BIVARIATE CUMULANTS OF A SINGLY TRUNCATED BIVARIATE NORMAL DISTRIBUTION

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SUMMARY.

A method of obtaining the bivariate cumulants of any order is given for a truncated bivariate normal distribution where one of the variates is truncated at w_o. Some representative values are displayed in tables.

Some key words: Bivariate cumulant; Singly truncated bivariate normal distribution.

1. INTRODUCTION

Let the joint density of a standardized bivariate normal distribution be given by

$$\phi(x,y;\rho) = (2\pi)^{-1}(1-\rho^2)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}(x^2 - 2\rho xy + y^2)/(1-\rho^2)\}$$

for $-\infty < x,y < +\infty$.

We will also use the notation

$$G'(x) = (2\pi)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}x^2\},$$

and $$G(x) = \int_{-\infty}^{x} G'(t)\,dt$$

for $-\infty < x < +\infty$,

for the standardized univariate density and cumulative, respectively.

Then if the X variate is truncated below w_o, the joint density of the singly truncated bivariate normal distribution (STBVND) is given by

$$f(x,y;\rho) = \phi(x,y;\rho)/G(-w_o) \quad \text{for} \quad w_o < x < +\infty, \; -\infty < y < +\infty.$$

Our purpose is to obtain the bivariate cumulants corresponding to $f(x,y;\rho)$.

2. BIVARIATE CUMULANTS

Cook (1951) illustrated three methods of deriving bivariate cumulants. Cumulants of all orders of the bivariate distribution may be worked out by choosing the appropriate operation. She gave all the formulae for bivariate cumulants, κ_{ij}, up to $i + j = 6$. As the order of the cumulants increases, the number of terms increases greatly. Johnson and Kotz (1972) gave bivariate cumulants only up to $i + j = 2$. Gajjar and Subrahmaniam (1978) obtained bivariate moments up to order 4. We give here a general formula for bivariate cumulants for any order.

The moment generating function of a STBVND is given by

$$M(t_1,t_2) = [G(-w_o)]^{-1} G(t_1 + \rho t_2 - w_o) \exp[\tfrac{1}{2}(t_1^2 + 2\rho t_1 t_2 + t_2^2)].$$

where t_1 corresponds to x and t_2 corresponds to y. Since the cumulant generating function is $\ln[M(t_1,t_2)]$, we have the following expression for the cumulant generating function $K(t_1,t_2)$

$$K(t_1,t_2) = -\ln[G(-w_o)] + \ln[G(t_1 + \rho t_2 - w_o)] + t_1^2/2 + \rho t_1 t_2 + t_2^2/2$$

and the cumulant κ_{ij} is obtained by taking the i-th partial derivative with respect to t_1 and the j-th partial derivative with respect to t_2 and setting $t_1 = t_2 = 0$.

We obtain

$$\kappa_{10} = G'(w_o)/[G(-w_o)]$$

$$\kappa_{01} = \rho\kappa_{10}$$

$$\kappa_{20} = w_o G'(w_o)/[G(-w_o)] - [G'(w_o)/G(-w_o)]^2 + 1$$

$$= w_o\kappa_{10} - \kappa_{10}^2 + 1$$

$$\kappa_{02} = \rho^2(\kappa_{20}-1) + 1$$

and

$$\kappa_{ij} = \rho^j \left. \frac{\partial^{i+j-1}}{\partial x^{i+j-1}} \left(\frac{G'(x)}{G(x)}\right) \right|_{x = -w_o} \qquad \text{for } i + j \neq 2$$

The expressions for $\frac{\partial^i}{\partial x^i}\left(\frac{G'(x)}{G(x)}\right)$ become very cumbersome as is illustrated by the following.

$$\frac{\partial}{\partial x}\left(\frac{G'(x)}{G(x)}\right) = -x\left(\frac{G'(x)}{G(x)}\right) - \left(\frac{G'(x)}{G(x)}\right)^2 ,$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{G'(x)}{G(x)}\right) = (x^2-1)\left(\frac{G'(x)}{G(x)}\right) + 3x\left(\frac{G'(x)}{G(x)}\right)^2 + 2\left(\frac{G'(x)}{G(x)}\right)^3 ,$$

$$\frac{\partial^3}{\partial x^3}\left(\frac{G'(x)}{G(x)}\right) = (-x^3 + 3x)\left(\frac{G'(x)}{G(x)}\right) - (7x^2 - 4)\left(\frac{G'(x)}{G(x)}\right)^2 - 12x\left(\frac{G'(x)}{G(x)}\right)^3 - 6\left(\frac{G'(x)}{G(x)}\right)^4 .$$

Hence, we look for a way to generate these derivatives recursively. We let $g = g(x) = G'(x)/[G(x)]$ and $h = 1/g = G(x)/[G'(x)]$ and note that the derivatives of h are easily obtained as

$$h' = 1 + xh$$

$$h^{(n)} = (n-1)h^{(n-2)} + xh^{(n-1)} \quad \text{for } n \geqslant 2, \text{ where } h^{(0)} \equiv h.$$

Since gh = 1, an application of Liebniz's rule gives

$$\sum_{j=0}^{n} \binom{n}{j} g^{(n-j)} h^{(j)} = 0$$

or

$$g^{(n)} = -g \sum_{j=1}^{n} \binom{n}{j} g^{(n-j)} h^{(j)} .$$

Hence, we can obtain any derivative of $G'(x)/G(x)$ in terms of lower order derivatives and derivatives of h which are given by the recursion formula for $h^{(n)}$.

3. APPLICATION

In the problem of screening based on a singly truncated bivariate normal distribution, one needs to know the distribution of the sample correlation coefficient. This can be achieved by supplying the bivariate cumulants given above to Gayen's (1951) results.

4. TABLE

In the accompanying table we give values of κ_{10}, κ_{20}, κ_{30}, κ_{40} and κ_{50}. These cumulants are independent of ρ. However, the remaining cumulants up to order 5 may be obtained from these, using the following simple formulas which were obtained from the general formula for κ_{ij} given above:

$$\kappa_{01} = \rho\kappa_{10}$$

$$\kappa_{02} = \rho^2(\kappa_{20} - 1) + 1$$

$$\kappa_{11} = \rho\, \kappa_{20}$$

$$\kappa_{ij} = \rho^j \kappa_{i+j,0} \quad \text{for} \quad i + j > 2$$

TABLE 1

BIVARIATE CUMULANTS OF A SINGLY TRUNCATED BIVARIATE NORMAL DISTRIBUTION

w_o	κ_{10}	κ_{20}	κ_{30}	κ_{40}	κ_{50}
-3.0	.004438	.986667	.035680	-.081046	.139672
-2.8	.007936	.977717	.054810	-.110766	.154800
-2.6	.013647	.964333	.080062	-.141556	.148857
-2.4	.022580	.945299	.111173	-.168418	.114778
-2.2	.035975	.919561	.146778	-.185535	.052033
-2.0	.055248	.886452	.184395	-.187855	-.031092
-1.8	.081893	.845887	.220752	-.172786	-.118818
-1.6	.117352	.798466	.252404	-.141250	-.192852
-1.4	.162881	.745436	.276436	-.097540	-.238752
-1.2	.219437	.688524	.291034	-.048050	-.250453
-1.0	.287600	.629686	.295718	.000547	-.230966
- .8	.367562	.570849	.291238	.042875	-.189622
- .6	.459147	.513695	.279206	.075707	-.137889
- .4	.561883	.459534	.261660	.098016	-.085824
- .2	.675073	.409261	.240659	.110477	-.040233
.0	.797885	.363380	.218015	.114769	-.004433
.2	.929416	.322069	.195158	.112946	.020990
.4	1.068757	.285262	.173110	.106994	.037137
.6	1.215026	.252727	.152523	.098587	.045869
.8	1.367403	.224132	.133752	.089010	.049170
1.0	1.525136	.199096	.116935	.079166	.048804
1.2	1.687553	.177229	.102061	.069643	.046173
1.4	1.854058	.158150	.089030	.060786	.042308
1.6	2.024130	.141506	.077687	.052765	.037920
1.8	2.197314	.126976	.067859	.045637	.033466
2.0	2.373217	.114276	.059367	.039387	.029222
2.2	2.551498	.103155	.052040	.033956	.025338
2.4	2.731863	.093396	.045722	.029266	.021881
2.6	2.914059	.084813	.040273	.025234	.018866
2.8	3.097868	.077244	.035568	.021774	.016280
3.0	3.283101	.070551	.031501	.018808	.014096

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