INVISIBLE PROOF: THE ROLE OF GESTURES AND ACTION IN PROOF

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The role of reasoning and proof in mathematics is undeniably crucial, and yet research in mathematics education has repeatedly indicated that students struggle with proof production. Our research shows that proof activities can be illuminated by newly considering action and gesture as a modality for crucial aspects of mathematical communication. We share two examples that highlight the importance of gesture and action in supporting students' mathematical proof. We conclude by discussing the implications of our research and proposed next steps.

Keywords: Geometry and Geometrical and Spatial Thinking; Reasoning and Proof; Instructional activities and practices

Research in mathematics education has consistently shown that students, as well as pre-service and in-service teachers, struggle with constructing, interpreting, and evaluating proofs (Knuth, 2002; Healy & Hoyles, 2000; Chazan, 1993). Such research is deeply concerning, as proof is "an essential component of doing, communicating, and recording mathematics" (Schoenfeld, 1994, p. 74). Yet these results are perhaps unsurprising, as only recently has mathematics education begun to emphasize proof in the curriculum (e.g., National Council of Teachers of Mathematics [NCTM], 1989, 2000). NCTM recommends that proof and reasoning be taught from prekindergarten to 12th grade. The reasoning and proof process standards most relevant to our research include: the investigation of mathematical conjectures; the development and evaluation of mathematical arguments and proofs; and the use of "various types of reasoning and methods of proof" (NCTM, 2000, p. 322). Building upon NCTM's work, the new but widely-adopted Common Core Standards identify constructing viable arguments and critiquing the reasoning of others as critical skills for students across grade levels to learn, while the standards for high school geometry specifically call for students to learn to construct mathematical proofs of theorems (Common Core State Standards Initiative, 2010).

As the field continues to struggle with teaching proof, alternative forms of supporting students when constructing justifications are worth considering. Healy and Hoyles (2000) found that correct proof production is easier for students when they can engage in building narrative forms, rather than algebraic ones. Various authors have examined the role of digital geometry environments in supporting the development of proof (Chazan, 1993; Hoyles & Jones, 1998).

Here, we take a novel approach by looking to theories of embodied cognition, examining alternative methods of supporting proof production and communication through the modalities of action and gesture. We place ourselves on two of PME-NA's continuums, as we are using an *innovative* method of examining *student learning* during proof production. This paper makes no claims as to the newness of action and gesture in supporting mathematical learning and communication—in fact, it is the overlooked role of action and gesture to which we wish to draw attention. Consequently, we contribute a new lens in order to reveal invisible proofs.

In the following paper, we begin by defining the practice of mathematical proof, and then explore relevant research on embodied cognition, action, simulated action, and gesture. We then share two excerpts of students using gesture and action to support proof from a recent study we conducted. In these excerpts and others, we find that consideration of the practice of proof is greatly enriched by considering both the verbal and the physical modalities. Finally, we share our future plans for this research, and examine the potential implications for teaching proof.

Theoretical Framework

Mathematical Proof and Justification

We conceptualize mathematical justification using Harel and Sowder's (1998) proof scheme, and our intended *mode of thought* (e.g., the modality of mathematical observation and reflection; p. 240) is body-based action and gesture. Harel and Sowder define *proving* as "the process employed by an individual to remove or create doubts about the truth of an observation" (p. 241). They further identify two subprocesses of proving: ascertaining as the proof activities an individual engages in when attempting to convince themselves; and persuading as the proof activities an individual engages in when attempting to convince others. As mathematical proof occurs in a social context where the argument must be communicated to an audience effectively and convincingly, we argue that each subprocess is essential when considering the learning of proof.

Harel and Sowder's (1998) proof scheme includes multiple categories and levels for classifying mathematical proofs. For our purposes, we focus on the *analytical>transformational* proof scheme, which involves "operations on objects and anticipations of the operations' results" (p. 259). In particular, when students are utilizing the analytical>transformational proof scheme, they are *transforming* a mathematical object or concept by varying some relationships purposefully in anticipation of certain results, observing the resulting changes, and deducing mathematical properties accordingly. Although this is a powerful and effective method of proving for students to learn, little research has examined how gesture and body-based action can play a role in supporting these dynamic transformations. We thus turn to a discussion of research on embodied cognition and gesture.

Gesture and Action

Theories of embodied cognition suggest that cognitive processes are not algorithms acting upon amodal mental systems, but rather they are bound up with the action and perceptual systems of the thinker (Barsalou, 1999; Barsalou, 2008; Glenberg & Robertson, 2000; Wilson, 2002). These action and perceptual systems, in turn, are not only guided by cognitive process but they also constitute and transform those processes. In other words, gestures and actions are not simply byproducts of cognition—they are coupled (Shapiro, 2011) to cognitive processes, feeding back into cognition just as they can be the result of cognition. For example, gesture accompanied by speech may elaborate upon the thoughts possessed by the speaker (contributing additional information not contained by the speech acts), as well as feed back into the cognitive acts and transform the gesturer's cognition (Goldin-Meadow & Beilock, 2010; Alibali & Kita, 2010).

Gestures are a particular form of action that represents the world, rather than acts upon the world directly (Goldin-Meadow & Beilock, 2010). Furthermore, gestures are more than mere movements; as McNeill (1992) says, they "can never be fully explained in purely kinesthetic terms" (p. 105). Gesture is tied tightly to action, in our view, following Hostetter and Alibali's (2008) conceptualization of gestures as "manifestations of the simulated actions and perceptions that underlie thinking" (p. 508). Gestures are symbols that serve both to communicate and to affect the gesturer's cognition. Whether the participants' gestures are produced as communicative or cognitive acts may appear to be a crucial distinction that we are hereby in need of making. However, Hostetter and Alibali determine such a distinction to be somewhat false:

> …gestures are a natural by-product of the cognitive processes that underlie speaking, and it is difficult to consider the two separately because both are expressions of the same simulation…. [G]esture and speech may express different aspects of that simulation…but they derive from a single simulation; thus, they are part of the same system. (2008, p. 508)

Consequently, we use verbal and gestural data side by side in our analysis for the sole purpose of triangulating in on participants' cognition. By considering multiple modalities in this fashion, we are able to gain access to elements of ascertaining and persuading proof activities that would otherwise remain hidden in plain sight.

Gesture, Action, and Mathematics

Learners' gestures and actions have been found to support mathematics learning in many previous studies (e.g., Glenberg et al., 2007; Nathan, Kintsch, & Young, 1992; JLS piece, in press; Alibali & Goldin-Meadow, 1993), and are "involved not only in processing old ideas, but also in creating new ones" (Goldin-Meadow, Cook, & Mitchell, 2009, p. 271). In our research, we build upon this prior work while venturing into new territory: the role of action and gesture in supporting proof production is a new arena of study. Consequently, in the following section, we discuss gesture and mathematics in a general fashion, and draw out some threads that are particularly relevant to the domain of proof.

In some cases, gesture or body-based action may allow students to manipulate conceptual objects in a fashion similar to dynamic geometry software. In these systems, students can build objects that maintain invariant relationships even as the object is manipulated and acted upon (e.g., when a single vertex is moved on a triangle, the connected sides will stretch to meet the new location of the vertex, always keeping a triangular shape) (Hoyles & Jones, 1998). These environments can support students in generating and verifying conjectures about the relationships contained within these objects. Similarly, the real-world context within which action and gesture are produced can give feedback about the legitimacy of the constructs evoked by the mathematical conjectures. Whereas with paper and pen impossible triangles can be constructed (e.g., a triangle where the hypotenuse is labeled as longer than the sum of the remaining two sides; see Table 1 for the relevant task), using the body to construct the triangle can constrain students from such impossibility.

One of the features of dynamic geometry software is that it affords the possibility of testing a large number of examples while maintaining the relevant invariant features. Gesture and simulated action may have a similar affordance. Hostetter and Alibali (2008) note that "Because mental images retain the spatial, physical, and kinesthetic properties of the events they represent, they are dependent on the same relationships between perceptual and motor processes that are involved in interacting with physical objects in the environment" (p. 499). In other words, mental simulation, as evidenced through action and gesture, also allows such testing. We hypothesize that action and gesture can influence cognition in a way that is soundly based upon the physical experience of space within the world. In this way, action and gesture may support analytical>transformational proof production.

Given the potential influence of action, gesture, and simulated action upon cognition, we designed an experiment to examine the role of action and gesture upon mathematical proof production and communication. In the following section, we share our methodology and mathematical conjecture tasks.

Methods

The data reported in this paper were drawn from a larger study of the role of action and gesture in proof. Participants were 36 students (22 F; 14 M) at a large midwestern university enrolled in a psychology course, and they received partial class credit for participating. The average age of our participants was 20 years old (15 freshmen; 9 sophomores; 7 juniors; 4 seniors; 1 part-time student). Participants reported their ethnicities as follows: 21 Caucasian; 8 Asian; 3 Hmong; and one each from four other ethnicities.

In one-on-one interviews, participants were asked to justify a variety of mathematical theorems from number theory and geometry; here we report on the two conjectures shown in Table 1. Our research question was: *How are gestures and actions related to the ascertaining and persuading phases of proof?* In the larger study (not reported here), we encouraged participants to produce particular sorts of gestures for some conjectures. However the analysis here only focuses on cases in which students spontaneously gestured and used action when proving (i.e., their behavior was not manipulated).

Table 1: Conjectures

Conjecture 1: Gears	Conjecture 2: Triangle Inequality
An unknown number of gears are connected in a chain. You know which direction the first gear turns. How can you determine what direction the last gear will turn? Provide a justification for your answer.	*Mary came up with the following conjecture: "For any triangle, the sum of the lengths of any two sides must be greater than the length of the remaining side." Provide a conceptual justification as to why Mary's conjecture is true or false.*

We intentionally created a context in which participants would be likely to produce gestures by setting up the discourse context (Hostetter & Alibali, 2008) in various ways. First, we emphasized the talk-aloud nature of the experiment to all of our participants, and we verbally prompted them to reason out loud if they fell silent. For some, we removed pen and paper; and for others we also removed the chair so that they (and the interviewer) had to remain standing. Given that our goal was to support mathematical cognition with gesture, and not to duplicate an educational setting, such choices seemed appropriate. Furthermore, by designing the protocol to give no feedback to the participants, we emphasized *innovative* proof instead of *imitative*, and shifted the *mode of thought* (Harel & Sowder, 1998) to mental and physical by lessening the availability of pen and paper.

Each participant was videotaped, with the camera(s) capturing both small and large gestures. The data were analyzed in Transana, a qualitative analysis software program, and team members selected various gesture segments with the units of analysis based on each mathematical conjecture. In particular, we used *multimodal analysis* techniques (Alibali & Nathan, in press; McNeill, 1992), and coded how each student used action and gesture during the ascertaining and persuading phases of their proof (Harel & Sowder, 1998). For the purposes of this paper, we selected two examples of justification that highlight the role of gesture in mathematical proof, and in our presentation we will provide additional examples.

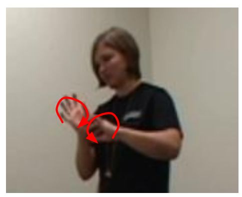
Results

We share two specific examples: an example of how simulated action (manifested through gesture) can illuminate the ascertaining phase of proof, and an example of embodied proof that demonstrates the role of paired gesture and speech during the persuading phase of proof.

1 **Well, if you know which direction the first gear is turning, then you also know that the second gear must to be turning in the opposite direction,**

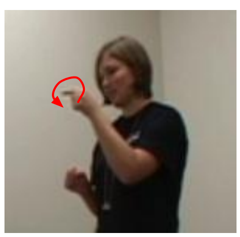
(Ascertaining) Uses her hands to replicate the turning of a gear

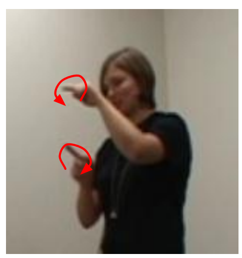
2 **Because one gear turning would push the other gear...**



(Ascertaining) Uses both her hands to simulate the movement of two gears turning against each other

3 **See if one gear was turning this way, then the spokes on it would push...**

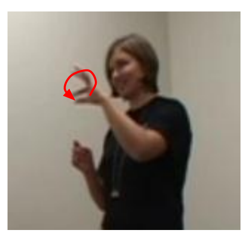




(Ascertaining) Uses her hands to simulate the spokes of one gear pushing the second gear

4 **If one gear was turning like that, then the spokes would push the other gear in the opposite direction, so you would know that every other gear from the first gear was turning the same direction as the first gear.**

(Persuading) Uses her hands to simulate the movement of two gears as she solves the conjecture.



5 **So all the odd gears would be turning in the original direction and all the even gears would be turning in the opposite direction. So if the last gear was odd it would be turning in the original direction and if the last gear was even it would be turning in the opposite of the original direction.**

Figure 1: Simulated Action Illuminates Ascertaining

Simulated Action Illuminates Ascertaining

To illustrate the illumination of proof by simulated action, we share an excerpt from the Gears conjecture (Table 1), as the participant leverages her body as a tool for simulating the actions of the gears and identifying parity (shown in Figure 1). The excerpt contains both ascertaining and persuading phases of the proof, and is annotated as such.

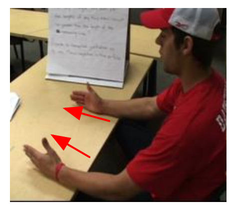
For each idea the speaker expresses verbally in the ascertaining phase, she also produces gestures that physically simulated the motions of the gears. By using her body, she is able to understand the relationship of individual gears to each other, and consequently to solve the conjecture. Her eyes remain focused on her hands, as she uses her body to understand *why* the conjecture is true. During the persuading phase, gesture is also critical; as she speaks, her gestural simulations *show* the audience how and why her verbal statement is true. However, a more powerful example of how action supports and constitutes mathematical proof during the persuading phase is given next.

1 Say one side's 5.

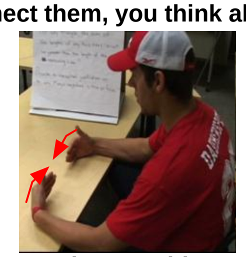
Uses his fingers to indicate a general side length

2 If the other two sides weren't at least 5 or even equal to 5,

Uses his hands to indicate the other two sides, of unspecified lengths but shorter than the general side length indicated in Line 1



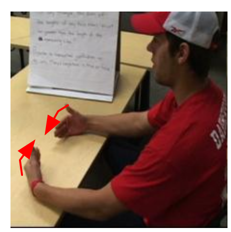
3 if you tried to connect them, you think about it, they'd be too short to touch.



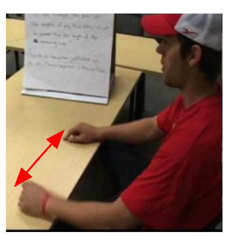
Curves his hands in to illustrate the two shorter sides trying to close the triangle

4 So they would have to-- the two sides would have to be longer

Repeats the gesture of the two shorter sides failing to meet; re-invokes the original conjecture



5 than the other remaining side.



Repeats the gesture of the first general side length; completes answering the conjecture.

Figure 2: Paired Gesture and Speech Persuading

Paired Gesture and Speech Persuading

To illustrate the pairing of gesture and speech to support persuading in embodied proof, we examine an excerpt that occurs after the participant has solved the Triangle Inequality conjecture (Table 1) and has shifted into the persuading phase of the proof (Figure 2). The verbal element of the proof provides a specific example, as simultaneously the gestural components communicate the generalizability of the participant's proof.

In Figure 2, the verbal and gestural components are woven together to provide a complete proof. Attending only to the verbal proof elements would result in an incomplete empirical justification, as the participant would appear to be basing his entire argument upon testing a single (and incompletely described) triangle. However, in considering the gestures, we gain insight into the participant's full argument that goes beyond empirical and into the realms of a mathematically legitimate proof. Consequently, it is through considering gesture and speech as paired that full insight into the participant's reasoning process and proof is possible.

Discussion

The two examples given are intended to highlight the multi-modal nature of proof, and to show that understanding proof production can require attending to more than just students' verbal and written work. In Figure 1, we share a proof in both the ascertaining and the persuading phase. Following solely the participant's verbal actions gives an incomplete account of the proof and, furthermore, gives only minimal insight into her mental actions. By attending to her gestures as well, we can see the role of the gear simulation in her solution, and gain considerable insight into *why* her solution is correct in the persuading phase. In Figure 2, we share a proof in the persuading phase that superficially (verbally) appears to be the mere testing of a single example, couched within language that can incorrectly indicate that the participant believes his single example to be sufficient proof. As soon as we attend to his accompanying gestures, however, a powerful transformational proof is revealed.

The two examples included in this paper are not rare examples from our data, but rather are characteristic of many other proofs we saw. Alongside our exploration of the different modalities of proof, we are examining various ways that gesture and action can support mathematical learning and, consequently, proof production. Although the data reported here come from the proofs that participants spontaneously generated, the data from conditions in which we manipulated action and gesture promise to reveal ways in which carefully designed action and gesture interventions can support proof production (Walkington et al., accepted). As many students have difficulty producing traditional deductive proofs, preferring inductive empirical reasoning (Chazan, 1993; Healy & Hoyles, 2000; Hoyles & Healy, 1999, 2007), gestures and actions may provide an accessible bridge between the two. The potential of simple physical movements to support mathematical understanding is vast—and a crucial new area of study, given the importance of proof to the mathematical community and the general difficulty of students engaging in proof practices.

Conclusions

The implications of emerging research on embodied cognition are profound for mathematics education in general, and the teaching of mathematical proof in particular. Action and gesture provide another modality for mathematics learning and expression, which may particularly support those students who struggle with the abstract traditional notation used with proof.

Extending the examination of proof production into gesture and action allows us to conceptualize a more complete model of cognition (Shapiro, 2011), and consequently allows us to design new activities that more coherently account for different strategies of proof production.

Our research provides a starting point for those examining mathematical proof through the modalities of action and gesture, and we continue to research the impact of action and gesture upon proof production (Walkington et al., accepted). An important question remaining is how does this research fit into or otherwise stretch existing proof frameworks, such as Harel & Sowder's (1998) and Healy & Hoyles' (2000)? Our next step is to answer exactly that question, and provide a multidimensional framework that incorporates a proof scheme with a spectrum of gesture, action, and proof.

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