well as the FORTRAN IV subroutines used in evaluating the functions, the partial derivatives and the weights.

The investigation lends some support to the assumption of a constant response region for a diversity of mixtures of stimulants. The analytic

model incorporating the bivariate Burr distribution is recommended for all cases unless the number of parameters to be estimated is a primary concern, in which case the analytic model utilizing the bivariate normal distribution is recommended. The bivariate Burr distribution developed in this paper is found to be more useful in application than that developed by Takahasi (1965).

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TABLE OF CONTENTS

CHAPTER I

1. Introduction

The joint action of mixtures of stimulants in a biological assay has been investigated by Bliss (1939), Finney (1942), Plackett and Hewlett (1967), Ashford and Smith (1966), and others. Plackett and Hewlett have made their investigations largely from the standpoint of biological considerations such as the physiology of the biological organism being used in experimentation. Ashford and Smith, on the other hand, have dealt with the problem somewhat more within a mathematical framework. In this paper, the problem will be approached mathematically.

For the purposes of this paper a biological assay of a mixture of two stimulants will be conducted as follows: A population of N organisms is divided at random into t groups, where the i^{th} group is of size n_i, $n_1 + n_2 + \cdots + n_t = N$. The i^{th} group receives a treatment of a predetermined mixture (z_{1i}, z_{2i}) of two drugs, where z_{ji} is the quantity of stimulant j measured in any convenient units. r_i is the observed number which manifest a prescribed quantal response. The observed relative frequency of response $p_i = r_i/n_i$ is an estimate of the probability of an organism responding if picked at random from the population. The probability that this organism picked at random will respond when treated by the mixture (z_{1i}, z_{2i}) may be assumed to take on a general form, say, $P(z_{1i}, z_{2i}, \underline{\Theta})$.

Now the probability of r_i responses with the i^{th} combination of levels of drugs can be written as

$$P(r_i) = \frac{n_i!}{r_i!(n_i-r_i)!}\left[P(z_{1i}\ ,\ z_{2i}\ ,\ \underline{\theta})\right]^{r_i}\left[1 - P(z_{1i}\ ,\ z_{2i}\ ,\ \underline{\theta})\right]^{n_i-r_i}$$

$$r_i = 0,\ 1,\ 2,\ \cdots\ ,\ t \qquad (1)$$

$$= 0 \qquad \text{elsewhere.}$$

A series of t combinations of doses is tested in an experiment. The probability of a particular set of r_i's is equal to exp(L) , the likelihood, where

$$L = \sum_{i=1}^{t} r_i \ln\ (P_i) + \sum_{i=1}^{t} (n_i-r_i)\ln(Q_i) + \sum_{i=1}^{t} \ln[n_i!/r_i!(n_i-r_i)!] \qquad (2)$$

and $P_i = P(z_{1i}\ ,\ z_{2i}\ ,\ \underline{\theta})$, $Q_i = 1 - P_i$. The maximum likelihood estimator $\hat{\theta}$ of a parameter θ , θ an element of $\underline{\theta}$, must satisfy the relation

$$0 = \frac{\partial L}{\partial \theta} = \sum_{i=1}^{t} \frac{r_i}{P_i}\frac{\partial P_i}{\partial \theta} + \sum_{i=1}^{t} \frac{(n_i-r_i)}{Q_i}\frac{\partial Q_i}{\partial \theta} = \sum_{i=1}^{t} \frac{n_i(p_i-P_i)}{P_iQ_i}\frac{\partial P_i}{\partial \theta}\ . \qquad (3)$$

Direct solution for θ is not in general possible, but iterative techniques are available which give a convergent series of approximations to the solution.

The following procedure for two parameters θ and ϕ , θ and ϕ elements of $\underline{\theta}$, is of completely general applicability and may easily be extended for the estimation of a greater number of parameters. By the Taylor-Maclaurin expansion of $\frac{\partial L}{\partial \theta}$, $\frac{\partial L}{\partial \phi}$ (See relation (3).) ignoring quantities containing terms of higher than the first degree

$$\frac{\partial L}{\partial \theta_1} + \delta_\theta \frac{\partial^2 L}{\partial \theta_1^2} + \delta_\phi \frac{\partial^2 L}{\partial \theta_1 \partial \phi_1} = 0 \qquad (4)$$

$$\frac{\partial L}{\partial \phi_1} + \delta_\phi \frac{\partial^2 L}{\partial \phi_1^2} + \delta_\Theta \frac{\partial^2 L}{\partial \Theta_1 \partial \phi_1} = 0 \quad ,$$

where the addition of the suffix 1 to Θ , ϕ indicated that the first approximations are to be substituted after differentiation. The solutions δ_Θ , δ_ϕ are adjustments to Θ_1 , ϕ_1 which give the improved approximations $\Theta_2 = \Theta_1 + \delta_\Theta$, $\phi_2 = \phi_1 + \delta_\phi$.

Equations (4) may be simplified through the following procedure which will be illustrated by means of the first of equations (4).

$$\frac{\partial L}{\partial \Theta_1} + \delta_\Theta \frac{\partial^2 L}{\partial \Theta_1^2} + \delta_\phi \frac{\partial^2 L}{\partial \Theta_1 \partial \phi_1} = 0 \quad ,$$

or

$$- \delta_\Theta \frac{\partial^2 L}{\partial \Theta_1^2} - \delta_\phi \frac{\partial^2 L}{\partial \Theta_1 \partial \phi_1} = \frac{\partial L}{\partial \Theta_1} \quad .$$

$$- \delta_\Theta \frac{\partial}{\partial \Theta_1}\left[\frac{\partial L}{\partial \Theta_1}\right] - \delta_\phi \frac{\partial}{\partial \phi_1}\left[\frac{\partial L}{\partial \Theta_1}\right] = \frac{\partial L}{\partial \Theta_1} \quad ;$$

$$- \delta_\Theta \frac{\partial}{\partial \Theta_1}\left\{\sum_{i=1}^{t} \frac{n_i p_i}{P_i Q_i} \frac{\partial P_i}{\partial \Theta_1} - \sum_{i=1}^{t} \frac{n_i}{Q_i} \frac{\partial P_i}{\partial \Theta_1}\right\}$$

$$- \delta_\phi \frac{\partial}{\partial \phi_1}\left\{\sum_{i=1}^{t} \frac{n_i p_i}{P_i Q_i} \frac{\partial P_i}{\partial \Theta_1} - \sum_{i=1}^{t} \frac{n_i}{Q_i} \frac{\partial P_i}{\partial \Theta_1}\right\} = \frac{\partial L}{\partial \Theta_1} \quad ;$$

$$- \delta_\Theta \left\{\sum_{i=1}^{t} \frac{n_i p_i}{P_i Q_i} \frac{\partial^2 P_i}{\partial \Theta_1^2} - \sum_{i=1}^{t} \frac{n_i p_i}{P_i^2 Q_i}\left(\frac{\partial P_i}{\partial \Theta_1}\right)^2 + \sum_{i=1}^{t} \frac{n_i p_i}{P_i Q_i^2}\left(\frac{\partial P_i}{\partial \Theta_1}\right)^2 \right.$$

$$\left. - \sum_{i=1}^{t} \frac{n_i}{Q_i} \frac{\partial^2 P_i}{\partial \Theta_1^2} - \sum_{i=1}^{t} \frac{n_i}{Q_i^2}\left(\frac{\partial P_i}{\partial \Theta_1}\right)^2 \right\}$$

$$- \delta_\phi \left\{ \sum_{i=1}^{t} \frac{n_i p_i}{P_i Q_i} \frac{\partial^2 P_i}{\partial\Theta_1 \partial\phi_1} - \sum_{i=1}^{t} \frac{n_i p_i}{P_i^2 Q_i} \frac{\partial P_i}{\partial\Theta_1} \frac{\partial P_i}{\partial\phi_1} \right.$$

$$+ \sum_{i=1}^{t} \frac{n_i p_i}{P_i Q_i^2} \frac{\partial P_i}{\partial\Theta_1} \frac{\partial P_i}{\partial\phi_1} - \sum_{i=1}^{t} \frac{n_i}{Q_i} \frac{\partial^2 P_i}{\partial\Theta_1 \partial\phi_1}$$

$$\left. - \sum_{i=1}^{t} \frac{n_i}{Q_i^2} \frac{\partial P_i}{\partial\Theta_1} \frac{\partial P_i}{\partial\phi_1} \right\} = \frac{\partial L}{\partial\Theta_1} \quad .$$

At this stage the equation may be simplified by putting $p_i = P_i$ in the coefficients of $\frac{\partial^2 L}{\partial\Theta_1^2}$, $\frac{\partial^2 L}{\partial\Theta_1 \partial\phi_1}$, i.e., on the left hand side of the last equation to give expected instead of empirical values. The last equation then reduces to

$$\delta_\Theta \sum_{i=1}^{t} \frac{n_i}{P_i Q_i} \left(\frac{\partial P_i}{\partial\Theta_1}\right)^2 + \delta_\phi \sum_{i=1}^{t} \frac{n_i}{P_i Q_i} \left(\frac{\partial P_i}{\partial\Theta_1}\right)\left(\frac{\partial P_i}{\partial\phi_1}\right) = \frac{\partial L}{\partial\Theta_1} \quad .$$

The latter equation in (4) can be reduced by means of a similar procedure, i.e., putting $p_i = P_i$ in the coefficients of $\frac{\partial^2 L}{\partial\phi_1^2}$, $\frac{\partial^2 L}{\partial\Theta_1 \partial\phi_1}$. Thus equations (4) are simplified to

$$\delta_\Theta \sum_{i=1}^{t} \frac{n_i}{P_{i1} Q_{i1}} \left(\frac{\partial P_i}{\partial\Theta_1}\right)^2 + \delta_\phi \sum_{i=1}^{t} \frac{n_i}{P_{i1} Q_{i1}} \left(\frac{\partial P_i}{\partial\Theta_1}\right)\left(\frac{\partial P_i}{\partial\phi_1}\right) = \sum_{i=1}^{t} \frac{n_i (p_i - P_{i1})}{P_{i1} Q_{i1}} \left(\frac{\partial P_i}{\partial\Theta_1}\right) ;$$

(5)

$$\delta_\Theta \sum_{i=1}^{t} \frac{n_i}{P_{i1} Q_{i1}} \left(\frac{\partial P_i}{\partial\Theta_1}\right)\left(\frac{\partial P_i}{\partial\phi_1}\right) + \delta_\phi \sum_{i=1}^{t} \frac{n_i}{P_{i1} Q_{i1}} \left(\frac{\partial P_i}{\partial\phi_1}\right)^2 = \sum_{i=1}^{t} \frac{n_i (p_i - P_{i1})}{P_{i1} Q_{i1}} \left(\frac{\partial P_i}{\partial\phi_1}\right) .$$

Here the addition of the suffix 1 to P_{i1} , Q_{i1} indicates that the first approximations are to be used in the evaluation of $P(z_{1i}, z_{2i}, \underline{\Theta})$. Equations (5) illustrate that only first derivatives are needed in this iterative procedure.

2. Methodology for Obtaining Estimates

Now, it will be seen from what follows that relation (3) can be solved by using a modified non-linear least squares [Moore and Zeigler (1967)]. Assume that the data corresponds to the mathematical model

$$y_i = h(\underline{z}_i , \underline{\alpha}) + \varepsilon_i \quad , \quad i = 1, 2, \cdots , t \tag{6}$$

where the y_i are observed random variables, $\underline{z}_i$ is a vector of known independent variables, $\underline{\alpha}$ is a vector of unknown parameters, and ε_i is a random variable such that $E(\varepsilon_i) = 0$, $E(\varepsilon_i^2) = \sigma_i^2$, and $E(\varepsilon_i \varepsilon_j) = 0$ for all $i \neq j$. Then the vector of unknown parameters may be estimated by minimizing the weighted sum of squares,

$$S = \sum_{i=1}^{t} \left(y_i - h(\underline{z}_i , \underline{\alpha})\right)^2 W_i \quad , \tag{7}$$

where W_i is an appropriate weight. If the usual procedure is modified so that the partial derivatives are taken ignoring W_i , the normal equations are

$$\frac{\partial S}{\partial \alpha_k} = - 2 \sum_{i=1}^{t} W_i [y_i - h(\underline{z}_i , \underline{\alpha})] \frac{\partial h(\underline{z}_i , \underline{\alpha})}{\partial \alpha_k} = 0 \quad , \tag{8}$$

for $k = 1, 2, \cdots , \ell$, where ℓ is the number of unknown parameters. Now by letting $W_i = n_i/P_iQ_i$ (the reciprocal of the variance of p_i) , $y_i = p_i$, $h(\underline{z}_i , \underline{\alpha}) = P_i$, $Q_i = 1 - P_i$, and $\alpha_k = \theta$ it can be seen that relation (3) and equation (8) are equivalent. Thus the maximum likelihood estimate can be obtained by means of a modified weighted non-linear least squares. Relations (5) and their equivalent extensions are used in the modified non-linear least squares fitting of equation (6).

3. Consideration of Necessary Conditions on $P(z_{1i}, z_{2i}, \underline{\Theta})$

From this point onward the vector $\underline{z}_i = \begin{pmatrix} z_{1i} \\ z_{2i} \end{pmatrix}$ will be considered from the standpoint of a mixture of stimulants where a transformation has been applied to the original dosage levels so that $-\infty$ is equivalent to zero dosage and $+\infty$ is equivalent to an infinite dosage. For the purposes of this paper $P(z_{1i}, z_{2i}, \underline{\Theta})$ must satisfy the following conditions

$$P(z_{1i}, +\infty, \underline{\Theta}) = P(+\infty, z_{2i}, \underline{\Theta}) = 1 \quad , \tag{9}$$

$$P(z_{1i}, -\infty, \underline{\Theta}) = P_1(z_{1i}, \underline{\Theta}_1) \quad , \tag{10}$$

and

$$P(-\infty, z_{2i}, \underline{\Theta}) = P_2(z_{2i}, \underline{\Theta}_2) \quad , \tag{11}$$

where $P_1(z_{1i}, \underline{\Theta}_1)$ and $P_2(z_{2i}, \underline{\Theta}_2)$ are not in general zero, but rather are marginal probabilities, i.e., the probability of a random individual biological organism responding if it is given a dosage of stimulant j corresponding to z_{ji}. Conditions (9), (10), and (11) imply the conditions

$$P(-\infty, -\infty, \underline{\Theta}) = 0 \quad , \tag{12}$$

and

$$P(+\infty, +\infty, \underline{\Theta}) = 1 \quad . \tag{13}$$

All five of these conditions are necessary in a bioassay of quantal response data involving a mixture of two stimulants. Natural extensions of these conditions for a mixture of more than two stimulants are now obvious.

Plackett and Hewlett (1967) proposed that

$$P = \int_R f(z_{1i}, z_{2i})dz_{1i}dz_{2i} \tag{14}$$

where $f(z_{1i}, z_{2i})$ is a bivariate density with the usual properties and R is defined on the basis of biological considerations, thus implying that the region of integration may, arbitrarily, be changed due to biological considerations. Their papers do not indicate any homogeneity in the regions. Nowhere is there a general formulation for $P(z_{1i}, z_{2i}, \underline{\theta})$ where the form of the region of integration is homogeneous, much less constant. It would appear that the region of integration should be constant except possibly for simple monotonic transformations of the original dosage levels, such as a logarithmic transformation. The bivariate function itself might be, in specific instances, of different types but still retaining a constant response region.

Let $F_1(z_1, \underline{\theta}_1)$, $F_2(z_2, \underline{\theta}_2)$ be univariate distributions where the parameter vectors $\underline{\theta}_1$, $\underline{\theta}_2$ are not, in general, equal. Note that $F_1(z_1, \underline{\theta}_1)$, $F_2(z_2, \underline{\theta}_2)$ are not necessarily even from the same family of distributions, e.g., the family of normal distributions. Let $F_3(z_1, z_2, \underline{\theta})$ be a bivariate distribution such that $F_3(z_1, +\infty, \underline{\theta}) = F_1(z_1, \underline{\theta}_1)$ and $F_3(+\infty, z_2, \underline{\theta}_2) = F_2(z_2, \underline{\theta}_2)$, where $\underline{\theta}' = (\underline{\theta}_1', \underline{\theta}_2', \underline{\theta}_3')$. Now, what is needed is a function which satisfies conditions (9) through (13).

Let

$$H(z_1, z_2, \underline{\theta}) = F_1(z_1, \underline{\theta}_1) + F_2(z_2, \underline{\theta}_2) - F_3(z_1, z_2, \underline{\theta}) \quad . \tag{15}$$

Then $H(-\infty, z_2, \underline{\theta}) = F_2(z_2, \underline{\theta}_2)$, $H(z_1, -\infty, \underline{\theta}) = F_1(z_1, \underline{\theta}_1)$, $H(+\infty, z_2, \underline{\theta}) = 1 = H(z_1, +\infty, \underline{\theta})$, $H(-\infty, -\infty, \underline{\theta}) = 0$, and $H(+\infty, +\infty, \underline{\theta}) = 1$. Thus $H(z_1, z_2, \underline{\theta})$ does satisfy conditions (9) through (13) which suggests that

$$P(z_{1i}, z_{2i}, \underline{\theta}) = F_1(z_1, \underline{\theta}_1) + F_2(z_2, \underline{\theta}_2) - F_3(z_1, z_2, \underline{\theta}) \tag{16}$$

is a general formulation for $P(z_{1i}, z_{2i}, \underline{\theta})$ where the response region is constant. Note that the forms $F_1(z_{1i}, \underline{\theta}_1)$, $F_2(z_{2i}, \underline{\theta}_2)$, $F_3(z_{1i}, z_{2i}, \underline{\theta})$

are completely general distributions whose forms can depend on biological, chemical, or other considerations. At the same time the region of integration is constant and easily understood from a geometrical standpoint as well as from other standpoints. It is noted here that the general formulation for $P(z_{1i}, z_{2i}, \underline{\theta})$ can easily be extended for a vector of more than two stimulants. The utility of this form is quite general; the only restrictions being conditions (9) through (13) which have been imposed in the development of the general form in equation (16).

CHAPTER II

It is natural in the study of a mixture of two stimulants to consider a bivariate probit or normit. Probit analysis has no advantage over a normit analysis if the analysis is run on a high-speed computer. Also, the analyses are equivalent. Almost all of the work that has been done to date has been along the lines of a bivariate normit.

Bliss (1939) was among the first to study the action of mixtures of two stimulants. He classified the joint action of two stimulants into three biological categories: independent joint action, similar joint action, and synergistic action. Independent joint action occurs whenever two components act on different vital systems in the organism and do not interact with one another. Similar joint action is observed whenever two components act independently of one another but on the same vital system. Synergistic action is characterized by a larger frequency of response than could be predicted from experiments using the individual stimulants. He mentions antagonistic action but did not treat this concept at all. He stated that it is the reverse of synergistic action. Finney (1942) suggested that antagonism is negative synergism and can thus be treated in the same category as synergism.

Bliss (1939), for the category of independent joint action, plotted expected response in probits against dosage of mixture in logarithms. At each point the ratio of the amount of a given stimulant to the amount of the other stimulant was held constant. These curves were not smooth but rather fell into two segments each of which appeared to be a straight line. The transition from one straight line to the other was relatively abrupt. He suggested the equation

$$p_C = p_A + p_B(1 - p_A)(1 - r) \tag{17}$$

where $p_A > p_B$, p_A is the probability of response due to the effect of stimulant A , p_B is the probability of response due to stimulant B , r is a measure of "association of susceptibilities," and p_C is the probability of death due to the combination of stimulants A and B . He did not indicate what, if any, relation he assumed between equation (17) and the plots of data.

For the category of similar joint action Bliss suggested equation

$$Y_C = a' + b \log(D_A + kD_B) \tag{18}$$

for the dosage response curve, where D_A and D_B are the respective doses of stimulants A and B in the mixture and k is the ratio of the frequency of response of the individual stimulants. The plots for this case are thus straight lines.

Bliss suggested two possible equations for synergistic action. The first, which relates the total amount of active material $(D_A + D_B)$ and the amount of the more active stimulant, say A , is

$$(D_A + D_B)D_A^i = k \quad , \tag{19}$$

where D_A and D_B are in original dosage units, which implies the probability of response to the combination of the two stimulants is determined by the sum of the ingredients multiplied by some power of the amount of the more active stimulant. The second equation, again with A being the more active stimulant, is

$$(1 + k_1 D_A) D_B^i = k_2 \tag{20}$$

which was suggested for the cases where the proportion of A approaches zero. It should be noted that these suggested equations do not bear any clear logical relation one to another.

Plackett and Hewlett (1961) utilize the following biological classification of joint drug actions:

	Similar	Dissimilar
Non-interactive	Simple Similar	Independent
Interactive	Complex Similar	Dependent

Here the suggestion is that the actions of the stimulants are similar or dissimilar respectively as the stimulants act on the same biological site or on different ones, and as interactive or non-interactive depending on the presence or absence of synergism (or antagonism). They, then, propose mathematical equations (some in an implicit form) based on the above biological classifications which, again, do not bear any clear logical relation one to another. Finally, they introduce a statistical concept into their presentation by making an assumption as to the bivariate distribution of $\tilde{z}_1$, $\tilde{z}_2$ where $\tilde{z}_1$, $\tilde{z}_2$ are the respective tolerances to stimulants A and B . They suggested that a reasonable assumption would be that the log tolerances $\log \tilde{z}_1$, $\log \tilde{z}_2$ are distributed bivariate normally. They did not give examples of data fitting any of the proposed models.

Ashford and Smith (1964) approached the problem somewhat differently. They classified the mathematical model as interactive or non-interactive rather than attempting to classify on the basis of biological considerations. They define non-interaction as being equivalent to the condition on $P = P(z_1, z_2, \underline{\theta})$, the probability of response, where z_1 and z_2 are the logarithms of dose, such that

$$W_{12}(P) = P_1P_2(P_2P_{12} - P_1P_{122}) + P_{12}(P_1^2P_{22} - P_2^2P_{11}) = 0 \quad (21)$$

where $P_\alpha = \frac{\partial P}{\partial z_\alpha}$, $P_{\alpha\beta} = \frac{\partial^2 P}{\partial z_\alpha \partial z_\beta}$, and $P_{\alpha\beta\gamma} = \frac{\partial^3 P}{\partial z_\alpha \partial z_\beta \partial z_\gamma}$. Their mathematical classification is not equivalent to Plackett and Hewlett's. Ashford and Smith remarked that no valid distinction can be made between similar and dissimilar action purely on the basis of quantal response data.

Ashford and Smith published some trivariate data on exposure to coal dust for which the response was the prevalence of pneumoconiosis for groups of mine workers. The three dosage variables, respectively, were the time spent in years at coalface coal-getting, coalface preparation, and elsewhere underground. They assumed that the tolerances were normally distributed. They then compared two models where the regions of response were not only different but were each complicated functions of the dosage levels. They applied chi-square goodness-of-fit tests (each with fifteen degrees of freedom) to the models obtaining chi-square values of 12.73 and 16.86, respectively, from which they quote the corresponding approximate significance levels. They do not indicate explicitly the form of the probability function used but rather only the functional forms indicating the response regions.

Zeigler and Moore (1966) presented a paper at the 126th Annual Meeting of the American Statistical Association on "Multivariate Quantal Response Analysis Using Regression Methods." In this paper, in addition to showing that weighted least squares can be used to converge on maximum likelihood estimates, they fitted a bivariate normal distribution to toxicity data involving the direct sprays of Pyrethrins and D.D.T. in Shell Oil P31 applied to flour beetles (Tribolium castaneum). Using a chi-square goodness-of-fit test with nineteen degrees of freedom, they obtained a value of 12.17 and reached the conclusion that the fit was satisfactory.

None of the investigations up to this point have utilized the general form suggested in Chapter I, although the specific form utilized by Zeigler and Moore (1966) is equivalent for the special case where the tolerances $\tilde{z}_1$ and $\tilde{z}_2$ to drugs A and B are each distributed normally.

It would seem useful to do some numerical studies utilizing some of the data in the literature with some analytic models which conform to the general form in equation (16). For this purpose, seven sets of data were utilized. Included among these were sets that have been classified in the following categories by previous investigators: synergistic action, simple similar action, independent action, and additive action.

Data set one, classified as synergistic by Bliss (1939), was first published by Kagy and Richardson (1936). This set is from a study of the combined action of 2-4-dinitro-6-cyclohexylphenol and petroleum oil sprayed in emulsions against eggs of a plant bug (Lygueus kalmii Stål.). Data set two, published by Plackett and Hewlett (1952), was classified by them as simple similar action. This data set is from a study of the combined action of D.D.T. and methoxychlor applied in Shell Oil P31 to flour beetles. Data set three, published by Hewlett and Plackett (1950), was classified by them as independent action. This is the data set which Zeigler and Moore (1966) fitted to a bivariate normal by means of weighted least squares. Data sets four, five, and six, published by Martin (1942), were not classified by the investigator into any category. Data set four is from a study of the toxicity of the combined action of rotenone and a dequelin concentrate in a medium of 0.5% saponin containing 5% of alcohol applied to chrysanthemum aphides (Macrosiphoniella sanborni). Data set five is from a study of the toxicity of the combined action of rotenone and ℓ-elliptone under the same laboratory conditions as data set four. Data set six is from a study of the toxicity of the combined action of rotenone and ℓ-α-toxicarol under like laboratory conditions. These three data sets showed some signs of synergism to the investigator, but he did not find it to be significant in any one of the data sets. Data set seven, published by Ashford and Smith (1964), is from a study of the prevalence of pneumoconiosis in groups of mine workers where the years spent on "coal-getting" is one imput and the other imput is years spent in "haulage." This data set was classified as an example of additive action by the investigators.

A bivariate normit analysis was run on the above seven sets of data. The analytic model for the bivariate normit analysis was

$$P(z_1 , z_2 , \underline{\Theta}) = \int_{-\infty}^{a_1+B_1z_1} (2\pi)^{-\frac{1}{2}} \exp(-\tfrac{1}{2} t^2)dt + \int_{-\infty}^{a_2+B_2z_2} (2\pi)^{-\frac{1}{2}} \exp(-\tfrac{1}{2} s^2)ds - \int_{-\infty}^{a_1+B_1z_1} \int_{-\infty}^{a_2+B_2z_2} (2\pi\sqrt{1 - \rho^2})^{-1} \exp[-(t^2 - 2\rho ts + s^2)/2(1 - \rho^2)]dtds, \quad -\infty < z_1 , z_2 < \infty \quad . \tag{22}$$

A modified least squares (see Chapter I) FORTRAN IV Computer program was utilized on a Model 44 IBM 360 system. A resumé of the results is given in Table 1.

The following is a brief explanation of the items listed in Table 1 as well as the next two tables: N is the number of stimulant combinations. SSE is the weighted sum of squares due to error which is approximately distributed as a chi-square. SSR is the weighted sum of squares due to regression and is computed as SST - SSE where SST is the weighted sum of squares adjusted for the weighted mean. SSR is approximately distributed as a chi-square. R^2 , which is computed as SSR/SST , tells what portion of SST is due to regression. Computing SSR as SST - SSE and R^2 as SSR/SST gives both a conservative estimate of the significance of regression and a conservative coefficient of determination R^2 . The column entitled "No. of significant chi-squares" tells how many of the chi-square statistics computed at each dosage level (stimulant combination) exceeded 3.84 , the .95 value of a chi-square with one degree of freedom.

Data Set No.	N	No. of Significant Chi-squares	SSE	d.f.	SSR	d.f.	R^2
1	18	5	67.029	13	66041	4	.99899
2	10	1	21.775	5	598.86	4	.96491
3	24	0	11.805	19	11176	4	.99894
4	17	2	27.147	12	1656.0	4	.98387
5	12	2	28.947	7	921.36	4	.96954
6	15	0	10.145	10	30672	4	.99967
7	40	2	38.141	35	217.75	4	.85095

TABLE 1

For all of the data sets the regression is found to be significant using SSR as the indicator. However, the chi-square for departure from the model is insignificant in only three of the cases, namely data sets three, six, and seven, which include the cases of independent action and additive action.

The synergistic data (data set 1) had sample sizes ranging from 240 to 479 (see Appendix I) at its eighteen data points. The bivariate normit analysis indicated that five of these points differed significantly from the bivariate normal model. Some of these points were marginal data points and some were not. One of the data points contributed 34.266 to the cumulative chi-square, slightly more than half of the total, but the chi-square would still be significant even without this particular data point. Upon examination of the residuals, the fit does look good with the exception of the one data point, but with the large sample size at each point, the fit would have to be extremely close in order for the cumulative chi-square to be insignificant. On the whole, it is felt that the bivariate normit analysis did quite well with the data and that the model does describe the phenomenon reasonably well, considering the significance of regression (SSR), the weighted sum of squares due to error (SSE), along with the sample sizes, and the coefficient of determination R^2 .

The simple similar action data (data set 2) had sample sizes ranging from 148 to 200 (see Appendix II) at its ten data points. The analysis indicated that one of these points differed significantly from the bivariate normal model. Again upon examination of the residuals, the fit does look good although not quite as good as the previous data set. The conclusion based on the analysis of the data is that the model does describe the phenomenon fairly well, with the exception of the one data point.

The independent action data (data set 3) had sample sizes ranging from 48 to 50 (see Appendix III) at its twenty four data points. The model does fit the data well and none of the data points differed significantly from the model. The weighted sum of squares due to error is 11.805 . Zeigler and Moore (1966) fitted this same data set and the weighted sum of squares due to error for their model is 12.17 , thus indicating the similarity of the fit.

Data sets four and five are quite similar. They had sample sizes ranging from 28 to 51 (see Appendices IV and V) at their data points. Each had two data points that differed significantly from the bivariate normal model and examination of the residuals does not indicate as good a fit as for any of the previous data sets. The model still does describe most of the data points well, but it does not seem to do as well as for the earlier cases.

Data set six had sample sizes ranging from 48 to 51 (see Appendix VI) at its fifteen data points. The model does fit the data well and none of the data points exhibit a significant deviation from the model. Two bivariate normit analyses were run on this data set using slightly different convergence criteria. The first run utilized the relative change in the unweighted sum of squares due to error and the second the relative change in the weighted sum of squares due to error. The first run after convergence had the sum of squares due to error as 0.031265, while the weighted sum of squares due to error was 0.78159×10^{15} . The second run after convergence had the sum of squares due to error as 0.032040 while the weighted sum of squares due to error was 10.145 . Which criteria produces the best fit becomes questionable at this point. It would seem that either set of parameter estimates would have to be considered acceptable despite the large chi-square value attributed to the first fit.

Data set seven had sample sizes ranging from 2 to 135 (see Appendix VII) at its forty data points. The model does fit the data well although there are two data points which deviate significantly from the model. Ashford and Smith (1964), who classified this data as an example of additive action, fitted the data to a model, assuming the marginals to be logistic, using a rather complicated response region which does not seem to have been necessary.

In general the bivariate normit analysis seems to do quite well with a diversity of mixtures of stimulants, as is evidenced by the seven sets of data analyses here. These analyses appear to lend support to the assumption that the form of the response region should remain constant irrespective of the biological considerations, at least in relation to a bivariate normit.

CHAPTER III

A bivariate logit is, perhaps, as natural to consider in the study of a mixture of two stimulants as a bivariate normit, even though very little work has been done along these lines.

Ashford and Smith (1964) ran an analysis on data set seven assuming the marginals to be logistic. They fitted the data to a model using a complicated response region without explicitly defining the mathematical model. There does not appear to have been any other examinations of data by means of a bivariate logit in the literature

In the case of a bivariate logit, the first consideration is the form of the bivariate distribution to be used. The bivariate logistic distribution utilized in this study was

$$F_3(x,y) = \begin{cases} [1 + \exp(-x)]^{-1}[1 + \exp(-y)]^{-1} \\ \cdot \{1 + a_0[1 + \exp(-x)]^{-1}[1 + \exp(-y)]^{-1} \\ \cdot \quad \exp(-x-y)\} \quad , \quad -\infty < x \, , y < \infty \end{cases} \tag{23}$$

which was developed by Gumbel (1961). The density function is

$$f_3(x,y) = \begin{cases} \{\exp(-x-y) \cdot [1 + \exp(-x)]^{-2} \\ \cdot \; [1 + \exp(-y)]^{-2}\} \cdot \{1 + a_0[1 - \exp(-x) \\ - \exp(-y) + \exp(-x-y)]/[1 + \exp(-x) \\ + \exp(-y) + \exp(-x-y)]\} \quad , -\infty < x \, , y < \infty \end{cases} \tag{24}$$

The correlation coefficient is

$$\rho = 3a_0/\pi^2 \quad , \qquad (25)$$

where $-1 \leq a_0 \leq 1$; thus $|\rho| \leq 3/\pi^2$.

A bivariate logit analysis was run on the seven data sets utilizing the Gumbel bivariate logistic distribution. The analytic model for the bivariate logit analysis was

$$P(z_1 , z_2 , \underline{\Theta}) = \{1 + \exp[B_1(z_1 + a_1)]\}^{-1} + \{1 + \exp[B_2(z_2 + a_2)]\}^{-1}$$
$$- \{1 + \exp[B_1(z_1 + a_1)]\}^{-1} \{1 + \exp[B_2(z_2 + a_2)]\}^{-1}$$
$$\cdot \Big\{1 + a_0\{1 + \exp[B_1(z_1 + a_1)]\}^{-1} \{1 + \exp[B_2(z_2 + a_2)]\}^{-1}$$
$$\cdot \quad \exp[B_1(z_1 + a_1) + B_2(z_2 + a_2)]\Big\} \ .$$
$$-\infty < z_1 , z_2 < \infty \quad . \qquad (26)$$

A resumé of the results is given in Table 2. The entries of Table 2 are the same as those of Table 1.

Data Set No.	N	No. of Significant Chi-squares	SSE	d.f.	SSR	d.f.	R^2
1	18	5	76.403	13	48773	4	.99844
2	10	4	29.262	5	574.90	4	.95157
3	24	0	15.603	19	3165.4	4	.99509
4	17	2	29.616	12	1365.2	4	.97877
5	12	3	30.169	7	675.69	4	.95724
6	15	0	11.976	10	20170	4	.99941
7	40	2	39.020	35	221.36	4	.85014

TABLE 2

As was the case for the bivariate normit, the regression was found to be significant for each of the seven data sets, and data sets three, six, and seven have nonsignificant chi-squares indicating no significant departure from regression.

For each data set, the SSE from the bivariate logit analysis was larger than the corresponding SSE from the bivariate normit analysis. Similarly R^2 from the bivariate logit analysis for each data set was smaller than the corresponding R^2 from the bivariate normit analysis. The bivariate logit analysis indicated that the same number of data points differed significantly from the bivariate logit model as was the case with bivariate normit model for each data set with the exception of data set two (simple similar action), and data set five. With data set two, the bivariate logit analysis indicated that four out of the ten data points differed significantly from the bivariate logit model as compared to one out of ten in the bivariate normit analysis. With data set five, the bivariate logit analysis indicated that three out of the twelve data points differed significantly from the bivariate logit model as compared to two out of twelve in the bivariate normit analysis.

On the whole, the bivariate logit analysis did not do as well as the bivariate normit analysis, although it did nearly as well with six out of seven of the data sets. It would seem likely that the main reason that the bivariate logit model did not do as well was due to the fact that the correlation coefficient of the model employed was restricted so that $|\rho| \leq 0.30396$, approximately, and $|\hat{\rho}| > 0.30396$ for all seven data sets in the bivariate normit analysis. It would be useful to extend this investigation to include a bivariate logit model where the correlation coefficient is not so restricted, i.e., where $-1 \leq \rho \leq 1$ inclusive.

CHAPTER IV

In this chapter the assumption that the marginals follow the Burr distribution will be made. This is a somewhat more general assumption than the assumption that the marginals are normal (or logistic).

The general system of distributions referred to here was first given by Burr (1942). Using as an expression for the distribution function

$$F(x) = \begin{cases} 1 - (1 + x^b)^{-p} & x \geq 0 \,; b \,, p > 0 \\ 0 & x < 0 \end{cases} \tag{27}$$

F(x) covers an important region of the standardized third and fourth central moments in the following sense. Figure 1 shows that the system covers a large portion of the curve-shape characteristics for Types I, III, IV, and VI of the Pearson system. Figure (1) is drawn with coordinates $\alpha_3^2 = \beta_1$ and $\delta = (2\alpha_4 - 3\alpha_3^2 - 6)/(\alpha_4 + 3)$, where α_i is the i^{th} standardized central moment. The regions covered by the Pearson Types I (or beta), IV, and VI are indicated, as well as Type III (or gamma) which lies on a curve, and the normal, logistic,rectangular, and exponential distributions which are represented by points. The subscript B refers to bell shaped functions and J to J shaped functions. It can be seen from Figure (1) that this system of distributions is quite general.

Takahasi (1965) developed a multivariate Burr distribution by using the fact that a Burr distribution is a compound Weibull distribution with a gamma-distribution as a compounder. That is, if

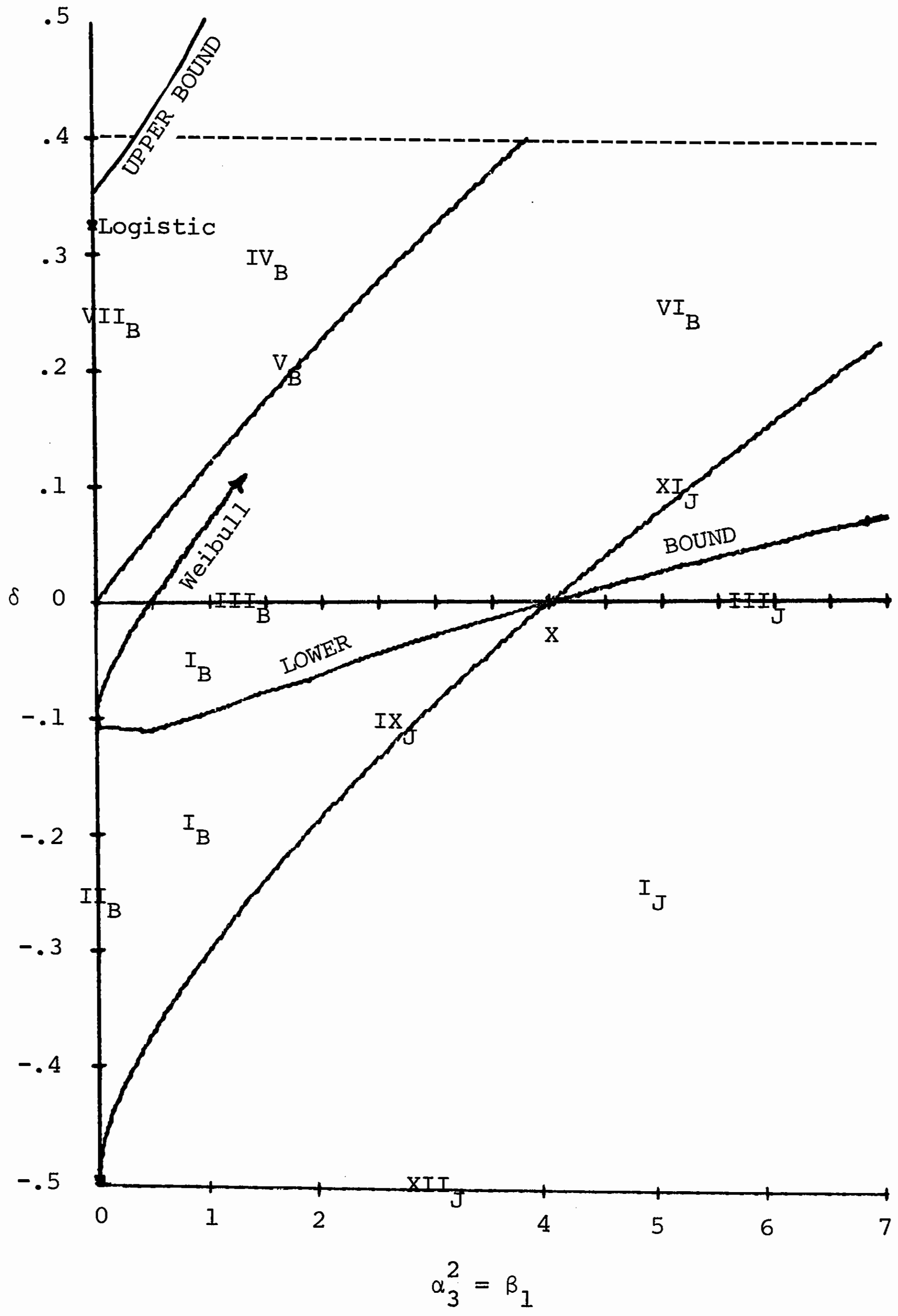

Figure 1. Upper and lower bounds of coverage in β_1 , δ space for the general system of distributions as given by Burr (1968).

$$w(x;b,\theta) = \begin{cases} \theta b x^{b-1} e^{-\theta x^b} & x > 0 \\ 0 & x \leq 0 \end{cases} \tag{28}$$

and θ is a random variable such that

$$g(\theta;p,1) = \begin{cases} \theta^{p-1} e^{-\theta}/\Gamma(p) & \theta > 0 \\ 0 & \theta \leq 0 \end{cases} \tag{29}$$

then the resultant probability density function is Burr. The special case of the bivariate density is

$$f(x_1, x_2) = \frac{\Gamma(p+2)}{\Gamma(p)} b_1 b_2 r_1 r_2 x_1^{b_1-1} x_2^{b_2-1} (1 + r_1 x_1^{b_1} + r_2 x_2^{b_2})^{-(p+2)} \quad x_i > 0 \ (i=1,2)$$

$$= 0 \qquad \text{elsewhere.} \tag{30}$$

The bivariate distribution is

$$F(x_1, x_2) = 1 - (1 + r_1 x_1^{b_1})^{-p} - (1 + r_2 x_2^{b_2})^{-p} + (1 + r_1 x^{b_1} + r_2 y^{b_2})^{-p} \quad x_i \leq 0$$

$$= 0 \qquad \text{elsewhere.} \tag{31}$$

It should be noted at this point that the r_i are equal to one in the Burr distribution as given by Burr (1942). If x_i is set equal to $B_i(z_i + a_i)$, it is easily seen that the r_i's are redundant. In addition, if the b_i's and p are held constant, e.g., the third and fourth standardized central moments can be set equal to those of the normal distribution by proper choice of the b_i's and p , then the correlation coefficient is a constant.

It was attempted to find a form of a bivariate Burr distribution such that the correlation coefficient would not be a fixed constant. The form developed by the author is

$$F(x_1, x_2) = 1 - (1 + x_1^{b_1})^{-p} - (1 + x_2^{b_2})^{-p} + (1 + x_1^{b_1} + x_2^{b_2} + rx_1^{b_1}x_2^{b_2})^{-p} .$$

$$x_i \geq 0$$

$$0 \leq r \leq p + 1$$

$$= 0 \qquad \text{elsewhere .} \qquad (32)$$

The bivariate density is

$$f(x_1, x_2) = p(p+1)(1 + rx_2^{b_2})(1 + rx_1^{b_1})b_1b_2 \; x_1^{b_1-1} x_2^{b_2-1} (1 + x_1^{b_1} + x_2^{b_2} + rx_1^{b_1}x_2^{b_2})^{-(p+2)} - prb_1b_2x_1^{b_1-1} x_2^{b_2-1} (1 + x_1^{b_1} + x_2^{b_2} + rx_1^{b_1}x_2^{b_2})^{-(p+1)} .$$

$$x_i \geq 0$$

$$0 \leq r \leq p + 1$$

$$= 0 \qquad \text{elsewhere .} \qquad (33)$$

The marginals are of the form given by Burr (1942). The conditional distribution of x_i given x_j $i \neq j$ is

$$F(x_i | x_j) = 1 - r \left(\frac{1 + x_j^{b_j}}{1 + rx_j^{b_j}} \right) \left[1 + \left(\frac{1 + rx_j^{b_j}}{1 + x_j^{b_j}} \right) x_i^{b_i} \right]^{-p} + \left(\frac{r - 1}{1 + rx_j^{b_j}} \right) \left[1 + \left(\frac{1 + rx_j^{b_j}}{1 + x_j^{b_j}} \right) x_i^{b_i} \right]^{-(p+1)} . \quad x_i \geq 0$$

$$= 0 \qquad \text{elsewhere .} \qquad (34)$$

The conditional density of x_i given x_j is

$$f(x_i | x_j) = (p+1)(1 + rx_i^{b_i}) \left(\frac{1 + rx_j^{b_j}}{1 + x_j^{b_j}} \right) b_i x_i^{b_i-1} \left[1 + \left(\frac{1 + rx_j^{b_j}}{1 + x_j^{b_j}} \right) x_i^{b_i} \right]^{-(p+2)} - rb_i x_i^{b_i-1} \left[1 + \left(\frac{1 + rx_j^{b_j}}{1 + x_j^{b_j}} \right) x_i^{b_i} \right]^{-(p+1)} . \qquad x_i \geq 0$$

$$= 0 \qquad \text{elsewhere.} \qquad (35)$$

The correlation coefficient is

$$\rho_{x_1x_2} = \Gamma\left(p - \frac{1}{b_1}\right)\Gamma\left(p - \frac{1}{b_2}\right)\Gamma\left(1 + \frac{1}{b_1}\right)\Gamma\left(1 + \frac{1}{b_2}\right) \cdot$$

$$\left\{\left(1 - \frac{1}{pb_1}\right) {}_2F_1\left(p + 1 - \frac{1}{b_1} , 1 + \frac{1}{b_2} ; p + 1 ; 1 - r\right)\right.$$

$$\left. + \frac{r}{pb_1} {}_2F_1\left(p - \frac{1}{b_1} , 1 + \frac{1}{b_2} ; p + 1 ; 1 - r\right)\right\} \Bigg/$$

$$\left\{\left[\Gamma\left(1 + \frac{2}{b_1}\right)\Gamma\left(p - \frac{2}{b_1}\right)\Gamma(p) - \Gamma^2\left(1 + \frac{1}{b_1}\right)\Gamma^2\left(p - \frac{1}{b_1}\right)\right] \cdot\right.$$

$$\left.\left[\Gamma\left(1 + \frac{2}{b_2}\right)\Gamma\left(p - \frac{2}{b_2}\right)\Gamma(p) - \Gamma^2\left(1 + \frac{1}{b_2}\right)\Gamma^2\left(p - \frac{1}{b_2}\right)\right]\right\}^{\frac{1}{2}} , \qquad (36)$$

where ${}_2F_1(\alpha,\beta;\gamma,z)$ is Gauss' hypergeometric function. If $r = 1$, then $\rho_{x_1x_2} = 0$.

A bivariate Burrit analysis was run on the seven sets of data using the bivariate Burr distribution described above. The analytic model for the bivariate Burrit analysis was

$$P(z_1 , z_2 , \underline{\theta}) = 1 - \left\{ 1 + [B_1(z_1 + a_1)]^{b_1} + [B_2(z_2 + a_2)]^{b_2} \right.$$

$$\left. + r[B_1(z_1 + a_1)]^{b_1}[B_2(z_2 + a_2)]^{b_2}\right\}^{-p} . \qquad \begin{array}{l} -a_1 \leq z_1 < \infty \\ -a_2 \leq z_2 < \infty \end{array}$$

$$= 0 \qquad \text{elsewhere} . \qquad (37)$$

A resume of the results is given in Table 3. The rows corresponding to eight parameters to be estimated constitute the general case of the Burrit

Data Set No.	No. of Parameters To be Estimated	N	No. of Significant Chi-squares	SSE	d.f.	SSR	d.f.	R^2
1	8	18	2	41.646	10	67964	7	.99939
	7		5	64.694	11	45745	6	.99859
	5		5	70.017	13	70566	4	.99901
	4		5	92.619	14	34582	3	.99733
2	8	10	3	23.027	2	573.70	7	.96146
	7		4	38.279	3	544.20	6	.93380
	5		3	26.296	5	577.20	4	.95643
	4		5	46.538	6	559.09	3	.92316
3	8	24	0	13.227	16	4262.0	7	.99690
	7		2	29.737	17	1178.8	6	.97539
	5		2	32.579	19	8668.4	4	.99696
	4		10	114.17	20	1309.0	3	.91978
4	8	17	2	27.125	9	1655.7	7	.98388
	7		3	32.782	10	1683.0	6	.98089
	5		2	27.130	12	1673.8	4	.98450
	4		3	34.593	13	1483.8	3	.97722
5	8	12	2	29.221	4	1081.9	7	.97370
	7		2	31.901	5	1508.3	6	.97929
	5		2	29.379	7	862.11	4	.96705
	4		3	36.603	8	905.50	3	.96115
6	8	15	1	9.660	7	67305	7	.99986
	7		2	13.648	8	13398	6	.99898
	5		0	12.047	10	23681	4	.99949
	4		1	23.480	11	6862.7	3	.99659
7	8	40	2	38.293	32	217.30	7	.85018
	7		3	38.765	33	219.84	6	.85010
	5		2	38.338	35	218.53	4	.85075
	4		2	38.766	36	219.06	3	.84964

TABLE 3

analysis; the rows corresponding to seven parameters to be estimated correspond to the special case with r = 0 which reduces to a Burrit analysis using the bivariate Burr distribution developed by Takahasi (1965). The rows corresponding to five and four parameters to be estimated have $\alpha_3 = 0$, $\alpha_4 = 3$ (the third and fourth standardized central moments), which are the same as the normal distributions' α_3 and α_4 . The first of these is a special case of the general Burrit analysis and the second, a special case of the Burrit analysis using the Takahasi bivariate Burr distribution.

As was the case for both the bivariate normit and the bivariate logit, the regression was found to be significant for each of the seven data sets for all of the bivariate Burrit analyses (four on each data set). The chi-square test was insignificant, indicating no significant departure from regression for data set three with the general Burrit analysis, for data set six for all but the analysis with four parameters to be estimated, and for data set seven for all four of the analyses.

The SSE from the general case of the bivariate Burrit analysis was significantly smaller than that from the bivariate normit analysis only with the synergistic data (data set one). In no other case is there any indication that the bivariate Burrit model is better than the bivariate normit model in the actual fitting of these data to a model.

Each SSE from the bivariate Burrit analyses utilizing the bivariate Burr developed in this paper is significantly smaller than the corresponding SSE from the analyses utilizing the Takahasi bivariate Burr distribution in all but three cases: both cases with data set seven, and the first case with data set five (the case corresponding to the two analyses with eight and seven parameters to be estimated). On the basis of these analyses it would seem that the bivariate Burr developed in this paper would be, in general, more useful in application than the form developed by Takahasi.

In addition it may be noted that the marginal distributions for the synergistic data, as characterized by α_3 and α_4 , do not lie in the same Pearson curve area (see figure 2). The marginals for data set three also display this characteristic but not to as high a degree. The marginals for data sets four through seven are all clustered around the normal distribution. The fact that the assumption that the marginals are Burr distribution does allow given marginal to have curve shape characteristics different from that of the other marginal suggests that the bivariate Burrit analysis may be well adapted for the analysis of data where the marginal distributions do not belong to the same family, e.g. the family of normal distributions.

In summary, the bivariate analyses utilizing the general form indicated by equation (16) seem to do quite well with a diversity of mixtures of stimulants as is evidenced by the seven sets of data which have been analyzed in this paper. The bivariate normit model and the bivariate Burrit model (general case, i.e., the case with eight parameters to be estimated) seem to be best suited for these types of analyses. The bivariate normit model would have to be recommended if the number of parameters to be estimated is of concern, but otherwise the bivariate Burrit model could well be the best model for these types of analyses.

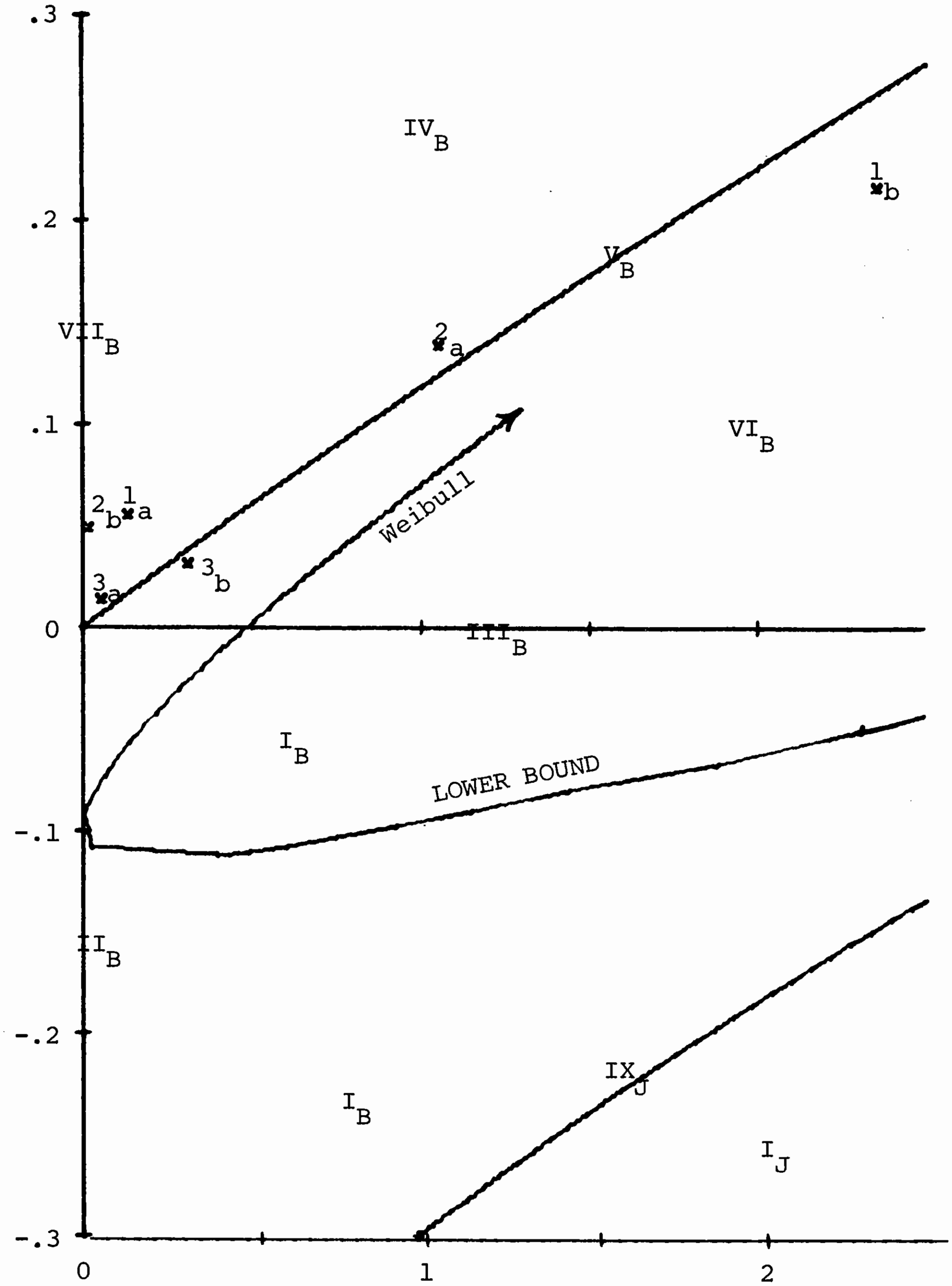

Figure 2. Expanded portion of the coverage β_1, δ space. The x's mark six of the sample population points (β_1, δ), from the data sets analyzed in this paper. N_i (N=1, 2, 3; i=a, b) refers to the i^{th} marginal of the N^{th} data set.

APPENDIX I

Data of Kagy and Richardson (1936): The combined action of 2-4 dinitro-6-cyclohexylphenol and petroleum oil sprayed in emulsions against eggs of a plant bug (*Lygaeus Kalmii Stål*). The data as described by Kagy and Richardson, the translated data, and the analyses on this set of data (data set one) are in this appendix.

DATA AS DESCRIBED IN TEXT

CONCENTRATION OF: Phenol in Oil Mixture %	CONCENTRATION OF: Mixture in Spray %	N_i Number of Eggs	Net Kill %
0	1	240	6.5
0	2	479	40.1
0	3	479	58.7
0.1	1	240	9.9
0.1	2	479	59.7
0.1	3	479	72.3
0.5	1	288	30.1
0.5	2	479	73.7
0.5	3	479	90.4
1.0	1	288	58.6
1.0	2	384	94.0
1.0	3	288	97.22
2.0	1	288	81.2
2.0	2	384	97.13
2.0	3	288	99.65
3.0	1	288	86.8
3.0	2	384	99.48
5.0	1	240	96.66

TRANSLATED DATA

X(1)	X(2)	p_i
0	.01	.0667
0	.02	.4008
0	.03	.5866
.00001	.00999	.1000
.00002	.01998	.5971
.00003	.02997	.7223
.00005	.00995	.3021
.0001	.0199	.7370
.00015	.02985	.9040
.0001	.0099	.5868
.0002	.0198	.9401
.0003	.0297	.9722
.0002	.0098	.8125
.0004	.0196	.9714
.0006	.0294	.9965
.0003	.0097	.8681
.0006	.0194	.9948
.0005	.0095	.9667

Here (N_i(i^{th} Net kill %)/100) was rounded off to the nearest integer -- which should be r_i, the number that responded to the i^{th} mixture of stimulants, and then p_i was computed as r_i/N_i.

BIVARIATE NORMIT ANALYSIS

Parameter Estimates

$\hat{a}_1 = 8.640$

$\hat{B}_1 = 0.935$

$\hat{a}_2 = 5.583$

$\hat{B}_2 = 1.489$

$\hat{\rho} = -0.379$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	4	66041
Departure from Model	SSE	13	67.029
TOTAL	SST	17	66107

Coefficient of Determination $R^2 = .99899$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0667	.1016	-.0349	3.201
.4008	.4049	-.0041	0.033
.5866	.6417	-.0551	6.317
.1000	.1180	-.0180	0.750
.5971	.4637	.1334	34.266
.7223	.7231	-.0008	0.002
.3021	.3596	-.0575	4.139
.7370	.7675	-.0305	2.490
.9040	.9277	-.0237	4.024
.5868	.5859	-.0009	0.001
.9401	.8986	.0415	7.266
.9722	.9768	-.0046	0.274
.8125	.7986	-.0139	0.346
.9714	.9691	.0023	0.069
.9965	.9950	.0015	0.137
.8681	.8865	-.0184	0.973
.9948	.9870	.0078	1.809
.9667	.9536	.0131	0.931

BIVARIATE LOGIT ANALYSIS

Parameter Estimates

$\hat{a}_0 = -0.938$

$\hat{a}_1 = 9.220$

$\hat{B}_1 = -1.690$

$\hat{a}_2 = 3.762$

$\hat{B}_2 = -2.413$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	4	48773
Departure from Model	SSE	13	76.402
TOTAL	SST	17	48849

Coefficient of Determination $R^2 = .99844$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0667	.1155	-.0488	5.597
.4008	.4103	-.0095	0.177
.5866	.6492	-.0626	8.246
.1000	.1352	-.0352	2.540
.5971	.4601	.1370	36.200
.7223	.7120	.0103	0.247
.3021	.3438	-.0417	2.220
.7370	.7627	-.0257	1.749
.9040	.9303	-.0263	5.117
.5868	.5837	.0031	0.011
.9401	.9013	.0388	6.503
.9722	.9774	-.0052	0.348
.8125	.8087	.0038	0.027
.9714	.9658	.0056	0.365
.9965	.9929	.0036	0.520
.8681	.8917	-.0236	1.654
.9948	.9821	.0127	3.527
.9667	.9504	.0163	1.354

BIVARIATE BURRIT ANALYSES

1: General Case - Eight Parameters to be Estimated

Parameter Estimates

$\hat{r} = 3.556$

$\hat{b}_1 = 9.643$

$\hat{b}_2 = 1.773$

$\hat{p} = 4.813$

$\hat{a}_1 = 18.073$

$\hat{B}_1 = 0.094$

$\hat{a}_2 = 4.877$

$\hat{B}_2 = 0.318$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	7	67964
Departure from Model	SSE	10	41.646
TOTAL	SST	17	68006

Coefficient of Determination $R^2 = .99939$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0667	.0602	.0065	0.181
.4008	.4273	-.0265	1.377
.5866	.6289	-.0423	3.676
.1000	.1018	-.0018	0.008
.5971	.5054	.0917	16.107
.7223	.7202	.0021	0.010
.3021	.3471	-.0450	2.575
.7370	.7743	-.0373	3.821
.9040	.9211	-.0171	1.935
.5868	.5627	.0241	0.682
.9401	.9010	.0390	6.580
.9722	.9755	-.0033	0.130
.8125	.7823	.0302	1.540
.9714	.9711	.0003	0.014
.9965	.9950	.0015	0.137
.8681	.8781	-.0100	0.271
.9948	.9883	.0066	1.420
.9667	.9516	.0151	1.194

2: Takahasi Burr - r = 0 ; Seven Parameters to be Estimated

Parameter Estimates

$\hat{b}_1 = 8.008$
$\hat{b}_2 = 1.799$
$\hat{p} = 6.239$
$\hat{a}_1 = 16.156$
$\hat{B}_1 = 0.113$
$\hat{a}_2 = 4.869$
$\hat{B}_2 = 0.291$

Chi-square Analysis Table

source		d.f.	
Due to model	SSR	6	45745
Departure from Model	SSE	11	64.694
TOTAL	SST	17	45810

Coefficient of Determination $R^2 = .99859$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0667	.0592	.0075	0.245
.4008	.4480	-.0472	4.318
.5866	.6599	-.0733	11.463
.1000	.0921	.0079	0.181
.5971	.5001	.0970	18.015
.7223	.7115	.0108	0.272
.3021	.3552	-.0531	3.544
.7370	.7466	-.0096	0.231
.9040	.8874	.0166	1.319
.5868	.5958	-.0090	0.097
.9401	.8844	.0557	0.001
.9722	.9581	.0141	1.420
.8125	.8261	-.0136	0.370
.9714	.9670	.0044	0.230
.9965	.9904	.0061	1.111
.8681	.9151	-.0469	8.167
.9948	.9875	.0073	1.656
.9667	.9733	-.0066	0.403

3: $\alpha_3 = 0$; $\alpha_4 = 3$ (Third and Fourth Standardized Central Moments) ;

$b_1 = b_2 = 4.874$; $p = 6.158$; Five Parameters to be Estimated

Parameter Estimates

$\hat{r} = 4.383$

$\hat{a}_1 = 13.485$

$\hat{B}_1 = 0.153$

$\hat{a}_2 = 6.426$

$\hat{B}_2 = 0.242$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	4	70566
Departure from Model	SSE	13	70.017
TOTAL	SST	17	70636

Coefficient of Determination $R^2 = .99901$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0667	.1057	-.0390	3.861
.4008	.4058	-.0050	0.049
.5866	.6443	-.0577	6.967
.1000	.1221	-.0221	1.090
.5971	.4603	.1368	36.090
.7223	.7168	.0056	0.073
.3021	.3594	-.0573	4.107
.7370	.7592	-.0222	1.296
.9040	.9271	-.0231	3.794
.5868	.5829	.0039	1.873
.9401	.8964	.0437	7.906
.9722	.9782	-.0060	0.483
.8125	.7985	.0140	0.353
.9714	.9691	.0023	0.068
.9965	.9954	.0011	0.073
.8681	.8876	-.0195	1.100
.9948	.9869	.0079	1.847
.9667	.9543	.0124	0.843

4: Takahasi Burr - r = 0 ; $\alpha_3 = 0$; $\alpha_4 = 3$ (Third and Fourth Standardized Central Moments); $b_1 = b_2 = 4.874$; p = 6.158 ; Four Parameters to be Estimated

Parameter Estimates

$\hat{a}_1 = 13.460$

$\hat{B}_1 = 0.160$

$\hat{a}_2 = 6.265$

$\hat{B}_2 = 0.259$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	3	34582
Departure from Model	SSE	14	92.619
TOTAL	SST	17	34674

Coefficient of Determination $R^2 = .99733$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0667	.0957	-.0290	2.333
.4008	.4125	-.0117	0.270
.5866	.6674	-.0808	14.102
.1000	.1140	-.0140	0.463
.5971	.4596	.1375	36.464
.7223	.7144	.0079	0.147
.3021	.3799	-.0778	7.395
.7370	.7378	-.0008	14.954
.9040	.8911	.0129	0.822
.5868	.6188	-.0319	1.246
.9401	.8789	.0612	13.499
.9722	.9564	.0158	1.723
.8125	.8321	-.0196	0.796
.9714	.9611	.0103	1.086
.9965	.9877	.0088	1.844
.8681	.9128	-.0446	7.208
.9948	.9830	.0118	3.208
.9667	.9680	.0013	0.013

APPENDIX II

Data of Plackett and Hewlett (1952): The toxity to *Tribolium castaneum* of D.D.T., methoxychlor (MOC), and combinations of the two applied in Shell Oil P31 as films on filter paper, six-day exposures. The data as described by Plackett and Hewlett, the translated data, and the analyses on this set of data (data set two) are in this appendix.

DATA AS DESCRIBED BY PLACKETT AND HEWLETT

D.D.T. Percent w/v	MOC Percent w/v	N_i Number of Beetles	Observed Mortality Percent
0.0	0.4	199	7.5
0.0	0.8	148	29.7
0.0	1.6	199	77.9
0.2	0.0	200	14.5
0.2	0.4	150	26.0
0.2	0.8	151	63.6
0.4	0	149	43.6
0.4	0.4	148	66.2
0.4	0.8	150	78.7
0.8	0.0	199	70.9

TRANSLATED DATA

X(1)	X(2)	p_i
0.0	0.004	.0754
0.0	0.008	.2973
0.0	0.016	.7789
0.002	0.0	.1450
0.002	0.004	.2600
0.002	0.008	.6358
0.004	0.0	.4362
0.004	0.004	.6622
0.004	0.008	.7867
0.008	0.0	.7085

BIVARIATE NORMIT ANALYSIS

Parameter Estimates

$\hat{a}_1 = 5.787$

$\hat{B}_1 = 1.071$

$\hat{a}_2 = 6.925$

$\hat{B}_2 = 1.503$

$\hat{\rho} = -0.9999$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	4	598.86
Departure from Model	SSE	5	21.775
TOTAL	SST	9	620.63

Coefficient of Determination $R^2 = .96491$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0754	.0844	-.0090	0.210
.2973	.3693	-.0720	3.292
.7789	.7606	.0183	0.365
.1450	.1920	.0470	2.848
.2600	.2764	-.0164	0.203
.6358	.5613	.0745	3.405
.4362	.4491	-.0129	0.100
.6622	.5335	.1287	9.845
.7867	.8184	-.0317	1.013
.7085	.7306	-.0221	0.494

BIVARIATE LOGIT ANALYSIS

Parameter Estimates

$\hat{a}_0 = -1.000$

$\hat{a}_1 = 5.448$

$\hat{B}_1 = -1.776$

$\hat{a}_2 = 4.645$

$\hat{B}_2 = -2.559$

Chi-Square Analysis Table

source		d.f.	
Due to Model	SSR	4	574.90
Departure from Model	SSE	5	29.262
TOTAL	SST	9	604.16

Coefficient of Determination $R^2 = .95157$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0754	.0958	-.0204	0.960
.2973	.3846	-.0873	4.760
.7789	.7865	-.0076	0.068
.1450	.2041	-.0591	4.297
.2600	.2944	-.0344	0.856
.6358	.5486	.0872	4.638
.4362	.4675	-.0313	0.586
.6622	.5401	.1221	8.811
.7867	.7312	.0555	2.351
.7085	.7504	-.0419	1.864

BIVARIATE BURRIT ANALYSES

1: General Case - Eight Parameters to be Estimated

Parameter Estimates

$\hat{r} = 5.271$

$\hat{b}_1 = 2.351$

$\hat{b}_2 = 6.755$

$\hat{p} = 4.271$

$\hat{a}_1 = 7.252$

$\hat{B}_1 = 0.270$

$\hat{a}_2 = 8.206$

$\hat{B}_2 = 0.216$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	7	573.70
Departure from Model	SSE	2	23.027
TOTAL	SST	9	596.73

Coefficient of Determination $R^2 = .96141$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0754	.1009	-.0255	1.430
.2973	.3814	-.0841	4.440
.7789	.7764	.0025	0.007
.1450	.1889	-.0439	2.513
.2600	.2862	-.0262	0.504
.6358	.5423	.0935	5.321
.4362	.4835	-.0473	1.332
.6622	.5643	.0979	5.768
.7867	.7559	.0308	0.770
.7085	.7387	-.0302	0.943

2: Takahasi Burr - r = 0 ; Seven Parameters to be Estimated

Parameter Estimates

$\hat{b}_1 = 0.961$

$\hat{b}_2 = 5.064$

$\hat{p} = 3.265$

$\hat{a}_1 = 6.399$

$\hat{B}_1 = 0.298$

$\hat{a}_2 = 7.263$

$\hat{B}_2 = 0.290$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	6	544.20
Departure from Model	SSE	3	38.249
TOTAL	SST	9	582.45

Coefficient of Determination $R^2 = .93433$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0754	.0987	-.0233	1.217
.2973	.4096	-.1123	7.723
.7789	.7928	-.0139	0.234
.1450	.1774	-.0324	1.440
.2600	.2542	.0058	0.026
.6358	.5004	.1354	11.071
.4362	.5490	-.1128	7.662
.6622	.5844	.0778	3.685
.7867	.7037	.0830	4.953
.7085	.7240	-.0155	0.239

3: $\alpha_3 = 0$; $\alpha_4 = 3$ (Third and Fourth Standardized Central Moments) ;

$b_1 = b_2 = 4.874$; $p = 6.158$; Five Parameters to be Estimated

Parameter Estimates

$\hat{r} = 7.158$

$\hat{a}_1 = 9.108$

$\hat{B}_1 = 0.176$

$\hat{a}_2 = 7.272$

$\hat{B}_2 = 0.245$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	4	577.20
Departure from Model	SSE	5	26.296
TOTAL	SST	9	603.50

Coefficient of Determination $R^2 = .95643$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0754	.0946	-.0192	0.854
.2973	.3869	-.0896	5.005
.7789	.7805	-.0016	0.003
.1450	.2018	-.0568	4.002
.2600	.2928	-.0328	0.780
.6358	.5586	.0772	3.653
.4362	.4628	-.0266	0.424
.6622	.5410	.1212	8.757
.7867	.7490	.0377	1.133
.7085	.7484	-.0399	1.685

4: Takahasi Burr - r = 0 ; α_3 = 0 ; α_4 = 3 (Third and Fourth Standardized Central Moments); $b_1 = b_2 = 4.874$; p = 6.158 ; Four Parameters to be Estimated

Parameter Estimates

$\hat{a}_1 = 9.196$

$\hat{B}_1 = 0.174$

$\hat{a}_2 = 7.270$

$\hat{B}_2 = 0.248$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	3	559.09
Departure from Model	SSE	6	46.537
TOTAL	SST	9	605.63

Coefficient of Determination $R^2 = .92316$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0754	.0990	-.0236	1.239
.2973	.4013	-.1040	6.666
.7789	.7952	-.0163	0.324
.1450	.2165	-.0715	6.027
.2600	.2912	-.0312	0.707
.6358	.5219	.1139	7.855
.4362	.4798	-.0436	1.135
.6622	.5264	.1358	10.953
.7867	.6727	.1140	8.857
.7085	.7590	-.0505	2.775

APPENDIX III

Data of Hewlett and Plackett (1950): A study of six day toxicity to beetles (*Tribolium castaneum*) of direct sprays of Pyrethins, D.D.T., and the two together in Shell Oil P31. The data as reproduced by Zeigler and Moore (1966), and the analyses on this set of data (data set three) are in this appendix.

DATA AS REPRODUCED BY ZEIGLER AND MOORE

Insecticide	DEPOSIT (mg./10 sq. cm.)	X(1)	X(2)	N_i	p_i
1.2% w/v Pyrethins	2.52	.03024	0	48	.0625
	3.30	.03960	0	48	.0625
	4.25	.05100	0	50	.1800
	5.33	.06396	0	50	.3200
	7.15	.08580	0	50	.4000
	9.53	.11436	0	50	.6000
	12.28	.14739	0	49	.7551
	15.58	.18696	0	50	.7000
2.0% w/v D.D.T.	2.45	0	.0490	49	.1633
	3.18	0	.0636	50	.1600
	4.25	0	.0850	50	.3200
	5.48	0	.1096	50	.4200
	7.24	0	.1448	50	.5000
	9.54	0	.1908	50	.5600
	12.36	0	.2472	50	.7000
	15.54	0	.3108	50	.7400
1.2% w/v Pyrethins plus 2.0% w/v D.D.T.	2.74	.02964	.0494	50	.2800
	3.20	.03840	.0640	49	.3673
	4.10	.04920	.0820	50	.4400
	5.34	.06408	.1068	50	.7200
	7.11	.08532	.1422	50	.8400
	9.60	.11520	.1920	50	.9000
	12.45	.14940	.2490	50	1.0000
	15.65	.18780	.3130	50	1.0000

BIVARIATE NORMIT ANALYSIS

Parameter Estimates

$\hat{a}_1 = 2.827$

$\hat{B}_1 = 1.231$

$\hat{a}_2 = 1.698$

$\hat{B}_2 = 0.882$

$\hat{\rho} = -0.686$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	4	11176
Departure from Model	SSE	19	11.805
TOTAL	SST	23	11188

Coefficient of Determination $R^2 = .99894$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0625	.0696	-.0071	0.037
.0625	.1257	-.0632	1.744
.1800	.2017	-.0217	0.146
.3200	.2888	.0312	0.237
.4000	.4225	-.0225	0.104
.6000	.5629	.0371	0.280
.7551	.6809	.0742	1.241
.7000	.7773	-.0773	1.727
.1633	.1677	-.0044	0.007
.1600	.2318	-.0718	1.447
.3200	.3166	.0034	0.003
.4200	.4002	.0198	0.082
.5000	.4972	.0028	0.002
.5600	.5934	-.0334	0.231
.7000	.6790	.0210	0.101
.7400	.7476	-.0076	0.015
.2800	.2358	.0442	0.542
.3673	.3506	.0167	0.060
.4400	.4896	-.0496	0.492
.7200	.6559	.0642	0.912
.8400	.8215	.0185	0.117
.9000	.9364	-.0364	1.109
1.0000	.9816	.0184	0.939
1.0000	.9954	.0046	0.231

BIVARIATE LOGIT ANALYSIS

Parameter Estimates

$\hat{a}_0 = -1.000$

$\hat{a}_1 = 2.338$

$\hat{B}_1 = -2.090$

$\hat{a}_2 = 1.960$

$\hat{B}_2 = -1.520$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	4	3165.4
Departure from Model	SSE	19	15.603
TOTAL	SST	23	3181.0

Coefficient of Determination $R^2 = .99509$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0625	.0813	-.0188	0.227
.0625	.1346	-.0720	2.140
.1800	.2087	-.0287	0.250
.3200	.2975	.0225	0.121
.4000	.4390	-.0390	0.309
.6000	.5879	.0121	0.030
.7551	.7079	.0472	0.528
.7000	.7994	-.0994	3.082
.1633	.1674	-.0041	0.006
.1600	.2301	-.0701	1.385
.3200	.3171	.0029	0.002
.4200	.4059	.0141	0.041
.5000	.5105	-.0105	0.022
.5600	.6133	-.0533	0.600
.7000	.7016	-.0016	0.001
.7400	.7690	-.0290	0.237
.2800	.2442	.0358	0.347
.3673	.3493	.0180	0.070
.4400	.4754	-.0354	0.252
.7200	.6266	.0934	1.865
.8400	.7816	.0584	0.999
.9000	.9002	-.0002	0.000
1.0000	.9578	.0422	2.201
1.0000	.9826	.0174	0.888

BIVARIATE BURRIT ANALYSES

1: General Case - Eight Parameters to be Estimated

Parameter Estimates

$\hat{r} = 6.323$

$\hat{b}_1 = 3.933$

$\hat{b}_2 = 2.941$

$\hat{p} = 5.323$

$\hat{a}_1 = 4.863$

$\hat{B}_1 = 0.239$

$\hat{a}_2 = 4.760$

$\hat{B}_2 = 0.183$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	7	4258.4
Departure from Model	SSE	16	13.227
TOTAL	SST	23	4271.6

Coefficient of Determination $R^2 = .99690$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0625	.0622	.0003	0.000
.0625	.1216	-.0591	1.569
.1800	.2025	-.0225	0.156
.3200	.2947	.0253	0.154
.4000	.4346	-.0346	0.244
.6000	.5784	.0216	0.095
.7551	.6963	.0588	0.802
.7000	.7898	-.0898	2.429
.1633	.1666	-.0033	0.004
.1600	.2384	-.0784	1.691
.3200	.3298	-.0098	0.022
.4200	.4161	.0039	0.003
.5000	.5120	-.0120	0.029
.5600	.6034	-.0434	0.393
.7000	.6820	.0180	0.075
.7400	.7438	-.0038	0.037
.2800	.2261	.0540	0.832
.3673	.3475	.0198	0.085
.4400	.4875	-.0475	0.451
.7200	.6463	.0737	1.187
.8400	.8007	.0393	0.484
.9000	.9135	-.0135	0.115
1.0000	.9660	.0340	1.759
1.0000	.9873	.0127	0.644

2: Takahasi Burr - r = 0 ; Seven Parameters to be Estimated

Parameter Estimates

$\hat{b}_1 = 3.831$

$\hat{b}_2 = 2.939$

$\hat{p} = 6.740$

$\hat{a}_1 = 4.675$

$\hat{B}_1 = 0.243$

$\hat{a}_2 = 4.701$

$\hat{B}_2 = 0.176$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	6	1178.8
Departure from Model	SSE	17	29.737
TOTAL	SST	23	1208.5

Coefficient of Determination $R^2 = .97539$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0625	.0540	.0085	0.067
.0625	.1148	-.0523	1.292
.1800	.2011	-.0211	0.138
.3200	.3017	.0184	0.080
.4000	.4552	-.0552	0.614
.6000	.6110	-.0109	0.025
.7551	.7346	.0205	0.106
.7000	.8282	-.1282	5.772
.1633	.1707	-.0074	0.019
.1600	.2469	-.0869	2.032
.3200	.3444	-.0244	0.132
.4200	.4361	-.0161	0.053
.5000	.5373	-.0373	0.280
.5600	.6326	-.0725	1.132
.7000	.7132	-.0132	0.042
.7400	.7752	-.0352	0.355
.2800	.2137	.0663	1.310
.3673	.3256	.0417	0.387
.4400	.4503	-.0103	0.022
.7200	.5888	.1312	3.554
.8400	.7267	.1134	3.234
.9000	.8406	.0594	1.318
1.0000	.9088	.0912	5.019
1.0000	.9478	.0522	2.754

3: $\alpha_3 = 0$; $\alpha_4 = 3$ (Third and Fourth Standardized Central Moments) ;

$b_1 = b_2 = 4.874$; $p = 6.158$; Five Parameters to be Estimated

Parameter Estimates

$\hat{r} = 7.129$

$\hat{a}_1 = 5.499$

$\hat{B}_1 = 0.200$

$\hat{a}_2 = 5.056$

$\hat{B}_2 = 0.207$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	4	8668.4
Departure from Model	SSE	19	32.579
TOTAL	SST	23	8701.0

Coefficient of Determination $R^2 = .99626$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0625	.0679	-.0054	0.022
.0625	.1215	-.0590	1.566
.1800	.1937	-.0137	0.060
.3200	.2771	.0429	0.460
.4000	.4076	-.0076	0.012
.6000	.5483	.0517	0.540
.7551	.6695	.0856	1.624
.7000	.7699	-.0699	1.378
.1633	.0884	.0749	3.408
.1600	.1525	.0075	0.022
.3200	.2532	.0668	1.178
.4200	.3654	.0546	0.644
.5000	.5046	-.0046	0.004
.5600	.6438	-.0838	1.530
.7000	.7602	-.0602	0.995
.7400	.8431	-.1031	4.014
.2800	.1542	.1258	6.062
.3673	.2656	.1018	2.601
.4400	.4094	.0306	0.194
.7200	.5903	.1297	3.477
.8400	.7803	.0597	1.041
.9000	.9197	-.0197	0.263
1.0000	.9767	.0233	1.193
1.0000	.9942	.0058	0.291

4: Takahasi Burr - r = 0 ; $\alpha_3 = 0$; $\alpha_4 = 3$ (Third and Fourth Standardized Central Moments); $b_1 = b_2 = 4.874$; p = 6.158 ; Four Parameters to be Estimated

Parameter Estimates

$\hat{a}_1 = 4.932$

$\hat{B}_1 = 0.246$

$\hat{a}_2 = 4.585$

$\hat{B}_2 = 0.237$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	3	1309.0
Departure from Model	SSE	20	114.16
TOTAL	SST	23	1423.2

Coefficient of Determination $R^2 = .91978$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0625	.0377	.0248	0.813
.0625	.0849	-.0224	0.309
.1800	.1588	.0212	0.168
.3200	.2533	.0667	1.176
.4000	.4114	-.0114	0.027
.6000	.5849	.0151	0.047
.7551	.7280	.0271	0.182
.7000	.8357	-.1357	6.706
.1633	.0482	.1151	14.133
.1600	.0989	.0611	2.094
.3200	.1906	.1294	5.423
.4200	.3043	.1157	3.162
.5000	.4569	.0431	0.374
.5600	.6174	-.0573	0.696
.7000	.7533	-.0533	0.764
.7400	.8480	-.1080	4.521
.2800	.0827	.1973	25.672
.3673	.1693	.1978	13.626
.4400	.2940	.1460	5.132
.7200	.4632	.2568	13.259
.8400	.6569	.1831	7.440
.9000	.8241	.0759	1.987
1.0000	.9172	.0828	4.513
1.0000	.9625	.0375	1.947

APPENDIX IV

Data of J. T. Martin (1942): The toxicities to *Macrosiphoniella sanborni* of rotenone, a deguelin concentrate, and of a mixture. Tests of 17 November 1938. Fivefold replication. Results one day after spraying. Medium 0.5% saponin, containing 5% alcohol. Tattersfield apparatus. The data as described by Martin, the translated data, and the analyses of this set of data (data set four) are in this appendix.

DATA AS DESCRIBED BY MARTIN — TRANSLATED DATA

CONCENTRATIONS (mg./ℓ.) X(1) Rotenone	X(2) Deguelin Concentrate	N_i Number of Insects Used	Percent Mortality	p_i
10.2	0.0	50	88.0	.8800
7.7	0.0	49	85.7	.8571
5.1	0.0	46	52.2	.5217
3.8	0.0	48	33.3	.3333
2.6	0.0	50	12.0	.1200
0.0	50.5	48	100.0	1.0000
0.0	40.4	50	94.0	.9400
0.0	30.3	49	95.9	.9592
0.0	20.2	48	70.8	.7083
0.0	10.1	48	37.5	.3750
0.0	5.1	49	32.6	.3265
5.1	20.3	50	96.0	.9600
4.0	16.3	46	93.5	.9348
3.0	12.2	48	79.2	.7917
2.0	8.1	46	58.7	.5870
1.0	4.1	46	47.8	.4783
0.5	2.0	47	14.9	.1489

BIVARIATE NORMIT ANALYSIS

Parameter Estimates

$\hat{a}_1 = -2.775$

$\hat{B}_1 = 1.762$

$\hat{a}_2 = -1.645$

$\hat{B}_2 = 0.823$

$\hat{\rho} = -0.530$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	4	1656.0
Departure from Model	SSE	12	27.146
TOTAL	SST	16	1683.1

Coefficient of Determination $R^2 = .98387$

Residual Analysis

p_i	P_i	Residual	Chi-square
.8800	.9059	-.0259	0.392
.8571	.7940	.0631	1.193
.5217	.5377	-.0160	0.047
.3333	.3359	-.0026	0.001
.1200	.1374	-.0174	0.128
1.0000	.9432	.0568	2.893
.9400	.9190	.0210	0.297
.9592	.8773	.0819	3.053
.7083	.7962	-.0879	2.284
.3750	.6017	-.2267	10.297
.3265	.3805	-.0540	0.605
.9600	.9649	-.0049	0.036
.9348	.9084	.0264	0.386
.7917	.7893	.0024	0.002
.5870	.5826	.0044	0.004
.4783	.3170	.1613	5.528
.1489	.1506	-.0017	0.001

BIVARIATE NORMIT ANALYSIS

Parameter Estimates

$\hat{a}_1 = -4.184$

$\hat{B}_1 = 2.545$

$\hat{a}_2 = -4.978$

$\hat{B}_2 = 1.618$

$\hat{\rho} = -0.743$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	4	921.36
Departure from Model	SSE	7	28.947
TOTAL	SST	11	950.30

Coefficient of Determination $R^2 = .96954$

Residual Analysis

p_i	P_i	Residual	Chi-square
1.0000	.9577	.0423	1.236
.8500	.8396	.0104	0.032
.3750	.4846	-.1096	2.306
.0816	.0904	-.0088	0.046
.9216	.9145	.0071	0.032
.9184	.8172	.1012	3.359
.3878	.5467	-.1589	4.993
.0652	.2830	-.2178	10.753
.9200	.9217	-.0017	0.002
.6667	.6173	.0492	0.491
.3182	.2258	.0924	2.149
.1020	.0458	.0562	3.548

BIVARIATE LOGIT ANALYSIS

Parameter Estimates

$\hat{a}_0 = -1.000$

$\hat{a}_1 = -1.611$

$\hat{B}_1 = -3.509$

$\hat{a}_2 = -3.136$

$\hat{B}_2 = -3.145$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	4	675.39
Departure from Model	SSE	7	30.169
TOTAL	SST	11	705.56

Coefficient of Determination $R^2 = .95724$

Residual Analysis

p_i	P_i	Residual	Chi-square
1.0000	.9240	.0760	2.303
.8500	.8159	.0341	0.310
.3750	.5164	-.1414	3.844
.0816	.1510	-.0694	1.841
.9216	.9221	-.0005	0.000
.9184	.8275	.0909	2.836
.3878	.5095	-.1217	2.904
.0652	.2133	-.1481	6.010
.9200	.8549	.0651	1.707
.6667	.5743	.0924	1.678
.3182	.2141	.1041	2.835
.1020	.0441	.0579	3.902

BIVARIATE BURRIT ANALYSES

1: General Case - Eight Parameters to be Estimated

Parameter Estimates

$\hat{r} = 7.345$

$\hat{b}_1 = 6.078$

$\hat{b}_2 = 5.388$

$\hat{p} = 6.345$

$\hat{a}_1 = 0.224$

$\hat{B}_1 = 0.373$

$\hat{a}_2 = -0.483$

$\hat{B}_2 = 0.260$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	7	1081.9
Departure from Model	SSE	4	29.221
TOTAL	SST	11	1111.1

Coefficient of Determination $R^2 = .97370$

Residual Analysis

p_i	P_i	Residual	Chi-square
1.0000	.9691	.0309	0.893
.8500	.8526	-.0025	0.002
.3750	.4718	-.0967	1.803
.0816	.0896	-.0080	0.039
.9216	.9361	-.0145	0.180
.9184	.8434	.0750	2.087
.3878	.5585	-.1707	5.794
.0652	.2823	-.2171	10.704
.9200	.8784	.0416	0.811
.6667	.5871	.0796	1.254
.3182	.2258	.0924	2.151
.1020	.0460	.0560	3.503

2: Takahasi Burr - r = 0 ; Seven Parameters to be Estimated

Parameter Estimates

$\hat{b}_1 = 9.296$

$\hat{b}_2 = 7.696$

$\hat{p} = 77.198$

$\hat{a}_1 = 1.655$

$\hat{B}_1 = 0.183$

$\hat{a}_2 = 0.700$

$\hat{B}_2 = 0.142$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	6	1508.3
Departure from Model	SSE	5	31.901
TOTAL	SST	11	1540.2

Coefficient of Determination $R^2 = .97929$

Residual Analysis

p_i	P_i	Residual	Chi-square
1.0000	.9804	.0196	0.559
.8500	.8623	-.0123	0.051
.3750	.4921	-.1171	2.631
.0816	.1317	-.0501	1.074
.9216	.9509	-.0293	0.935
.9184	.8434	.0751	2.089
.3878	.5262	-.1384	3.767
.0652	.2614	-.1962	9.171
.9200	.7874	.1326	5.253
.6667	.5364	.1304	3.280
.3182	.2376	.0806	1.577
.1020	.0602	.0418	1.515

3: α_3 = ; α_4 = 3 (Third and Fourth Standardized Central Moments) ;

$b_1 = b_2 = 4.874$; p = 6.158 ; Five Parameters to be Estimated

Parameter Estimates

$\hat{r}$ = 7.158

$\hat{a}_1$ = -0.085

$\hat{B}_1$ = 0.414

$\hat{a}_2$ = -0.588

$\hat{B}_2$ = 0.260

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	4	862.11
Departure from Model	SSE	7	29.379
TOTAL	SST	11	891.49

Coefficient of Determination R^2 = .96705

Residual Analysis

p_i	P_i	Residual	Chi-square
1.0000	.9598	.0402	1.174
.8500	.8426	.0074	0.016
.3750	.4811	-.1061	2.166
.0816	.0925	-.0109	0.069
.9216	.9181	.0035	0.009
.9184	.8217	.0697	3.126
.3878	.5500	-.1622	5.205
.0652	.2887	-.2235	11.188
.9200	.8762	.0438	0.883
.6667	.5942	.0725	1.046
.3182	.2333	.0849	1.773
.1020	.0504	.0516	2.723

4: Takahasi Burr - $r = 0$; $\alpha_3 = 0$; $\alpha_4 = 3$ (Third and Fourth Standardized Central Moments); $b_1 = b_2 = 4.874$; $p = 6.158$; Four Parameters to be Estimated

Parameter Estimates

$\hat{a}_1 = -0.080$

$\hat{B}_1 = 0.418$

$\hat{a}_2 = -0.488$

$\hat{B}_2 = 0.254$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	3	905.59
Departure from Model	SSE	8	36.603
TOTAL	SST	11	942.19

Coefficient of Determination $R^2 = .96115$

Residual Analysis

p_i	P_i	Residual	Chi-square
1.0000	.9659	.0341	0.989
.8500	.8588	-.0088	0.025
.3750	.5037	-.1287	3.179
.0816	.0993	-.0177	0.172
.9216	.9207	.0009	0.001
.9184	.8294	.0890	2.741
.3878	.5697	-.1819	6.615
.0652	.3118	-.2466	13.032
.9200	.7923	.1277	4.954
.6667	.5576	.1091	2.317
.3182	.2484	.0698	1.147
.1020	.0611	.0409	1.430

APPENDIX VI

Data of J. T. Martin (1942): The toxicities to _Macrosiphoniella sanborni_ of rotenone, ℓ-α-toxicarol, and of a mixture. Tests of 24 September 1941. Fivefold replication. Results one day after spraying. Medium 0.5% saponin, containing 5% of alcohol. Tattersfield apparatus. The data as described by Martin, the translated data, and the analyses of this set of data (data set six) are in this appendix.

DATA AS DESCRIBED BY MARTIN | TRANSLATED DATA

CONCENTRATIONS (mg./ℓ.) X(1) Rotenone	X(2) ℓ-α-Toxicarol	N_i Number of Insects Used	Percent Mortality	p_i
1.06	0.0	51	100.0	1.0000
0.85	0.0	48	97.9	.9792
0.64	0.0	48	93.8	.9375
0.42	0.0	48	62.5	.6250
0.21	0.0	48	12.5	.1250
0.0	9.75	49	100.0	1.0000
0.0	7.80	48	97.9	.9792
0.0	5.85	52	98.1	.9808
0.0	3.90	49	87.7	.8776
0.0	1.95	48	50.0	.5000
0.53	4.88	48	100.0	1.0000
0.42	3.90	48	100.0	1.0000
0.32	2.93	49	89.8	.8980
0.21	1.95	50	82.0	.8200
0.11	0.98	50	30.0	.3000

BIVARIATE NORMIT ANALYSIS

1: Using Relative Change in Weighted Sum of Squares Due To Error as Part of the Convergence Criteria

Parameter Estimates

$\hat{a}_1 = 2.372$

$\hat{B}_1 = 2.212$

$\hat{a}_2 = -0.580$

$\hat{B}_2 = 1.328$

$\hat{\rho} = -0.432$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	4	30672
Departure from Model	SSE	10	10.144
TOTAL	SST	14	30682

Coefficient of Determination $R^2 = .99967$

Unweighted Sum of Squares Due to Error = .03204

Residual Analysis

p_i	P_i	Residual	Chi-square
1.0000	.9938	.0062	0.318
.9792	.9779	.0013	0.004
.9375	.9169	.0206	0.267
.6250	.6747	-.0497	0.541
.1250	.1401	-.0151	0.090
1.0000	.9928	.0072	0.357
.9792	.9845	-.0053	0.089
.9808	.9614	.0194	0.528
.8776	.8903	-.0127	0.081
.5000	.6207	-.1207	2.971
1.0000	.9985	.0015	0.072
1.0000	.9893	.0107	0.518
.8980	.9385	-.0405	1.391
.8200	.7130	.1070	2.796
.3000	.2780	.0220	1.210

BIVARIATE NORMIT ANALYSIS

2: Using Relative Change in Unweighted Sum of Squares Due to Error as Part of the Convergence Criteria

Parameter Estimates

$\hat{a}_1 = 2.279$

$\hat{B}_1 = 2.063$

$\hat{a}_2 = -0.589$

$\hat{B}_2 = 1.290$

$\hat{\rho} = -0.9999$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	4	$.19 \cdot 10^{18}$
Departure from Model	SSE	10	$.78 \cdot 10^{15}$
TOTAL	SST	14	$.19 \cdot 10^{18}$

Coefficient of Determination $R^2 = .99592$

Unweighted Sum of Squares Due to Error = .03127

Residual Analysis

p_i	P_i	Residual	Chi-square
1.0000	.9918	.0082	0.422
.9792	.9740	.0052	0.050
.9375	.9129	.0246	0.367
.6250	.6878	-.0628	0.882
.1250	.1736	-.0496	0.790
1.0000	.9906	.0094	0.465
.9792	.9808	-.0016	0.006
.9808	.9545	.0263	0.828
.8776	.8784	-.0008	0.000
.5000	.6074	-.1074	2.321
1.0000	1.0000	.0000	0.000
1.0000	1.0000	.0000	0.000
.8980	1.0000	-.1020	$0.78 \cdot 10^{15}$
.8200	.7810	.0390	0.445
.3000	.2806	.0194	0.928

BIVARIATE LOGIT ANALYSIS

Parameter Estimates

$\hat{a}_0 = -1.000$

$\hat{a}_1 = 1.053$

$\hat{B}_1 = -4.027$

$\hat{a}_2 = -0.436$

$\hat{B}_2 = -2.332$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	4	20170
Departure from Model	SSE	10	11.975
TOTAL	SST	14	20182

Coefficient of Determination $R^2 = .99941$

Residual Analysis

p_i	P_i	Residual	Chi-square
1.0000	.9888	.0113	0.580
.9792	.9731	.0061	0.069
.9375	.9201	.0174	0.198
.6250	.6786	-.0536	0.632
.1250	.1146	.0104	0.051
1.0000	.9865	.0135	0.670
.9792	.9779	.0014	0.004
.9808	.9570	.0238	0.717
.8776	.8962	-.0186	0.183
.5000	.6317	-.1317	3.579
1.0000	.9979	.0021	0.102
1.0000	.9869	.0131	0.635
.8980	.9286	-.0306	0.690
.8200	.6976	.1225	3.554
.3000	.2652	.0348	0.311

BIVARIATE BURRIT ANALYSES

1: General Case - Eight Parameters to be Estimated

Parameter Estimates

$\hat{r} = 8.817$

$\hat{b}_1 = 5.045$

$\hat{b}_2 = 5.469$

$\hat{p} = 7.826$

$\hat{a}_1 = 2.869$

$\hat{B}_1 = 0.339$

$\hat{a}_2 = 3.055$

$\hat{B}_2 = 0.184$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	7	67305
Departure from Model	SSE	7	9.6604
TOTAL	SST	14	67314

Coefficient of Determination $R^2 = .99986$

Residual Analysis

p_i	P_i	Residual	Chi-square
1.0000	.9948	.0052	0.266
.9792	.9797	-.0005	0.001
.9375	.9144	.0231	0.327
.6250	.6434	-.0184	0.070
.1250	.1202	.0049	0.011
1.0000	.9932	.0068	0.335
.9792	.9851	.0059	0.113
.9808	.9612	.0196	0.536
.8776	.8850	-.0074	0.026
.5000	.6025	-.1025	2.107
1.0000	.9994	.0006	0.030
1.0000	.9920	.0080	0.385
.8980	.9355	-.0375	1.140
.8200	.6866	.1334	4.135
.3000	.2734	.0266	0.179

2: Takahasi Burr - r = 0 ; Seven Parameters to be Estimated

Parameter Estimates

$\hat{b}_1 = 4.451$

$\hat{b}_2 = 5.250$

$\hat{p} = 7.940$

$\hat{a}_1 = 2.746$

$\hat{B}_1 = 0.346$

$\hat{a}_2 = 2.909$

$\hat{B}_2 = 0.192$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	6	13398
Departure from Model	SSE	8	13.648
TOTAL	SST	14	13412

Coefficient of Determination $R^2 = .99898$

Residual Analysis

p_i	P_i	Residual	Chi-square
1.0000	.9932	.0068	0.349
.9792	.9769	.0023	0.011
.9375	.9139	.0236	0.339
.6250	.6637	-.0387	0.321
.1250	.1385	-.0135	0.073
1.0000	.9953	.0047	0.229
.9792	.9894	-.0102	0.472
.9808	.9708	.0100	0.183
.8776	.9067	-.0291	0.492
.5000	.6402	-.1402	4.095
1.0000	.9854	.0146	0.711
1.0000	.9590	.0410	2.051
.8980	.8865	.0115	0.064
.8200	.6844	.1356	4.255
.3000	.2967	.0033	0.003

3: $\alpha_3 = 0$; $\alpha_4 = 3$ (Third and Fourth Standardized Central Moments) ;

$b_1 = b_2 = 4.874$; $p = 6.158$; Five Parameters to be Estimated

Parameter Estimates

$\hat{r} = 5.265$

$\hat{a}_1 = 2.907$

$\hat{B}_1 = 0.357$

$\hat{a}_2 = 3.215$

$\hat{B}_2 = 0.179$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	4	23681
Departure from Model	SSE	10	12.047
TOTAL	SST	14	23693

Coefficient of Determination $R^2 = .99949$

Residual Analysis

p_i	P_i	Residual	Chi-square
1.0000	.9943	.0057	0.294
.9792	.9808	-.0016	0.006
.9375	.9267	.0108	0.083
.6250	.6934	-.0684	1.056
.1250	.1561	-.0311	0.353
1.0000	.9826	.0174	0.869
.9792	.9697	.0095	0.148
.9808	.9397	.0411	1.549
.8776	.8637	.0139	0.081
.5000	.6229	-.1229	3.086
1.0000	.9980	.0020	0.095
1.0000	.9868	.0132	0.642
.8980	.9297	-.0317	0.755
.8200	.7131	.1069	2.792
.3000	.3326	.0326	0.240

4: Takahasi Burr - r = 0 ; $\alpha_3 = 0$; $\alpha_4 = 3$ (Third and Fourth Standardized Central Moments); $b_1 = b_2 = 4.874$; p = 6.158 ; Four Parameters to be Estimated

Parameter Estimates

$\hat{a}_1 = 2.741$
$\hat{B}_1 = 0.357$
$\hat{a}_2 = 3.171$
$\hat{B}_2 = 0.189$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	3	6862.7
Departure from Model	SSE	11	23.480
TOTAL	SST	14	6886.2

Coefficient of Determination $R^2 = .99659$

Residual Analysis

p_i	P_i	Residual	Chi-square
1.0000	.9857	.0143	0.741
.9792	.9567	.0225	0.586
.9375	.8604	.0771	2.374
.6250	.5544	.0706	0.969
.1250	.0863	.0387	0.911
1.0000	.9914	.0086	0.423
.9792	.9840	-.0048	0.070
.9808	.9648	.0160	0.393
.8776	.9088	-.0312	0.575
.5000	.6934	-.1934	8.444
1.0000	.9773	.0227	1.114
1.0000	.9479	.0521	2.640
.8980	.8801	.0179	0.148
.8200	.7154	.1046	2.685
.3000	.3814	-.0814	1.405

APPENDIX VII

Data of Ashford and Smith (1964): Exposure to dust and prevalence of pneumoconiosis for groups of mine workers. The data as described by Ashford and Smith, the computed p_i , and the analyses on this set of data (data set seven) are in this appendix.

DATA AS PRESENTED BY ASHFORD AND SMITH | COMPUTED DATA

Period Spent (Years) X(1) Coal-Getting	Period Spent (Years) X(2) Haulage	N_i Number of Men	r_i Number Observed With Pneumoconiosis	p_i
2.1	0.5	135	3	.0222
1.9	6.6	18	2	.1111
1.6	12.0	16	1	.0625
1.4	16.9	17	3	.1765
0.7	21.6	14	2	.1429
1.1	27.6	12	3	.2500
1.2	32.4	22	5	.2273
1.5	37.2	31	7	.2258
2.4	41.6	25	5	.2000
1.4	47.1	17	5	.2941
6.6	0.4	80	7	.0875
6.3	6.7	10	1	.1000
7.1	12.0	14	5	.3571
6.4	17.5	8	2	.2500
6.3	21.9	21	11	.5238
6.9	27.2	14	5	.3571
6.2	32.3	13	7	.5385
7.2	37.3	10	7	.7000
12.2	0.2	71	19	.2676
12.0	6.9	8	1	.1250
11.8	11.8	4	2	.5000
11.0	16.7	7	2	.2857
11.5	22.5	6	3	.5000
12.8	29.5	10	6	.6000
12.5	37.8	4	2	.5000
17.0	0.3	106	53	.5000
16.2	6.6	5	2	.4000
16.8	13.2	5	2	.4000
19.5	17.0	6	4	.6667
17.2	21.5	4	1	.2500
21.8	0.2	58	34	.5862
24.7	7.7	3	0	0.0000
26.0	10.8	4	1	.2500
22.0	23.7	3	1	.3333
26.8	0.2	66	43	.6515
27.5	18.2	4	3	.7500
32.5	13.0	2	2	1.0000
31.7	0.2	33	22	.6667
36.8	0.2	20	11	.5500
42.2	1.0	10	8	.8000

BIVARIATE NORMIT ANALYSIS

Parameter Estimates

$\hat{a}_1 = -2.818$

$\hat{B}_1 = 0.937$

$\hat{a}_2 = -2.446$

$\hat{B}_2 = 0.501$

$\hat{\rho} = -0.320$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	4	217.75
Departure from Model	SSE	35	38.141
TOTAL	SST	39	255.89

Coefficient of Determination $R^2 = .85095$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0222	.0195	.0027	0.053
.1111	.0800	.0311	0.236
.0625	.1236	-.0611	0.551
.1765	.1579	.0186	0.044
.1429	.1834	-.0405	0.153
.2500	.2200	.0300	0.063
.2273	.2451	-.0179	0.038
.2258	.2703	-.0445	0.311
.2000	.3030	-.1030	1.256
.2941	.3091	-.0150	0.018
.0875	.1488	-.0613	2.375
.1000	.2025	-.1025	0.651
.3571	.2719	.0852	0.514
.2500	.2886	-.0386	0.058
.5238	.3120	.2118	4.390

Residual Analysis--*Continued*

p_i	P_i	Residual	Chi-square
.3571	.3573	-.0002	0.000
.5385	.3611	.1773	1.772
.7000	.4099	.2901	3.480
.2676	.3185	-.0509	0.847
.1250	.3735	-.2485	2.112
.5000	.4047	.0953	0.151
.2857	.4146	-.1289	0.479
.5000	.4580	.0420	0.043
.6000	.5186	.0815	0.266
.5000	.5417	-.0417	0.028
.5000	.4365	.0635	1.737
.4000	.4717	-.0717	0.103
.4000	.5267	-.1267	0.322
.6667	.5947	.0720	0.129
.2500	.5744	-.3243	1.721
.5862	.5287	.0575	0.771
0.0000	.6260	-.6260	5.022
.2500	.6594	-.4094	2.985
.3333	.6586	-.3253	1.412
.6515	.6047	.0468	0.606
.7500	.7069	.0431	0.036
1.0000	.7376	.2624	0.712
.6667	.6638	.0029	0.001
.5500	.7132	-.1632	2.603
.8000	.7591	.0410	0.092

BIVARIATE LOGIT ANALYSIS

Parameter Estimates

$\hat{a}_0 = -1.000$

$\hat{a}_1 = -2.996$

$\hat{B}_1 = -1.627$

$\hat{a}_2 = -4.527$

$\hat{B}_2 = -1.049$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	4	221.36
Departure from Model	SSE	35	39.020
TOTAL	SST	39	260.38

Coefficient of Determination $R^2 = .85014$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0222	.0291	-.0069	0.226
.1111	.0802	.0309	0.233
.0625	.1211	-.0586	0.516
.1765	.1567	.0197	0.050
.1429	.1828	-.0399	0.150
.2500	.2280	.0220	0.033
.2273	.2592	-.0320	0.117
.2258	.2912	-.0654	0.643
.2000	.3297	-.1297	1.903
.2941	.3416	-.0475	0.170
.0875	.1446	-.0571	2.111
.1000	.1909	-.0909	0.535
.3571	.2575	.0996	0.727
.2500	.2787	-.0287	0.033
.5238	.3063	.2175	4.674

Residual Analysis--*Continued*

p_i	P_i	Residual	Chi-square
.3571	.3564	.0007	0.000
.5385	.3674	.1710	1.636
.7000	.4202	.2796	3.209
.2676	.3106	-.0429	0.612
.1250	.3586	-.2336	1.898
.5000	.3897	.1103	0.205
.2857	.4020	-.1163	0.394
.5000	.4515	.0485	0.057
.6000	.5213	.0787	0.248
.5000	.5533	-.0533	0.046
.5000	.4362	.0638	1.753
.4000	.4631	-.0631	0.080
.4000	.5200	-.1200	0.288
.6667	.5945	.0722	1.298
.2500	.5748	-.3248	1.727
.5862	.5361	.0501	0.586
0.0000	.6290	-.6290	5.086
.2500	.6632	-.4132	3.058
.3333	.6666	-.3333	1.500
.6515	.6178	.0338	0.318
.7500	.7145	.0355	0.025
1.0000	.7447	.2553	0.686
.6667	.6798	-.0132	0.026
.5500	.7302	-.1802	3.295
.8000	.7746	.0254	0.037

BIVARIATE BURRIT ANALYSES

1: General Case - Eight Parameters to be Estimated

Parameter Estimates

$\hat{r} = 8.582$

$\hat{b}_1 = 4.714$

$\hat{b}_2 = 5.933$

$\hat{p} = 7.582$

$\hat{a}_1 = 1.127$

$\hat{B}_1 = 0.147$

$\hat{a}_2 = 3.533$

$\hat{B}_2 = 0.082$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	7	217.30
Departure from Model	SSE	32	38.293
TOTAL	SST	39	255.59

Coefficient of Determination $R^2 = .85018$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0222	.0183	.0039	0.113
.1111	.0734	.0377	0.377
.0625	.1170	-.0545	0.459
.1765	.1528	.0236	0.073
.1429	.1809	-.0381	0.137
.2500	.2208	.0292	0.060
.2273	.2489	-.0217	0.055
.2258	.2774	-.0516	0.412
.2000	.3145	-.1145	1.520
.2941	.3223	-.0282	0.062
.0875	.1500	-.0625	2.448
.1000	.1997	-.0997	0.622
.3571	.2711	.0860	0.524
.2500	.2908	-.0408	0.064
.5238	.3171	.2067	4.142

Residual Analysis--*Continued*

p_i	P_i	Residual	Chi-square
.3571	.3670	-.0099	0.006
.5385	.3737	.1648	1.509
.7000	.4274	.2726	3.037
.2676	.3174	-.0498	0.811
.1250	.3704	-.2454	2.065
.5000	.4050	.0950	0.150
.2857	.4188	-.1331	0.510
.5000	.4677	.0323	0.025
.6000	.5355	.0645	0.167
.5000	.5653	-.0653	0.069
.5000	.4344	.0656	1.856
.4000	.4689	-.0689	0.095
.4000	.5303	-.1303	0.341
.6667	.6034	.0632	0.100
.2500	.5866	-.3366	1.869
.5862	.5275	.0587	0.802
0.0000	.6271	-.6271	5.045
.2500	.6642	-.4142	3.076
.3333	.6754	-.3420	1.601
.6515	.6048	.0467	0.603
.7500	.7200	.0300	0.018
1.0000	.7466	.2534	0.679
.6667	.6653	.0013	0.000
.5500	.7160	-.1660	2.712
.8000	.7619	.0381	0.080

2: Takahasi Burr - r = 0 ; Seven Parameters to be Estimated

Parameter Estimates

$\hat{b}_1 = 5.074$

$\hat{b}_2 = 5.026$

$\hat{p} = 6.351$

$\hat{a}_1 = 1.369$

$\hat{B}_1 = 0.149$

$\hat{a}_2 = 3.850$

$\hat{B}_2 = 0.075$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	6	219.84
Departure from Model	SSE	33	38.765
TOTAL	SST	39	258.60

Coefficient of Determination $R^2 = .85010$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0222	.0222	.0000	0.000
.1111	.0980	.0131	0.035
.0625	.1436	-.0811	0.856
.1765	.1779	-.0014	0.000
.1429	.2034	-.0605	0.317
.2500	.2383	.0117	0.009
.2273	.2622	-.0349	0.139
.2258	.2857	-.0598	0.544
.2000	.3144	-.1144	1.517
.2941	.3224	-.0282	0.062
.0875	.1510	-.0635	2.519
.1000	.2117	-.1117	0.748
.3571	.2752	.0819	0.471
.2500	.2908	-.0408	0.064
.5238	.3117	.2121	4.402

Residual Analysis--*Continued*

p_i	P_i	Residual	Chi-square
.3571	.3507	.0064	0.003
.5385	.3554	.1831	1.903
.7000	.3960	.3040	3.865
.2676	.3165	-.0489	0.784
.1250	.3683	-.2433	2.036
.5000	.3937	.1064	0.190
.2857	.4001	-.1144	0.382
.5000	.4364	.0636	0.099
.6000	.4876	.1125	0.506
.5000	.5059	-.0059	0.001
.5000	.4342	.0658	1.867
.4000	.4615	-.0615	0.076
.4000	.5057	-.1057	0.223
.6667	.5663	.1004	0.246
.2500	.5428	-.2928	1.382
.5862	.5276	.0586	0.798
0.0000	.6110	-.6110	4.712
.2500	.6388	-.3888	2.620
.3333	.6218	-.2885	1.062
.6515	.6056	.0460	0.584
.7500	.6760	.0740	0.100
1.0000	.7149	.2851	0.798
.6667	.6666	.0001	0.000
.5500	.7176	-.1676	2.773
.8000	.7632	.0368	0.075

3: $\alpha_3 = 0$; $\alpha_4 = 3$ (Third and Fourth Standardized Central Moments) ;

$b_1 = b_2 = 4.874$; $p = 6.158$; Five Parameters to be Estimated

Parameter Estimates

$\hat{r} = 4.742$

$\hat{a}_1 = 1.259$

$\hat{B}_1 = 0.151$

$\hat{a}_2 = 3.405$

$\hat{B}_2 = 0.078$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	4	218.53
Departure from Model	SSE	35	38.338
TOTAL	SST	39	256.87

Coefficient of Determination $R^2 = .85075$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0222	.0211	.0011	0.008
.1111	.0905	.0206	0.093
.0625	.1340	-.0715	0.704
.1765	.1670	.0094	0.011
.1429	.1913	-.0485	0.212
.2500	.2261	.0239	0.039
.2273	.2499	-.0226	0.060
.2258	.2740	-.0481	0.361
.2000	.3061	-.1061	1.326
.2941	.3103	-.0162	0.021
.0875	.1528	-.0653	2.634
.1000	.2139	-.1139	0.772
.3571	.2820	.0752	0.391
.2500	.2978	-.0478	0.087
.5238	.3199	.2039	4.013

Residual Analysis--*Continued*

p_i	P_i	Residual	Chi-square
.3571	.3631	-.0060	0.002
.5385	.3660	.1725	1.666
.7000	.4127	.2873	3.405
.2676	.3187	-.0511	0.854
.1250	.3793	-.2543	2.198
.5000	.4103	.0897	0.133
.2857	.4198	-.1341	0.517
.5000	.4615	.0385	0.036
.6000	.5201	.0800	0.256
.5000	.5420	-.0419	0.028
.5000	.4353	.0647	1.806
.4000	.4749	-.0749	0.112
.4000	.5294	-.1294	0.336
.6667	.5964	.0703	0.123
.2500	.5760	-.3260	1.740
.5862	.5273	.0589	0.807
0.0000	.6278	-.6278	5.059
.2500	.6612	-.4112	3.018
.3333	.6598	-.3265	1.425
.6515	.6039	.0477	0.626
.7500	.7086	.0414	0.033
1.0000	.7396	.2604	0.704
.6667	.6638	.0029	0.001
.5500	.7139	-.1639	2.632
.8000	.7605	.0395	0.086

4: Takahasi Burr - $r = 0$; $\alpha_3 = 0$; $\alpha_4 = 3$ (Third and Fourth Standardized Central Moments); $b_1 = b_2 = 4.874$; $p = 6.158$; Four Parameters to be Estimated

Parameter Estimates

$\hat{a}_1 = 1.313$
$\hat{B}_1 = 0.150$
$\hat{a}_2 = 3.783$
$\hat{B}_2 = 0.074$

Chi-square Analysis Table

source		d.f.	
Due to Model	SSR	3	219.06
Departure from Model	SSE	36	38.766
TOTAL	SST	39	257.83

Coefficient of Determination $R^2 = .84964$

Residual Analysis

p_i	P_i	Residual	Chi-square
.0222	.0240	-.0018	0.018
.1111	.1007	.0104	0.021
.0625	.1456	-.0831	0.888
.1765	.1791	-.0026	0.001
.1429	.2037	-.0608	0.319
.2500	.2380	.0120	0.010
.2273	.2613	-.0341	0.132
.2258	.2845	-.0587	0.524
.2000	.3135	-.1135	1.496
.2941	.3200	-.0259	0.052
.0875	.1566	-.0691	2.894
.1000	.2179	-.1179	0.815
.3571	.2809	.0763	0.403
.2500	.2956	-.0455	0.080
.5238	.3159	.2080	4.202

Residual Analysis--*Continued*

p_i	P_i	Residual	Chi-square
.3571	.3542	.0029	0.001
.5385	.3581	.1804	1.841
.7000	.3983	.3017	3.799
.2676	.3210	-.0534	0.930
.1250	.3735	-.2485	2.110
.5000	.3984	.1016	0.172
.2857	.4045	-.1188	0.410
.5000	.4399	.0601	0.088
.6000	.4899	.1101	0.485
.5000	.5075	-.0075	0.001
.5000	.4363	.0637	1.750
.4000	.4645	-.0645	0.084
.4000	.5079	-.1079	0.233
.6667	.5669	.0997	0.243
.2500	.5441	-.2941	1.395
.5862	.5271	.0591	0.814
0.0000	.6096	-.6096	4.685
.2500	.6369	-.3869	2.588
.3333	.6209	-.2876	1.054
.6515	.6027	.0489	0.658
.7500	.6734	.0766	0.107
1.0000	.7108	.2892	0.814
.6667	.6619	.0048	0.003
.5500	.7115	-.1615	2.542
.8000	.7563	.0437	0.104

APPENDIX VIII

Listings of FORTRAN subroutines used in evaluating the functions, partial derivatives, and the weights are in this appendix.

1. Bivariate Normal Subroutines

```
      DOUBLE PRECISION FUNCTION F(X,A)
C     BIVARIATE NORMAL FUNCTION
      DIMENSION X(1),A(1)
      DOUBLE PRECISION X,A,AA,B,R,S,GOFU,THA ,C,D
      AA=A(1)+A(2)*X(1)
      B=A(3)+A(4)*X(2)
      IF (DABS(A(5)).LE.0.9999D+00) GO TO 16
      A(5)=DSIGN(0.9999D+00,A(5))
   16 R=A(5)
      S=DSQRT(1.-R*R)
      C=GOFU(AA)
      D=GOFU(B)
      F=C+D-THA(AA,B/AA)-THA(B,AA/B)
     1+THA(AA,(B-R*AA)/(AA*S))+THA(B,(AA-R*B)/(B*S))-C*D
      RETURN
      END
```

```
      SUBROUTINE PD(X,A,FXA,P)
      DIMENSION X(1),A(1),P(1)
      DOUBLE PRECISION X,A,AA,R,S,B,FXA,P,WATE,GOFU,GPRIME,C,D
C     A1=ALPHA1,A2=BETA1,A3=ALPHA2,A4=BETA2,A5=RHO
C     BIVARIATE NORMAL PARTIALS
      AA=A(1)+A(2)*X(1)
      B=A(3)+A(4)*X(2)
      R=A(5)
      S=DSQRT(1.-R*R)
      C=GPRIME(AA)
      D=(B-R*AA)/S
      P(1)=C*(1.-GOFU(D))
      P(2)=X(1)*P(1)
      P(3)=GPRIME(B)*(1.-GOFU((AA-R*B)/S))
      P(4)=X(2)*P(3)
      P(5)=-C*GPRIME(D)/S
      RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION GPRIME(X)
      DOUBLE PRECISION X,A1,A2,A3
      A1 =X*X*(-0.5D 0)
      IF (A1.LE.-150) GO TO 15
      A2=DEXP(A1)
      A3 = .398942280401433D+00
      GPRIME = A2*A3
      GO TO 16
   15 GPRIME=0
   16 RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION GOFU(U)
      DIMENSION Y(160)
      DOUBLE PRECISION U,GU,X,XSET,DELTA,GPR,DELK,DELTAX,Y,SUM,TOP
      IF(Y(17)-.69146246127D+00)2,5,2
C
 2    Y(  1) =       .50000000000D+00
      Y(  2) =       .39894228040D+00
      Y(  3) =       .00000000000D+00
      Y(  4) =      -.66490380066D-01
      Y(  5) =       .00000000000D+00
      Y(  6) =       .99735570100D-02
      Y(  7) =       .00000000000D+00
      Y(  8) =      -.11873282155D-02
      Y(  9) =       .59870632568D+00
      Y( 10) =       .38666811680D+00
      Y( 11) =      -.48333514600D-01
      Y( 12) =      -.60416893250D-01
      Y( 13) =       .11831641595D-01
      Y( 14) =       .84709519078D-02
      Y( 15) =      -.19305085421D-02
      Y( 16) =      -.93949992204D-03
      Y( 17) =       .69146246127D+00
      Y( 18) =       .35206532676D+00
      Y( 19) =      -.88016331691D-01
      Y( 20) =      -.44008165845D-01
      Y( 21) =       .20170409346D-01
      Y( 22) =       .45841839423D-02
      Y( 23) =      -.30714032413D-02
      Y( 24) =      -.32635023779D-03
      Y( 25) =       .77337264763D+00
      Y( 26) =       .30113743216D+00
      Y( 27) =      -.11292653706D+00
      Y( 28) =      -.21957937761D-01
      Y( 29) =       .22938202840D-01
      Y( 30) =      -.14703976180D-03
```

```
Y( 31) =    -.30400470751D-02
Y( 32) =     .34322406302D-03
Y( 33) =     .84134474606D+00
Y( 34) =     .24197072452D+00
Y( 35) =    -.12098536226D+00
Y( 36) =     .00000000000D+00
Y( 37) =     .20164227043D-01
Y( 38) =    -.40328454087D-02
Y( 39) =    -.20164227043D-02
Y( 40) =     .76816103022D-03
Y( 41) =     .89435022633D+00
Y( 42) =     .18264908539D+00
Y( 43) =    -.11415567837D+00
Y( 44) =     .17123351755D-01
Y( 45) =     .13674898971D-01
Y( 46) =    -.59872275061D-02
Y( 47) =    -.57598079906D-03
Y( 48) =     .81561889341D-03
Y( 49) =     .93319279874D+00
Y( 50) =     .12951759567D+00
Y( 51) =    -.97138196750D-01
Y( 52) =     .26982832430D-01
Y( 53) =     .60711372969D-02
Y( 54) =    -.58687660536D-02
Y( 55) =     .65770654049D-03
Y( 56) =     .55772550961D-03
Y( 57) =     .95994084314D+00
Y( 58) =     .86277318827D-01
Y( 59) =    -.75492653974D-01
Y( 60) =     .29657828346D-01
Y( 61) =    -.39319090611D-03
Y( 62) =    -.43110574348D-02
Y( 63) =     .13098172060D-02
Y( 64) =     .18576682170D-03
Y( 65) =     .97724986805D+00
Y( 66) =     .53990966513D-01
Y( 67) =    -.53990966513D-01
Y( 68) =     .26995483257D-01
Y( 69) =    -.44992472094D-02
Y( 70) =    -.22496236047D-02
Y( 71) =     .13497741628D-02
Y( 72) =    -.11783742691D-03
Y( 73) =     .98777552735D+00
Y( 74) =     .31739651836D-01
Y( 75) =    -.35707108315D-01
Y( 76) =     .21490389264D-01
Y( 77) =    -.61371592417D-02
Y( 78) =    -.46183673081D-03
Y( 79) =     .99147667294D-03
Y( 80) =    -.26370836740D-03
```

```
Y( 81) =    .99379033467D+00
Y( 82) =    .17528300494D-01
Y( 83) =   -.21910375617D-01
Y( 84) =    .15337262932D-01
Y( 85) =   -.59340600629D-02
Y( 86) =    .66644059169D-03
Y( 87) =    .51352442853D-03
Y( 88) =   -.26273974729D-03
Y( 89) =    .99702023677D+00
Y( 90) =    .90935625017D-02
Y( 91) =   -.12503648440D-01
Y( 92) =    .99460839861D-02
Y( 93) =   -.47539913338D-02
Y( 94) =    .11227826357D-02
Y( 95) =    .11925680315D-03
Y( 96) =   -.18051548644D-03
Y( 97) =    .99865010197D+00
Y( 98) =    .44318484120D-02
Y( 99) =   -.66477726180D-02
Y(100) =    .59091312160D-02
Y(101) =   -.33238863090D-02
Y(102) =    .11079621030D-02
Y(103) =   -.11079621030D-03
Y(104) =   -.84416160226D-04
Y(105) =    .99942297496D+00
Y(106) =    .20290480573D-02
Y(107) =   -.32972030931D-02
Y(108) =    .32337953413D-02
Y(109) =   -.20779248659D-02
Y(110) =    .86558186169D-03
Y(111) =   -.19180019295D-03
Y(112) =   -.13995370141D-04
Y(113) =    .99976737091D+00
Y(114) =    .87268269505D-03
Y(115) =   -.15271947163D-02
Y(116) =    .16362800532D-02
Y(117) =   -.11772125938D-02
Y(118) =    .57860680771D-03
Y(119) =   -.18055895865D-03
Y(120) =    .21397716502D-04
Y(121) =    .99991158271D+00
Y(122) =    .35259568237D-03
Y(123) =   -.66111690444D-03
Y(124) =    .76763018349D-03
Y(125) =   -.60946714628D-03
Y(126) =    .34195583218D-03
Y(127) =   -.13246010895D-03
Y(128) =    .30251745009D-04
Y(129) =    .99996832876D+00
Y(130) =    .13383022576D-03
```

```
      Y(131) =     -.26766045153D-03
      Y(132) =      .33457556441D-03
      Y(133) =     -.28996548916D-03
      Y(134) =      .18178605667D-03
      Y(135) =     -.82528639221D-04
      Y(136) =      .25518025191D-04
      Y(137) =      .99998931147D+00
      Y(138) =      .47718636540D-04
      Y(139) =     -.10140210265D-03
      Y(140) =      .13569987266D-03
      Y(141) =     -.12728076426D-03
      Y(142) =      .87833668723D-04
      Y(143) =     -.45244746779D-04
      Y(144) =      .17013635696D-04
      Y(145) =      .99999660233D+00
      Y(146) =      .15983741107D-04
      Y(147) =     -.35963417490D-04
      Y(148) =      .51281169385D-04
      Y(149) =     -.51697412642D-04
      Y(150) =      .38835495972D-04
      Y(151) =     -.22233633626D-04
      Y(152) =      .96697768581D-05
      Y(153) =      .99999898292D+00
      Y(154) =      .50295072886D-05
      Y(155) =     -.11945079810D-04
      Y(156) =      .18074791818D-04
      Y(157) =     -.19472968649D-04
      Y(158) =      .15788101444D-04
      Y(159) =     -.99025178234D-05
      Y(160) =      .48400297796D-05
C
    5 IF(U) 11,10,12
   10 GU=.5D+00
      GO TO 100
   11 X=DABS(U)
      GO TO 13
   12 X=U
   13 IF(X-7.0D+00)15,14,14
   14 IF(U)141,10,142
  141 GU=0.0D+00
      GO TO 100
  142 GU=1.0D+00
  100 GOFU=GU
      RETURN
   15 IF(X-4.87499D+00)16,16,40
   16 XSET=X*4.0D+00
      XSST=XSET+.5D+00
      I=IFIX(XSST)
      XSET=DFLOAT(I)
      DELTA=X-(XSET*.25D+00)
  201 K=I*8+1
      I=K+7
      SUM=0.0D+00
```

```
      DO 20 J=K,I,1
      L=K+I-J
      SUM=SUM*DELTA
      SUM=Y(L)+SUM
   20 CONTINUE
      IF(U) 21,10,22
   21 GU=1.0D+00 -SUM
      GO TO 100
   22 GU=SUM
      GO TO 100
   40 XSET=-X*X
      IF (XSET.LE.-300) GO TO 101
      GPR=(DEXP(XSET*.5D+00))*.398942280401433D+00
      GO TO 102
  101 GPR=0
  102 DELTA=1.0D+00/X
      SUM=DELTA
      TOP=1.0D+00
   41 DELK=TOP/XSET
      DELTAX=DELTA*DELK
      IF (DABS(DELTA)-DABS(DELTAX))45,45,43
   43 DELTA=DELTAX
      SUM=SUM+DELTA
      IF(DABS(GPR*DELTA)-.5D-9)45,45,42
   42 TOP=TOP+2.0D+00
      GO TO 41
   45 SUM=GPR*SUM
      IF(U) 22,10,21
      END
```

```
      DOUBLE PRECISION FUNCTION THA(HX,AX)
      DOUBLE PRECISION HX,AX,AA,U,H,A,SUM,C,DA,TA,TX,X,Y,Z ,GOFU
      DIMENSION AA(9),U(9)
      DATA AA(1),AA(9),AA(2),AA(8),AA(3),AA(7),AA(4),AA(6),AA(5)/2*.4063
     17194181E-1,2*.90324080347E-1,2*.1303053482,2*.15617353852,.1651196
     2775/,U(1),U(2),U(3),U(4),U(5),U(6),U(7),U(8),U(9)/.15919880246E-1,
     3.81984446337E-1,.19331428365,.3378732883,.5,.6621267117,.806685716
     435,.91801555366,.98408011975/
      H=HX
      A=AX
      IF(DABS(H).LE.5.77)GO TO 10
   11 THA=0.
      RETURN
   10 IF (DABS(A).LE.1.)GO TO 13
   12 H=A*H
      IF (DABS(H).LE.5.77)GO TO 15
      GO TO 16
   15 A=1./A
```

```
13 SUM=0.
   DO 61 M=1,9,1
   DA=1.+A**2*U(M)**2
   C=-.5*H**2*DA
   TA=DEXP (C)/DA * AA(M)
   SUM=SUM+TA
61 CONTINUE
   TX=A/6.2831853072*SUM
   GO TO 17
16 TX=0.
17 IF (DABS (AX).LE.1.)GO TO 20
14 X=GOFU(HX)
   Y=GOFU(H)
   Z=X*Y
   TX=.5*X+.5*Y-Z-TX
18 IF(AX)21,20,20
21 TX=TX-.5D+00
20 THA=TX
   RETURN
   END
```

2. Bivariate Logistic Subroutines

```
      DOUBLE PRECISION FUNCTION F(X,A)
C     BIVARIATE LOGISTIC (GUMBEL) FUNCTION
      DIMENSION X(1),A(1)
      DOUBLE PRECISION X,A,C,CC,B,BB,D,E,BC,DE
      IF (DABS(A(1)).GE.1.0D+00) A(1)=DSIGN(1.0D+00,A(1))
      C=A(3)*(X(1)+A(2))
      IF (DABS(C).GT.150.0D+00) C=DSIGN(150.0D+00,C)
      CC=DEXP(C)
      B=A(5)*(X(2)+A(4))
      IF (DABS(B).GT.150.0D+00) B=DSIGN(150.0D+00,B)
      BB=DEXP(B)
      D=1./(1.+CC)
      E=1./(1.+BB)
      BC=BB*CC
      DE=D*E
      F=D+E-DE*(1.+A(1)*BC*DE)
      RETURN
      END
```

```
      SUBROUTINE PD(X,A,FXA,P)
C     BIVARIATE LOGISTIC (GUMBEL) PARTIALS
C     A(1)=ALPHA 0,A(2)=ALPHA1,A(3)=BETA1,A(4)=ALPHA2,A(5)=BETA2
      DIMENSION X(1),A(1),P(1)
      DOUBLE PRECISION X,A,P,FXA,WATE,C,CC,B,BB,D,DD,E,EE,BC,DE,R,S,T,Z
      C=A(3)*(X(1)+A(2))
      IF (DABS(C).GT.150.0D+00) C=DSIGN(150.0D+00,C)
      CC=DEXP(C)
      B=A(5)*(X(2)+A(4))
      IF (DABS(B).GT.150.0D+00) B=DSIGN(150.0D+00,B)
      BB=DEXP(B)
      D=1./(1.+CC)
      DD=D*D
      E=1./(1.+BB)
      EE=E*E
      BC=BB*CC
      DE=D*E
      Z=1.+A(1)*BC*DE
      P(1)=-DD*EE*BC
      T=A(1)*P(1)
      R=DD*CC*(E*Z-1.)+T*(1.-D*CC)
      P(2)=A(3)*R
      P(3)=(X(1)+A(2))*R
      S=EE*BB*(D*Z-1.)+T*(1.-E*BB)
      P(4)=A(5)*S
      P(5)=(X(2)+A(4))*S
      RETURN
      END
```

3. Bivariate Burr Subroutines

```
      DOUBLE PRECISION FUNCTION F(X,A)
C     BIVARIATE BURR R (FD) FUNCTION
C     A1=R,A2=B1,A3=B2,A4=P,A5=ALPHA1,A6=BETA1,A7=ALPHA2,A8=BETA2
      DOUBLE PRECISION X,A,B,BB,C,CC,D,DD
      DIMENSION X(1),A(1)
      IF(A(1).GT.A(4)+1.0D+00) A(1)=A(4)+1.0D+00
      IF(A(1).LE.0.0D+00) A(1)=0.0
      B=A(6)*(X(1)+A(5))
      IF (B.GT.0.0D+00) GO TO 17
      BB=0
      GO TO 18
   17 IF (A(2)*DLOG(B).LT.150.0) GO TO 19
      BB=DEXP(150.0D+00)
      GO TO 18
   19 IF (A(2))13,14,15
   13 BB=1./(B**DABS(A(2)))
      GO TO 18
   14 BB=1.0
      GO TO 18
   15 BB=B**A(2)
   18 C=A(8)*(X(2)+A(7))
      IF (C.GT.0.0D+00) GO TO 16
      CC=0
      GO TO 10
   16  IF(A(3)*DLOG(C).LT.150.0) GO TO 20
      CC=DEXP(150.0D+00)
      GO TO 10
   20 IF (A(3))9,11,12
    9 CC=1./(C**DABS(A(3)))
      GO TO 10
   11 CC=1.0
      GO TO 10
   12 CC=C**A(3)
   10 D=(1.0+BB+CC+A(1)*BB*CC)
      IF (A(4)*DLOG(D).LT.150.0) GO TO 21
      DD=DEXP(-150.0D+00)
      GO TO 22
   21 DD=1.0/(D**A(4))
   22 F=1.0D+00-DD
      RETURN
      END
```

```
      SUBROUTINE PD(X,A,FXA,P)
C     BIVARIATE BURR R PARTIALS
C     A1=R,A2=B1,A3=B2,A4=P,A5=ALPHA1,A6=BETA1,A7=ALPHA2,A8=BETA2
      DIMENSION X(1),A(1),P(1)
      DOUBLE PRECISION X,A,FXA,P,WATE,B,BB,C,CC,D,DD,E,EE,G,GG,H,HH,I
      IF(A(1).GT.A(4)+1.0D+00) A(1)=A(4)+1.0D+00
      IF(A(1).LE.0.0D+00) A(1)=0.0
      B=A(6)*(X(1)+A(5))
      IF (B.GT.0.0D+00) GO TO 17
      BB=0
      GO TO 18
   17 IF (A(2)*DLOG(B).LT.150.0) GO TO 30
      BB=DEXP(150.0D+00)
      GO TO 18
   30 IF (A(2))13,14,15
   13 BB=1./(B**DABS(A(2)))
      GO TO 18
   14 BB=1.0
      GO TO 18
   15 BB=B**A(2)
   18 C=A(8)*(X(2)+A(7))
      IF (C.GT.0.0D+00) GO TO 16
      CC=0
      GO TO 10
   16  IF(A(3)*DLOG(C).LT.150.0) GO TO 31
      CC=DEXP(150.0D+00)
      GO TO 10
   31 IF (A(3))9,11,12
    9 CC=1./(C**DABS(A(3)))
      GO TO 10
   11 CC=1.0
      GO TO 10
   12 CC=C**A(3)
   10 D=(1.0+BB+CC+A(1)*BB*CC)
      IF((A(4)+1.0)*DLOG(D).LT.150.0) GO TO 32
      DD=A(4)*DEXP(-150.0D+00)
      GO TO 33
   32 DD=A(4)*((1.0/D)**(A(4)+1.0))
   33 E=DD*BB
      G=1.0+A(1)*BB
      GG=1.0+A(1)*CC
      H=DD*A(2)*GG
      HH=DD*A(3)*G
      P(1)=E*CC
      IF (B.GT.0.0D+00) GO TO 1
      P(2)=0
      GO TO 2
    1 P(2)=E*DLOG(B)*GG
    2 IF (C.GT.0.0D+00) GO TO 3
      P(3)=0
      GO TO 4
    3 P(3)=DD*CC*DLOG(C)*G
    4 IF (A(4)*DLOG(D).LT.150.0) GO TO 34
      P(4)=(DLOG(D))*(DEXP(-150.0D+00))
```

```
      GO TO 35
   34 P(4)=DLOG(D)/(D**A(4))
   35 IF (B.GT.0.0D+00) GO TO 23
      P(5)=0.0
      P(6)=0.0
      GO TO 24
   23 IF ((A(2)-1.0)*DLOG(B).LT.150.0) GO TO 36
      EE=H*DEXP(150.0D+00)
      GO TO 8
   36 IF (A(2)-1.0)5,6,7
    5 EE=H/(B**DABS(A(2)-1.0))
      GO TO 8
    6 EE=H
    7 EE=H*(B**(A(2)-1.0))
    8 P(5)=EE*A(6)
      P(6)=EE*X(1)
   24 IF (C.GT.0.0D+00) GO TO 25
      P(7)=0.0
      P(8)=0.0
      GO TO 26
   25 IF ((A(3)-1.0)*DLOG(C).LT.150.0) GO TO 37
      I=HH*DEXP(150.0D+00)
      GO TO 22
   37 IF (A(3)-1.0)19,20,21
   19 I=HH/(C**DABS(A(3)-1.0))
      GO TO 22
   20 I=HH
      GO TO 22
   21 I=HH*(C**(A(3)-1.0))
   22 P(7)=I*A(8)
      P(8)=I*X(2)
   26 CONTINUE
      RETURN
      END
```

4. Weight Subroutine

```
DOUBLEPRECISIONFUNCTIONWATE(X,FXA)
DOUBLEPRECISIONX,FXA
DIMENSIONX(1)
WATE=DABS(X(3)/((1.0D+00-FXA)*FXA))
RETURN
END
```

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13. ABSTRACT

The problem of a mixture of two stimulants in a biological quantal assay is investigated from a mathematical standpoint. The basic assumption is made that the response region does not depend on biological considerations - i.e., given a specified mixture of stimulants $\underline{z}$, the response region is defined by the point $\underline{z}'$ in the p-variate space where there are p stimulants under consideration; instead, the probability functions, themselves, may take on different forms. A general form is proposed and investigated. Three analytic models (one utilizing the bivariate normal distribution, one a bivariate logistic distribution developed by Gumbel (1961), and one a bivariate Burr distribution developed by this author) are employed in this investigation. The investigation includes the analysis of data, under the three analytic models, which had been classified by previous investigators as examples of synergistic action, simple similar action, independent action, and additive action. The residual analyses are included as well as the FORTRAN IV subroutines used in evaluating the functions, the partial derivatives and the weights.

The investigation lends some support to the assumption of a constant response region for a diversity of mixtures of stimulants. The analytic model incorporating the bivariate Burr distribution is recommended for all cases unless the number of parameters to be estimated is a primary concern, in which case the analytic model utilizing the bivariate normal distribution is recommended. The bivariate Burr distribution developed in this paper is found to be more useful in application than that developed by Takahasi (1965).

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