TESTS WHEN ERRORS ARE CORRELATED IN A RANDOMIZED BLOCK DESIGN

by

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IN A RANDOMIZED BLOCK DESIGN

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Consider a randomized block design where the errors are correlated within a block but are independent from block to block. The theory for testing the significance of the treatment effects was done by Box [1954] and Geisser and Greenhouse [1958], and a partial solution was given by Graybill [1954]. A more general solution to this problem is now presented and several test procedures are derived.

The variance-covariance matrix for the above design can have two forms. When the correlation coefficient, ρ_j, differs from block to block, an exact test of reduced dimension is proposed which can be used in solving problems in growth studies. When ρ_j is identical to ρ for each block, two tests are presented. One is exact when ρ is known; both are approximate when ρ is unknown. In this latter case, comparisons are made between the two tests using a specified form for the covariance matrix and estimating ρ . For this example a Satterthwaite test is most accurate; but, the usual F-test, which ignores the correlation, performs well when $|\rho|$ varies somewhat from zero.

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CHAPTER I

INTRODUCTION

A basic assumption in the model for a randomized block design (RBD) having t treatments and b blocks is that the errors are normally and independently distributed. The physical nature of some experimental situations, however, offers considerable doubt as to the validity of this assumption of independent errors. Data occur in cases where there is no possibility of introducing randomization because the factor to be studied is the effect of time or position. Box [1954_a, 1954_b] and Geisser and Greenhouse [1958] have shown that these correlated errors can seriously affect the probability of the Type I error of certain tests of hypothesis from the standard Analysis of Variance.

Correlated errors are particularly prevalent when repeated measurements are made on one experimental unit (e.g., growth curves); in general, when randomization of experimental units to treatment levels is restricted. In some cases, correct tests on the significance of treatment contrasts have been achieved through insight as Yates [1937] (theory for this type of solution can be found in Chakrabarti [1962] p. 62 ff), or through the use of Hotelling's T^2 test [1931] (if $b > t$), but no general approach has been given.

The purpose of this paper is to give some exact and approximate criterions for testing the effects of independent sets of treatment contrasts in a RBD when the errors are correlated within a block but are independent from block to block. For testing the equality of all the treatment means, Graybill [1954] has given an exact test using Hotelling's T^2. But this is useful only when $b > t$ and the covariance matrix is the same within each block; it also involves considerable computation when t is large. The applied statistician, however, is sometimes confronted with the case where $b \leq t$, or situations where adequate means are not available for computing large-order inverses. He might even be interested in a specified set of treatment comparisons. It is these areas that are to be studied in this work.

Cases are considered where the correlations within a block are a function of a single unknown parameter, ρ_j, and the structure of the covariance matrix is the same within each block. The problem is then approached from two avenues (which in some cases may lead to the same solution):

1) Break down the variance-covariance matrix into an additive decomposition as illustrated by Good [1969] where the correlation coefficient within a block is dominant in only a few multipliers. Transformations orthogonal to their corresponding vectors would lead to less (usually zero) correlated data. Yates' [1937] solution is essentially this; where ρ_j is the multiplier of only one additive component (in fact, this is a latent root and vector of the covariance matrix).

2) When ρ_j is identical to ρ for each block, make an exact F-test if ρ is known. Otherwise, estimate the unknown correlation parameter and through this make an approximate F-test. A solution similar to Satterthwaite [1946] would result when this is substituted into Box's theory.

An example with a common form of the covariance matrix will be considered for both the above. In the latter case, a Monte Carlo study will be made comparing the exact test with the approximate one using this example. An easily computed estimate for ρ will also be given.

CHAPTER II

THE C-METHOD

Consider a class of randomized block designs with b blocks and t treatments and let Y_{ij} be the observation in the j^{th} block on the i^{th} treatment. Assume that Y_{ij} may be represented by a linear model

$$Y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij} \quad , \quad i = 1, \cdots , t; \ j = 1, \cdots , b$$

where μ is the grand mean, τ_i reflects the fixed effect of the i^{th} treatment subject to the condition $\sum_{i=1}^{t} \tau_i = 0$, β_j reflects the effect of the j^{th} block, and ε_{ij} reflects the error effect. Alternatively, denote the model for all the elements of the j^{th} block by

$$\underline{Y}_j = (\mu + \beta_j)\underline{1} + \underline{\tau} + \underline{\varepsilon}_j \quad , \quad j = 1, \cdots , b \tag{1}$$

where

$$\underline{Y}_j' = (Y_{1j}, Y_{2j}, \cdots , Y_{tj})$$

$$\underline{1}' = (1, 1, \cdots , 1)$$

$$\underline{\tau}' = (\tau_1, \tau_2, \cdots , \tau_t)$$

$$\underline{\varepsilon}_j' = (\varepsilon_{1j}, \varepsilon_{2j}, \cdots , \varepsilon_{tj}) \quad .$$

It is also assumed that

$$\underline{\varepsilon}_j \sim N_t(\underline{0}, \Sigma_j), \text{ independently, } j = 1, \cdots , b \tag{2}$$

where $\underline{0}$ is the null vector and Σ_j is the variance-covariance matrix of $\underline{\varepsilon}_j$.

Suppose it is desired to test the significance of the treatment effects in the model of (1). While no exact test exists that is applicable in all cases for an unknown Σ_j, it is possible to find an exact method for testing certain sets of treatment contrasts provided Σ_j is of a special form. But information on other sets may not be attainable, and the test statistics for these sets will usually be correlated. So the overall power of the test is often reduced. Such a loss, however, might be tolerated at times in order to avoid cumbersome approximations and difficult computations.

Consider a covariance matrix that can be expressed as

$$\Sigma_j = \sigma^2(I_t + \rho_j M) \quad , \quad j = 1, \cdots , b \tag{3}$$

where σ^2 and ρ_j are unknown constants and M is a known matrix, $t \times t$, with zeros along its diagonal, i.e., all the Y_{ij}'s have equal variances. It is assumed that Σ_j is positive definite. If Σ_j has the form

$$\Sigma_j = \sigma^2(I_t + M_0 + \rho_j M)$$

it can be transformed to

$$\Sigma_j^* = \sigma^2(I + \rho_j M^*)$$

as in (3), where $\Sigma_j^* = L\Sigma_j L'$, $LL' = \frac{1}{2} I$, $LM_0L' = \frac{1}{2} I$, $LML' = M^*$. And if

$$\Sigma_j = \sigma^2[I_t + M(\rho_j)]$$

i.e., if M is a function of ρ_j, express Σ_j as

$$\Sigma_j = \sigma^2(I_t + \rho_j M_1 + \rho_j^2 M_2 + \cdots) .$$

Then it might be possible to use

$$\Sigma_j^* = \sigma^2(I_t + \rho_j M_1)$$

as an approximation for Σ_j since powers of ρ_j greater than one may be negligible.

Assuming Σ_j has the form in (3) when using the RBD of (1), in general, greatly restricts the randomization of treatments to blocks. In fact the treatments must be positioned in a certain order in each block so as to guarantee that the errors within a block are properly correlated, unless the correlation is related to the treatments rather than the plots, e.g., see Geisser and Greenhouse [1958]. At times, however, the treatments in certain sets, e.g., the odd-numbered treatments, can be randomly assigned to certain plots, e.g., the odd-numbered plots. The example at the end of this chapter will better illustrate this idea.

With the Σ_j given in (3), it can be shown that there exists a matrix, C, $t \times q$, of rank q such that

$$C'MC = \Phi \quad , \quad C'C = I_q \tag{4}$$

i.e.,

$$C'\Sigma_j C = \sigma^2 I_q \quad .$$

By transforming $\underline{Y}_j$ to $K\underline{Y}_j = \underline{Z}_j$ the design matrix becomes one in which the errors are independently and normally distributed. This transformation then leads to the necessary statistic for testing the hypothesis:

$$\begin{aligned} &H_o: \; K\,\underline{\tau} = \underline{0} \\ \text{vs} \quad &H_a: \; K\,\underline{\tau} \neq \underline{0} \end{aligned} \tag{5}$$

where

$$K = \begin{cases} C' - C'\underline{1}[\underline{1}'CC'\underline{1}]^{-1}\underline{1}'CC' , & C'\underline{1} \neq \underline{0} \\ C' , & C'\underline{1} = \underline{0} \end{cases} \quad . \tag{6}$$

Notice in (6) that when $C'\underline{1} \neq \underline{0}$, C must be adjusted for the effects of $C'\underline{1}$. The initial problem then is one of choosing the appropriate matrix, C. Although the following approach has many good properties, there is no unique way of constructing C.

Using the technique illustrated by Good [1969], it is possible to break down M into an additive decomposition, i.e.,

$$M = \sum_{i=1}^{t} \lambda_i \, \underline{\alpha}_i \underline{\alpha}_i' \tag{7}$$

where λ_i is a latent root and $\underline{\alpha}_i$ is the corresponding orthonormal latent vector of M. Since $\ddagger_j$ has equal variances, M has zeroes along its diagonal and the trace of M is zero, i.e., the sum of the λ_i is zero. Therefore, there exists at least one negative root. Consider pairing each of these negative roots, say λ_k, with a positive root, say λ_ℓ, to form the separate ratios

$$R_{k,\ell} = \sqrt{\frac{\lambda_\ell}{|\lambda_k|}} \tag{8}$$

and let p be the number of these pairs of negative-positive roots. It is suggested that λ_k and λ_ℓ be chosen in such a manner that $R_{k,\ell}$ is as close to one as possible. Re-label the $R_{k,\ell}$ so that R_1 is the smallest ratio, R_2 is the next smallest ratio, $\cdots$, and R_p is the largest ratio. Now construct the orthonormal vectors

$$\underline{\beta}_v = (1 + R_v^2)^{-\frac{1}{2}}(\underline{\alpha}_\ell + R_v\,\underline{\alpha}_k) \ , \quad v = 1, \cdots , p \tag{9}$$

where $\underline{\alpha}_\ell$ and $\underline{\alpha}_k$ are the latent vectors corresponding to the roots, λ_ℓ and λ_k , in R_v . There is at least one of these vectors since there is at least one pair of positive-negative roots. At the most there are as many such vectors as one-half the rank of M (if the rank of M is even), i.e., there are as many such vectors as there are possible ratios, R_v . Each vector, $\underline{\beta}_v$, is a column of C.

It is important to realize the limitations and assets of the above method of construction. If M is non-singular and $C'\underline{1} \neq \underline{0}$, then C will be useless unless M has more than one pair of positive-negative roots, for one such pair leaves no degrees of freedom in testing the hypothesis of (5). When M is singular, there are no restrictions as C can be augmented by those latent vectors, $\underline{\alpha}_i$, orthogonal to M. Necessarily the $\underline{\alpha}_i$ will be orthogonal to the $\underline{\beta}_v$. Then the columns of C consist of the constructed vectors, $\underline{\beta}_v$, of (9), and the latent vectors, $\underline{\alpha}_i$, orthogonal to M, i.e.,

$$C = [\underline{\beta}_1, \underline{\beta}_2, \cdots , \underline{\beta}_p \,|\, \underline{\alpha}_i\text{'s orthogonal to M}] \ . \tag{10}$$

Hence,

$$q \leq \begin{cases} t - \dfrac{r}{2} \ , \text{ if } r = \text{rank (M) is even} \\ t - \dfrac{r+1}{2} \ , \text{ if } r = \text{rank (M) is odd} \end{cases} \ .$$

To show that the conditions of (4) hold, let the columns of C equal $\underline{\gamma}_c$, $c = 1, \cdots , q$. Then

$$\underline{\gamma}_c \underline{\gamma}'_{c'} = \begin{cases} 1 , & c = c' \\ 0 , & c \neq c' \end{cases} \qquad \text{from (7) and (9)}$$

so that

$$C'C = I_q \quad .$$

Also,

$$\underline{\gamma}'_c M = \begin{cases} \dfrac{\lambda_\ell \underline{\alpha}_\ell + \lambda_k R_c \underline{\alpha}_k}{\sqrt{1+R_c^2}} , & \text{if } \underline{\gamma}'_c = \underline{\beta}_c' \\ 0 , & \text{if } \underline{\gamma}'_c = \underline{\alpha}'_c \end{cases}$$

so that

$$\underline{\gamma}'_c M \underline{\gamma}_{c'} = 0 \quad , \quad \text{for all } c, c' \ ;$$

therefore,

$$C'MC = \Phi$$

and

$$C' \Sigma_j C = \sigma^2 I_q \quad .$$

Recall that in constructing C, the ratios R_v were chosen to be as close to one as possible. This was done for several reasons. If some R_v equal one, it is possible to construct a second matrix, C_2, orthogonal to $C_1 \equiv C$, which satisfies the conditions of (4), i.e.,

$$C_i' M C_i = \Phi \quad , \quad C_i' C_i = I_{q_i} \quad , \quad i = 1, 2$$

and

$$C_i' C_j = \Phi \quad , \quad i \neq j ; \ i,j = 1, 2 \quad . \tag{11}$$

Then two sets of contrasts as given in (5) can be tested instead of one, and more degrees of freedom are involved in the test. The columns of C_1 are the same as in C, replacing ℓ, k, v, q, and R_v by ℓ_1, k_1, v_1, q_1, and ${}_1R_{v_1}$. The columns of C_2 are the orthonormal vectors

$$\underline{\delta}_{v_2} = \frac{1}{\sqrt{2}} (\underline{\alpha}_{\ell_2} - \underline{\alpha}_{k_2}) \quad , \quad v_2 = 1, \cdots , q_2 \qquad (12)$$

where $\underline{\alpha}_{\ell_2}$ and $\underline{\alpha}_{k_2}$ correspond to the roots, λ_{ℓ_2} and λ_{k_2}, in ${}_2R_{v_2}$, i.e., R_{ℓ_2,k_2}, which represents the ratios, R_v, that equal one; and q_2 is the number of these ratios identical to one. Hence, the rank of C_1 is q_1, i.e., q, and the rank of C_2 is q_2 . If $C_2'\underline{1} \neq 0$, then q_2 must exceed one or there will be no degrees of freedom available for testing the hypothesis of (5); and the restrictions that held for C can be applied to C_1 .

Note that condition (4) holds using C_1 since $C_1 = C$. For C_2 the column vectors, $\underline{\delta}_{v_2}$, are orthonormal so that $C_2'C_2 = I_{q_2}$ and

$$\underline{\delta}_{v_2}' M \underline{\delta}_{v_2'} = \frac{1}{2} (\lambda_{\ell_2} \underline{\alpha}_{\ell_2}' - \lambda_{k_2} \underline{\alpha}_{k_2}') (\underline{\alpha}_{\ell_2'} - \underline{\alpha}_{k_2'})$$

$$= 0$$

since the $\underline{\alpha}_i$ are orthonormal and $\lambda_{\ell_2} = -\lambda_{k_2}$. Thus, $C_2'MC_2 = \Phi$ and condition (4) follows. Further, since ${}_2R_{v_2} = 1$, $\underline{\delta}_{v_2}' \underline{\beta}_{v_1} = 0$, so that $C_1'C_2 = \Phi$. Hence, all conditions are satisfied and C_1 and C_2 can be used to test the hypothesis of the nature of (5).

For the ratios, R_v, unequal to one it is possible to construct a third matrix, C_3^*, with the properties that

$$C_3^{*\prime} C_i = \begin{cases} I_{q_3} & , \quad i = 3 \\ \Phi & , \quad i = 1, 2 \end{cases}$$

and

$$C_3^{*\prime} M C_3^* = \mathrm{diag}(a_{v_3}) \quad , \quad v_3 = 1, \cdots , q_3$$

where a_{v_3} is some constant greater than zero and q_3 is the number of ratios, R_v, not equal to one. The columns of C_3^* are the orthonormal vectors

$$\eta_{v_3} = \sqrt{\frac{{}_3R^2_{v_3}}{1 + {}_3R^2_{v_3}}} \; (\underline{\alpha}_{\ell_3} - {}_3R^{-2}_{v_3} \underline{\alpha}_{k_3}) \quad , \quad v_3 = 1, \cdots , q_3 \qquad (13)$$

where $\underline{\alpha}_{\ell_3}$ and $\underline{\alpha}_{k_3}$ are the latent vectors corresponding to the latent roots, λ_{ℓ_3} and λ_{k_3}, of ${}_3R_{v_3}$, i.e., R_{ℓ_3, k_3}, which represents the ratios, R_v, that are unequal to one. The rank of C_3^* is q_3 and the same restriction holds for q_3 as did for q_2 when $C_3^{*\prime}\underline{1} \neq \underline{0}$. Notice now that

$$a_{v_3} = \frac{{}_3R^2_{v_3}}{1 + {}_3R^2_{v_3}} (\underline{\alpha}_{\ell_3} - {}_3R^{-1}_{v_3} \underline{\alpha}_{k_3})' \sum_{i=1}^{t} \lambda_i \underline{\alpha}_i \underline{\alpha}_i' \; (\underline{\alpha}_{\ell_3} - {}_3R^{-1}_{v_3} \underline{\alpha}_{k_3})$$

$$= \frac{{}_3R^2_{v_3}}{1 + {}_3R^2_{v_3}} [\lambda_{\ell_3} \underline{\alpha}_{\ell_3} - {}_3R^{-1}_{v_3} \lambda_{k_3} \underline{\alpha}_{k_3}]' [\underline{\alpha}_{\ell_3} - {}_3R^{-1}_{v_3} \underline{\alpha}_{k_3}]$$

$$= \frac{{}_3R^2_{v_3}}{1 + {}_3R^2_{v_3}} [\lambda_{\ell_3} + {}_3R^{-2}_{v_3} \lambda_{k_3}]$$

$$= \left[1 + {}_3R^2_{v_3}\right]^{-1} \left[{}_3R^2_{v_3} \lambda_{\ell_3} + \lambda_{k_3}\right]$$

$$= -\lambda_{k_3} \left[\frac{{}_3R^4_{v_3} - 1}{{}_3R^2_{v_3} + 1}\right]$$

$$= |\lambda_{k_3}| \cdot ({}_3R^2_{v_3} - 1) \quad .$$

Hence, a_{v_3} is approximately zero when ${}_3R^2_{v_3}$ is near one, or λ_{k_3} is near zero. This implies that in these cases

$$C_3^{*\prime}MC_3^* = \text{diag}(a_{v_3}) \doteq \Phi$$

so that

$$C_3^{*\prime}\ddagger_j C_3^* = \sigma^2[I_{q_3} + \rho_j \text{ diag}(a_{v_3})]$$
$$\doteq \sigma^2 I_{q_3} \quad .$$

So if the a_{v_3} are small, $C_3^{*\prime}MC_3^*$ is near the null matrix, and it might prove feasible to ignore these contrasts. Then C_3^* could be used along with C_1, or C_1 and C_2, to form another set of contrasts orthogonal to the others. It is easy to verify that the conditions of (4) hold with this approximation.

If the a_{v_3} vary greatly so that $C_3^{*\prime}MC_3^*$ is not near Φ it would be advantageous to examine this matrix using the first method developed above. Let

$$M^* = C_3^{*\prime}MC_3^* = \text{diag}(a_{v_3}) \quad , \quad v_3 = 1, \cdots , q_3$$

so that

$$\ddagger_j^* = \sigma^2(I_{q_3} + \rho_j M^*)$$
$$= C_3^{*\prime}\ddagger_j C_3^* \quad .$$

A matrix, C_4, similar to C_1 is sought. First, notice that the a_{v_3} are latent roots of the diagonal matrix M^* . And if ${}_3R^2_{v_3}$ is near one,

${}_3R^2_{v_3} - 1$ might be negative or positive implying that there might be a negative and positive a_{v_3} . If this is true there is a matrix, C_4, that can be constructed with columns similar to the vectors of (9). Now the ratios, R_v, are formed using the a_{v_3}'s, and the $\underline{\alpha}_i$ are unit vectors, i.e., the latent vectors of M* . Letting

$$C_3 = C_3^* C_4 \quad ,$$

it follows from previous results that

$$C_3' M C_3 = \Phi \quad ; \quad C_3' C_3 = I$$

and

$$C_3' C_i = \Phi \quad , \quad i = 1, 2 \quad .$$

Also, $C_3^{*\prime} M C_3^*$ is nonsingular, as $a_{v_3} \neq 0$ except when ${}_3R^2_{v_3} = 1$, and this is not possible by the manner in which C_3^* was constructed. Therefore, $C_3^{*\prime} M C_3^*$ must have at least two pair of negative-positive a_{v_3} or the matrix, C_3, will be useless since there will be no degrees of freedom available for testing the hypothesis of (5), based on C_3 . Of course, this restriction does not hold if $C_3'\underline{1} = \underline{0}$.

Hence, it has been shown that by keeping the R_v near one there may exist as many as three orthogonal matrices, C_1, C_2, C_3, or C_1, C_2, C_3^*, with the properties that

$$C_i' M C_i = \Phi \quad , \quad i = 1, 2, 3$$

$$C_i' C_j = \begin{cases} I_{q_i} & , \quad i = j \\ \Phi & , \quad i \neq j \end{cases}$$

so that

$$C_i' \Sigma_j C_i = \sigma^2 I_{q_i} \quad , \quad i = 1, 2, 3 \quad .$$

In terms of tests of hypotheses this means that three different sets of contrasts can be examined instead of one. However, $C_i' \Sigma_j C_k$ is not necessarily zero except when i = 2 and k = 3, so these sets will generally not be independent. The result is several dependent tests.

To derive the test statistics for the hypothesis of (5), using the above matrices, recall the model given in (1), i.e.,

$$\underline{Y}_j = (\mu + \beta_j)\underline{1} + \underline{\tau} + \underline{\varepsilon}_j \quad , \quad j = 1, \cdots , b$$

where

$$\underline{Y}_j \sim N_t[E(\underline{Y}_j), \Sigma_j] \quad , \quad \text{independently} .$$

Transform $\underline{Y}_j$ to $K\underline{Y}_j = \underline{Z}_j$ using the matrix, K, given in (6). Since $K\underline{1}$ is $\underline{0}$, $\underline{Z}_j$ is given by

$$\underline{Z}_j = K\underline{Y}_j = K\underline{\tau} + K\underline{\varepsilon}_j \tag{14}$$

with

$$\begin{aligned} E(\underline{Z}_j) &= K\underline{\tau} \\ V(\underline{Z}_j) &= \sigma^2 K \Sigma_j K' \\ &= \sigma^2 KK' \quad , \quad \text{from (4)} \quad . \end{aligned} \tag{15}$$

Note that, if $C'\underline{1} \neq \underline{0}$,

$$KK' = \left[C' - \frac{C'\underline{1}\underline{1}'CC'}{\underline{1}'CC'\underline{1}} \right] \left[C - \frac{CC'\underline{1}\underline{1}'C}{\underline{1}'CC'\underline{1}} \right]$$

$$= C'C - \frac{C'\underline{1}\underline{1}'C}{\underline{1}'CC'\underline{1}} \quad ;$$

therefore,

$$KK' = \begin{cases} I_q & , \quad C'\underline{1} = \underline{0} \\ I_q - \dfrac{C'\underline{1}\underline{1}'C}{\underline{1}'CC'\underline{1}} & , \quad C'\underline{1} \neq \underline{0} \end{cases} \quad . \tag{16}$$

Since the $\underline{Y}_j$ are i.i.d. normal variates, it follows that

$$\underline{Z}_j \sim N_t(K\underline{\tau}, \sigma^2 KK'), \text{ independently} \quad , \quad j = 1, \cdots , b;$$

therefore,

$$\bar{\underline{Z}}_{\cdot} = \frac{1}{b} \sum_{j=1}^{b} \underline{Z}_j \sim N_t(K\underline{\tau}, \frac{1}{b}\sigma^2 KK') \quad . \tag{17}$$

Consider the quadratic form

$$SST = b\bar{\underline{Z}}_{\cdot}'\bar{\underline{Z}}_{\cdot}$$

$$= b\bar{\underline{Y}}_{\cdot}'K'K\bar{\underline{Y}}_{\cdot} \tag{18}$$

where $\bar{\underline{Y}}_{\cdot} = \frac{1}{b} \sum_{j=1}^{b} \underline{Y}_j$. From (17) it follows that $\frac{SST}{\sigma^2}$ is distributed as a chi-square since KK' is idempotent, i.e.,

$$\frac{SST}{\sigma^2} \sim \chi_d^2(\lambda) \tag{19}$$

where

$$d = tr(KK')$$

$$= \begin{cases} tr(I_q) & , \quad C'\underline{1} = \underline{0} \\ tr(I_q) - tr\left(\dfrac{C'\underline{1}\underline{1}'C}{\underline{1}'CC'\underline{1}}\right) & , \quad C'\underline{1} \neq \underline{0} \end{cases}$$

$$= \begin{cases} q & , \quad C'\underline{1} = \underline{0} \\ q-1 & , \quad C'\underline{1} \neq \underline{0} \end{cases} \qquad . \tag{20}$$

Also,

$$\lambda = \frac{b}{2\sigma^2} E(\underline{\bar{Z}}_{\cdot}')E(\underline{\bar{Z}}_{\cdot})$$

$$= \frac{b}{2\sigma^2} \underline{\tau}'K'K\underline{\tau} \qquad .$$

The $K\underline{\tau}$ are a set of independent contrasts in τ since the rows of K are mutually orthogonal and $K\underline{1} = \underline{0}$. It will be shown below that there exists a test statistic for testing the hypothesis

$$\begin{aligned} H_o: &\ \lambda = 0 \\ \text{vs} \quad H_a: &\ \lambda \neq 0 \end{aligned} \qquad . \tag{21}$$

But this is equivalent to the hypothesis of (5), i.e.,

$$\begin{aligned} H_o: &\ K\underline{\tau} = \underline{0} \\ \text{vs} \quad H_a: &\ K\underline{\tau} \neq \underline{0} \end{aligned}$$

or,

$$\begin{aligned} H_o: &\ \underline{k}_i'\underline{\tau} = 0 \qquad i = 1, \cdots, d \\ H_a: &\ \underline{k}_i'\underline{\tau} \neq 0 \end{aligned}$$

where the $\underline{k}_i$ are linear combinations of the rows of K and are mutually orthogonal, or, these vectors are the basis for the vector space of K and are orthogonal to $\underline{1}$.

The results of Appendix A will now be used in deriving the test statistic for (21) given by

$$F = \frac{MST}{MSE} \tag{22}$$

with

$$\begin{aligned} MST &= \frac{1}{d}\, SST \\ &= \frac{1}{d}\, \underline{Y}' Q_t(A) \underline{Y} \end{aligned} \tag{23}$$

where $Q_t(A)$ is given in (A1), $A = K'K$, and

$$\underline{Y}' = [\underline{Y}_1', \cdots, \underline{Y}_b'] \; ;$$

and

$$\begin{aligned} MSE &= \frac{1}{(b-1)d}\, SSE \\ &= \frac{1}{(b-1)d}\, \underline{Y}' Q(A) \underline{Y} \end{aligned} \tag{24}$$

where $Q(A)$ is given in (A2) and A is the same as above. Recall that it was assumed in (1) that

$$\underline{Y} \sim N_{bt}[E(\underline{Y}), \Sigma) \quad , \quad \Sigma = \mathrm{diag}(\ddagger_j) \quad .$$

So $\frac{SSE}{\sigma^2}$ is distributed as a chi-square if $\frac{1}{\sigma^2} Q(A)\Sigma$ is idempotent. Now

$$\begin{aligned} A\ddagger_j A &= K'K\ddagger_j K'K \\ &= \sigma^2 A \quad , \quad \text{from (16)} \end{aligned} \tag{25}$$

and this result with that of (A4) implies that

$$\left[\frac{1}{\sigma^2} Q(A)\Sigma\right]^2 = \frac{1}{\sigma^2} Q(A)\Sigma \quad .$$

Hence,

$$\frac{SSE}{\sigma^2} \sim \chi_e^2(\lambda_e) \tag{26}$$

where

$$e = tr\left[\frac{1}{\sigma^2} Q(A)\Sigma\right]$$

$$= \frac{1}{\sigma^2}\frac{1}{b}(b-1)\sum_{j=1}^{b} tr(K'K\Phi_j), \qquad \text{from (A5)}$$

$$= (b-1)tr(K'K), \qquad \text{as } K\Phi_j = \sigma^2 K$$

$$= (b-1)d, \qquad \text{from (20)}$$

and

$$\lambda_e = 0, \qquad \text{from (A7)}$$

since

$$E(\underline{Y}_j')A = \underline{\tau}'K'K \quad , \quad \text{for all } j \quad .$$

Therefore,

$$\frac{SSE}{\sigma^2} \sim \chi^2_{(b-1)d}(0)$$

and from (19)

$$\frac{SST}{\sigma^2} \sim \chi^2_d(0) \quad ,$$

if H_o of (21) is true. Further, SSE and SST are independent since (A3) and (25) imply that $Q_t(A)\Sigma Q(A) = \Phi$. Therefore, $\frac{SST}{\sigma^2}$ and $\frac{SSE}{\sigma^2}$ are independent chi-squares and their ratio divided by their respective degrees of freedom yield

$$F = \frac{MST}{MSE} \sim F_{[d,(b-1)d]} \quad ,$$

if H_o is true. And the hypothesis given in (5) can be tested using the above result.

In cases where there are two orthogonal matrices, C_1 and C_2, fulfilling the conditions of (4), the above argument can again be used on each C_i . The only difference is that the variables of C are now subscripted. This results in two non-centrality parameters,

$$\lambda_1 = \frac{b}{2\sigma^2} \underline{\tau}'K_1'K_1\underline{\tau} \;, \quad \text{with } K_1 \text{ based on } C_1$$

and

$$\lambda_2 = \frac{b}{2\sigma^2} \underline{\tau}'K_2'K_2\underline{\tau} \;, \quad \text{with } K_2 \text{ based on } C_2$$

which lead to two dependent tests, one on λ_1 and one on λ_2 . The hypotheses are:

$$\left.\begin{array}{llll} & H_{01}:\ \lambda_1 = 0 & \text{and} & H_{02}:\ \lambda_2 = 0 \\ \text{vs} & H_{11}:\ \lambda_1 \neq 0 & \text{vs} & H_{12}:\ \lambda_2 \neq 0 \\ \text{or,} & H_{01}:\ K_1\underline{\tau} = \underline{0} & \text{and} & H_{02}:\ K_2\underline{\tau} = \underline{0} \\ \text{vs} & H_{11}:\ K_1\underline{\tau} \neq \underline{0} & \text{vs} & H_{12}:\ K_2\underline{\tau} \neq \underline{0} \end{array}\right\} \quad . \qquad (27)$$

The test statistics are

$$F_1 = \frac{MST_1}{MSE_1} \sim F_{[d_1,(b-1)d_1]} \;, \quad \text{if } H_{01} \text{ is true}$$

and

$$F_2 = \frac{MST_2}{MSE_2} \sim F_{[d_2,(b-1)d_2]} \;, \quad \text{if } H_{02} \text{ is true}$$

where d_1 and d_2 are similar to d, i.e.,

$$d_1 = \begin{cases} q_1 - 1 \;, & C_1'\underline{1} \neq \underline{0} \\ q_1 \;, & C_1'\underline{1} = \underline{0} \end{cases}$$

$$d_2 = \begin{cases} q_2 - 1 \;, & C_2'\underline{1} \neq \underline{0} \\ q_2 \;, & C_2'\underline{1} = \underline{0} \end{cases}$$

and SST_1, SSE_1, and SST_2, SSE_2, are similar to SST, SSE; here K is replaced by K_1 and K_2, respectively, i.e.,

$$MST_1 = \frac{1}{d_1} SST_1$$

$$MST_2 = \frac{1}{d_2} SST_2$$

$$MSE_1 = \frac{1}{(b-1)d_1} \underline{Y}'Q(A_1)\underline{Y} , \quad A_1 = K_1'K_1$$

$$MSE_2 = \frac{1}{(b-1)d_2} \underline{Y}'Q(A_2)\underline{Y} , \quad A_2 = K_2'K_2 .$$

As would be expected, the results proved for C hold in the cases of C_1 and C_2 since only a label has been changed. But MST_1 and MST_2 are not independent since

$$K_1'K_1 \ddagger_j K_2'K_2 \neq \Phi$$

i.e.,

$$C_1' \ddagger_j C_2 \neq \Phi$$

and MSE_1 and MSE_2 are not independent since

$$Q(A_1)\Sigma Q(A_2) \neq \Phi$$

i.e.,

$$C_1' \ddagger_j C_2 \neq \Phi .$$

Therefore, F_1 and F_2 are not independent. However, these are marginally exact tests under H_{01} and H_{02} and each can be individually tested.

This analogy can be further extended to cases where there exist three matrices, C_1, C_2, C_3 . The argument is the same only there will be three hypotheses:

$$H_{01}: \lambda_1 = 0 \ , \quad \text{based on } K_1 \text{ using } C_1$$

$$H_{02}: \lambda_2 = 0 \ , \quad \text{based on } K_2 \text{ using } C_2$$

and

$$H_{03}: \lambda_3 = 0 \ , \quad \text{based on } K_3 \text{ using } C_3 \ .$$

The test statistics are F_i, $i = 1, 2, 3$, where

$$F_i = \frac{MST_i}{MSE_i} \sim F_{[d_i, (b-1)d_i]} \ , \quad \text{if } H_{0i} \text{ is true} \ , \quad i = 1, 2, 3$$

and

$$MST_i = \frac{1}{d_i} SST_i \ , \quad \text{using } K_i$$

$$MSE_i = \frac{1}{(b-1)d_i} \underline{Y}'Q(A_i)\underline{Y} \ , \quad A_i = K_i'K_i$$

and

$$d_i = \begin{cases} q_i & , \ C_i'\underline{1} = \underline{0} \\ q_i - 1 \ , & C_i'\underline{1} \neq \underline{0} \end{cases} \ .$$

Notice that F_1 and F_2 are dependent tests as are F_1 and F_3 since

$$C_1' \ddagger_j C_i \neq \Phi \ , \quad i = 2, 3 \ .$$

But F_2 and F_3 are independent tests since

$$C_2' \ddagger_j C_3 = C_2' \ddagger_j C_3^* C_4$$

$$= \Phi \ , \quad \text{as } C_2' \ddagger_j C_3^* = \Phi \ .$$

This is expected since F_2 and F_3 are functions of entirely different $\underline{\alpha}_i$'s, which are mutually orthogonal, while F_1 has $\underline{\alpha}_i$'s in common with F_2 and F_3 . Thus, the result is three marginally exact tests of which two are dependent.

Additional tests of single degrees of freedom may exist provided the $\ddagger_j$ are identical, i.e., $\ddagger_j = \ddagger$, or $\rho_j = \rho$. The resulting contrasts can be constructed to be orthogonal to one another and to any of the above sets. Unfortunately, these tests are dependent on the others. The hypothesis to consider has the form:

$$H'_{0k}: \ \underline{a}'_k \underline{\tau} = \underline{0} \ , \quad k = 1, \cdots , d' \tag{28}$$

where the $\underline{a}_k$ are a set of orthonormal vectors such that

$$\underline{a}'_k \underline{1} = 0 \ ; \quad \underline{a}'_k K'_i = \underline{0}' \ , \quad i = 1, 2, 3; \ k = 1, \cdots , d' \quad .$$

Also,

$$d' = t - 1 - \sum_{i}^{3} d'_i$$

where

$$d'_i = \begin{cases} d_i \ , & \text{if the matrix } C_i \text{ is used} \\ 0 \ , & \text{otherwise} \end{cases} \quad .$$

Then the $\underline{a}'_k \underline{\tau}$ are a set of independent contrasts and

$$\underline{a}'_k \underline{Y}_j = \underline{a}'_k \underline{\tau} + \underline{a}'_k \underline{\varepsilon}_j \ , \quad \text{as } \underline{a}'_k \underline{1} = 0$$

so

$$\begin{aligned} E(\underline{a}'_k \underline{Y}_j) &= \underline{a}'_k \underline{\tau} \\ &= 0 \ , \quad \text{if } H'_{0k} \text{ is true} \end{aligned}$$

and

$$\begin{aligned} V(\underline{a}'_k \underline{Y}_j) &= \underline{a}'_k \Sigma \underline{a}_k \\ &= \underline{a}'_k (I_t + \rho M) \underline{a}_k \sigma^2 \\ &= \sigma^2 (1 + \rho c_k) \ , \quad c_k = \underline{a}'_k M \underline{a}_k \quad . \end{aligned}$$

Further, since the $\underline{Y}_j$ are i.i.d. normal variates,

$$\underline{a}'_k \underline{Y}_j \sim \text{N.I.D.}(\underline{a}'_k \underline{\tau} \ , \ \underline{a}'_k \Sigma \underline{a}_k) \ , \quad j = 1, \cdots , b$$

and

$$\underline{a}_k'\underline{\bar{Y}}_{\cdot} \sim N(\underline{a}_k'\underline{\tau}\ ,\ \frac{1}{b}\ \underline{a}_k'\ddagger\underline{a}_k) \quad .$$

Now it is well known that if $x \sim N(\mu, \sigma^2)$, then $\frac{n\bar{x}^2}{\sigma^2} \sim \chi_1^2\left(\frac{\mu^2}{2\sigma^2}\right)$ and $\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \bar{x})^2 \sim \chi_{n-1}^2(0)$, independently. So it follows that

$$\frac{SST_k'}{\underline{a}_k'\ddagger\underline{a}_k} = \frac{b(\underline{a}_k'\underline{\bar{Y}}_{\cdot})^2}{\underline{a}_k'\ddagger\underline{a}_k} \sim \chi_1^2(\lambda_k')$$

where

$$\lambda_k' = \frac{1}{2\sigma^2}\ (\underline{a}_k'\underline{\tau})^2$$

and

$$\frac{SSE_k'}{\underline{a}_k'\ddagger\underline{a}_k} = \frac{\sum_{j=1}^{b}(\underline{a}_k'\underline{Y}_j - \underline{a}_k'\underline{\bar{Y}}_{\cdot})^2}{\underline{a}_k'\ddagger\underline{a}_k}$$

$$\sim \chi_{b-1}^2(0) \quad .$$

Hence, the test statistic for testing

$$H_{0k}' : \lambda_k' = 0$$

$$\text{vs} \quad H_{1k}' : \lambda_k' \neq 0$$

i.e.,

$$H_{0k}' : \underline{a}_k'\underline{\tau} = 0$$

$$H_{1k}' : \underline{a}_k'\underline{\tau} \neq 0$$

is given by

$$F_k' = (b-1)\ \frac{SST_k'}{SSE_k'} \sim F_{[1,b-1]}\ ,\ \text{if } H_{0k}' \text{ is true .} \tag{29}$$

Since $\underline{a}_k'\ddagger C_i$ and $\underline{a}_k'\ddagger\underline{a}_k$ are not necessarily equal to the null matrix, these tests are dependent, as would be expected. Further, using these single degree of freedom tests with the tests based on the C_i , $i = 1, 2, 3$, leads to sets of contrasts using a maximum number of degrees of freedom. The combined set spans the parameter space of $\underline{\tau}$ and has a total rank of $t - 1$, as this space is restricted by the fact that $\underline{\tau}'\underline{1} = 0$. But it is desired that the contrasts obtained using only C_1, C_2, and C_3 span this space as there would be fewer dependent tests and, hence, an increase in the overall power using this approach besides not needing $\rho_j = \rho$, for all j .

In summary, it has been shown that, depending on the form of the matrix, M, there may exist as many as three matrices, C_1, C_2, C_3, but no fewer than one, C_1, with the properties that

$$C_i'MC_i = \Phi \ , \quad C_i'C_i = I_{q_i} \ , \quad i = 1, 2, 3$$

and

$$C_i'C_j = \Phi \ , \quad i \neq j, \ i,j = 1, 2, 3 \ .$$

These matrices can be used to test hypothesis of the form given in (5), and the derived test statistics have exact distributions, i.e., the F-distribution is used. Vectors have also been shown to exist which can be used in testing hypotheses of the form given in (28); likewise, these tests are exact. Finally the sets of contrasts formed when $\ddagger_j \equiv \ddagger$ will span the space of $\underline{\tau}$ and have a total rank equal to $t - 1$.

The above tests, unfortunately, were shown to be dependent as was expected, due to the form of $\ddagger_j$. But each individual test is exact and if any hypothesis happens to be of interest, it can be easily tested at the required α-significance level with the use of a set of F tables. The actual sets of contrasts that can be analyzed will be determined by the form of M. Due to the dependency of the resulting statistics, a joint test of significance using all the sets is not known. Such a test would have an α-level which would be quite difficult to compute. But bounds may exist and this fact needs further investigation (this will not be done here). For the individual tests there is an obvious loss in power; but, if b is large, this might not be too noticeable.

It is of interest to point out that the dependency above can partially be eliminated, but there is a sacrifice of the power of the test. To do this it is necessary to make a transformation of the form (C_3 can also be added)

$$\underline{Z} = \begin{bmatrix} C_1' & & & & & \\ & \ddots & & & \Phi & \\ & & C_1' & & & \\ & \Phi & & C_2' & & \\ & & & & \ddots & \\ & & & & & C_2' \end{bmatrix} \begin{bmatrix} \underline{Y}_1 \\ \vdots \\ \underline{Y}_b \end{bmatrix}$$

This might increase the degrees of freedom in MST, which is formed by pooling MST_1 and MST_2 (now independent) but a good deal of the information on $\underline{Y}$ would be ignored. Also, the degrees of freedom for MSE, formed by pooling MSE_1 and MSE_2 (which are also independent) could be reduced by as much as $\frac{1}{2}$. The procedure here would be similar to the previous one. If b were large this method could be used with much success but in general it has little value and thus will not be analyzed.

As a final point recall that it might not be possible to construct more than the one matrix, C_1. In this case there is only one test, other than the single degree of freedom tests which require the $\ddagger_j$ to be identical. Hence, in this case there is no need to consider problems of dependency.

To understand more fully the advantages of the method just derived, it will be helpful to analyze an example. Consider a randomized block experiment which uses the model in (1) and has a variance-covariance matrix of the form

$$\Sigma_j = \sigma^2(I_t + \rho_j M) \ , \quad j = 1, \cdots , b$$

$$= \sigma^2 \begin{bmatrix} 1 & \rho_j & & \Phi \\ \rho_j & \ddots & \ddots & \\ & \ddots & \ddots & \rho_j \\ \Phi & & \rho_j & 1 \end{bmatrix} \tag{30}$$

so that

$$M = \begin{bmatrix} 0 & 1 & & \Phi \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ \Phi & & 1 & 0 \end{bmatrix} \ . \tag{31}$$

Such a covariance matrix could occur in growth studies where the treatments are applied to each individual at specified times. For adjacent time periods one would expect a certain correlation between error effects; but, as time passes and other treatments are applied, there should be little or no correlation between the former and these latter errors. Hence, the order in which treatments are given to an individual is not as restricted as before. And, as is evident from growth curves, the treatment effects decrease with time so that a typical hypothesis might be

$$\begin{aligned} & H_o: \ \tau_i = \tau_{i'} \\ \text{vs} \quad & H_1: \ \tau_i > \tau_{i'} \end{aligned} \qquad \text{for } i < i' \ .$$

Also, under these circumstances, ρ_j usually varies from individual to individual so it is correct in using a different value for each person.

Note that ρ_j, the serial correlation between experimental units in the same block, is restricted by the condition that

$$|\rho_j| < \left\{2 \cos\left(\frac{\pi}{t+1}\right)\right\}^{-1} \tag{32}$$

which guarantees that Σ_j will be positive definite. The eigenroots and eigenvectors of M are given by Anderson [1948] as

$$\lambda_i = 2 \cos\left(\frac{\pi i}{t+1}\right), \quad i = 1, \cdots, t \tag{33}$$

and

$$\underline{\alpha}_i = \frac{1}{\sqrt{L_i}} \begin{bmatrix} \sin\left(\frac{\pi i}{t+1}\right) \\ \sin\left(\frac{2\pi i}{t+1}\right) \\ \vdots \\ \sin\left(\frac{t\pi i}{t+1}\right) \end{bmatrix}, \quad i = 1, 2, \cdots, t \tag{34}$$

where

$$L_i = \sum_{k=1}^{t} \sin^2\left(\frac{\pi i k}{t+1}\right) .$$

Notice that

$$|M| = \begin{cases} 0, & \text{when } t \text{ is odd} \\ 1, & \text{when } t \text{ is even} \end{cases} .$$

Also,

$$\lambda_i = 2\cos\left(\frac{\pi i}{t+1}\right)$$

$$= -2\cos\left(\frac{\pi(t+1-i)}{t+1}\right)$$

$$= -\lambda_{t+1-i}$$

so when t is even each eigenroot can be matched with the negative of another, implying that

$$R_v = 1 \ , \quad v = 1, 2, \cdots , \frac{t}{2}$$

and M is nonsingular. When t = odd, M becomes singular so that

$$R_v = 1 \ , \quad v = 1, 2, \cdots , \frac{t-1}{2}$$

and there remains one extra latent root corresponding to the latent vector, say $\underline{\alpha}_{(t+1)/2}$, orthogonal to M . Hence, it is possible to construct the two matrices, C_1 and C_2, which were defined earlier. Since all the R_v's are identical to one, there can be no C_3 matrix, as its construction requires ratios that are not equal to one.

The columns of C_1 are given by

$$\underline{\beta}_{v_1} = \frac{1}{\sqrt{2}}\left(\underline{\alpha}_{v_1} + \underline{\alpha}_{t+1-v_1}\right) \ , \quad v_1 = 1, \cdots , p_1$$

where

$$p_1 = \begin{cases} \frac{t}{2} \ , & \text{if t is even} \\ \frac{t-1}{2} \ , & \text{if t is odd} \end{cases} .$$

Therefore,

$$C_1 = \begin{cases} [\underline{\beta}_1, \cdots, \underline{\beta}_{t/2}] & , \ t \text{ even} \\ [\underline{\beta}_1, \cdots, \underline{\beta}_{(t-1)/2}, \underline{\alpha}_{(t+1)/2}] & , \ t \text{ odd} \end{cases} \quad . \tag{35}$$

Likewise, the columns of C_2, are given by

$$\underline{\delta}_{v_2} = \frac{1}{\sqrt{2}} (\underline{\alpha}_{v_2} - \underline{\alpha}_{t+1-v_2}) \ , \quad v_2 = 1, \cdots, q_2$$

where

$$q_2 = \begin{cases} \frac{t}{2} & , \text{ if } t \text{ is even} \\ \frac{t-1}{2} & , \text{ if } t \text{ is odd} \end{cases}$$

so that

$$C_2 = [\underline{\delta}_1, \underline{\delta}_2, \cdots, \underline{\delta}_{q_2}] \quad . \tag{36}$$

Then

$$q_1 = \text{Rank } (C_1) = \begin{cases} \frac{t}{2} & , \text{ if } t \text{ is even} \\ \frac{t+1}{2} & , \text{ if } t \text{ is odd} \end{cases}$$

and

$$q_2 = \text{Rank } (C_2) = \begin{cases} \frac{t}{2} & , \text{ if } t \text{ is even} \\ \frac{t-1}{2} & , \text{ if } t \text{ is odd} \end{cases}$$

Together C_1 and C_2 span the vector space of M, with C_1 containing the vector of the null space when M is singular; thus, there is no C_3 matrix.

Although it would now be an easy task to derive K_1 and K_2 and, hence, the F-statistic, this will not be done. Trial and error has revealed that it is not necessary to recompute C_1 and C_2 using (35) and (36) each time t changes in value. Since the C_i's are not unique, there is no loss in generality in using the bases of the vector spaces spanned by the columns of the C_i's as the columns of the C_i's. The result is a patterned matrix for C_1 and C_2 which holds in all cases, i.e.,

$$C_1' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & \end{bmatrix} , \quad C_1'\underline{1} \neq \underline{0} \tag{37}$$

and

$$C_2' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \end{bmatrix} , \quad C_2'\underline{1} \neq \underline{0} \quad . \tag{38}$$

Notice that the rows of C_1 and C_2 are orthogonal so that

$$\underline{C}_i'\underline{C}_j = \begin{cases} I_{q_i} , & i = j \\ \Phi , & i \neq j \end{cases}$$

and the conditions of (4) hold so that

$$C_i'MC_i = \Phi , \quad i = 1, 2 \quad ;$$

therefore,

$$C_i'\ddagger_j C_i = \sigma^2 I_{q_i} , \quad i = 1, 2 \quad .$$

The formula in (6) then yields

$$K_1 = \frac{1}{q_1}\begin{bmatrix} q_1-1 & 0 & -1 & 0 & \cdots \\ -1 & 0 & q_1-1 & 0 & \cdots \\ -1 & 0 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ -1 & 0 & -1 & 0 & \cdots \end{bmatrix} \tag{39}$$

and

$$K_2 = \frac{1}{q_2}\begin{bmatrix} 0 & q_2-1 & 0 & -1 & \cdots \\ 0 & -1 & 0 & q_2-1 & \cdots \\ 0 & -1 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & -1 & 0 & -1 & \cdots \end{bmatrix} \tag{40}$$

so that

$$A_1 = K_1'K_1 = \frac{1}{q_1}\begin{bmatrix} q_1-1 & 0 & -1 & 0 & -1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 0 & q_1-1 & 0 & -1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 0 & -1 & 0 & q_1-1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{41}$$

and

$$A_2 = K_2'K_2 = \frac{1}{q_2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & q_2-1 & 0 & -1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & q_2-1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{42}$$

It is obvious from (41) that SST_1 is

$$SST_1 = b\bar{\underline{Y}}_{\cdot}' K_1' K_1 \bar{\underline{Y}}_{\cdot}$$

$$= b \sum_{i=1}^{q_1} \left(\bar{Y}_{2i-1,\cdot} - \bar{\bar{Y}}_{\cdot\cdot}^{(1)} \right)^2$$

where

$$\bar{\bar{Y}}_{\cdot\cdot}^{(1)} = \frac{1}{bq_1} \sum_{j=1}^{b} \sum_{i=1}^{q_1} Y_{2i-1,j}, \bar{Y}_{2i-1,\cdot} = \frac{1}{b} \sum_{j=1}^{b} Y_{2i-1,j}$$

and from (42), SST_2 is

$$SST_2 = b\bar{\underline{Y}}_{\cdot}' K_2' K_2 \bar{\underline{Y}}_{\cdot}$$

$$= b \sum_{i=1}^{q_2} \left(\bar{Y}_{2i,\cdot} - \bar{\bar{Y}}_{\cdot\cdot}^{(2)} \right)^2$$

where

$$\bar{\bar{Y}}_{\cdot\cdot}^{(2)} = \frac{1}{bq_2} \sum_{j=1}^{b} \sum_{i=1}^{q_2} Y_{2i,j} \ , \ \bar{Y}_{2i,\cdot} = \frac{1}{b_2} \sum_{j=1}^{b} Y_{2i,j} \ .$$

Also,

$$SSE_1 = \sum_{j=1}^{b} \sum_{i=1}^{q_1} \left(Y_{2i-1,j} - \bar{Y}_{2i-1,\cdot} - \bar{Y}_{\cdot j}^{(1)} + \bar{\bar{Y}}_{\cdot\cdot}^{(1)} \right)^2 \qquad (43)$$

where

$$\bar{Y}_{\cdot j}^{(1)} = \frac{1}{q_1} \sum_{i=1}^{q_1} Y_{2i-1,j}$$

and

$$SSE_2 = \sum_{j=1}^{b} \sum_{i=1}^{q_2} \left(Y_{2i,j} - \bar{Y}_{2i,\cdot} - \bar{Y}_{\cdot j}^{(2)} + \bar{\bar{Y}}_{\cdot\cdot}^{(2)} \right)^2 \qquad (44)$$

where

$$\bar{Y}^{(2)}_{\cdot j} = \frac{1}{q_2} \sum_{i=1}^{q_2} Y_{2i,j} \quad .$$

It now becomes an easy task to test separately the two hypotheses:

$$\begin{aligned} & H_{01}: \ K_1\underline{\tau} = \underline{0} \\ \text{vs} \quad & H_{11}: \ K_1\underline{\tau} \neq \underline{0} \end{aligned}$$

and

$$\begin{aligned} & H_{02}: \ K_2\underline{\tau} = \underline{0} \\ \text{vs} \quad & H_{12}: \ K_2\underline{\tau} \neq \underline{0} \quad . \end{aligned}$$

Using (39) and (40) and relating this problem to growth studies, these hypotheses become:

$$\begin{aligned} & H_{01}: \ \tau_1 = \tau_3 = \cdots = \tau_{2q_1-1} \\ \text{vs} \quad & H_{11}: \ \tau_1 > \tau_3 > \cdots > \tau_{2q_1-1} \end{aligned}$$

and

$$\begin{aligned} & H_{02}: \ \tau_2 = \tau_4 = \cdots = \tau_{2q_2} \\ & H_{12}: \ \tau_2 > \tau_4 > \cdots > \tau_{2q_2} \quad . \end{aligned}$$

If, however, the physical nature of the problem allows for randomization, order the treatments from 1 to t to correspond to the ordering of the plots in each block so that $\not{t}_j$ has the form in (30). The odd-numbered treatments can be randomly assigned to the odd-numbered plots; likewise, the even-numbered treatments are randomly assigned to the even-numbered plots. Then collect the data in two parts; one containing the observations on the even-numbered treatments and one containing the observations on the odd-numbered treatments. Compute SST and SSE in the usual manner for each set of data using SST_1, SSE_1, SST_2, and SSE_2 . Finally calculate the test statistics

$$F_1 = (b-1)\,\frac{SST_1}{SSE_1}$$

and

$$F_2 = (b-1)\,\frac{SST_2}{SSE_2} \quad .$$

To test H_{01} compare F_1 with a tabled $F_{[q_1-1,(b-1)(q_1-1)]}$ at some α-level of significance; to test H_{02} compare F_2 with a tabled $F_{[q_2-1,(b-1)(q_2-1)]}$ at an α-level of significance.

As an example of this result consider the case where $t = 8$. Then

$$C_1' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$C_2' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

so that

$$q_1 = \frac{t}{2} = 4 \quad , \quad q_2 = \frac{t}{2} = 4$$

and

$$K_1'K_1 = \frac{1}{4}\begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad K_2'K_2 = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix} \quad .$$

Then

$$SST_1 = b\sum_{i=1}^{4}\left(\bar{Y}_{2i-1,\cdot} - \bar{\bar{Y}}^{(1)}_{\cdot\cdot}\right)^2$$

and

$$SSE_1 = \sum_{j=1}^{b}\sum_{i=1}^{4}\left(Y_{2i-1,j} - \bar{Y}_{2i-1,\cdot} - \bar{Y}^{(1)}_{\cdot j} + \bar{\bar{Y}}^{(1)}_{\cdot\cdot}\right)^2 .$$

Also,

$$SST_2 = b\sum_{i=1}^{4}\left(\bar{Y}_{2i,\cdot} - \bar{\bar{Y}}^{(2)}_{\cdot\cdot}\right)^2$$

and

$$SSE_2 = \sum_{j=1}^{b}\sum_{i=1}^{4}\left(Y_{2i,j} - \bar{Y}_{2i,\cdot} - \bar{Y}^{(2)}_{\cdot j} + \bar{\bar{Y}}^{(2)}_{\cdot\cdot}\right)^2 .$$

Hence,

$$F_1 = b\,\frac{SST_1}{SSE_1} \sim F_{[3,\ 3b]}\ , \quad \text{if } H_{01} \text{ is true}$$

and

$$F_2 = b\,\frac{SST_2}{SSE_2} \sim F_{[3,\ 3b]}\ , \quad \text{if } H_{02} \text{ is true}$$

where

$$H_{01}:\ K_1\underline{\tau} = \underline{0}$$
$$\text{vs}\quad H_{11}:\ K_1\underline{\tau} \neq \underline{0}$$

i.e.,

$$H_{01}:\ \tau_1 = \tau_3 = \tau_5 = \tau_7$$
$$\text{vs}\quad H_{11}:\ \tau_1 > \tau_3 > \tau_5 > \tau_7$$

and

$$H_{02}:\ K_2\underline{\tau} = \underline{0}$$
$$H_{12}:\ K_2\underline{\tau} \neq \underline{0}$$

i.e.,

$$H_{02}:\ \tau_2 = \tau_4 = \tau_6 = \tau_8$$
$$\text{vs}\quad H_{12}:\ \tau_2 > \tau_4 > \tau_6 > \tau_8 \; .$$

If $\ddagger_j$ is identical for each block, i.e., $\ddagger_j = \ddagger$, where $\ddagger$ has the form in (30), a single degree of freedom test can be made on the remaining degree of freedom in the above example. Notice that

$$t - 1 - (q_1 - 1) - (q_2 - 1) = 1 \;, \quad \text{as } q_1 + q_2 = t$$

so there always remains one degree of freedom untested using the $\ddagger$ in (30). A general contrast to use in this test is

$$\underline{a}_1' = \begin{cases} \dfrac{1}{\sqrt{t}}\ [1, -1, 1, -1, \cdots, 1, -1] \;, & \text{if } t \text{ is even} \\[2ex] \sqrt{\dfrac{t-1}{t(t+1)}} \left[1, \dfrac{-(t+1)}{t-1}, 1, -\dfrac{(t+1)}{t-1}, \cdots, 1\right] \;, & \text{if } t \text{ is odd} \end{cases}$$

and the hypothesis becomes

$$H_{01}':\ \underline{a}_1'\underline{\tau} = 0$$
$$\text{vs}\quad H_{11}':\ \underline{a}_1'\underline{\tau} \neq 0 \; .$$

The test statistic to use has been given in (29) and is relatively easy to compute. For the example where $t = 8$, this hypothesis would become

$$H_{01}':\ \tau_1 + \tau_3 + \tau_5 + \tau_7 = \tau_2 + \tau_4 + \tau_6 + \tau_8$$
$$\text{vs}\quad H_{11}':\ \tau_1 + \tau_3 + \tau_5 + \tau_7 > \tau_2 + \tau_4 + \tau_6 + \tau_8$$

Hence, if $\ddagger_j = \ddagger$, there exists three tests which together test for the effects of the t-1 independent treatment contrasts. And, with the $\ddagger_j$ given in (30), it is always possible using this method to find two test statistics which will test the significance of $t-2$ of the $t-1$ independent contrasts. However, these tests, although exact, are dependent and a joint test of significance using them is not known.

The value of this chapter consists in the derivation of an exact test for testing sets of treatment contrasts when the variance-covariance matrix has the form given in (3) or a form that can be transformed to that of (3). Although randomization of treatments to plots is restricted, no approximations are necessary and this is an advantage. The tests, in general, are dependent. Thus, no joint test is available and, at times, some degrees of freedom are analyzed individually causing the power of each test to be diminished. But in situations where M has many pairs of positive-negative roots that are identical, as in the given example, or when M is singular and of small rank, the method of this chapter is extremely valuable. Of special interest is the analysis given in the above example as the covariance matrix of (30) is one that furnishes a good approximation to many real-life problems.

CHAPTER III

THE D-METHOD

In the previous chapter a class of randomized block designs was analyzed where the model was given by

$$\underline{Y}_j = (\mu + \beta_j)\underline{1} + \underline{\tau} + \underline{\varepsilon}_j \,, \quad j = 1, \cdots, b \tag{1}$$

and it was assumed that

$$\underline{\varepsilon}_j \sim N_t(\underline{0}, \Sigma_j), \text{ independently}, \quad j = 1, \cdots, b$$

with the restriction that Σ_j could be written as

$$\Sigma_j = \sigma^2 (I_t + \rho_j M) \tag{2}$$

where σ^2 and ρ_j are unknown constants while M is a known matrix. An exact method for testing the significance of certain sets of treatment contrasts was devised that required the construction of a matrix, C, possessing certain desired properties. And in this approach, M could be either singular or non-singular. In particular, if M was singular and also of small rank, C consisted of the latent vectors orthogonal to M and the vectors formed using pairs of positive-negative latent roots of M, if there were any available. The present chapter attempts to give an alternative approach for these situations, i.e., cases where the rank of M is small in comparison to t. This new method is limited to testing one set of contrasts in the τ's besides the usual single degree of freedom tests. And its test statistics are much easier to compute than those for the C method. It is noted that in this context, Σ_j may be of the form

$$\Sigma_j = \sigma^2[I_t + M(\rho_j)]$$

where the latent roots of M are functions of the unknown ρ_j, but the latent vectors are known constants.

As before, break down M into an additive decomposition, i.e.,

$$M = \sum_{i=1}^{r} \lambda_i \underline{\alpha}_i \underline{\alpha}_i' \tag{3}$$

where λ_i is a non-zero latent root and $\underline{\alpha}_i$ is the corresponding orthonormal vector of M, with r being the rank of M. If one of the $\underline{\alpha}_i$, say $\underline{\alpha}_r$, is $\underline{1}$, then the remaining $\underline{\alpha}_i$ are a set of orthonormal vectors orthogonal to $\underline{1}$. So let

$$\begin{aligned} \underline{\alpha}_i^* &= \underline{\alpha}_i \ , \quad i = 1, \cdots, r-1 \\ \underline{\alpha}_r^* &= \underline{1} \quad . \end{aligned} \tag{4}$$

If none of the $\underline{\alpha}_i$ are $\underline{1}$, adjust the $\underline{\alpha}_i$ so that they are orthogonal to $\underline{1}$, i.e., let

$$\begin{aligned} \underline{x}_i &= (I_t - \frac{1}{t}\underline{1}\underline{1}')\underline{\alpha}_i \\ &= \underline{\alpha}_i - \bar{\alpha}_i \underline{1} \ , \quad i = 1, \cdots, r \end{aligned}$$

where

$$\bar{\alpha}_i = \frac{1}{t}\underline{\alpha}_i' \underline{1} \quad .$$

Now adjust the $\underline{x}_i$ so that they are orthogonal to one another, i.e., let

$$\underline{\alpha}_1^* = \underline{x}_1$$
$$\underline{\alpha}_2^* = h_{12}\underline{x}_1 + \underline{x}_2$$
$$\vdots \qquad \vdots$$
$$\underline{\alpha}_r^* = h_{1r}\underline{x}_1 + h_{2r}\underline{x}_2 + \cdots + \underline{x}_r \tag{5}$$

where the h_{ij} are found by solving equations of the form

$$\underline{\alpha}_i^{*\prime}\underline{\alpha}_i^* = 0 \ , \quad i \neq j,\ i,j = 1, \cdots , r \quad .$$

Finally, orthonormalize the $\underline{\alpha}_i^*$ so that

$$\underline{\alpha}_i^{*\prime}\underline{\alpha}_i^* = 1 \ , \quad i = 1, \cdots , r \quad .$$

Then the $\underline{\alpha}_i^*$ are a set of orthonormal vectors orthogonal to $\underline{1}$. Construct the matrix

$$M^* = \sum_{i=1}^{m} \underline{\alpha}_i^*\underline{\alpha}_i^{*\prime} \tag{6}$$

where

$$m = \begin{cases} r \ , & \text{if no } \underline{\alpha}_i = \underline{1} \\ r-1 \ , & \text{if one } \underline{\alpha}_i\text{, say } \underline{\alpha}_r\text{, is } \underline{1} \end{cases} \tag{7}$$

so that M^* is idempotent and $M^*\underline{1} = \underline{0}$.

Consider now the matrix

$$D = I_t - \frac{1}{t}\,\underline{1}\underline{1}' - M^* \quad . \tag{8}$$

Then

$$\begin{aligned} D^2 &= (I_t - \frac{1}{t}\,\underline{1}\underline{1}' - M^*)(I_t - \frac{1}{t}\,\underline{1}\underline{1}' - M^*) \\ &= (I_t - \frac{1}{t}\,\underline{1}\underline{1}')^2 + (M^*)^2 - 2M^*(I_t - \frac{1}{t}\,\underline{1}\underline{1}') \\ &= D \ , \quad \text{as } (M^*)^2 = M^*,\ M^*\underline{1} = \underline{0} \end{aligned}$$

implying that D is idempotent. The rank of D is given by

$$d = tr(D)$$

$$= tr(I_t - \frac{1}{t}\underline{1}\underline{1}' - M^*)$$

$$= t - 1 - tr(M^*)$$

$$= \begin{cases} t-1-r , & \text{if no } \underline{\alpha}_i = \underline{1} \\ t-r , & \text{if one } \underline{\alpha}_i = \underline{1} \end{cases} \quad . \tag{9}$$

Also,

$$D\underline{1} = (I_t - \frac{1}{t}\underline{1}\underline{1}' - M^*)\underline{1}$$

$$= \underline{0} \;, \quad \text{as } M^*\underline{1} = \underline{0} \tag{10}$$

and

$$D\Sigma_j D = \sigma^2 D(I_t + \rho_j M)D$$

$$= \sigma^2(D^2 + \rho_j DMD)$$

$$= \sigma^2[D + \rho_j(M - \frac{1}{t}\underline{1}\underline{1}'M - M^*M)D] \quad . \tag{11}$$

Note that

$$M^* = \alpha^* \alpha^{*\prime}$$

where

$$\alpha^* = [\underline{\alpha}_1^*, \underline{\alpha}_2^*, \cdots , \underline{\alpha}_m^*]$$

so that

$$M^* + \frac{1}{t}\underline{1}\underline{1}' = \left(\alpha^*, \frac{1}{\sqrt{t}}\underline{1}\underline{1}'\right)\left(\alpha^*, \frac{1}{\sqrt{t}}\underline{1}\underline{1}'\right)' \quad .$$

Let B, (m+1) × (m+1), be an orthogonal transformation such that

$$\left(\alpha^*, \frac{1}{\sqrt{t}}\underline{1}\underline{1}'\right)B = (\alpha, \underline{a})$$

where

$$\alpha = [\underline{\alpha}_1, \underline{\alpha}_2, \cdots, \underline{\alpha}_m]$$

and $\underline{a}$ is some constant vector such that

$$\alpha'\underline{a} = \underline{0} \ .$$

Then

$$M^* + \frac{1}{t}\,\underline{1}\underline{1}' = \left(\alpha^*, \frac{1}{\sqrt{t}}\,\underline{1}\underline{1}'\right)BB'\left(\alpha^*, \frac{1}{\sqrt{t}}\,\underline{1}\underline{1}'\right)', \quad \text{as } BB' = I_{m+1}$$

$$= (\alpha, \underline{a})(\alpha, \underline{a})'$$

$$= \alpha\alpha' + \underline{a}\underline{a}' \quad .$$

Therefore,

$$M^* = \alpha\alpha' + \underline{a}\underline{a}' - \frac{1}{t}\,\underline{1}\underline{1}'$$

and

$$M^*M = \alpha\alpha'M + \underline{a}\underline{a}'M - \frac{1}{t}\,\underline{1}\underline{1}'M \quad .$$

But

$$M = \alpha(\text{diag } \lambda_i)\alpha'$$

so that

$$\alpha\alpha'M = \alpha\alpha'\alpha \ \text{diag}(\lambda_i)\alpha'$$

$$= \alpha \ \text{diag}(\lambda_i)\alpha' \quad , \quad \text{as } \alpha'\alpha = I_t$$

$$= M$$

and

$$\underline{a}\underline{a}'M = \underline{a}\underline{a}' \ \alpha \ \text{diag}(\lambda_i)\alpha$$

$$= \underline{\Phi} \quad , \quad \text{as } \underline{a}'\alpha = \underline{0}' \quad .$$

Hence,

$$M^*M = M - \frac{1}{t}\,\underline{1}\underline{1}'M$$
$$= (I - \frac{1}{t}\,\underline{1}\underline{1}')M$$

so that

$$(I - \frac{1}{t}\,\underline{1}\underline{1}')M - M^*M = \Phi$$

and (11) become

$$D\Sigma_{\cdot j}D = \sigma^2 D \quad . \qquad (12)$$

Consider transforming $\underline{Y}_{\cdot j}$ to $D\underline{Y}_{\cdot j} = \underline{Z}_{\cdot j}$ so that

$$\underline{Z}_{\cdot j} = D\underline{Y}_{\cdot j} = D\underline{\tau} + D\underline{\varepsilon}_{\cdot j} \ , \quad j = 1, \cdots, b \ , \quad \text{by } (10)$$

with

$$E(\underline{Z}_{\cdot j}) = D\underline{\tau}$$

and

$$V(\underline{Z}_{\cdot j}) = D\Sigma_{\cdot j}D$$
$$= \sigma^2 D \ , \quad \text{by } (12) \quad .$$

Since $\underline{Y}_{\cdot j}$ are i.i.d. normal variates, it follows that

$$\underline{Z}_{\cdot j} \sim N_t(D\underline{\tau}, \sigma^2 D), \text{ independently} \ , \quad j = 1, \cdots, b$$

and

$$\bar{\underline{Z}}_{\cdot} = \frac{1}{b}\sum_{j=1}^{b} \underline{Z}_{\cdot j} \sim N_t(D\underline{\tau}, \frac{1}{b}\sigma^2 D) \quad . \qquad (13)$$

This leads to the quadratic form

$$SST_0 = b\bar{\underline{Z}}_{\cdot}'D\bar{\underline{Z}}_{\cdot}$$

$$= b\bar{\underline{Y}}_{\cdot}'DD\bar{\underline{Y}}_{\cdot} \quad , \quad \bar{\underline{Y}}_{\cdot} = \frac{1}{b}\sum_{j=1}^{b} \underline{Y}_{j}$$

$$= b\bar{\underline{Y}}_{\cdot}'D\bar{\underline{Y}}_{\cdot} \quad , \quad \text{as } D^2 = D$$

$$= SST - \sum_{k=1}^{m} SST_k \qquad (14)$$

where SST is the usual sum of squares of treatments in the model of (1), i.e.,

$$SST = b\sum_{i=1}^{t} (\bar{Y}_{i\cdot} - \bar{\bar{Y}}_{\cdot\cdot})^2$$

with

$$\bar{Y}_{i\cdot} = \frac{1}{b}\sum_{j=1}^{b} Y_{ij} \quad \text{and} \quad \bar{\bar{Y}}_{\cdot\cdot} = \frac{1}{bt}\sum_{j=1}^{b}\sum_{i=1}^{t} Y_{ij}$$

and

$$SST_k = b(\bar{\underline{Y}}_{\cdot}'\underline{\alpha}_k^*)^2$$

From (13) it follows that $\frac{SST_0}{\sigma^2}$ is distributed as a chi-square since D is idempotent, i.e.,

$$\frac{SST_0}{\sigma^2} \sim \chi_d^2(\lambda) \qquad (15)$$

where

$$d = tr(D)$$

$$= \begin{cases} t-1-r , & \text{if no } \underline{\alpha}_i = \underline{1} \\ t-r , & \text{if one } \underline{\alpha}_i = \underline{1} \end{cases} \qquad \text{from (9)}$$

and

and $$\lambda = \frac{b}{2\sigma^2} \underline{\tau}'\mathrm{DDD}\underline{\tau}$$

$$= \frac{b}{2\sigma^2} \underline{\tau}'\mathrm{D}\underline{\tau}$$

$$= \frac{b}{2\sigma^2} \underline{\tau}'(\mathrm{I} - \frac{1}{t}\,\underline{1}\,\underline{1}' - \mathrm{M}^*)\underline{\tau} \quad , \quad \text{by (8)}$$

$$= \frac{b}{2\sigma^2} \left\{ \sum_{i=1}^{t} \tau_i^2 - \sum_{k=1}^{m} (\underline{\tau}'\underline{\alpha}_i^*)^2 \right\}, \quad \text{as } \underline{\tau}'\underline{1} = 0 \quad .$$

Since $\mathrm{D}\underline{1} = \underline{0}$, $\mathrm{D}\underline{\tau}$ is a system of contrasts in the τ's. And it will be shown below that there exists a test statistic for testing the hypothesis

$$H_0: \lambda = 0$$
$$\text{vs} \quad H_1: \lambda \neq 0 \quad .$$

But this is equivalent to the hypothesis

$$H_0: \mathrm{D}\underline{\tau} = \underline{0}$$
$$\text{vs} \quad H_1: \mathrm{D}\underline{\tau} \neq \underline{0} \qquad (16)$$

which can be written as

$$H_0: \underline{d}_i'\underline{\tau} = 0$$
$$\text{vs} \quad H_1: \underline{d}_i'\underline{\tau} \neq 0$$

where the $\underline{d}_i$ are linear combinations of the rows of D and are mutually orthogonal, or they are the basis of the vector space of D and are necessarily orthogonal to $\underline{1}$.

Using the results of Appendix A, the test statistic for H_0 is

$$F_0 = (b-1)\,\frac{SST_0}{SSE_0} \qquad (17)$$

where

$$SST_0 = \underline{Y}'Q_t(A)\underline{Y} \ , \quad \text{using (A1)}$$

with

$$A = B = D$$

and

$$\underline{Y}' = (\underline{Y}_1', \underline{Y}_2', \cdots, \underline{Y}_b')$$

and

$$SSE_0 = \underline{Y}'Q(A)\underline{Y} \ , \quad \text{using (A2)} \tag{18}$$

i.e.,

$$SSE_0 = SSE - \sum_{k=1}^{m} SSE_k \tag{19}$$

where SSE is the usual sum of squares of error in the model of (1), i.e.,

$$SSE = \sum_{j=1}^{b} \sum_{i=1}^{t} (Y_{ij} - \bar{Y}_{.j} - \bar{Y}_{i.} + \bar{\bar{Y}}_{..})^2$$

and

$$SSE_k = \sum_{j=1}^{b} [(\underline{Y}_j - \bar{\underline{Y}}_.)' \underline{\alpha}_k^*]^2 \quad .$$

Recall that it was assumed in (1) that

$$\underline{Y} \sim N_{bt}[E(\underline{Y}), \Sigma] \ , \quad \Sigma = \text{diag}(\ddagger_j)$$

so $\frac{SSE_0}{\sigma^2}$ is distributed as a chi-square if $\frac{1}{\sigma^2} Q(A)\Sigma$ is idempotent. But this follows from (A4) and (12) and

$$\frac{SSE_0}{\sigma^2} \sim \chi_e^2(\lambda_e)$$

where

$$e = tr\left(\frac{1}{\sigma^2} Q(A)\Sigma\right)$$

$$= \frac{1}{\sigma^2}(b-1)b\ tr(A\ddagger_j) \ , \quad \text{from (A6)}$$

$$= (b-1)tr(D) \ , \quad \text{from (12)}$$

$$= (b-1)d \ , \quad \text{from (9)}$$

and

$$\lambda_e = 0 \ , \quad \text{from (A7)}$$

since

$$E(\underline{Y}_j')D = \underline{\tau}'D = \text{constant} \ , \quad \text{for all } j \quad .$$

Therefore,

$$\frac{SSE_0}{\sigma^2} \sim \chi^2_{(b-1)d}(0) \tag{20}$$

and from (15)

$$\frac{SST_0}{\sigma^2} \sim \chi^2_d(0) \ , \quad \text{if } H_0 \text{ of (16) is true} \ .$$

Further, SST_0 and SSE_0 are independent since

$$Q_t(A)\Sigma Q(A) = \Phi \ , \quad \text{using (A3) and (12)} \quad .$$

Therefore, the ratio of SST_0 to SSE_0 divided by their respective degrees of freedom yields

$$F_0 = (b-1)\frac{SST_0}{SSE_0} \sim F_{[d,(b-1)d]} \ , \quad \text{if } H_0 \text{ is true} \quad . \tag{21}$$

And H_0 can be tested by evaluating

$$SST_0 = SST - \sum_{k=1}^{m} SST_k$$

and

$$SSE_0 = SSE - \sum_{k=1}^{m} SSE_k \quad .$$

Since d should be close to t-1 there will remain only a few contrasts in the τ's that are not included in (16). If Σ_j is identical from block to block, i.e., $\Sigma_j = \Sigma$, or $\rho_j = \rho$, for all j, it will be possible to test for the significance of these contrasts. Consider hypotheses of the form

$$H_{0k}: \quad \underline{\alpha}_k^{*\prime}\underline{\tau} = 0 \ , \quad k = 1, \cdots , m \tag{22}$$

and recall that the $\underline{\alpha}_k^*$ are orthonormal vectors with

$$\underline{\alpha}_k^{*\prime}\underline{1} = 0 \ , \quad k = 1, \cdots , m$$

$$\underline{\alpha}_k^{*\prime}D = \underline{\alpha}_k^{*\prime}(I - \frac{1}{t}\,\underline{1}\underline{1}' - M^*)$$

$$= \underline{0}' \ , \quad \text{as } \underline{\alpha}_k^{*\prime}M = \underline{\alpha}_k^{*\prime} \quad . \tag{23}$$

Then

$$\underline{\alpha}_k^{*\prime}\underline{Y}_j \sim N.I.D.(\underline{\alpha}_k^{*\prime}\underline{\tau},\ \underline{\alpha}_k^{*\prime}\Sigma\underline{\alpha}_k^*)$$

so that

$$\underline{\alpha}_k^{*\prime}\underline{\bar{Y}}_{\cdot} \sim N(\underline{\alpha}_k^{*\prime}\underline{\tau},\ \frac{1}{b}\,a_k)$$

where

$$a_k = \underline{\alpha}_k^{*\prime}\ \Sigma\ \underline{\alpha}_k^* \quad .$$

It is well known that if $x \sim N(\mu, \sigma^2)$, then $\frac{n\bar{x}^2}{\sigma^2} \sim \chi_1^2\left(\frac{\mu^2}{2\sigma^2}\right)$ and $\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \bar{x})^2 \sim \chi_{n-1}^2$, independently. Thus,

$$\frac{SST_k}{a_k} \sim \chi_1^2(\lambda_k)$$

where

$$SST_k = b(\underline{\bar{Y}}_{\cdot}'\underline{\alpha}_k^*)^2 \quad , \quad \text{from (14)}$$

$$= \underline{Y}'Q_t(A_k)\underline{Y} \quad , \quad \text{using (A1)}$$

with

$$A_k = \underline{\alpha}_k^* \underline{\alpha}_k^{*\prime}$$

and

$$\lambda_k = \frac{b}{2\sigma^2}(\underline{\tau}'\underline{\alpha}_k^*)^2 \quad .$$

Also,

$$SSE_k = \sum_{j=1}^{b} [(\underline{Y}_j - \underline{\bar{Y}}_{\cdot})'\underline{\alpha}_k^*]^2 \quad , \quad \text{from (19)}$$

$$= \underline{Y}'Q(A_k)\underline{Y} \quad .$$

Then the test statistic for testing (22) becomes

$$F_k = (b-1)\frac{SST_k}{SSE_k} \sim F_{(1,b-1)} \quad , \quad \text{if } H_{0k} \text{ is true} \quad . \tag{24}$$

Hence, tests of single orthogonal contrasts are possible. And each of these tests are independent of the statistic given in (21).

Note that SST_0 and SSE_k are independent as

$$Q_t(A)\Sigma Q(A_k) = \Phi \quad , \quad \text{for all } k$$

using (A3) and the fact that

$$A \ddagger A_k = D \ddagger \underline{\alpha}^*_k \underline{\alpha}^{*\prime}_k$$

$$= \Phi \ , \quad \text{for all } k, \text{ from (12) and (23)} \ . \qquad (25)$$

Also, SST_k and SSE_0 are independent since

$$Q_t(A_k)\Sigma\, Q(A) = \Phi \ , \quad \text{for all } k$$

using (A3) and the result that

$$A_k \ddagger A = (A \ddagger A_k)' = \Phi \ , \quad \text{from above} \ .$$

Further, SSE_0 and SSE_k are independent since

$$Q(A)\Sigma Q(A_k) = \Phi \ , \quad \text{using (A9) and (25)} \ ,$$

and SST_0 and SST_k are independent since

$$Q_t(A)\Sigma Q_t(A_k) = \Phi \ , \quad \text{using (A10) and (23)} \ .$$

Therefore, in all cases, F_0 and F_k are independent. However, the F_k are usually dependent since $\underline{\alpha}^{*\prime}_k M \underline{\alpha}^*_{k'}$, $k \neq k'$, is not necessarily zero. The result is a set of correlated tests in addition to the independent test using F_0 . Because of this, it is possible to combine F_0 with any of the F_k, say F_1, to form a single test on the joint hypothesis. And this can be done in several ways.

Consider the hypothesis

$$H_0': \begin{cases} \underline{\alpha}^{*\prime}_1 \underline{\tau} = 0 \\ D\, \underline{\tau} = \underline{0} \end{cases} \ . \qquad (26)$$

One simple method of testing H_0' would be to test H_0 of (16) at a significance level of

$$\alpha_0 = \Pr\left\{F_0 \geq F_{[d,(b-1)d]} \;\middle|\; H_0\right\}$$

which is the probability of rejecting H_0 when H_0 is true, and test H_{01} of (22) at a significance level of

$$\alpha_1 = \Pr\left\{F_1 \geq F_{[1,b-1]} \;\middle|\; H_{01}\right\} \quad .$$

Then reject H_0' if either H_0 or H_{01} is rejected, and do not reject H_0' if both H_0 and H_{01} are not rejected. Since F_0 and F_1 are independent

$$\Pr\left\{F_0 \leq F_{[d,(b-1)d]},\ F_1 \leq F_{[1,b-1]}\right\} = \Pr\left\{F_0 \leq F_{[d,(b-1)d]}\right\} \cdot \Pr\left\{F_1 \leq F_{[1,b-1]}\right\}$$

$$= (1 - \alpha_0)(1 - \alpha_1)$$

and it follows that

$$\alpha = \Pr\left\{\text{reject } H_0' \middle| H_0'\right\}$$

$$= 1 - (1-\alpha_0)(1-\alpha_1)$$

and α will be the significance level of this test. Note, however, that for fixed α there exist an infinite number of choices for α_0 and α_1 .

To eliminate this disadvantage consider combining F_0 and F_1 into a single test so that the power of the combined test is greater than either of the individual tests. The method was developed by Zelen and Joel [1959]. Let

$$P_j = \Pr\left\{F \geq F_j \;\middle|\; H_0'\right\}, \quad j = 0, 1$$

which is the probability of the F-ratio exceeding the calculated F_j if H_0' is true. Then the critical region for this hypothesis is given by

$$\omega: \left\{P_0 P_1^{\theta} \le C_\alpha\right\} \tag{27}$$

where C_α is a constant depending on an α-level of significance and θ is a weighting factor $(0 \le \theta \le 1)$ which weights F_0 relative to F_1 . When $\theta = 1$, both tests are given equal weight and this is equivalent to the method of Fisher [1954] for combining independent tests of significance. In this case the probability of a Type I error for the combined test is

$$\alpha = \Pr\left\{\text{reject } H_0' \middle| H_0'\right\}$$

$$= \Pr\left\{P_0 P_1 \le C_\alpha \middle| H_0'\right\}$$

$$= \iint_\omega dP_0\, dP_1 , \qquad \text{given in (27)}$$

since P_j is distributed uniform when H_0' is true. By fixing α it is possible to solve for C_α to find the critical value of this test.

Since F_0 has more degrees of freedom than F_1 it should be weighted more than F_1, i.e., there should be a better choice for θ than $\theta = 1$. In H_0 there are d independent contrasts being considered each having the form $\underline{d}_i'\underline{\tau} = \tau_i^*$, where $\underline{d}_i'\underline{1} = 0$, $i = 1, \cdots, d$; and in H_{01} there is one contrast, say $\underline{\alpha}_1^{*\prime}\underline{\tau} = \tau_{d+1}^*$. In totality there are t-1 possible contrasts so the given τ's span a certain portion of the parameter space of $\underline{\tau}$. It seems appropriate then to weight the statistics for H_0 and H_{01} in proportion to their influence on this space. A good choice for θ would be

$$\theta = d^{-1}$$

so that

$$\omega : \left\{ P_0 \, P_1^{\frac{1}{d}} \leq C_\alpha \right\}$$

and

$$\alpha = \iint_{\omega} dP_0 \, dP_1 \quad .$$

On the average this result will lead to more power in the combined test if θ has been chosen correctly and it eliminates having to choose α_0 and α_1 .

In summary, when the rank of M is small a method for testing the significance of the d contrasts in the τ's as given in (16) was developed. The derived test is unique and is based on an F-statistic, i.e., the F-distribution is used. For all other contrasts single degree of freedom tests exist provided $\Sigma_j = \Sigma$. And these individual tests, although usually correlated among themselves, are all independent of the test for (16). Each test is exact and easily performed, but a joint statistic would be difficult to find. In situations where $\Sigma_j = \Sigma$, for all j, one of the single degree of freedom tests can be combined with the formulated tests, and a combined test results. It is then an easy task to analyze the combined hypothesis at any fixed α-level of significance. This set and the other sets of contrasts formed, span the space of $\underline{\tau}$ and have a total rank of t-1 .

To better understand the advantages of this method, it will be helpful to sketch an example. Consider a randomized block experiment using the model in (1) with a variance-covariance matrix, Σ, similar to the dispersion matrix proposed by Williams [1970] for the offspring in an animal breeding experiment. There is a slight modification in that

$$\ddagger = \sigma^2\left(I_t + \sum_{i=1}^{3} \lambda_i \underline{\alpha}_i \underline{\alpha}_i'\right)$$

where λ_i is a function of an unknown ρ and the $\underline{\alpha}_i$ are known vectors of constants such that $\ddagger$ is positive definite. Then

$$M = \sum_{i=1}^{3} \lambda_i \underline{\alpha}_i \underline{\alpha}_i'$$

and by constructing the vectors in (5), M* can be obtained using

$$M^* = \sum_{k=1}^{m} \underline{\alpha}_k^* \underline{\alpha}_k^{*\prime}$$

where

$$m = \begin{cases} 3 , & \text{if one } \underline{\alpha}_i = \underline{1} \\ 2 , & \text{if no } \underline{\alpha}_i = \underline{1} \end{cases} \quad ;$$

therefore,

$$D = I_t - \frac{1}{t} \underline{1}\underline{1}' - \sum_{k=1}^{m} \underline{\alpha}_k^* \underline{\alpha}_k^{*\prime} \quad .$$

Compute

$$SST_0 = SST - \sum_{k=1}^{m} SST_k , \quad \text{from (14)}$$

and

$$SSE_0 = SSE - \sum_{k=1}^{m} SSE_k , \quad \text{from (19)} \quad .$$

Then one can test

$$H_0: \; D\underline{\tau} = \underline{0}$$

$$\text{vs} \quad H_1: \; D\underline{\tau} \neq \underline{0}$$

using $F_0 = (b-1)\,\frac{SST_0}{SSE_0}$ as a test statistic. Compare F_0 to a tabled $F_{[d,(b-1)d]}$ where

$$d = \begin{cases} t-4\ , & \text{if no } \underline{\alpha}_i = \underline{1} \\ t-3\ , & \text{of one } \underline{\alpha}_i = \underline{1} \end{cases} .$$

Single degree of freedom tests having the form of (22) can also be made since $\ddagger_j \equiv \ddagger$. Further, instead of using the test above for H_0, one can make a combined test on the hypothesis

$$H_0': \begin{cases} \underline{\alpha}_k^{*\prime}\underline{\tau} = 0 \\ D\,\underline{\tau} = \underline{0} \end{cases}$$

which considers d+1 independent contrasts. The method to use has been outlined above.

In this chapter a new criterion has been derived for testing sets of treatment contrasts for the model in (1) with the covariance matrix of (2) where the matrix M is assumed to be singular and of small rank. Since this method requires only the calculation of the latent vectors of the non-zero roots of M, it has much value in the above situations. It is extremely easy to compute the test statistics, as is evident from the formulaes in (14) and (19), and single degree of freedom tests require little additional work. The method of Chapter I, however, can also be used. But this approach requires the derivation of all the latent vectors and roots of M, and necessitates much more time and effort in obtaining its test statistics. So the present D-method, due to its ease and simplicity, would usually be preferred over the C-method when M has the above properties.

CHAPTER IV

TWO TESTS FOR EQUALITY OF TREATMENT MEANS

In Chapter II a randomized block experiment was introduced using the model

$$\underline{Y}_j = (\mu + \beta_j)\underline{1} + \underline{\tau} + \underline{\varepsilon}_j \ , \quad j = 1, \cdots , b \tag{1}$$

where

$$\underline{\varepsilon}_j \sim N_t(\underline{0}, \Sigma_j) \text{, independently,} \quad j = 1, \cdots , b \tag{2}$$

and Σ_j could be transformed to the special form

$$\Sigma_j = \sigma^2(I_t + \rho_j M) \quad . \tag{3}$$

Consider now the case where $\rho_j = \rho$, for all j, so that $\Sigma_j = \Sigma$, for all j. Of interest is the hypothesis

$$H_0: \ \tau_i = 0 \ , \quad i = 1, \cdots , t$$
$$\text{vs} \quad H_1: \ \text{at least one } \tau_i \neq 0 \tag{4}$$

which is equivalent to the hypothesis

$$H_0: \ H'\underline{\tau} = \underline{0}$$
$$\text{vs} \quad H_1: \ H'\underline{\tau} \neq \underline{0}$$

where H is a t × (t-1) matrix satisfying

$$HH' = I_t - \frac{1}{t}\underline{1}\underline{1}' = Q_t \tag{5}$$

$$H'H = I_{t-1} \tag{6}$$

and

$$H'\underline{1} = \underline{0} \quad .$$

This chapter derives an exact as well as an approximate test statistic for testing this hypothesis. If ρ is known the exact test should be made; and if ρ is unknown, either test can be made provided there exists an estimate for ρ. In either case one can analyze the effects of all the treatment contrasts.

Consider now the derivation of the exact test statistic. Since the $\underline{Y}_j$ are i.i.d. normal variates it follows that

$$H'\bar{\underline{Y}}_{\cdot} \sim N(H'\underline{\tau}, \frac{1}{b} H'\Sigma H) \quad . \tag{7}$$

Let

$$SST^* = \sigma^2 b\bar{\underline{Y}}_{\cdot}'H(H'\Sigma H)^{-1}H'\bar{\underline{Y}}_{\cdot} \quad . \tag{8}$$

Then $\frac{SST^*}{\sigma^2}$ is a chi-square variate if $(H'\Sigma H)(H'\Sigma H)^{-1}$ is idempotent. This holds since

$$[(H'\Sigma H)(H'\Sigma H)^{-1}]^2 = I_{t-1} = (H'\Sigma H)(H'\Sigma H)^{-1} \quad .$$

Therefore,

$$\frac{SST^*}{\sigma^2} \sim \chi^2_{t^*}(\lambda^*) \tag{9}$$

with

$$t^* = tr[(H'\Sigma H)(H'\Sigma H)^{-1}]$$

$$= t-1 \tag{10}$$

and

$$\lambda^* = \frac{b}{2}\,\underline{\tau}'H(H'\Sigma H)^{-1}H'\underline{\tau} \quad .$$

Notice that $(H'\Sigma H)^{-1}$ can be written as

$$(H'\Sigma H)^{-1} = \sum_{i=1}^{t-1} \omega_i\, \underline{\gamma}_i\, \underline{\gamma}_i'$$

where ω_i is a latent root (positive) and $\underline{\gamma}_i$ is the corresponding latent vector of $(H'\Sigma H)^{-1}$. Hence,

$$\lambda^* = \frac{b}{2} \sum_{i=1}^{t-1} \omega_i (\underline{\gamma}_i'H'\underline{\tau})^2 \quad .$$

So the hypothesis

$$H_0: \ \lambda^* = 0$$

$$\text{vs} \quad H_1: \ \lambda^* \neq 0 \qquad (11)$$

is equivalent to the hypothesis

$$H_0: \ \underline{\gamma}_i'H'\underline{\tau} = 0 \ , \quad i = 1, \cdots , t-1$$

$$\text{vs} \quad H_1: \ \text{at least one } \underline{\gamma}_i'H'\underline{\tau} \neq 0 \quad .$$

But the $\underline{\gamma}_i'H'$ are linear combinations of the rows of H' and are mutually orthogonal since

$$\underline{\gamma}_i'H'H\underline{\gamma}_{i'} = \underline{\gamma}_i'\,\underline{\gamma}_{i'} \ , \quad \text{as } H'H = I_{t-1}$$

$$= 0 \ , \quad i \neq i' \ , \quad \text{as } \underline{\gamma}_i \text{ are orthogonal} \quad .$$

So this hypothesis is equivalent to the one in (4), i.e.,

$$H_0: \; H'\underline{\tau} = \underline{0} \;, \quad \text{or, } \tau_i = 0, \text{ for all } i$$

$$\text{vs} \quad H_1: \; H'\underline{\tau} \neq \underline{0} \;, \quad \text{or, at least one } \tau_i = 0$$

It will be shown below that there exists a test statistic for testing this hypothesis.

Using Appendix A, let

$$A = H(H'\ddagger H)^{-1}H'\sigma^2$$

so that

$$SST^* = \underline{Y}'Q_t(A)\underline{Y} \;, \quad \text{using (A1)}$$

and let

$$SSE^* = \underline{Y}'Q(A)\underline{Y} \;, \quad \text{using (A2)} \quad .$$

Recall that it was assumed that

$$\underline{Y} \sim N_{bt}[E(\underline{Y}), \Sigma] \quad , \quad \Sigma = \text{diag}(\ddagger) \quad .$$

Then $\frac{SSE^*}{\sigma^2}$ is distributed as a chi-square if $\frac{1}{\sigma^2} Q(A)\Sigma$ is idempotent. Now,

$$A\ddagger A = \sigma^2 H(H'\ddagger H)^{-1}H'\ddagger H(H'\ddagger H)^{-1}H'\sigma^2$$

$$= \sigma^2 A \tag{12}$$

and this result with that of (A4) implies that

$$\left(\frac{1}{\sigma^2} Q(A)\Sigma\right)^2 = \frac{1}{\sigma^2} Q(A)\Sigma \quad .$$

Hence,

$$\frac{SSE^*}{\sigma^2} \sim \chi^2_e(\lambda_e)$$

where

$$e = tr\left(\frac{1}{\sigma^2} Q(A)\Sigma\right)$$

$$= \frac{1}{\sigma^2} (b-1)tr(A\ddagger) , \quad \text{from (A6)}$$

$$= \frac{1}{\sigma^2} (b-1)tr[(H'\ddagger H)^{-1}(H'\ddagger H)\sigma^2] , \quad \text{by cyclic permutation}$$

$$= (b-1)(t-1)$$

and

$$\lambda_e = 0 , \quad \text{from (A7)}$$

since

$$E(\underline{Y}_j')A = \underline{\tau}'H(H'\ddagger H)^{-1}H' , \quad \text{for all } j .$$

Therefore,

$$\frac{SSE^*}{\sigma^2} \sim \chi^2_{(b-1)(t-1)}(0) \tag{13}$$

and from (9) $\frac{SST^*}{\sigma^2} \sim \chi^2_{(t-1)}(0)$, if H_0 of (4) is true. Further, SST* and SSE* are independent since (A3) and (12) imply that $Q_t(A)\Sigma Q(A) = \Phi$. Therefore, SSE* and SST* are independent chi-squares and their ratio divided by their respective degrees of freedom yield

$$F^* = (b-1)\frac{SST^*}{SSE^*} \sim F_{[(t-1),(b-1)(t-1)]} , \quad \text{if } H_0 \text{ of (4) is true} . \tag{14}$$

In order to evaluate $(H'\ddagger H)^{-1}$ consider the orthogonal matrix, H*, t × t, where

$$H^* = \left[H \;\middle|\; \frac{1}{\sqrt{t}}\underline{1}\right] . \tag{15}$$

Then

$$H^{*\prime}H^{*} = \begin{bmatrix} H'H & \underline{0} \\ \underline{0}' & \frac{1}{t}\,\underline{1}'\underline{1} \end{bmatrix} \quad , \text{ as } H'\underline{1} = \underline{0}$$

$$= I_t \qquad , \text{ from (6)}$$

and

$$H^{*}H^{*\prime} = HH' - \frac{1}{t}\,\underline{1}\underline{1}'$$

$$= I_t \qquad , \text{ from (5)} \quad .$$

Therefore,

$$H^{*\prime}H^{*} = H^{*}H^{*\prime} = I_t \quad . \tag{16}$$

So

$$(H^{*\prime}\Sigma H^{*})^{-1} = (H^{*})^{-1}\Sigma^{-1}(H^{*\prime})^{-1}$$

$$= H^{*\prime}\Sigma^{-1}H^{*} \quad , \quad \text{from (16)}$$

where

$$\Sigma^{-1} = \frac{1}{\sigma^2}\left[I_t + \rho M^{\frac{1}{2}} M'^{\frac{1}{2}}\right]^{-1} \quad , \quad M = M^{\frac{1}{2}} M'^{\frac{1}{2}}$$

$$= \frac{1}{\sigma^2}\left[I_t - \rho M^{\frac{1}{2}}\left(I + \rho M'^{\frac{1}{2}} M^{\frac{1}{2}}\right)^{-1} M'^{\frac{1}{2}}\right]$$

since

$$(I + PQ)^{-1} = I - P(I + PQ)^{-1}Q \quad .$$

Now equation (7) of Chapter II implies that

$$M = [\underline{\alpha}_1, \cdots, \underline{\alpha}_t] \operatorname{diag}(\lambda_i) \begin{bmatrix} \underline{\alpha}_1' \\ \cdot \\ \cdot \\ \cdot \\ \underline{\alpha}_t' \end{bmatrix} \tag{17}$$

so that

$$M^{\frac{1}{2}} = [\underline{\alpha}_1, \cdots, \underline{\alpha}_t] \operatorname{diag}\left(\sqrt{\lambda_i}\right) .$$

Therefore,

$$M'^{\frac{1}{2}} M^{\frac{1}{2}} = \operatorname{diag}\left(\sqrt{\lambda_i}\right) \begin{bmatrix} \underline{\alpha}_1' \\ \cdot \\ \cdot \\ \cdot \\ \underline{\alpha}_t' \end{bmatrix} [\underline{\alpha}_1, \cdots, \underline{\alpha}_t] \operatorname{diag}\left(\sqrt{\lambda_i}\right)$$

$$= \operatorname{diag}(\lambda_i) \quad , \quad \text{as } \underline{\alpha}_i' \underline{\alpha}_j = 0, \ i \neq j$$

and

$$\left(I + \rho M'^{\frac{1}{2}} M^{\frac{1}{2}}\right)^{-1} = [I + \rho \operatorname{diag}(\lambda_i)]^{-1}$$

$$= \operatorname{diag}(1 + \rho\lambda_i)^{-1} .$$

Then

$$M^{\frac{1}{2}} \left(I + \rho M'^{\frac{1}{2}} M^{\frac{1}{2}}\right)^{-1} M'^{\frac{1}{2}} = [\underline{\alpha}_1, \cdots, \underline{\alpha}_t] \operatorname{diag}\left(\lambda_i (1 + \rho\lambda_i)^{-1}\right) \begin{bmatrix} \underline{\alpha}_1' \\ \cdot \\ \cdot \\ \cdot \\ \underline{\alpha}_t' \end{bmatrix}$$

$$= \sum_{i=1}^{t} \lambda_i^* \underline{\alpha}_i \underline{\alpha}_i'$$

with

$$\lambda_i^* = \frac{\lambda_i}{1 + \rho\lambda_i} \tag{18}$$

and

$$\Sigma^{-1} = \frac{1}{\sigma^2}\left[I_t - \rho \sum_{i=1}^{t} \lambda_i^* \underline{\alpha}_i \underline{\alpha}_i'\right] .$$

Therefore,

$$\sigma^2 (H^{*\prime}\Sigma H^*)^{-1} = \sigma^2 H^{*\prime}\Sigma^{-1} H^*$$

$$= I_t - \rho \sum_{i=1}^{t} \lambda_i^* H^{*\prime} \underline{\alpha}_i \underline{\alpha}_i' H^* , \quad \text{from (16)}$$

$$= \begin{bmatrix} I_{t-1} - \rho \sum_1^t \lambda_i^* H' \underline{\alpha}_i \underline{\alpha}_i' H & -\frac{1}{\sqrt{t}} \rho \sum_1^t \lambda_i^* (\underline{1}'\underline{\alpha}_i) H' \underline{\alpha}_i \\ -\frac{1}{\sqrt{t}} \rho \sum_1^t \lambda_i^* (\underline{1}'\underline{\alpha}_i) \underline{\alpha}_i' H & \frac{1}{t}\left(t - \rho \sum_1^t \lambda_i^* (\underline{1}'\underline{\alpha}_i)^2\right) \end{bmatrix} . \tag{19}$$

But

$$\sigma^2 (H^{*\prime}\Sigma H^*) = \sigma^2 \begin{bmatrix} H'\Sigma H & \frac{1}{\sqrt{t}} H'\Sigma \underline{1} \\ \frac{1}{\sqrt{t}} \underline{1}'\Sigma H & \frac{1}{t} \underline{1}'\Sigma \underline{1} \end{bmatrix}^{-1} . \tag{20}$$

Equating (19) and (20) yields

$$\sigma^2 (H'\Sigma H)^{-1} = I_{t-1} - \rho \sum_1^t \lambda_i^* H' \underline{\alpha}_i \underline{\alpha}_i' H - \rho^2 \frac{\left(\sum_1^t \lambda_i^* \underline{1}'\underline{\alpha}_i H' \underline{\alpha}_i\right)\left(\sum_1^t \lambda_i^* \underline{1}'\underline{\alpha}_i \underline{\alpha}_i' H\right)}{t - \rho \sum_1^t \lambda_i^* (\underline{1}'\underline{\alpha}_i)^2}$$

so that

$$\sigma^2 H(H'\ddagger H)^{-1}H' = \underset{\sim}{Q}_t - \rho \sum_1^t \lambda_i^* \underset{\sim}{Q}_t \underline{\alpha}_i \underline{\alpha}_i' \underset{\sim}{Q}_t - \rho^2 \frac{\left(\sum_1^t \lambda_i^* \underline{1}'\underline{\alpha}_i \underset{\sim}{Q}_t \underline{\alpha}_i\right)\left(\sum_1^t \lambda_i^* \underline{1}'\underline{\alpha}_i \underline{\alpha}_i' \underset{\sim}{Q}_t\right)}{t - \rho \sum_1^t \lambda_i^* (\underline{1}'\underline{\alpha}_i)^2} .$$

Therefore,

$$SST^* = \sigma^2 b \bar{\underline{Y}}_{\cdot}' H(H'\ddagger H)^{-1} H' \bar{\underline{Y}}_{\cdot}$$

$$= b \left\{ \bar{\underline{Y}}_{\cdot}' \underset{\sim}{Q}_t \bar{\underline{Y}}_{\cdot} - \rho \sum_1^t \lambda_i^* (\bar{\underline{Y}}_{\cdot}' \underset{\sim}{Q}_t \underline{\alpha}_i)^2 - \rho^2 \frac{\left[\sum_1^t \lambda_i^* \, \underline{1}' \, \underline{\alpha}_i \, \bar{\underline{Y}}_{\cdot}' \, \underset{\sim}{Q}_t \, \underline{\alpha}_i\right]^2}{t - \rho \sum_1^t \lambda_i^* (\underline{1}' \, \underline{\alpha}_i)^2} \right\}$$

$$= SST - \rho SST_1 - \rho^2 SST_2 \tag{21}$$

where SST is the usual sum of squares treatment in a randomized block experiment, i.e.,

$$\left. \begin{aligned} SST &= b \sum_{i=1}^t (\bar{Y}_{i\cdot} - \bar{\bar{Y}}_{\cdot\cdot})^2 \\ SST_1 &= b \sum_{i=1}^t \lambda_i^* (\bar{\underline{Y}}_{\cdot}' \underset{\sim}{Q}_t \underline{\alpha}_i)^2 \\ SST_2 &= b \frac{\left[\sum_1^t \lambda_i^* (\underline{1}'\underline{\alpha}_i)(\bar{\underline{Y}}_{\cdot}' \underset{\sim}{Q}_t \underline{\alpha}_i)\right]^2}{t - \rho \sum_1^t \lambda_i^* (\underline{1}'\underline{\alpha}_i)^2} \end{aligned} \right\} \tag{22}$$

Further, recall

$$SSE^* = \underline{Y}'Q(A)\underline{Y}$$

$$= \sum_{j=1}^{b} \underline{Y}_j' A \underline{Y}_j - b\bar{\underline{Y}}_{\cdot}' A \bar{\underline{Y}}_{\cdot}$$

$$= \sum_{j=1}^{b} (\underline{Y}_j - \bar{\underline{Y}}_{\cdot})' A (\underline{Y}_j - \bar{\underline{Y}}_{\cdot})$$

$$= SSE - \rho \sum_{j=1}^{b} \sum_{i=1}^{t} \lambda_i^* [(\underline{Y}_j - \bar{\underline{Y}}_{\cdot})' Q_t \underline{\alpha}_i]^2 - \rho^2 \sum_{j=1}^{b} \frac{\left[\sum_{i=1}^{t} \lambda_i^* \underline{1}' \underline{\alpha}_i (\underline{Y}_j - \bar{\underline{Y}}_{\cdot})' Q_t \underline{\alpha}_i\right]^2}{t - \rho \sum_{i=1}^{t} \lambda_i^* (1' \underline{\alpha}_i)^2}$$

$$= SSE - \rho SSE_1 - \rho^2 SSE_2 \qquad (23)$$

where SSE is the usual sum of squares of error in a RBD, i.e.,

$$\left.\begin{aligned} SSE &= \sum_{j=1}^{b} \sum_{i=1}^{t} (Y_{ij} - \bar{Y}_{i\cdot} - \bar{Y}_{\cdot j} + \bar{\bar{Y}}_{\cdot\cdot})^2 \\ SSE_1 &= \sum_{j=1}^{b} \sum_{i=1}^{t} \lambda_i^* [(\underline{Y}_j - \bar{\underline{Y}}_{\cdot})' Q_t \underline{\alpha}_i]^2 \\ SSE_2 &= \sum_{j=1}^{b} \frac{\left[\sum_{i=1}^{t} \lambda_i^* \underline{1}' \underline{\alpha}_i (\underline{Y}_j - \bar{\underline{Y}}_{\cdot})' Q_t \underline{\alpha}_i\right]^2}{t - \rho \sum_{i=1}^{t} \lambda_i^* (\underline{1}' \underline{\alpha}_i)^2} \end{aligned}\right\} . \qquad (24)$$

Using (21) and (23) with (14) it would now be possible to test the hypothesis of (4) provided ρ is known or can be estimated. However, the applied statistician may desire an easier approach to this problem that, although not exact, can still be used with a high degree of success. It is natural then to turn to an approximate F-statistic. In this method SST from (22) and SSE from (24) are used to form the original F-ratio and formulaes are then necessary to obtain the adjusted degrees of freedom.

The following method is generally referred to as a "Satterthwaite method" through reference to the work of Satterthwaite [1946]. Consider any quadratic form, say SSC, and let

$$u = \frac{nSSC}{E(SSC)}$$

with

$$n = \frac{2E^2(SSC)}{V(SSC)} \quad .$$

Then

$$E(u) = n$$

and

$$V(u) = \frac{n^2}{E^2(SSC)} \; V(SSC)$$

$$= \frac{4E^4(SSC)}{V^2(SSC)} \; \frac{V(SSC)}{E^2(SSC)}$$

$$= 2n \quad .$$

Then u has the same first two moments as a chi-square distribution with n degrees of freedom so that, approximately,

$$u \mathrel{\dot{\sim}} \chi^2_n \quad .$$

This method can now be applied to SST of (22) and SSE of (24) and the results will be similar to those of Box [1954^b]. The same approximate F-test will be derived. So, using SST above, let

$$n = \frac{2E^2(SST)}{V(SST)}$$

$$= (t-1)\varepsilon \quad , \quad \text{in Box's notation,}$$

where

$$\varepsilon = \frac{2E^2(SST)}{(t-1)V(SST)} \quad . \tag{25}$$

Then

$$u_1 = \frac{(t-1)\varepsilon SST}{E(SST)} \stackrel{.}{\sim} \chi^2_{(t-1)} \quad . \tag{26}$$

Recall that, with the Q_t given in (5), SST can be written as

$$SST = b\underline{\bar{Y}}_{.}'Q_t\underline{\bar{Y}}_{.}$$

where

$$\underline{\bar{Y}}_{.} \sim N_t(\underline{\mu}, \frac{1}{b}\Sigma)$$

and

$$\underline{\mu} = \mu^*\underline{1} + \underline{\tau} \quad , \quad \mu^* = \mu + \frac{1}{b}\sum_{j=1}^{b} \beta_j \quad .$$

Also,

$$\underline{\mu}Q_t = (\mu^*\underline{1}' + \underline{\tau}')(I - \frac{1}{t}\underline{1}\underline{1}')$$

$$= \mu^*\underline{1}' + \underline{\tau}' - \frac{1}{t}\mu^*\underline{1}'\underline{1}\underline{1}' - \frac{1}{t}\underline{\tau}'\underline{1}\underline{1}'$$

$$= \underline{\tau}' \quad , \quad \text{as } \underline{\tau}'\underline{1} = 0 \quad . \tag{27}$$

Then

$$E(SST) = b[tr(\frac{1}{b}Q_t\Sigma) + \underline{\mu}'Q_t\underline{\mu}]$$

$$= tr(\Sigma - \frac{1}{t}\underline{1}\underline{1}'\Sigma) + b\,\underline{\tau}'\underline{\tau} \quad , \quad \text{from (27)}$$

$$= tr(\Sigma) - \frac{1}{t}\underline{1}'\Sigma\underline{1} + b\,\underline{\tau}'\underline{\tau}$$

$$= tr(\Sigma) - \frac{1}{t}\underline{1}'\Sigma\underline{1} \quad , \quad \text{if } H_0 \text{ of (4) is true} \quad . \tag{28}$$

Further,

$$V(SST) = 2tr(Q_t \ddagger Q_t \ddagger) + 4\underline{\mu}' Q_t \ddagger Q_t \underline{\mu}$$

$$= 2tr[(\ddagger - \frac{1}{t}\,\underline{1}\underline{1}'\ddagger)(\ddagger - \frac{1}{t}\,\underline{1}\underline{1}'\ddagger)] + 4\underline{\tau}'\ddagger\underline{\tau} \quad , \quad \text{from (27)}$$

$$= 2tr(\ddagger^2 - \frac{1}{t}\,\underline{1}\underline{1}'\ddagger^2 - \frac{1}{t}\ddagger\,\underline{1}\underline{1}'\ddagger + \frac{1}{t^2}\,\underline{1}\underline{1}'\ddagger\underline{1}\underline{1}'\ddagger) + 4\underline{\tau}'\ddagger\underline{\tau}$$

$$= 2\left\{tr(\ddagger^2) - \frac{2}{t}\,\underline{1}'\ddagger^2\underline{1} + \frac{1}{t^2}\,(\underline{1}'\ddagger\underline{1})^2\right\} + 4\underline{\tau}'\ddagger\underline{\tau}$$

$$= 2\left\{tr(\ddagger^2) - \frac{2}{t}\,\underline{1}'\ddagger^2\underline{1} + \frac{1}{t^2}\,(\underline{1}'\ddagger\underline{1})^2\right\} , \quad \text{if } H_0 \text{ of (4) is true} \quad . \quad (29)$$

Substituting (28) and (29) in (25) yields

$$\varepsilon = \frac{[tr(\ddagger) - \frac{1}{t}\,\underline{1}'\ddagger\underline{1}]^2}{(t-1)\left\{tr(\ddagger^2) - \frac{2}{t}\,\underline{1}'\ddagger^2\underline{1} + \frac{1}{t^2}\,(\underline{1}'\ddagger\underline{1})^2\right\}} , \quad \text{if } H_0 \text{ of (4) is true} \quad . \quad (30)$$

Now it is well known in Analysis of Variance that SSE can be expressed as

$$SSE = \underline{Y}'Q(A)\underline{Y} \ , \quad \text{using (A2) of the Appendix}$$

where

$$A = Q_t = I_t - \frac{1}{t}\,\underline{1}\underline{1}' \quad .$$

Also,

$$\underline{Y}' \sim N_{bt}[E(\underline{Y}'), \Sigma] \quad , \quad \ddagger = diag(\ddagger)$$

so that

$$E(SSE) = tr[Q(A)\Sigma] + E(\underline{Y}')Q(A)E(\underline{Y})$$

$$= (b-1)tr(Q_t\ddagger) + E(\underline{Y}')Q(A)E(\underline{Y}) \ , \quad \text{from (A6)}$$

$$= (b-1)tr(Q_t\ddagger) \quad , \quad \text{from (A7)}$$

since

$$E(\underline{Y}_j')A = [(\mu + \beta_{.j})\underline{1}' + \underline{\tau}'](I - \frac{1}{t}\,\underline{1}\underline{1}')$$

$$= (\mu + \beta_{.j})\underline{1}' + \underline{\tau}' - (\mu + \beta_{.j})\,\frac{1}{t}\,\underline{1}'\underline{1}\underline{1}' - \frac{1}{t}\,\underline{\tau}'\underline{1}\underline{1}'$$

$$= \underline{\tau}' \quad , \quad \text{for all } j \quad . \tag{31}$$

Therefore,

$$E(SSE) = (b-1)E(SST) \quad , \quad \text{if } H_0 \text{ of (4) is true} \quad .$$

Further,

$$V(SSE) = 2tr[Q(A)\Sigma Q(A)\Sigma] + 4E(\underline{Y}')Q(A)\Sigma Q(A)E(\underline{Y}')$$

$$= 2tr[Q(A)\Sigma Q(A)\Sigma] \quad , \quad \text{from (A7) and (31)}$$

$$= (b-1)2tr(A\ddagger A\ddagger) \quad , \quad \text{from (A8)}$$

$$= (b-1)V(SST) \quad , \quad \text{if } H_0 \text{ of (4) is true} \quad .$$

So let

$$n_2 = \frac{2E^2(SSE)}{V(SSE)}$$

$$= \frac{2(b-1)^2E^2(SST)}{(b-1)\;V(SST)}$$

$$= (b-1)(t-1)\varepsilon \quad , \quad \text{using (25)} \quad .$$

Hence,

$$u_2 = \frac{n_2 SSE}{E(SSE)}$$

$$= \frac{\varepsilon (b-1)(t-1) SSE}{(b-1)\ E(SST)}$$

$$= \frac{(t-1)\varepsilon\ SSE}{E(SST)}$$

and

$$u_2 \overset{\cdot}{\underset{\cdot}{\sim}} \chi^2_{(b-1)(t-1)\varepsilon} \quad . \tag{32}$$

Now

$$SST = \underline{Y}'Q_t(A)\underline{Y} \ , \quad \text{using (A1)}$$

and

$$SSE = \underline{Y}'Q(A)\underline{Y} \ , \quad \text{using (A2)}$$

where

$$A = Q_t = I_t - \frac{1}{t}\,\underline{1}\underline{1}' \quad .$$

Then

$$A\ddagger_j A = Q_t \ddagger Q_t \ , \quad \text{for all } j$$

so this result and (A3) imply that

$$Q_t(A)\Sigma Q(A) = \Phi$$

and SST and SSE are independent. It then follows from (26) and (32) that

$$F = \frac{u_1}{(t-1)\varepsilon} \Big/ \frac{u_2}{(b-1)(t-1)\varepsilon}$$

$$= (b-1)\left(\frac{SST}{E(SST)} \Big/ \frac{SSE}{E(SST)}\right)$$

$$= (b-1)\,\frac{SST}{SSE}$$

$$\overset{\cdot}{\underset{\cdot}{\sim}} F_{[(t-1)\varepsilon,(b-1)(t-1)\varepsilon]} \ , \quad \text{if } H_0 \text{ of (4) is true} \quad . \tag{33}$$

If $(t-1)\varepsilon$, where ε is given in (30), is not an integer, interpolate in the F-table to find the critical point. Note that equation (30) and (33) are the same as Box's equation (4.4).

Recall now the example presented in Chapter II using the model of (1) with a variance-covariance matrix

$$\underset{\sim}{\Sigma} = \sigma^2 \begin{bmatrix} 1 & \rho & & \Phi \\ \rho & 1 & \ddots & \\ & \ddots & \ddots & \rho \\ \Phi & & \rho & 1 \end{bmatrix} . \qquad (34)$$

Of interest is the hypothesis given in (4) on the significance of the t-1 independent contrasts in the τ's. The first approach will be the exact one using the statistic given in (14). To compute SST* of (21) and SSE* of (23) it will be necessary to have values for λ_i^* of (18), $\underline{\alpha}_i'\underline{1}$, and $Q_t\underline{\alpha}_i$, where λ_i and $\underline{\alpha}_i$ are given in equations (33) and (34) of Chapter II. The values for these are given in Table I for $t = 2, \cdots , 6$ and $t = 8$.

Hence, find a value for ρ or compute an estimate of it. Then, to obtain SST*, calculate $\bar{\underline{Y}}_{.}$ and use the tables with the formula given in (21). To find SSE* compute $\underline{Y}_j - \bar{\underline{Y}}_{.}$ for each j and use the tables with the equation in (23). Finally calculate

$$F^* = (b-1)\,\frac{SST^*}{SSE^*}$$

and compare this with a tabled $F_{[t-1,(b-1)(t-1)]}$ at some α-level of significance.

Table I

Coefficients for Calculating F*

t = 2 (divisor = $\sqrt{2}$)

	i=1	i=2
$Q_t \underline{\alpha}_1$, $Q_t \underline{\alpha}_2$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
λ_i^*	$(1+\rho)^{-1}$	$-(1-\rho)^{-1}$
$\underline{\alpha}_i'\underline{1}$	$\sqrt{2}$	0

t = 3 (divisor = 6)

	i=1	i=2	i=3
$Q_t \underline{\alpha}_i$	$\begin{bmatrix} 1-\sqrt{2} \\ -2(1-\sqrt{2}) \\ 1-\sqrt{2} \end{bmatrix}$	$\begin{bmatrix} 3\sqrt{2} \\ 0 \\ -3\sqrt{2} \end{bmatrix}$	$\begin{bmatrix} 1+\sqrt{2} \\ -2(1+\sqrt{2}) \\ 1+\sqrt{2} \end{bmatrix}$
λ_i^*	$\frac{2}{\sqrt{2}+2\rho}$	0	$\frac{-2}{\sqrt{2}-2\rho}$
$\underline{\alpha}_i'\underline{1}$	$1+\frac{\sqrt{2}}{2}$	0	$1-\frac{\sqrt{2}}{2}$

t = 4 (divisor = $\sqrt{10.0009}$)

	i=1	i=2	i=3	i=4
$Q_t \underline{\alpha}_i$	$\begin{bmatrix} -.36327 \\ .36327 \\ .36327 \\ -.36327 \end{bmatrix}$	$\begin{bmatrix} 1.90212 \\ 1.17558 \\ -1.17558 \\ -1.90212 \end{bmatrix}$	$\begin{bmatrix} 1.53885 \\ -1.53885 \\ -1.53885 \\ 1.53885 \end{bmatrix}$	$\begin{bmatrix} 1.17558 \\ -1.90212 \\ 1.90212 \\ -1.17558 \end{bmatrix}$
λ_i^*	$\frac{1.61804}{1+1.61804\rho}$	$\frac{0.61804}{1+0.61804\rho}$	$\frac{-0.61804}{1-0.61804\rho}$	$\frac{-1.61804}{1-1.61804\rho}$
$\underline{\alpha}_i'\underline{1}$	$\frac{6.15540}{\sqrt{10.0009}}$	0	$\frac{1.45308}{\sqrt{10.0009}}$	0

Table I--continued

t = 5

	i=1	i=2	i=3	i=4	i=5
	$Q_t \underline{\alpha}_1$	$Q_t \underline{\alpha}_2$	$Q_t \underline{\alpha}_3$	$Q_t \underline{\alpha}_4$	$Q_t \underline{\alpha}_5$
	$\begin{bmatrix} 1-2\sqrt{3} \\ 4+3\sqrt{3} \\ 6-2\sqrt{3} \\ -4+3\sqrt{3} \\ 1-2\sqrt{3} \end{bmatrix}$	$\begin{bmatrix} 5\sqrt{3} \\ 5\sqrt{3} \\ 0 \\ -5\sqrt{3} \\ -5\sqrt{3} \end{bmatrix}$	$\begin{bmatrix} 8 \\ -2 \\ -12 \\ -2 \\ 8 \end{bmatrix}$	$\begin{bmatrix} 5\sqrt{3} \\ -5\sqrt{3} \\ 0 \\ 5\sqrt{3} \\ -5\sqrt{3} \end{bmatrix}$	$\begin{bmatrix} 1+2\sqrt{3} \\ -4-3\sqrt{3} \\ 6+2\sqrt{3} \\ -4-3\sqrt{3} \\ 1+2\sqrt{3} \end{bmatrix}$
λ_i^*	$\frac{3}{\sqrt{3}+3\rho}$	$\frac{1}{1+\rho}$	0	$\frac{-1}{1-\rho}$	$\frac{-3}{\sqrt{3}-3\rho}$
$\underline{\alpha}_i'\underline{1}$	$1+\frac{2}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	0	$-1+\frac{2}{\sqrt{3}}$

divisor = $10\sqrt{3}$

t = 6

	i=1	i=2	i=3	i=4	i=5	i=6
	$Q_t \underline{\alpha}_1$	$Q_t \underline{\alpha}_2$	$Q_t \underline{\alpha}_3$	$Q_t \underline{\alpha}_4$	$Q_t \underline{\alpha}_5$	$Q_t \underline{\alpha}_6$
	$\begin{bmatrix} -.88900 \\ .15488 \\ .73412 \\ .73412 \\ .15488 \\ -.88900 \end{bmatrix}$	$\begin{bmatrix} 2.34564 \\ 2.92488 \\ 1.30176 \\ -1.30176 \\ -2.92488 \\ -2.34564 \end{bmatrix}$	$\begin{bmatrix} 2.29788 \\ 0.67476 \\ -2.97264 \\ -2.97264 \\ 0.67476 \\ 2.29788 \end{bmatrix}$	$\begin{bmatrix} 2.92488 \\ -1.30176 \\ -2.34564 \\ 2.34564 \\ 1.30176 \\ -2.92488 \end{bmatrix}$	$\begin{bmatrix} 2.10480 \\ -3.16572 \\ 1.06092 \\ 1.06092 \\ -3.16572 \\ 2.10480 \end{bmatrix}$	$\begin{bmatrix} 1.30176 \\ -2.34564 \\ 2.92488 \\ -2.92488 \\ 2.34564 \\ -1.30176 \end{bmatrix}$
λ_i^*	$\frac{1.80180}{1+1.80180\rho}$	$\frac{1.24684}{1+1.24684\rho}$	$\frac{0.44480}{1+0.44480\rho}$	$\frac{-0.44480}{1-0.44480\rho}$	$\frac{-1.24684}{1-1.24684\rho}$	$\frac{-1.80180}{1-1.80180\rho}$
$\underline{\alpha}_i'\underline{1}$	2.34191	0.0	0.67026	0.0	0.2576	0.0

divisor = $\sqrt{31.50306}$

Table I--continued

t = 8

	i=1	i=2	i=3	i=4
	$Q_t \underline{\alpha}_1$	$Q_t \underline{\alpha}_2$	$Q_t \underline{\alpha}_3$	$Q_t \underline{\alpha}_4$
	-0.172954	0.030314	0.306186	0.464243
	-0.031170	0.464243	0.306186	0.161229
	0.074064	0.408248	-0.102062	-0.408248
	0.130060	0.161229	-0.510310	-0.303014
	0.130060	-0.161229	-0.510310	0.303014
	0.074064	-0.408248	-0.102062	0.408248
	-0.031170	-0.464243	0.306186	-0.161229
	-0.172954	-0.303014	0.306186	-0.464243
λ_i^*	$\frac{1.87938}{1+1.87938\rho}$	$\frac{1.53208}{1+1.53208\rho}$	$\frac{1}{1+\rho}$	$\frac{0.34730}{1+0.34730\rho}$
$\underline{\alpha}_i'\underline{1}$	2.67347	0.0	0.81650	0.0

	i=5	i=6	i=7	i=8
	$Q_t \underline{\alpha}_5$	$Q_t \underline{\alpha}_6$	$Q_t \underline{\alpha}_7$	$Q_t \underline{\alpha}_8$
	0.414798	0.408248	0.281567	0.161229
	-0.210674	-0.408248	-0.485690	-0.303014
	-0.457692	0.0	0.386800	0.408248
	0.253569	0.408248	-0.182677	-0.464243
	0.253569	-0.408248	-0.182677	0.464243
	-0.457692	0.0	0.386800	-0.408248
	-0.210674	0.408248	-0.485690	0.303014
	0.414798	-0.408248	0.281567	-0.161229
λ_i^*	$\frac{-0.34730}{1-0.34730\rho}$	$\frac{-1}{1-\rho}$	$\frac{-1.53208}{1-1.53208\rho}$	$\frac{-1.87938}{1-1.87938\rho}$
$\underline{\alpha}_i'\underline{1}$	0.39556	0.0	0.17158	0.0

divisor = 1

If the approximate method is to be used all that is needed is SST of (22) and SSE of (24). Then calculate

$$F = (b-1)\frac{SST}{SSE} \quad . \tag{35}$$

However, to obtain the degrees of freedom for the tabled F one must calculate ε of (30). In this example,

$$\begin{aligned} tr(\Sigma) &= \sigma^2\, tr(I + \rho M) \\ &= \sigma^2(t + \rho\cdot 0) \\ &= t\sigma^2 \end{aligned}$$

and

$$\begin{aligned} \underline{1}'\Sigma\underline{1} &= \sigma^2\, \underline{1}'(I + \rho M)\underline{1} \\ &= \sigma^2[t + 2\rho(t-1)] \\ &= \sigma^2 t + 2\rho\sigma^2(t-1) \quad . \end{aligned}$$

Therefore,

$$\begin{aligned} tr(\Sigma) - \frac{1}{t}\underline{1}'\Sigma\underline{1} &= \sigma^2 t - \sigma^2 - 2\rho\sigma^2\frac{t-1}{t} \\ &= \sigma^2[t - 1 - \frac{2\rho}{t}(t-1)] \end{aligned}$$

i.e.,

$$E(SST) = \frac{\sigma^2(t-1)}{t}(t - 2\rho) \quad . \tag{36}$$

Also,

$$\begin{aligned} tr(\Sigma^2) &= \sigma^4\, tr(I + 2\rho M + \rho^2 M^2) \\ &= \sigma^4[t + 2\rho\cdot 0 + \rho^2 2(t-1)] \\ &= \sigma^4[t + 2\rho^2(t-1)] \end{aligned}$$

and

$$\underline{1}'\Sigma^2\underline{1} = \sigma^4[2(1+\rho)^2 + (t-2)(1+2\rho)^2]$$

$$= \sigma^4[t + 4\rho(t-1) + 2\rho^2(2t-3)] \quad .$$

Therefore,

$$\mathrm{tr}(\Sigma^2) - \frac{2}{t}\underline{1}'\Sigma^2\underline{1} + \frac{1}{t^2}(\underline{1}'\Sigma\underline{1})^2$$

$$= \sigma^4\left\{t + 2\rho^2(t-1) - \frac{2}{t}[t + 4\rho(t-1) + 2\rho^2(2t-3)] + \frac{1}{t^2}[t + 2\rho(t-1)]^2\right\}$$

$$= \sigma^4 \frac{(t-1)(t-2\rho)^2}{t^2}\left[1 + \frac{2\rho^2(t+1)(t-2)^2}{(t-1)(t-2\rho)^2}\right] \quad .$$

Hence,

$$\varepsilon = \frac{(\mathrm{tr}\,\Sigma - \frac{1}{t}\underline{1}'\Sigma\underline{1})^2}{(t-1)\left\{\mathrm{tr}(\Sigma^2) - \frac{2}{t}\underline{1}'\Sigma^2\underline{1} + \frac{1}{t^2}(\underline{1}'\Sigma\underline{1})^2\right\}}$$

$$= \frac{\sigma^4(t-1)^2}{t^2(t-1)}(t-2\rho)^2 \Bigg/ \sigma^4\frac{(t-1)(t-2\rho)^2}{t^2}\left(1 + 2\rho^2\frac{(t+1)(t-2)^2}{(t-1)(t-2\rho)^2}\right)$$

$$= \left[1 + 2\rho^2\frac{(t+1)(t-2)^2}{(t-1)(t-2\rho)^2}\right]^{-1} \qquad (37)$$

which is equation (6.10) in Box's notation. Now compare the F-statistic in (35) with a tabled $F_{[(t-1)\varepsilon,(b-1)(t-1)\varepsilon]}$ at some α-level of significance. Of course, the value of ρ or an estimate of ρ will be needed to compute ε, and if $(t-1)\varepsilon$ is not an integer then one must interpolate in the F tables.

In summary, there has been derived both an exact and approximate method for testing the hypothesis of (4). The exact test statistic is given in (14) and requires much time and labor unless tables of latent roots and vectors of M are available. The approximate test statistic is given in (33) and is the usual F-statistic of a RBD where the errors are normally and independently distributed; so it is relatively easy to compute. However, it may be necessary to interpolate in the F table to find the appropriate critical point in testing H_0 . In each of the above cases one must either know the value of ρ or be able to find an estimate of it. Which method is best will depend on this estimate. The next chapter will discuss in detail a Monte Carlo study comparing these tests when one such estimate of ρ is used.

CHAPTER V

A MONTE CARLO STUDY

In order to compare the two test statistics given in equations (14) and (33) of Chapter IV it is necessary to find an estimate of ρ, say $\hat{\rho}$. This will now be done for the example given in the last chapter, using the variance-covariance matrix

$$\Sigma = \sigma^2 \begin{bmatrix} 1 & \rho & & & \Phi \\ \rho & 1 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \rho \\ \Phi & & & \rho & 1 \end{bmatrix} . \quad (1)$$

A Monte Carlo study is made comparing different known significance levels, α, with the significance levels using the 'exact' test statistic in equation (14) above, i.e.,

$$F(\text{'exact'}) = (b-1)\,\frac{SST^*}{SSE^*} \sim F_{[(t-1)(b-1)(t-1)]}\,, \quad \text{if } H_0 \text{ is true} \qquad (2)$$

the approximate test statistic in equation (33) above, i.e.,

$$F(\text{approx.}) = (b-1)\,\frac{SST}{SSE} \sim F_{[(t-1)\varepsilon,(b-1)(t-1)\varepsilon]}\,, \text{ if } H_0 \text{ is true} \qquad (3)$$

and the usual F-statistic which can be computed using equation (33) above, i.e.,

$$F(\text{usual}) = (b-1)\frac{SST}{SSE} \sim F_{[(t-1),(b-1)(t-1)]}\ , \quad \text{if } H_0 \text{ is true} \quad . \tag{4}$$

The term 'exact' will be used to designate the exact test of the last chapter when an estimate for ρ is used; this, of course, will not be an exact test of significance but will be referred to as the 'exact' test in order to keep in mind its structure. Notice that the only difference between the usual statistic and the approximate statistic is the degrees of freedom used when finding the critical region. The results of the study prove to be helpful in determining which of the above statistics is appropriate when the $\hat{\rho}$ in (1) is used.

In the example of Chapter IV it was shown in equation (36) that

$$E(SSE) = (b-1)E(SST)\ , \quad \text{if } H_0 \text{ is true}$$

$$= \sigma^2(b-1)(t-1)\left(\frac{t-2\rho}{t}\right)\ , \quad \text{if } H_0 \text{ is true}$$

so that

$$\rho = \frac{t}{2}\left[1 - \frac{1}{\sigma^2}\frac{E(SSE)}{(b-1)(t-1)}\right] \quad . \tag{5}$$

Now if E(SSE) is replaced by

$$SSE = \sum_{j=1}^{b}\sum_{i=1}^{t}[Y_{ij} - \bar{Y}_{i\cdot} - \bar{Y}_{\cdot j} + \bar{\bar{Y}}_{\cdot\cdot}]^2$$

in equation (5) the result is an unbiased estimator of ρ when σ^2 is known, i.e.,

$$\hat{\rho} = \frac{t}{2}\left[1 - \frac{1}{\sigma^2}\frac{SSE}{(b-1)(t-1)}\right]\ , \quad \sigma^2 \text{ is known} \tag{6}$$

with

$$E(\hat{\rho}) = \rho \quad .$$

If σ^2 is unknown it can be estimated using the C method derived in Chapter II. In equation (43) of that section

$$SSE_1 = \sum_{j=1}^{b} \sum_{i=1}^{q_1} [Y_{2i-1,j} - \bar{Y}_{2i-1,\cdot} - \bar{Y}_{\cdot j}^{(1)} + \bar{\bar{Y}}_{\cdot\cdot}^{(1)}]^2$$

where

$$\frac{SSE_1}{\sigma^2} \sim \chi^2_{(b-1)(q_1-1)}$$

so that

$$E(SSE_1) = (b-1)(q_1-1)\sigma^2 \quad . \qquad (7)$$

Also, from equation (44) of that section

$$SSE_2 = \sum_{j=1}^{b} \sum_{i=1}^{q_2} [Y_{2i,j} - \bar{Y}_{2i,\cdot} - \bar{Y}_{\cdot j}^{(2)} + \bar{\bar{Y}}_{\cdot\cdot}^{(2)}]^2$$

where

$$\frac{SSE_2}{\sigma^2} \sim \chi^2_{(b-1)(q_2-1)}$$

so that

$$E(SSE_2) = (b-1)(q_2-1)\sigma^2 \quad . \qquad (8)$$

Combining (7) and (8) yields

$$E(SSE_1 + SSE_2) = (b-1)(q_1 + q_2 - 2)\sigma^2$$

$$= (b-1)(t-1)\sigma^2 \ , \quad \text{as } q_1 + q_2 = t \quad .$$

Hence,

$$\hat{\sigma}^2 = \frac{SSE_1 + SSE_2}{(b-1)(t-1)}$$

is an unbiased estimator for σ^2 . An easier method for obtaining

$SSE_1 + SSE_2$ would be to first calculate SSE and then subtract off SSE_3 where

$$SSE_3 = \sum_{j=1}^{b} (w_j - \bar{w}_.)^2 \tag{9}$$

with

$$\bar{w}_. = \frac{1}{b} \sum_{j=1}^{b} w_j$$

and

$$w_j = \begin{cases} \frac{\sqrt{t}}{2}\left(\bar{Y}^{(1)}_{.j} - \bar{Y}^{(2)}_{.j}\right) & , \quad t \text{ is even} \\ \frac{\sqrt{2(t^2-1)}}{2(t-2)}\left(\bar{Y}^{(1)}_{.j} - \bar{Y}^{(2)}_{.j}\right) & , \quad t \text{ is odd} \end{cases}$$

so that

$$\hat{\sigma}^2 = \frac{SSE - SSE_3}{(b-1)(t-2)} \quad . \tag{10}$$

Substituting the above estimate in (6) yields

$$\hat{\rho} = \frac{t}{2}\left[1 - \left(\frac{t-2}{t-1}\right)\frac{SSE}{SSE - SSE_3}\right] , \quad \sigma^2 \text{ unknown} \quad . \tag{11}$$

Notice that when σ^2 is known, $\hat{\rho}$ is an unbiased estimator of ρ, but when σ^2 is unknown this is not true. However, in (11) σ^2 is replaced by the unbiased estimator in (10). It is also relatively easy to compute $\hat{\rho}$ of (6) and (11). Because of these two points, i.e., pseudo-unbiasedness and ease of computation, this estimate of ρ was used in the Monte Carlo study given below. Other estimates of ρ could be devised but none are as simple to compute as this one.

Consider now generating a random sample from a multivariate normal population having mean $\underset{\sim}{0}$ and the variance-covariance matrix given in (1), where ρ and σ^2 are specified. Using this sample it would then be easy to compute the $\hat{\rho}$ of (11), assuming σ^2 is unknown, and the test statistics in (2), (3) and (4), assuming ρ is unknown. Comparisons could be drawn between the real value for ρ and the estimated values using $\hat{\rho}$. Also, one could compare the three statistics above to see which appears to be most correct when $\hat{\rho}$ is used. This has been done in a series of experiments using a UNIVAC 1108 computer where t=3 and b=3, 5; t=5 and b=3, 5, 7; t=8 and b=3, 5, 8, with $\sigma^2 = 1$, and $\rho = 0.45, 0.22, 0.0, -0.22, -0.45$. Each experiment was run 1500 times, varying t, b, and ρ and using different samples for each replication. In each replication, $\hat{\rho}$ of (11) was computed and the resulting value was used to calculate the test statistics in (2), (3) and (4). If the value of $\hat{\rho}$ ever exceeded the limits on ρ as given in equation (32) of Chapter II, i.e., $\left\{2 \cos\left(\frac{\pi}{t+1}\right)\right\}^{-1}$, then this end value was used instead of $\hat{\rho}$. This resulted in a partially biased estimate of ρ but a more correct one. Counts were then made of the number of times a certain test statistic fell in the critical region using three different significance levels, $\alpha = .10, .05, .025$. Table II below lists these counts in terms of probabilities.

A large number of values for $\hat{\rho}$ of (11) were also printed out. These indicated that this estimator was fair in that region where ρ was positive; but with a negative ρ, $\hat{\rho}$ performed poorly. Consequently, as the tables indicate, the test statistic, F('exact'), is not good when $\hat{\rho}$ is used. A better estimate of ρ, however, might improve this test greatly. Surprisingly, F(usual) was relatively accurate, even when $|\rho|$ varied from zero. The statistic that was most consistent over the values for ρ, t, and b was F(approx.) which turned out to be somewhat conservative. Another estimate of ρ might also improve this test.

Table II

Comparisons of F('exact'), F(usual), and F(approx.)

t=3, b=5

ρ	0.45			0.22			0.0			-0.22			-0.45		
α	.10	.05	.025	.10	.05	.025	.10	.05	.025	.10	.05	.025	.10	.05	.025
F('exact')	.1547	.0913	.0560	.1627	.0953	.0633	.1520	.0920	.0540	.1413	.0853	.0487	.1340	.0800	.0527
F(usual)	.1073	.0560	.0300	.1107	.0600	.0353	.1020	.0520	.0307	.1013	.0480	.0267	.1047	.0593	.0267
F(approx.)	.0940	.0487	.0240	.1027	.0527	.0287	.0967	.0480	.0267	.0947	.0427	.0220	.1027	.0513	.0247

t=3, b=7

ρ	0.45			0.22			0.0			-0.22			-0.45		
α	.10	.05	.025	.10	.05	.025	.10	.05	.025	.10	.05	.025	.10	.05	.025
F('exact')	.1320	.0780	.0447	.1293	.0813	.0460	.1487	.0773	.0467	.1427	.0767	.0393	.1333	.0740	.0427
F(usual)	.1040	.0600	.0307	.0880	.0460	.0240	.1133	.0573	.0267	.1033	.0473	.0207	.1013	.0473	.0227
F(approx.)	.0913	.0533	.0260	.0827	.0400	.0207	.1087	.0527	.0253	.0973	.0440	.0187	.0993	.0433	.0200

t=5, b=3

ρ	0.45			0.22			0.0			-0.22			-0.45		
α	.10	.05	.025	.10	.05	.025	.10	.05	.025	.10	.05	.025	.10	.05	.025
F('exact')	.1927	.1333	.0907	.1960	.1367	.0947	.1940	.1327	.0940	.1873	.1233	.0840	.1627	.1173	.0873
F(usual)	.1180	.0707	.0400	.1033	.0527	.0300	.1027	.0553	.0273	.1007	.0513	.0240	.1053	.0607	.0320
F(approx.)	.0927	.0480	.0260	.0853	.0413	.0213	.0880	.0407	.0207	.0800	.0387	.0180	.0853	.0467	.0247

Table II--continued

t=5, b=5

ρ	0.45			0.22			0.0			-0.22			-0.45		
α	.10	.05	.025	.10	.05	.025	.10	.05	.025	.10	.05	.025	.10	.05	.025
F('exact')	.1593	.0993	.0627	.1547	.0940	.0587	.1413	.0860	.0507	.1393	.0813	.0460	.1287	.0747	.0440
F(usual)	.1360	.0760	.0420	.1033	.0453	.0253	.1007	.0520	.0273	.1013	.0507	.0260	.1040	.0560	.0320
F(approx.)	.1080	.0600	.0293	.0900	.0400	.0213	.0893	.0447	.0207	.0933	.0427	.0133	.0893	.0473	.0233

t=5, b=7

ρ	0.45			0.22			0.0			-0.22			-0.45		
α	.10	.05	.025	.10	.05	.025	.10	.05	.025	.10	.05	.025	.10	.05	.025
F('exact')	.1460	.0933	.0560	.1260	.0700	.0420	.1347	.0780	.0513	.1247	.0707	.0433	.1160	.0667	.0360
F(usual)	.1273	.0827	.0507	.0960	.0453	.0227	.1027	.0433	.0220	.1067	.0567	.0307	.1007	.0600	.0280
F(approx.)	.1133	.0607	.0367	.0853	.0387	.0187	.0913	.0360	.0173	.0980	.0433	.0273	.0913	.0460	.0213

t=8, b=3

ρ	0.45			0.22			0.0			-0.22			-0.45		
α	.10	.05	.025	.10	.05	.025	.10	.05	.025	.10	.05	.025	.10	.05	.025
F('exact')	.2140	.1533	.1153	.2100	.1493	.1160	.1927	.1400	.1053	.2147	.1433	.1000	.1713	.1147	.0813
F(usual)	.1333	.0760	.0467	.1193	.0607	.0347	.1047	.0540	.0280	.1173	.0567	.0260	.1153	.0700	.0373
F(approx.)	.1020	.0547	.0287	.0973	.0467	.0233	.0787	.0387	.0187	.0893	.0373	.0167	.0973	.0500	.0260

Table II--continued

t=8, b=5

ρ	0.45			0.22			0.0			-0.22			-0.45		
α	.10	.05	.025	.10	.05	.025	.10	.05	.025	.10	.05	.025	.10	.05	.025
F('exact')	.1693	.1040	.0733	.1407	.0900	.0627	.1380	.0827	.05[illegible]7	.1460	.0847	.0507	.1380	.0813	.0513
F(usual)	.1393	.0800	.0413	.1127	.0633	.0267	.0960	0513	.0240	.1120	.0633	.0340	.1193	.0700	.0340
F(approx.)	.1113	.0500	.0267	.0960	.0480	.0160	.0807	.0420	.0173	.0940	.0547	.0293	.1067	.0560	.0253

t=8, b=8

ρ	0.45			0.22			0.0			-0.22			-0.45		
α	.10	.05	.025	.10	.05	.025	.10	.05	.025	.10	.05	.025	.10	.05	.025
F('exact')	.1233	.0760	.0487	.1193	.0660	.0427	.1373	.0867	.0[illegible]20	.1267	.0740	.0493	.1167	.0493	.0253
F(usual)	.1140	.0653	.0353	.0947	.0480	.0260	.1073	.0613	.0333	.1087	.0580	.0313	.1067	.0627	.0353
F(approx.)	.0833	.0413	.0187	.0833	.0433	.0220	.0960	.0533	.0293	.0980	.0493	.0253	.0947	.0507	.0287

It is suggested then that when $\ddagger$ has the form of (1) and ρ is known, one should use F('exact') of Chapter IV to test H_0 . When ρ is unknown, estimate ρ using (6) if σ^2 is known and (11) if σ^2 is unknown. Then to test H_0, evaluate F(approx.) of (3) and this $\hat{\rho}$. If one knows that $|\rho|$ is not too far from zero, but the actual value of ρ is unknown, calculate F(usual) of (4) and do not even estimate ρ . Finally, use another estimate of ρ if a better one is found.

CHAPTER VI

SUMMARY

In this paper methods have been proposed for testing the effects of certain sets of treatment contrasts in a randomized block experiment where the errors are not independently distributed but have, instead, a variance-covariance matrix of the form

$$\ddagger_j = \sigma^2 (I_t + \rho_j M) \quad , \quad j = 1, \cdots , b \quad . \tag{1}$$

These tests require neither that the number of blocks exceed the number of treatments nor the computation of large order inverse matrices, as does Hotelling's T^2 test. In fact, some of them use the usual test ratio

$$F = \frac{SST}{SSE} \quad .$$

Chapter II presents the C-method which transforms the original design into one in which the errors are independently distributed. An example on serial correlation within blocks is examined using this approach. Although not unique, the test statistic developed here has an exact distribution, namely, the F-distribution, and is not too difficult to derive. Unfortunately, it is useful in testing only sets of treatment contrasts and not in testing the equality of all the treatment means.

In Chapter III the D-method is proposed which can be used as an alternative to the C-method when the rank of M is small. This section also analyzes an example in animal breeding where this method appears to be very useful. The test statistic derived is quite easy to obtain and the sets of treatment contrasts considered almost span the parameter space of τ .

Chapter IV gives two methods which can be used in testing all t-1 independent contrasts. Both require that ρ_j be identical to ρ, for all j, and either that ρ is known or an estimate of ρ can be obtained. If ρ is known, one approach is exact while the other is approximate; if ρ is unknown, both are approximate. The example of Chapter II is studied in detail and some tables are given which are useful in deriving the test statistic of the exact method.

In Chapter V a Monte Carlo study is made on the methods of Chapter IV, using an easily computed estimate of ρ and the example of Chapter II. The results indicate that the approximate test is quite accurate while the 'exact' one does not perform well due to the inaccuracy of the estimator of ρ . Surprisingly, the F-test used when the errors are independently distributed performs quite well for this example.

In conclusion, if one is interested in testing the equality of all the treatment means, use

(1) the exact method of Chapter IV, if ρ_j is identical to ρ, for all j, and ρ is known;

(2) the approximate method of Chapter IV, if ρ_j is identical to ρ, for all j, and ρ can be estimated;

(3) Hotelling's T^2 if $b > t$; ρ_j is identical to ρ, for all j; and the necessary inverse matrix is easier to compute than (1) or (2) above;

(4) the D-method of Chapter III, if (1), (2), and (3) do not hold and the rank of M is small enough. In this case ρ_j does not have to be identical from block to block.

If one is satisfied with testing certain sets of treatment contrasts, use

(1) the C-method of Chapter II, if these sets can be obtained;

(2) the D-method of Chapter III, if the rank of M is small and these sets can be derived;

(3) Hotelling's T^2 if $b > t$; ρ_j is identical to ρ, for all j; and the inverse matrix is easier to compute than (1) or (2);

(4) the single degree of freedom tests of Chapter II, if ρ_j is identical to ρ, for all j, and individual treatment comparisons are of interest.

APPENDIX A

SOME RESULTS ON MATRICES

Let A and B be any $t \times t$ matrices and let $Q_t(A)$ and $Q(A)$ be $tb \times tb$ matrices such that

$$Q_t(A) = \frac{1}{b}\begin{bmatrix} I_t \\ \cdot \\ \cdot \\ \cdot \\ I_t \end{bmatrix} A \, [I_t \cdots I_t] \tag{A1}$$

and

$$Q(A) = \frac{1}{b}\begin{bmatrix} (b-1)A & -A & \cdots & -A \\ -A & (b-1)A & \cdots & -A \\ \vdots & \vdots & & \vdots \\ -A & -A & \cdots & (b-1)A \end{bmatrix} . \tag{A2}$$

Then

$$Q_t(A)\Sigma Q(B) = \frac{1}{b}\begin{bmatrix} I_t \\ \cdot \\ \cdot \\ \cdot \\ I_t \end{bmatrix} A \, [I_t \cdots I_t] \, \mathrm{diag}(\Sigma_j)\frac{1}{b}\begin{bmatrix} (b-1)B & \cdots & -B \\ \vdots & & \vdots \\ -B & \cdots & (b-1)B \end{bmatrix}$$

$$= \frac{1}{b^2}\begin{bmatrix} I_t \\ \cdot \\ \cdot \\ \cdot \\ I_t \end{bmatrix} A[\Sigma_1 \, \Sigma_2 \cdots \Sigma_b] \begin{bmatrix} (b-1)B & \cdots & -B \\ \vdots & & \vdots \\ -B & \cdots & (b-1)B \end{bmatrix}$$

$$= \frac{1}{b^2}\begin{bmatrix} I_t \\ \vdots \\ I_t \end{bmatrix}\left[(b-1)A\ddagger_1 B - \sum_{j=2}^{b} A\ddagger_j B, \cdots, (b-1)A\ddagger_b B - \sum_{j=1}^{b-1} A\ddagger_j B\right]$$

$$= \Phi \quad , \quad \text{if } A\ddagger_j B = \text{constant, for all } j \quad . \tag{A3}$$

Also,

$$\left(\frac{1}{\sigma^2} Q(A)\Sigma\right)^2 = \frac{1}{\sigma^2}\frac{1}{b}\begin{bmatrix} (b-1)A & \cdots & -A \\ \vdots & & \vdots \\ -A & \cdots & (b-1)A \end{bmatrix} \text{diag}(\ddagger_j)\frac{1}{\sigma^2}\frac{1}{b}\begin{bmatrix} (b-1)A & \cdots & -A \\ \vdots & & \vdots \\ -A & \cdots & (b-1)A \end{bmatrix}\Sigma$$

$$= \frac{1}{b}\begin{bmatrix} (b-1)A\ddagger_1 & \cdots & -A\ddagger_b \\ \vdots & & \vdots \\ -A\ddagger_1 & \cdots & (b-1)A\ddagger_b \end{bmatrix}\frac{1}{b}\begin{bmatrix} (b-1)A & \cdots & -A \\ \vdots & & \vdots \\ -A & \cdots & (b-1)A \end{bmatrix}\Sigma\frac{1}{\sigma^4}$$

$$= \frac{1}{b^2}\begin{bmatrix} b(b-1)A & \cdots & -bA \\ \vdots & & \vdots \\ -bA & \cdots & b(b-1)A \end{bmatrix}\Sigma\frac{1}{\sigma^2} \, , \quad \text{if } A\ddagger_j A = \sigma^2 A$$

$$= \frac{1}{\sigma^2} Q(A)\Sigma \quad , \quad \text{if } A\ddagger_j A = \sigma^2 A \tag{A4}$$

and

$$\text{tr}[Q(A)\Sigma] = \frac{1}{b}(b-1)\sum_{j=1}^{b}\text{tr}(A\ddagger_j) \tag{A5}$$

$$= (b-1)\text{tr}(A\ddagger) \quad , \quad \text{if } \ddagger_j = \ddagger, \text{ for all } j \quad . \tag{A6}$$

Further,

$$\lambda_e = \frac{1}{2} E(\underline{Y}')Q(A)E(\underline{Y})$$

$$= \frac{1}{2} [E(\underline{Y}_1'), \cdots , E(\underline{Y}_b')]\frac{1}{b}\begin{bmatrix} (b-1)A & \cdots & -A \\ \vdots & & \vdots \\ -A & \cdots & (b-1)A \end{bmatrix} E(\underline{Y})$$

$$= \frac{1}{2b}\left[(b-1)E(\underline{Y}_1')A - \sum_{j=2}^{b} E(\underline{Y}_j')A, \cdots , (b-1)E(\underline{Y}_b')A - \sum_{j=1}^{b-1} E(\underline{Y}_j')A\right] E(\underline{Y})$$

$$= 0, \quad \text{if } E(\underline{Y}_j')A = \text{constant, for all } j \tag{A7}$$

and

$$\mathrm{tr}[Q(A)\Sigma]^2 = \mathrm{tr}\left\{\frac{1}{b}\begin{bmatrix} (b-1)A\Sigma_1 & \cdots & -A\Sigma_b \\ \vdots & & \vdots \\ -A\Sigma_1 & \cdots & (b-1)A\Sigma_b \end{bmatrix}\frac{1}{b}\begin{bmatrix} (b-1)A\Sigma_1 & \cdots & -A\Sigma_b \\ \vdots & & \vdots \\ -A\Sigma_1 & \cdots & (b-1)A\Sigma_b \end{bmatrix}\right\}$$

$$= \mathrm{tr}\left\{\frac{1}{b}\begin{bmatrix} (b-1)A\Sigma A\Sigma & \cdots & -A\Sigma A\Sigma \\ \vdots & & \vdots \\ -A\Sigma A\Sigma & \cdots & (b-1)A\Sigma A\Sigma \end{bmatrix}\right\} \quad \text{if } \Sigma_j = \Sigma, \text{ for all } j$$

$$= (b-1)\mathrm{tr}(A\Sigma^2) \quad , \quad \text{if } \Sigma_j = \Sigma, \text{ for all } j \ . \tag{A8}$$

Now

$$Q(A)\Sigma Q(B) = \frac{1}{b}\begin{bmatrix} (b-1)A & \cdots & -A \\ \vdots & & \vdots \\ -A & \cdots & (b-1)A \end{bmatrix} \mathrm{diag}(\ddagger_j)\ \frac{1}{b}\begin{bmatrix} (b-1)B & \cdots & -B \\ \vdots & & \vdots \\ -B & \cdots & (b-1)B \end{bmatrix}$$

$$= \frac{1}{b}\begin{bmatrix} (b-1)A\ddagger_1 & \cdots & -A\ddagger_b \\ \vdots & & \vdots \\ -A\ddagger_1 & \cdots & (b-1)A\ddagger_b \end{bmatrix} \frac{1}{b}\begin{bmatrix} (b-1)B & \cdots & -B \\ \vdots & & \vdots \\ -B & \cdots & (b-1)B \end{bmatrix}$$

$$= \Phi \ , \quad \text{if } A\ddagger_j B = \Phi \ , \quad \text{for all } j \tag{A9}$$

and

$$Q_t(A)\Sigma Q_t(B) = \frac{1}{b}\begin{bmatrix} I_t \\ \vdots \\ I_t \end{bmatrix} A\ [I_t \cdots I_t]\ \mathrm{diag}(\ddagger_j)\ \frac{1}{b}\begin{bmatrix} I_t \\ \vdots \\ I_t \end{bmatrix} B\ [I_t \cdots I_t]$$

$$= \frac{1}{b^2}\begin{bmatrix} I_t \\ \vdots \\ I_t \end{bmatrix} \sum_{j=1}^{b} A\ddagger_j B\ [I_t \cdots I_t]$$

$$= \Phi \ , \quad \text{if } A\ddagger_j B = \Phi \ , \text{ for all } j \ . \tag{A10}$$

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13. ABSTRACT

Consider a randomized block design where the errors are correlated within a block but are independent from block to block. The theory for testing the significance for the treatment effects was done by Box [1954] and Geisser and Greenhouse [1958], and a partial solution was given by Gray ill [1954]. A more general solution to this problem is now presented and several test procedures are derived.

The variance-covariance matrix for the above design can have two forms. When the correlation coefficient, ρ_j, differs from block to block, an exact test of reduced dimension is proposed which can be used in solving problems in growth studies. When ρ_j is identical to ρ for each block, two tests are presented. One is exact when ρ is known; both are approximate when ρ is unknown. In this latter case, comparisons are made between the two tests using a specified form for the covariance matrix and estimating ρ. For this example a Satterthwaite test is most accurate; but, the usual F-test, which ignores the correlation, performs well when $|\rho|$ varies somewhat from zero.

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