Exploring Research in Algebra:
Algebra Readiness in the Elementary Classroom

February 15th, 2013

Research in Mathematics Education
The share of American males studying math-intensive subjects has fallen; it declined dramatically when the share completing college rose, but recently has slid downward along with college completion rates.

**Math Slide (Figure 1)**

**Source:** American Community Survey of 2009 and 2010
Research on Algebraic Understanding

- Algebra has often been characterized as developmentally constrained due to its inherent abstractness (e.g., Kieran, 1981, 1985; Vergnaud, 1985)

- Research in the former Soviet Union suggested that young children could generalize arithmetic, moving from particular to generalized numbers, learning to use variables and covariation in word problems, and focusing on the concept of function (Davydov, 1991, Bodanskii, 1991)
Recent research suggests that inappropriate instruction may have had a decisive role in the poor results from early studies of algebraic reasoning among adolescents (Booth, 1988; Schliemann & Carraher, 2002).

Studies of systemic algebra instruction have provided equivocal findings (Clotfelter, Ladd, &Vigdor, 2012; Cortes, Goodman, & Nomi, 2013).
Effects of Accelerating Algebra

Course Reversal (Figure 3)

In 2001 and 2002, Charlotte-Mecklenburg’s algebra acceleration policy expanded access to Algebra I by 8th grade for less-skilled students, but the change was short-lived.

Note: Figure shows the share of Charlotte-Mecklenburg students taking Algebra I by 8th grade, by 6th-grade math test-score quintile and year entering 7th grade.

Impact of Double-Doses of Algebra

Test-Score Boost (Figure 1)

Students who doubled up on algebra had higher scores on standardized tests taken after 10th grade.

Effect of double-dose algebra on standardized test scores in math, algebra, and geometry

<table>
<thead>
<tr>
<th></th>
<th>PLAN Math</th>
<th>PLAN Algebra</th>
<th>PLAN Geometry</th>
<th>PLAN Math</th>
<th>PLAN Algebra</th>
<th>PLAN Geometry</th>
<th>ACT Math</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fall 10th grade</td>
<td>0.09</td>
<td>0.09</td>
<td>0.01</td>
<td>0.16*</td>
<td>0.15*</td>
<td>0.10</td>
<td>0.15*</td>
</tr>
<tr>
<td>Fall 11th grade</td>
<td>0.09</td>
<td>0.09</td>
<td>0.01</td>
<td>0.16*</td>
<td>0.15*</td>
<td>0.10</td>
<td>0.15*</td>
</tr>
</tbody>
</table>

* indicates statistical significance at the .05 level

NOTE: PLAN is a test students take in 10th grade in preparation for taking the ACT college-entrance exam the following year.

SOURCE: Authors' calculations based on Chicago Public Schools data
Double-dose algebra increased the percentage of students who graduated from high school and of those who enrolled in college, with most choosing two-year institutions.

**High Impact** (Figure 2)

Effect of double-dose algebra on high school graduation and college-enrollment rates

- Graduated within 5 years: 7.9*
- Enrolled in college: 8.6*
- Enrolled in two-year college: 7.9*

* indicates statistical significance at the .05 level

**SOURCE:** Authors' calculations based on Chicago Public Schools data and National Student Clearinghouse data
Reading and Writing in Algebra (Figure 3)

Students with weak reading skills benefited more from the algebra support class than otherwise similar students, perhaps because reading and writing were central to the instructional model.

Effect of double-dose algebra on high school graduation and college enrollment rates, by reading skill level

- Graduated within five years
  - Below-median reader: 12.7*
  - Above-median reader: 3.1

- Enrolled in college
  - Below-median reader: 13.3*
  - Above-median reader: 3.9

* indicates statistical significance at the .05 level

SOURCE: Authors' calculations based on Chicago Public Schools data and National Student Clearinghouse data
Critical Topics for Teaching and Learning Algebra

(1) Variables and constants
(2) Decomposing and setting up word problems
(3) Symbolic manipulation
(4) Functions
(5) Inductive reasoning and mathematical induction

Milgram (2005)
“A good teacher walks the edge between the structure of mathematics and the development of a child by considering a progression of strategies, the big ideas involved, and the emergent models.”

Fosnot and Jacob, 2010
Developing an Essential Understanding of Algebraic Thinking

Arithmetic as a Context for Algebraic Thinking
  - 5 Essential Understandings

Equations
  - 3 Essential Understandings

Variables
  - 5 Essential Understandings

Quantitative Reasoning
  - 2 Essential Understandings

Functional Thinking
  - 6 Essential Understandings

Blanton, Levi, Crites, Dougherty, 2011
Arithmetic as a context for algebraic thinking

- The Fundamental Properties of number and operations govern how operations behave and relate to one another
- The Fundamental Properties are essential to computation
- The Fundamental Properties are used more explicitly in some computation strategies than in others
- Simplifying algebraic expressions entails decomposing quantities in insightful ways
- Generalizations in arithmetic can be derived from the fundamental properties.

Blanton et al., 2011
• “Historically, arithmetic and algebra were treated as distinct fields of study.”

• However, a true understanding of arithmetic also includes reasoning about the fundamental properties.

• Generalizations can be formed through exploration:
  – If you add a number to a given number and then subtract that same number, the given number stays the same.
    \[ a + b - b = a \]
  – An odd number plus an odd number is an even number

Blanton et. al, 2011
Fundamental Properties

Properties of Addition
- Associative
- Commutative
- Additive Identity
- Additive Inverse

Properties of Multiplication
- Associative
- Commutative
- Multiplicative Identity
- Multiplicative Inverse

Distributive Property of Multiplication over Addition
- Distributive

Blanton et al., 2011
Property significance when learning combinations

- The number of addition combinations and multiplication combinations to learn are cut in half when the commutative property is applied.

\[ 8 + 5 = 5 + 8 \quad 9 \times 2 = 2 \times 9 \]

- When combinations to learn are “chunked” and combined with the commutative and associative properties, students can compute long strings of numbers more efficiently.

---

**Addition**

<table>
<thead>
<tr>
<th>Adding 0</th>
<th>+1 or +2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Make 10</td>
<td>Up Over 10</td>
</tr>
<tr>
<td>Doubles</td>
<td>Near Doubles</td>
</tr>
</tbody>
</table>

**Multiplication**

<table>
<thead>
<tr>
<th>Zeros</th>
<th>Ones</th>
</tr>
</thead>
<tbody>
<tr>
<td>Doubles</td>
<td>Fives</td>
</tr>
<tr>
<td>Nifty Nines</td>
<td>Use the facts you know</td>
</tr>
</tbody>
</table>

Van de Walle, Karp, Bay-Williams, 2013
Use Mini-Lessons

• 10 minutes a day
• Focus on computational strategies and forming generalizations
• Select problems carefully
• Different types of structures
  – String of problems presented individually but share a relationship
  – Greater than, less than, or equal to
  – True or False
• All answers are valued and explored.
Reasoning with Fundamental Properties

\[ 1 + 2 = 2 + 1 \]
9 + 1

9 + 7 + 1

1 + 6 + 9
Reasoning with Fundamental Properties

(4 + 9) + 2 \[=\] 4 + (9 + 2)

43 + 17 \[=\] 17 + 33

(568 + 153) + 468 \[=\] 658 + (153 + 468)
Reasoning with Fundamental Properties

59 \times 16 \quad \square \quad 16 \times 15

4 \times 5 \quad \square \quad 5 \times 20

(65 \times 2) \times 1 \quad \square \quad 5 \times (2 \times 1)

13 \ (15 \times 10) \quad \square \quad 13 \times 130
The equals sign is a symbol that represents a relationship of equivalence.

\[ 9 + 5 = 8 + 6 \]

\[ 13 + 8 + 6 = 5 + 9 + 13 \]

\[ n + 13 + 9 + 5 = 6 + 8 + 13 + n \]

\[ 8 + 6 = 5 + 9 + n \]

Blanton et. al, 2011
The double number line

\[ 8 + 6 = 9 + 5 \]
The double number line

\[ 5 + 4 + 10 \neq 10 + 5 + 5 \]

Fosnot and Jacob, 2010
Equations

- Equations can be reasoned about in their entirety rather than as a series of computations to execute

<table>
<thead>
<tr>
<th>Column A</th>
<th>Column B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$48 \times 67 \times 6 = k$</td>
<td>$347 \times 25 \times 4 = k$</td>
</tr>
<tr>
<td>$346 \times 398 \div 42 = t$</td>
<td>$398 \times 746 \div 746 = d$</td>
</tr>
<tr>
<td>$978 + 778 = 394 + y$</td>
<td>$378 + 794 = 778 + j$</td>
</tr>
<tr>
<td>$475 \times 2365 = 352 \times w$</td>
<td>$8790 \times 598 = 879 \times n$</td>
</tr>
</tbody>
</table>
Equations

- Equations can be used to represent problem situations
  - The way we solve a problem does not always match the equation that represents the situation in the problem.

JaeQwan is making flowerpots. One flowerpot takes \( \frac{3}{4} \) of a pound of clay. How many flowerpots can JaeQwan make with \( 4\frac{1}{2} \) pounds of clay?

<table>
<thead>
<tr>
<th>Representation</th>
<th>Ways to solve</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m \times \frac{3}{4} = 4\frac{1}{2} )</td>
<td>( \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} = 4\frac{1}{2} )</td>
</tr>
<tr>
<td></td>
<td>( 4\frac{1}{2} ÷ \frac{3}{4} = 6 )</td>
</tr>
</tbody>
</table>

Blanton et. al, 2011
Functional Thinking

- Expressing those relationships in multiple ways

Symbolic Equation
Table
Context or pattern
Verbal Description
Graph

Blanton et. al, 2011
Van de Walle et. al, 2013
The number of circles is 2 times the position in the pattern.

\[ y = 2x \]

<table>
<thead>
<tr>
<th>Position</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td># of circles</td>
<td>2</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>

Graph showing points (1,2), (2,4), (3,6) on a grid.
Functions

\[ y = 2x \]

\[ y = x + 2 \]

\[ y = x \]
Functional Thinking

- Generalizing relationships
- Reasoning about those generalizations

Explore

Make conjectures

Build arguments to establish or refute conjectures

Make the generalization a piece of shared classroom knowledge

Revise

Blanton et. al, 2011

Kaput et. al, 2008
4 Instructional Goals

• **Represent**: Provide multiple ways for children to systematically represent algebraic situations.

• **Question**: Ask questions that encourage children to think algebraically.

• **Listen**: Listen to build on children’s thinking

• **Generalize**: Help children develop and justify their own conjectures

Kaput et. al, 2008
“When I’m working on a problem it’s like climbing a mountain. Sometimes I can’t even see where I’m going. It is one foot in front of another. And then I reach a point where all of a sudden the vistas open up and I can go down easily for a while, only to eventually reach another climb.”

Fosnot and Jacob, 2010
Build the learners capacity to make the climb

categorize

Develop the mathematician

evaluate

examine

Don’t fix the mathematician

build relations

cohesive structures

compare

Every action we take should develop the novice mathematicians in front of us

Fosnot and Jacob, 2010
References


