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PROBABILITY OF LOCATING A SUBMARINE WITHIN A STATED DISTANCE ON THE BASIS OF TWO DIRECTIONAL SENSORS

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ABSTRACT

The general problem is to estimate the surface position under which a submarine is located. The information for this estimation is provided by two directional sensors whose locations are known. The observed directions, in combination with the sensor locations, are combined to yield the estimated position. The specific problem is to approximately determine the probability that the true submarine position is within a stated distance (on the ocean surface) of the estimated position. This article identifies the parameters involved and, in terms of these parameters, develops an approximate expression for the probability value.

Introduction

The general problem is to estimate the surface position (on the ocean) under which a submarine is located. Here, all positions considered are on the ocean surface and, for brevity, the word "surface" is deleted in referring to positions or location of the submarine or sensors.

The information for estimating the submarine position is provided by two directional sensors whose locations are known. That is, at a single fixed time, each sensor furnishes an observed direction (along the ocean surface) for the submarine location. These observed directions, in combination with the known locations of the sensors, yield an estimated location for the submarine at the time considered.

Let a circle with given radius be centered at the estimated submarine location. The specific problem is to approximately determine the probability that, at the time considered, the true submarine location is contained in this circle. This probability depends on the locations of the sensors, the radius of the circle, and on the probability distributions of the angular errors for the sensors. The purpose of this article is to identify the parameters involved

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and, in terms of these parameters, to develop an approximate expression for the probability that the circle contains the true (but unknown) location of the submarine.

Assumptions and Simplifications

1. The ocean surface is considered to be flat (a geometric plane).
2. The submarine and the sensors can be represented as points.
3. The locations of the sensors are exactly known at the single time that is considered.
4. Time coordination is such that the angle readings for the location of the submarine position, as provided by the sensors, correspond to the observed submarine position at the time considered.
5. The observed directions furnished by the two sensors are statistically independent.
6. The observed direction provided by a sensor, at the time considered, has a normal (Gaussian) probability distribution with mean equal to the true direction of the submarine.
7. The sensors are positioned so that true submarine direction from any sensor is substantially different from the direction of the other sensor (with respect to this sensor).
8. The standard deviations of the angular probability distributions for the sensors are known and small (say, at most 0.03 radians). This combined with assumption 7, is considered to imply that second and higher order terms in angular errors (deviations from true direction) can be neglected in mathematical expressions.

Notation and Relationships

For standardization purposes, and ease of analysis, one sensor is considered to be located at the origin of the (x,y) rectangular coordinate system that is

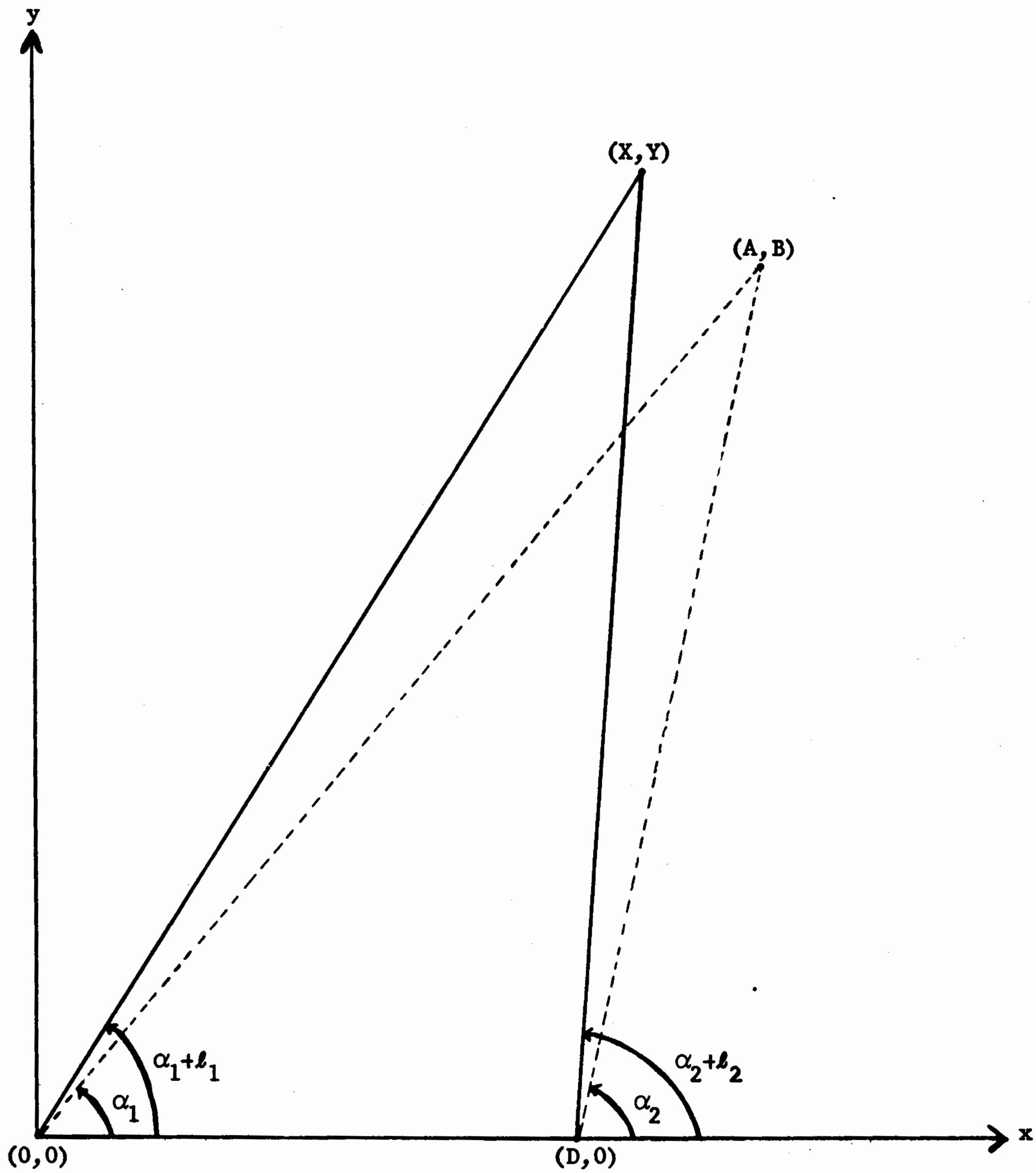

FIGURE 1. Schematic diagram for standardized representation and associated notation

used to represent the ocean surface. The second sensor is considered to be located at (D,0), where $D > 0$ is the distance between the sensors. Also, the submarine is considered to be located in the first quadrant. These standardizations do not represent any loss of generality for the type of analysis that is made.

The true location of the submarine is denoted by (A,B), while the position estimated on the basis of the directional observations is (X,Y). With respect to the first sensor, the true angular direction of the submarine is α_1 radians, while the observed angle is $\alpha_1+\ell_1$ radians. For the second sensor, the true direction is α_2 radians, and the observed direction is $\alpha_2+\ell_2$ radians. Figure 1 contains a schematic diagram that illustrates the standardization and notation used.

Now, consider some relationships that occur for this situation. First, α_1 and α_2 are directly determined by A,B, and D. That is, $\tan \alpha_1 = B/A$ and $\tan \alpha_2 = B/(A-D)$, where A-D can be negative. Second,

$$X = D\{1 + \cot(\alpha_2 + \ell_2)/[\cot(\alpha_1 + \ell_1) - \cot(\alpha_2 + \ell_2)]\}$$

$$= D \cot(\alpha_1 + \ell_1)/[\cot(\alpha_1 + \ell_1) - \cot(\alpha_2 + \ell_2)]$$

$$Y = D/[\cot(\alpha_1 + \ell_1) - \cot(\alpha_2 + \ell_2)].$$

In addition, the following relations are useful in the derivations:

$$\cot(\alpha + \varepsilon) = (1 - \tan \alpha \tan \varepsilon)/(\tan \alpha + \tan \varepsilon),$$

and, for ε reasonably small (ε in radians), $\tan \varepsilon \doteq \varepsilon$, so that

$$\cot(\alpha + \varepsilon) \doteq (1 - \varepsilon \tan \alpha)/(\varepsilon + \tan \alpha) \doteq [1 - \varepsilon(\tan \alpha + \cot \alpha)]/\tan \alpha,$$

where α is not small.

Estimate of Submarine Position

The method used to develop the approximation to the probability being sought consists in first approximating X and Y by linear functions of ℓ_1 and ℓ_2. The derivation of these linear functions (justified by assumptions 7 and 8) is given in detail.

First, consider the linear expression for Y. Examination shows that $(Y/D)^{-1}$ approximately equals

$$[1 - \ell_1(\tan\alpha_1 + \cot\alpha_1)]/\tan\alpha_1 - [1 - \ell_2(\tan\alpha_2 + \cot\alpha_2)]/\tan\alpha_2$$

$$= \frac{\tan\alpha_2 - \tan\alpha_1}{\tan\alpha_1\tan\alpha_2}\left\{1 - \ell_1\left[\frac{(\tan\alpha_1 + \cot\alpha_1)\tan\alpha_2}{\tan\alpha_2 - \tan\alpha_1}\right] + \ell_2\left[\frac{(\tan\alpha_2 + \cot\alpha_2)\tan\alpha_1}{\tan\alpha_2 - \tan\alpha_1}\right]\right\}.$$

Thus, Y approximately equals

$$\frac{D\tan\alpha_1\tan\alpha_2}{\tan\alpha_2 - \tan\alpha_1}\left\{1 + \ell_1\left[\frac{(\tan\alpha_1 + \cot\alpha_1)\tan\alpha_2}{\tan\alpha_2 - \tan\alpha_1}\right] - \ell_2\left[\frac{(\tan\alpha_2 + \cot\alpha_2)\tan\alpha_1}{\tan\alpha_2 - \tan\alpha_1}\right]\right\}.$$

Hence, Y is approximately expressed in the form $a' + b'\ell_1 + c'\ell_2$, where a', b', and c' can be expressed as functions of A, B, D, on the basis of the derived relation. For example it is easily verified that $a' = B$.

Finally, consider the approximate linear expression for X. This can be developed through multiplication of the approximate expression for Y by the approximating linear expression (linear in ℓ_1) for cot $(\alpha_1 + \ell_1)$. This yields

$$\frac{D \tan \alpha_2}{\tan \alpha_2 - \tan \alpha_1}\left\{1 + \ell_1\left[\frac{1 + (\tan \alpha_1)^2}{\tan \alpha_2 - \tan \alpha_1}\right] - \ell_2\left[\frac{(\tan \alpha_2 + \cot \alpha_2)\tan \alpha_1}{\tan \alpha_2 - \tan \alpha_1}\right]\right\}$$

as the approximate expression for X. Thus, X is expressed in the form $a + b\ell_1 + c\ell_2$, where it can be shown that a = A. Also b and c can be expressed as functions of A, B, D that are determined by the derived expression.

Relation for Inclusion in Circle

It is easily verified that the submarine location is contained in the circle of radius R centered at (X,Y) if and only if

$$(X - A)^2 + (Y - B)^2 \le R^2 .$$

Stated in terms of the linear approximations to X and Y, this relation becomes

$$(1) \qquad (b\ell_1 + c\ell_2)^2 + (b'\ell_1 + c'\ell_2)^2 \le R^2 ,$$

or equivalently

$$\sigma_1^2[b^2 + (b')^2](\ell_1/\sigma_1)^2 + 2\sigma_1\sigma_2(bc + b'c')(\ell_1/\sigma_1)(\ell_2/\sigma_2)$$

$$+ \sigma_2^2[c^2 + (c')^2](\ell_2/\sigma_2)^2 \le R^2 ,$$

where σ_1 is the standard deviation of ℓ_1, and σ_2 is the standard deviation of ℓ_2. The problem is to evaluate the probability that this relation is satisfied.

The first step in this evaluation is to change, by linear transformations, from the random variables ℓ_1, ℓ_2 to new random variables t_1, t_2. This is done so that t_1 and t_2 are standardized normal and independent. Also, so that relation (1) takes the form

$$v_1 t_1^2 + v_2 t_2^2 \leq R^2 .$$

In this form, existing methods are directly usable for approximately evaluating the probability that this relation holds.

Statement of Transformations

The relation (1) can be expressed as

$$\sum_{i,j=1}^{2} A_{ij}(\ell_i/\sigma_i)(\ell_j/\sigma_j) \leq R^2 ,$$

where the matrix $||A_{ij}||$ equals

$$\begin{Vmatrix} \sigma_1^2[b^2 + (b')^2] & \sigma_1\sigma_2(bc + b'c') \\ \sigma_1\sigma_2(bc + b'c') & \sigma_2^2[c^2 + (c')^2] \end{Vmatrix}$$

Let $||A^{ij}|| = ||A_{ij}||^{-1}$, and use $\lambda_1 \geq \lambda_2$ to denote the characteristic roots of $||A^{ij}||$. Since $A^{12} = A^{21}$, the values of λ_1 and λ_2 are the roots of

$$(A^{11} - \lambda)(A^{22} - \lambda) - (A_{12})^2 = 0 .$$

Specifically,

$$\lambda_1 = \tfrac{1}{2}\{A^{11} + A^{22} + [(A^{11} - A^{22})^2 + 4(A_{12})^2]^{1/2}\} ,$$

$$\lambda_2 = \tfrac{1}{2}\{A^{11} + A^{22} - [(A^{11} - A^{22})^2 + 4(A_{12})^2]^{1/2}\} ,$$

are values for λ_1 and λ_2, where both values should be positive.

Let the y_{gi} be any set of four numbers (not all zero) satisfying the four equations.

$$-\lambda_g y_{gi} + \sum_{j=1}^{2} A^{ij} y_{gj} = 0, \qquad (i,g = 1,2),$$

(there are an infinite number of possible choices). Define the numbers C_{gi} by

$$C_{gi} = y_{gi}[\sum_{j=1}^{2} y_{gj}^2]^{-1/2} ,$$

and let

$$w_g = \sum_{i=1}^{2} C_{gi}(\ell_i/\sigma_i) .$$

Here, w_1 and w_2 are independent and $Ew_1 = Ew_2 = 0$. Then,

$$\sum_{i,j=1}^{2} A_{ij}(\ell_i/\sigma_i)(\ell_j/\sigma_j) = \sum_{g=1}^{2} w_g^2/\lambda_g .$$

It is easily verified that the variance of $w_g/\sqrt{\lambda_g}$ is

$$v_g = \sum_{i=1}^{2} C_{gi}^2/\lambda_g , \qquad (g = 1,2) .$$

Let t_g equal $w_g/\sqrt{\lambda_g v_g}$. Then,

$$\sum_{i,j=1}^{2} A_{ij}(\ell_i/\sigma_i)(\ell_j/\sigma_j) = \sum_{g=1}^{2} v_g t_g^2 ,$$

where t_1, t_2 are independent and standardized normal. Thus,

$$P(v_1 t_1^2 + v_2 t_2^2 \leq R^2) = P(v_2 t_1^2 + v_1 t_2^2 \leq R^2)$$

approximately equals the probability that the submarine position is within a distance R (on the ocean surface) from the estimated position (X,Y).

The procedure used to make these transformations is the method of principal components (for example, see ref. 1).

Evaluation of Probability

Let $v = \min(v_1,v_2)$ and $V = \max(v_1,v_2)/v$. The probability to be approximately evaluated can be expressed in the form

$$P(t_1^2 + Vt_2^2 \leq R^2/v) .$$

One way of approximately evaluating probabilities of this form is by the method of ref. 2. Specifically, the cumulative distribution function of $t_1^2 + Vt_2^2$ can be expressed as

$$\sum_{w=0}^{\infty} H_w C_{2w+2}(x) ,$$

where $C_{2w+2}(x)$ is the cumulative distribution function for the continuous χ^2-distribution with 2w+2 degrees of freedom. The H_w, which are nonnegative numbers and satisfy $\sum_{w=0}^{\infty} H_w = 1$, are determined by the algebraic identity (in z)

$$V^{-1/2}[1 - (1 - 1/V)z]^{-1/2} \equiv \sum_{w=0}^{\infty} H_w z^w ,$$

for the expression that occurs for $|z|$ sufficiently small. Here,

$$[1 - (1 - 1/V)z]^{-1/2} = 1 + \tfrac{1}{2}(1 - 1/V)z + (3/8)(1 - 1/V)^2 z^2$$

$$+ (5/16)(1 - 1/V)^3 z^3 + (35/128)(1 - 1/V)^4 z^4 + \ldots .$$

Thus,

$$H_0 = V^{-1/2}, \quad H_1 = \tfrac{1}{2}(1 - 1/V)V^{-1/2}, \quad H_2 = (3/8)(1 - 1/V)^2 V^{-1/2} ,$$

$$H_3 = (5/16)(1 - 1/V)^3 V^{-1/2}, \quad H_4 = (35/128)(1 - 1/V)^4 V^{-1/2}, \text{ etc.}$$

For practical applications, the inequalities ($W \geq 0$)

$$\sum_{w=0}^{W} H_w C_{w+2}(x) \leq \sum_{w=0}^{\infty} H_w C_{w+2}(x) \leq \sum_{w=0}^{W} H_w C_{w+2}(x) + [1 - \sum_{w=0}^{W} H_w] C_{2W+4}(x)$$

are helpful. Using these results,

$$\sum_{w=0}^{W} H_w C_{w+2}(R^2/v) \leq P(t_1^2 + Vt_2^2 \leq R^2/v) \leq \sum_{w=0}^{W} H_w C_{2w+2}(R^2/v)$$

$$+ [1 - \sum_{w=0}^{W} H_w] C_{2W+4}(R^2/v) .$$

Ordinarily, the value of $P(t_1^2 + Vt_2^2 \leq R^2/v)$ can be approximated to reasonable accuracy without using a very large value for W.

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