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RESEARCH IN MATHEMATICS EDUCATION

Numeric Relational Reasoning: Learning Progressions Development

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Numeric Relational Reasoning: Learning Progressions Development

Eloise Aniag Kuehnert • Josh Geller • Lindsey Perry • Cassandra Hatfield • Leanne Ketterlin Geller

Southern Methodist University

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PO Box 750114
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Contact information: rme@smu.edu

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Abstract

The purpose of this technical report is to describe the development of the Numeric Relational Reasoning (NRR) Learning Progressions for the Measuring Early Mathematics Reasoning Skills (MMaRS) project aimed for grades K-2. The NRR Learning Progressions focus on essential content and reasoning skills necessary to address three Targeted Learning Goals of NRR: (1) Relations, (2) Composition and Decomposition, and (3) Properties of Operations. The learning progressions describe a projected path both across and within grade levels. This report describes the rigorous rounds of internal and external reviews along with anticipated next steps.

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Numeric Relational Reasoning: Learning Progressions Development

Introduction

The purpose of this technical report is to describe the development process for the Numeric Relational Reasoning (NRR) learning progression. The learning progression is organized in a hierarchical structure that represents cascading levels of specificity. The specificity is often referred to as grain size, and represents the level of detail in the content specification. The structure is representing in Figure 1.

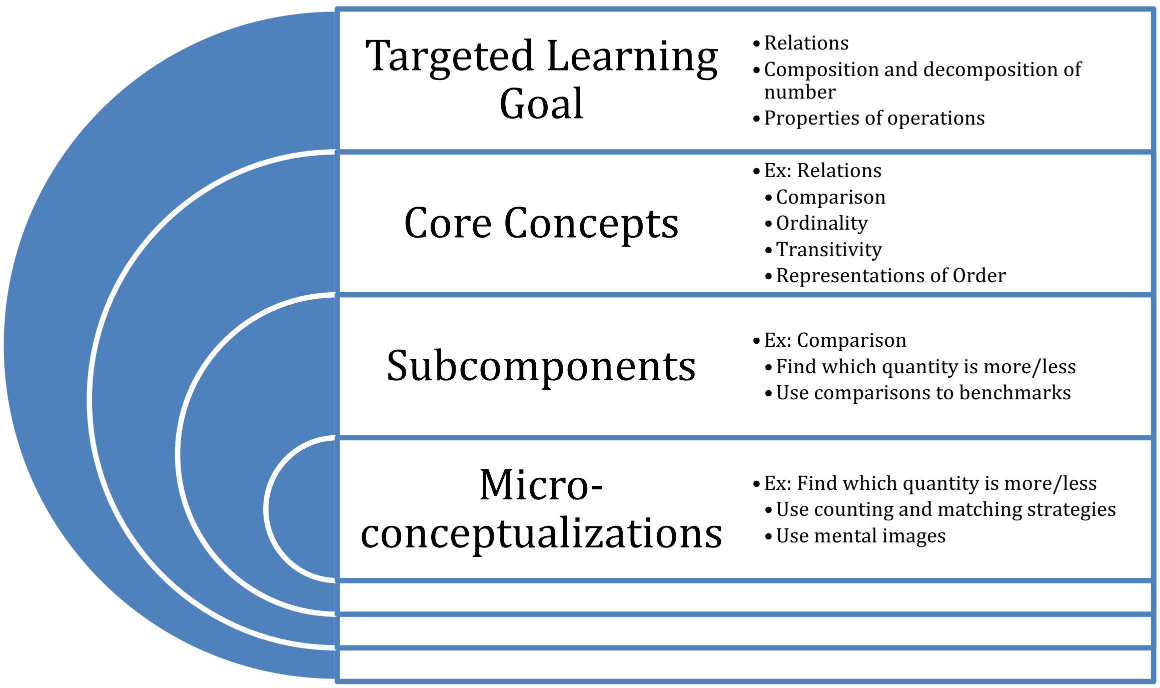


Figure 1. Structure of the MMaRS Learning Progressions.

The general process we used to specify the learning progression included:

1. a thorough review of the research literature on how children learn to reason relationally with numbers,

2. an extended meeting with external consultants to discuss the literature review and determine the Targeted Learning Goals, Core Concepts, and Subcomponents within the NRR learning progression,

3. iterative development and review cycles involving mathematics education researchers and mathematicians, and

4. collection of empirical evidence through cognitive interviews with students ranging from kindergarten to grade 3 and surveys of teachers in grades kindergarten to grade 3.

These data sources contributed to the final revisions of the NRR learning progressions.

Theoretical Framework: Numeric Relational Reasoning

Numeric relational reasoning (NRR) is defined as the ability to mentally analyze relationships between numbers or expressions, often using knowledge of properties of operations, decomposition, and known facts (Baroody, Purpura, Eiland, Reid, & Paliwal, 2016; Carpenter, Franke, & Levi, 2003; Farrington-Flint, Canobi, Wood, & Faulkner, 2007; Jacobs, Franke, Carpenter, Levi, & Battey, 2007). NRR, sometimes referred to as relational thinking, embodies flexibility with numbers and requires an acute "awareness of relations among numbers and the fundamental properties of number operations" (Jacobs et al., 2007). Because of this, NRR is not a procedure, nor a method with prescribed steps; it is instead a reasoning process that requires "strategic" decision making based on one's understanding of number relations (Whitacre, Schoen, Champagne, & Goddard, 2017). NRR focuses on mathematical relationships, whether it be between numbers or expressions. A person may engage their NRR skills as they solve a problem mentally by using number relations, preventing the need for lengthy mathematical calculations. For example, the commutative property [a + b = b + a] could be used to solve 15 + 28 = ☐ + 15, preventing the need to calculate 15 + 28 and then subtract 15. Furthermore, an expression such as 22 + 13 + 8 could be solved more easily by first recognizing the relationship between 22 and 8, which adds to a multiple of 10, and then adding 13.

In the MMaRS project, we specified a learning progression for NRR to serve as the content basis for classroom assessment resources and instructional resources.

Development Activities

This section describes the research activities we conducted to specify the NRR learning progression. In this report, we describe the extended meeting with external consultants and the iterative development and review cycles. Other technical reports document the processes we used to collect the empirical evidence through cognitive interviews with students and surveys of teachers.

Figure 2 outlines these major activities along with their products and purpose within the NRR learning progression development process. The development process for the NRR learning progressions spanned a total of six months.

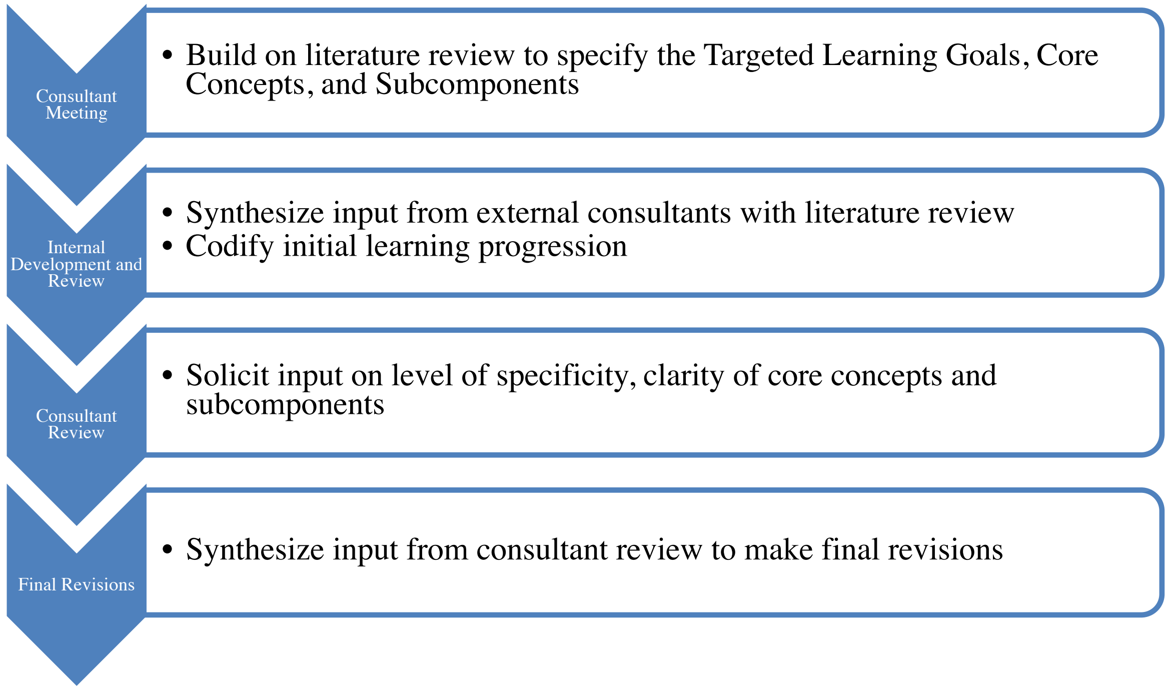


Figure 2. Systematic and Iterative Nature of the Internal and External Review and Development Process

External Consultant Meeting

Purpose. The primary goal of the external consultant meeting was to specify the Targeted Learning Goals, Core Concepts, and Subcomponents within Numeric Relational Reasoning (NRR) across grades K-2 in order from least complex to most complex. The three Targeted Learning Goals identified within NRR are Relations, Composition, and Decomposition, and Properties of Operations. Supporting literature includes the Common Core State Standards ([CCSS], Common Core State Standards Initiative, 2010) learning progressions (e.g., Confrey, 2012) and general trajectories (e.g., Sarama & Clements, 2009).

Method. Over a period of two days, the MMaRS research team met two external consultants at the RME offices in Dallas, TX.

<u>Consultant 1</u>

Juanita Copley is a Professor Emerita from the University of Houston where she served as the Chair of the Curriculum and Instruction Department in the College of Education. Dr. Copley has written and edited five books about early childhood mathematics co-published by the National Association for the Education of the Young Child (NAEYC) and the National Council Teachers of Mathematics (NCTM). She has also authored several national mathematics programs. Dr. Copley wrote and presented several Institutes and Academies for NCTM and was the primary teacher for National Head Start on the Webcast series, “Where’s the Math?” Over the past few years, she has worked with more than 20 states focusing primarily on early childhood mathematics and correlating state objectives with Common Core Standards for kindergarten.

She has also presented in international settings; Japan, Spain, Australia, and Bulgaria. Most importantly, she teaches in prekindergarten, kindergarten, and grade 1 settings weekly in high-need classrooms.

Consultant 2

Jane F. Schielack is Professor Emerita in the Department of Mathematics at Texas A&M University. A former elementary teacher, Dr. Schielack has built her career around working with teachers and students to enhance mathematics learning in the elementary grades. She has focused her activities on improving elementary mathematics education in two main areas: (1) teacher education and professional development and (2) curriculum development.

As a teacher educator, she has taught mathematics for elementary teachers for over 30 years and co-authored a textbook for the courses. Her professional development work has included the design of multiple sets of workshops, both face-to-face and web-based, addressing the mathematical knowledge of elementary and middle school teachers. Dr. Schielack participated on the writing committee of the National Council of Teachers of Mathematics (NCTM) *Professional Standards for Teaching Mathematics*.

Her curriculum development work began at the Texas Education Agency where she participated in the development of the Texas Essential Knowledge and Skills for K-8 Mathematics. At the national level, she was the chair of the writing committee for the NCTM *Curriculum Focal Points* in 2006 and editor of the NCTM *Teaching with Curriculum Focal Points* series for Grades 3-8. She was a member of the NCTM Review Team of the Common Core State Standards and was an input group contributor. Through consultation, she contributes to curriculum development and implementation at both the state and national levels. She is an author of *Scott Foresman-Addison Wesley Mathematics*, *Scott Foresman-Addison Wesley enVisionMATH*, and *enVisionMATH Common Core*. She is currently an author of *digits* and *enVisionMATH 2.0*.

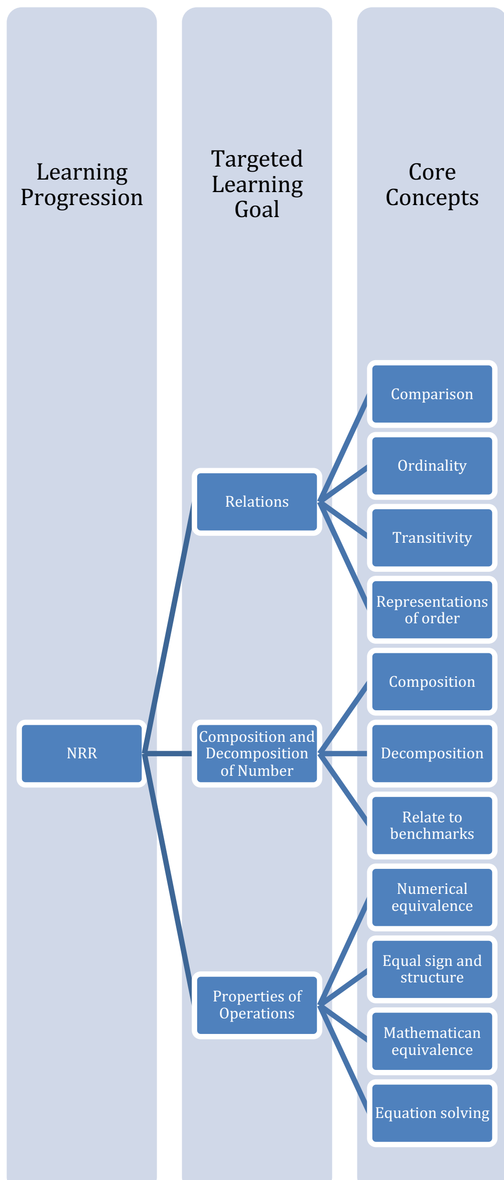
To begin the meeting, we reviewed the goals of the project and the purpose of the assessment; specifically explaining how the intended outcome includes classroom assessment resources that can be used to inform instruction. During Day 1, the meeting focused on specifying the Targeted Learning Goals and the Core Concepts. The team referenced existing learning progressions and learning trajectories from the research literature (Confrey, 2012; Sarama & Clements, 2009) as a starting point. Once the Targeted Learning Goals and the Core Concepts were specified, the team ordered the Core Concepts with increasing levels of sophistication. The ordered lists were examined to address any gaps. The same process was used for each core concept. During Day 2, we focused on specifying the subcomponents that make reasoning visible within the Core Concepts.

Questions to facilitate the conversations included:

- What is the definition of reasoning as elaborated from Sarama and Clements (2009) and the National Research Council (NRC, 2011)?

- How does reasoning progress?
- What evidence of reasoning (e.g., verbalizations, actions, expressions) would we expect from K-2 students from least to most complex reasoning for these subcomponents?
- How is reasoning operationalized?
- What are proposed levels of progression complexity?

Results. The following Targeted Learning Goals and Core Concepts were identified through this process:



The following descriptions detail the Targeted Learning Goals and the Core Concepts:

Relations

The operational definition for Relations is:

> *Numbers have an order that can be used to make comparisons*. Central to numeric relational reasoning is knowledge of the relationships between numbers (e.g., more, less, equal). In other words, in order to use number relationships for reasoning processes, one must first understand number relations. This includes understanding relative size and being able to compare quantities using a variety of methods (e.g., mental number lines, matching objects in two sets). Number relations is a core foundational mathematics topic (NRC, 2009) and supports students' numeric relational reasoning.

During the consultant meeting, discussion around Relations focused on "using order to compare quantities." Conceptually, the salient ideas are that each number in the counting sequence is one more than the previous number and that numbers have an order that can be used to make comparisons. The developmental progression of Relations begins with knowing the meaning of greater than and less than. This could be approached by comparing quantities using the counting sequence, comparing quantities using one-to-one matching, or estimating by relating to benchmarks.

Number order was identified as a slightly more complex understanding of Relations. At this point, number lines are introduced. Students could be given a number line with two numbers plotted and asked about the relationship between them (2 more/2 less; 10 more/10 less).

Moving toward the target that numbers have an order and can be used to make comparisons, we then discussed the idea of transitivity. Greater reasoning skills are elicited when a student can recognize order without relying on the specific quantities or the counting sequence. For example, knowing that if A<B and B<C, then A must also be less than C. Items within this stage of number relationships can vary in complexity with less complex items involving small numbers and more complex items using unknown quantities.

After transitivity, problems can progress into using representations of order to solve problems. For example, can students use the tools that represent order (i.e., 100s chart, bar diagrams, open number lines, box diagrams) to solve problems? At this level, symbolic representations and situated story problems are introduced.

Table 1 presents the core concepts within Relations and a brief description of the subcomponents that were generated from the meeting with the consultants.

Table 1

Core Concepts within Relations

Relations	*Numbers have an order that can be used to make comparisons (less than or greater than).*
Initial Core Concept	**Description**
Concept of comparison	- Greater than/less than: compare 2 quantities - Less complex: one-to-one correspondence, estimate related to benchmarks - More complex: using counting sequence
Concept that numbers have order	- Number line is a generalized representation of the order - Less complex: Given a quantity, compare more or less - Less complex: more/less (K: One more/one less; G1: ten more/ten less; G2: 100 more/less) - Given a quantity, relationship between benchmarks and intervals - More complex: Given two quantities, identify how many more/less (may/may not be related) - Given two quantities, relationship between benchmarks and intervals
Transitivity (if A<B and B<C, then A<C): Process of ordering	- Reasoning without relying on the specific quantities or the counting sequence - Less complex: with numbers - More complex: with unknown quantities but not relying on the counting sequence - More than, less than language vs. using “in between”
Use representations of order to solve problems.	- Progression of problems from story problems to symbolic representations - Can students use the tools that represent order to solve problems? - Possible tools: Bar diagrams (strip diagrams, 100 chart, open number line)

Note: K indicates Kindergarten; G1 indicates Grade 1; G2 indicates Grade 2.

Composition and Decomposition

The operational definition of composition and decomposition of number is:

> *Any number can be composed in many different ways*. Additive composition refers to the concept that any number can be composed and also decomposed into smaller numbers (e.g., 12 can be decomposed as 10 + 2 or 5 + 4 + 3). Additive composition is critical to numeric relational reasoning because it allows students to

> manipulate values within symbolic or visual tasks. For example, when asked, "How many more is 12 than 7?" students can utilize their knowledge of ways to make 10 to reason that "7 + 3 is 10, and 2 more is 12, so 12 is 5 (i.e., 3 + 2) more than 7." Knowledge of composition is rooted in students' understanding of part–whole relationships (i.e., a whole can be broken into parts), and students' ability to decompose and compose numbers stems from their experiences combining sets of concrete objects (Canobi et al., 2003). Because of this, these composition skills exist prior to formal schooling (Resnick, 1983, 1992), and, as students age, these skills continue to develop in complexity.

During the consultant meeting, we discussed how composition begins with children. One of the consultants noted that children begin putting together the quantity five. Next, children must recognize the quantity in many different arrangements. Children then develop an ability to subitize, knowing the quantity without counting (e.g., dominoes and dice).

The concept of place value was discussed in relation to composition and decomposition. The importance of making a number using parts that relate to place value units was discussed. However, because the concept of place value is unique to composition and decomposition, the discussion focused on using benchmark numbers when composing and decomposing. Place value may have a role within this Targeted Learning Goal, and could be integrated through models (e.g., concrete objects, set models, tape or bar diagrams, number symbols and mental representation). Operations were not included in the discussion about composition and decomposition.

The following developmental progression of students' ability to compose and decompose numbers was discussed in the following steps.

- Step 1: knowledge about a number or composition of a number,
- Step 2: ability to compose a number with parts,
- Step 3: ability to decompose into parts (equipartitioning may not be essential for successful performance),
- Step 4: ability to compose using part-part-whole representations whereby a child can find an unknown part when given one part
- Step 5: knowledge of benchmark numbers related to composition and decomposition

Table 2 presents the core concepts within Relations and a brief description of the subcomponents that were generated from the meeting with the consultants.

Table 2

Core Concepts within Composition and Decomposition

Composition and Decomposition		*Any number can be composed in many different ways.*
Initial Core Concept	**Label**	**Description**
Compose a number using single objects	Composition	Use similar concrete objects to compose a number
Compose a number using parts		More than 1 way
Decompose into parts	Decomposition	Possibly with G2: fair sharing situations
Decompose into parts given a part		Representations could be: bar diagrams. Part-part-whole
Relate to benchmark numbers	Relate to Benchmarks	Skill subsumed into Relations: Representations of Order in Comparison Situations

Note: K indicates Kindergarten; G1 indicates Grade 1; G2 indicates Grade 2.

Properties of Operations

The overarching goal of this Targeted Learning Goal is to "use properties of operations to maintain equality or solve equations." The operational definition is:

> *Operations have properties that can be used to solve equations*. Properties of operations are frequently used when reasoning relationally (Koehler, 2004; Molina et al., 2006; Nunes et al., 2007). Much of numeric relational reasoning involves restructuring or transforming number sentences to aid in mental calculation, both of which require the use of properties of operations (Koehler, 2004), just as the commutative property of addition ($a + b = b + a$) and the associative property of addition ($[a + b] + c = a + [b + c]$) (Gelman & Gallistel, 1978; Farrington-Flint et al., 2007; Piaget, 1952; Resnick, 1983; 1992). Applying properties of operations, such as the associative and commutative properties, assists students in manipulating expressions or visual representations in order to see the relationships between them, and enables students to recognize when a mathematical calculation is or isn't needed. While properties of operations are typically recognized in their symbolic form, they can be modeled and applied using concrete objects, visual representations, and story problems (e.g., four students get in a train car, followed by three more students. In the next train car, three students get in, followed by four more students. What is the relationship between the number of students in each train car?).

During the meeting, we discussed that Properties of Operations builds in complexity from the basic understanding of the equal sign, to using one property to solve a situation, to employing the properties of operations to keep both sides of an equation equal. At the most complex level, students solve equations for an unknown value using two or more properties. Difficulty, as opposed to complexity, increases when larger numbers are introduced, number of steps needed to complete are increased, and more properties are used per item.

Because of the intertwined nature of complexity and difficulty, care was taken to carefully sequence the learning progression to build in complexity and difficulty. It was discussed that children in Kindergarten could grasp the concept of equality while working with objects. Children in Grade 1 could work with scales and balances and the equal sign. Finally, children in Grade 2 students could work with equations and missing addend in story problems. Equations with larger numbers could be used with the intent that students may use properties of operations to solve them and not compute.

When examining the inter-relations among the targeted learning goals, we reconceptualized the placement of inequalities. Originally, inequalities was associated with Properties of Operations. However, because Relations focuses on the idea that numbers have order that could be used for comparisons, we moved the concept of inequality to Relations. As such, Properties of Operations focuses primarily on the concept of equality.

Properties of Operations focuses on the concept of equality and inequality; however, inequality was later removed.

Table 3 highlights the core concepts and subcomponents that were generated from the meeting with the consultants.

Table 3

Core Concepts within Properties of Operations

Properties of Operations	*Operations have properties that can be used to solve equations.*
Initial Core Concept	**Description**
Understand the concept of equality and inequality	- K: size, color, and arrangement does not impact equality - G1: equal sign
Use properties of operations to maintain equality with known starts	- Two representations mean the same thing - Less complex: Using one property or step to solve - More complex: Use two or more properties to solve
Use properties of operations to maintain equality with unknown starts	- No calculations of missing term
Use properties of operations to solve for unknown values	- Less complex: Using one property or step to solve - More complex: Use two or more properties to solve

Note: K indicates Kindergarten; G1 indicates Grade 1; G2 indicates Grade 2.

Reasoning

External consultants cautioned that reasoning is different for each Targeted Learning Goal and Core Concept, and noted that children need probes to provide their reasoning. Although the literature might reveal how students are thinking and their behaviors used to identify how students are interacting with the concepts, it would be important to not generalize the findings to the other categories.

In regards to linking reasoning to content knowledge, reasoning must be contextually based on students' ability to reason within the content areas. Reasoning measured in this way is more meaningful because it informs the teacher about students' ability to move to a new understanding. Content provides the context for the reasoning because without this link, the information provided to the teacher cannot simply be that the student is not reasoning (writ large) – it needs a context to inform.

Next, the discussion shifted to defining levels of reasoning as "novice to expert." By defining the continuum of reasoning, we could clarify what a student needs in order to attain each level. This may address an instructional goal by teaching students how to ask themselves probing questions about what they know and what they need to know. The proposed 5-point scale (0-4) follows.

0 – Lacks basic prerequisite knowledge – doesn't know how to reason about this

1 – Novice: rote knowledge; minimal understanding

2 – Using correct tools, starting an explanation; but not reaching valid conclusions

3 – A valid explanation but only one way; may not be the most efficient way

4 – Expert: uses prior understanding in efficient ways, flexible, more than one way to explain

Lastly, there was some discussion (and further questions raised) about designing assessments to elicit reasoning at the various levels. For one-to-one administration, the levels of prompting could change. Prompting could vary based on the amount of prompting needed to complete the problem. To elicit more reasoning, what types of techniques could be used to get students to demonstrate their thinking or facility with mathematical reasoning? And, if someone tops out, can we go up to the next grade level?

Discussion. The resulting learning progression was conceptualized as three nested learning goals: Relations, Composition and Decomposition, and Properties of Operations. The nesting orientation highlights the connections among the three concepts. Skills associated with Relations are prerequisite to skills within Composition and Decomposition. Likewise, Properties of Operations skills build upon Composition and Decomposition skills. This nesting structure also implies that skills and concepts in a foundational learning goal (e.g., Relations) may develop in conjunction with broader skills and concepts in later learning goals (e.g., Properties of Operations).

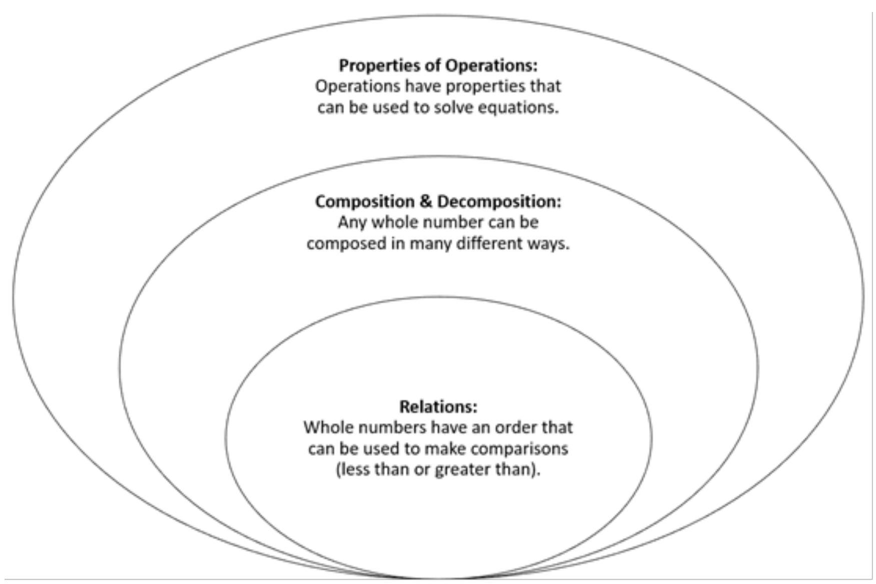


Figure 3. Targeted Learning Goals in Numeric Relational Reasoning, for grades K-2, connected within a nested diagram.

During the meeting, the consultants expressed that assessment results should indicate students' readiness for the next concept or experience, as opposed to a risk for failure. The question was posed, "Is there a progression of reasoning or levels?" Discussion ensued about the importance of life experiences leading to greater reasoning ability and also the importance of non-examples. To fully understand something, you need to know what it is not (e.g., red/not red, odds/evens, primes/composites, like/not like). Knowing whether or not something is reasonable is also an important concept.

Internal Development and Review

Purpose. As a result of the meeting with the consultants, we had identified the Targeted Learning Goals and the Core Concepts for the NRR learning progression. The next steps was adding specificity to the Subcomponents of the Core Concepts. As such, the MMaRS research team engaged in an iterative development process to specify the subcomponents.

As an initial step, additional details were added to the structure of the Core Concepts. Notably, to add specificity about when skills develop, each grade level was broken into a foundational, bridging, and target level. This progression represented when students would develop proficiency in the specified skills and concepts. Students should have foundational skills when entering a grade level, bridging skills to attain a new skill and target skills are mastered by the end of a grade level. Table 4 illustrates the progression in complexity from least complex (Subcomponent 1) to most complex (Subcomponent 4) and across the grades K-2.

Table 4

Initial structure of learning progressions

NRR Component	**Kindergarten**			**Grade 1**			**Grade 2**		
	F	**B**	**T**	**F**	**B**	**T**	**F**	**B**	**T**
Subcomponent 4						Skill 12	Skill 13	Skill 14	Skill 15
Subcomponent 3				Skill 8	Skill 9	Skill 10	Skill 11		
Subcomponent 2		Skill 4	Skill 5	Skill 6	Skill 7				
Subcomponent 1	Skill 1	Skill 2	Skill 3						

Note: F indicates Foundational; B indicates Bridging; T indicates Target.

Method. Two MMaRS team members began with the Core Concepts as established from the external consultant meeting. The team members reviewed literature (Sarama & Clements, 2009, NRC, 2011, NCTM, 2000, Baroody, 2006) specific to skills within concepts of relations, composition and decomposition, and properties of operations to support the development and inclusion of skills within each Core Concept. In addition to the established body of literature in early elementary mathematics, standards-based documents such as the CCSS-M (2010), TEKS (TEA, 2013), and Confrey's Learning Trajectories were used to substantiate grade level designations of the included skills.

Next, an iterative review process was used in which the full MMaRS research team engaged in revising the learning progressions.

Results. A condensed and consolidated version of the NRR learning progressions was developed through this process. Figure 6 displays a sample version for Composition and Decomposition that was sent to the external consultants. Within each grade level, timepoints of beginning, middle, and end of year were labeled as foundational, bridging, and target to classify the progression of a skill within a grade level.

	Core Concept		Kindergarten			Grade 1			Grade 2		
		Code	F	B	T	F	B	T	F	B	T
Essentialized Statements: Thirteen statements from least complex (top) to most complex (bottom)	Composition	NRR.B.5.a.	Compose a quantity with single								
		NRR.B.5.b.	Compose a number with two parts.								
		NRR.B.5.c.						Compose a number with three or more parts.			
		NRR.B.5.d.			Compose a number with two or more parts using place value.						
		NRR.B.5.e.						Compose a number with two or more parts in			
	Decomposition	NRR.B.6.a.	Decompose a number into two parts.								
		NRR.B.6.b.						Decompose a number into three or more			
		NRR.B.6.c.	Decompose a number into two or more parts using equipartitioning.								
		NRR.B.6.d.			Decompose a number into two or more parts using place value.						
		NRR.B.6.e.						Decompose a number into two or more parts			
	Decompose Part Unknown	NRR.B.7.a.	Given a unit, identify the missing part(s).								
		NRR.B.7.b.	Given one part of a number, identify the missing part(s).								
		NRR.B.7.c.	Given one part of a number, identify the missing part(s) in more than one way.								

Figure 6. Sample of Composition and Decomposition before sending out for external review.

Note: F indicates Foundational; B indicates Bridging; T indicates Target.

Discussion. Several key decisions were made to specify the subcomponents.

1. Number systems. The original learning progression referenced "numbers" without specifying if numbers were intended to include only "counting" numbers or if the number zero was intended to be included. All statements were updated to reflect whole numbers.

    a. Whole numbers have an order that can be used to make comparison (less than or greater than).

    b. Whole numbers can be composed in many different ways.

    c. Operations have properties that can be used to solve equations.

2. Number ranges. Number ranges were not originally specified in the learning progressions. To reflect the development of sophistication and difficulty discussed earlier, number ranges were then outlined specifying number ranges by proficiency levels within a grade. Figure 5 illustrates this progression.

Composition and Decomposition	**Kindergarten**		
	F	**B**	**T**

Compose a number using single objects.	Compose a number up to and including 5 using single objects.	Compose a number up to and including 10 using single objects.	Compose a number up to and including 19 using single objects.

Figure 5. Example of skills across proficiency levels within a grade
Note: F indicates Foundational; B indicates Bridging; T indicates Target.

3. Representation of Elements that Vary. To address issues of complexity and difficulty, we identified specific elements vary across and within grades. Varying elements increase in number, quantity, benchmark, or unit across foundational, bridging, and target sections within each grade and spanning across kindergarten, first, and second grades.

4. Level of Specificity of Mathematical Models/Representations. To further illustrate the scaffolding children might need to develop sophistication in their understanding of the Targeted Learning Goals, specific scaffolding skills were noted. For example, the content skill of "compose a number" included four scaffolding skills: (i) using single objects, (ii) using two parts, (iii) using place value, and (iv) using three parts.

Consultant Review

Purpose. The purpose of the external consultants' review was to solicit feedback from the original consultants about the synthesized NRR learning progression. The goal was to finalize the ordering of skill statements and confirm the developmental appropriateness of the subcomponents within the Core Concepts.

Method and Results. The external consultants who initially contributed to the specification of the learning progression reviewed the progressions for each of the three Targeted Learning Goals. They reviewed prepared materials and provided written feedback to specific questions. The consultants were also provided with a detailed primer document to aid in the review. The primer document can be found in Appendix B.

There were three parts to the review process: Part 1 was examining the vertical progression of the subcomponents, Part 2 was evaluating the elements that vary, and Part 3 was evaluating representations (e.g., patterned dots,10-frames, linking cubes, etc.) used. For this report, the results are presented separately by part.

Methods Part 1

Since the subcomponents were listed vertically from least complex to most complex, and also horizontally by kindergarten, grade 1, and grade 2, consultants were asked to comment on this arrangement. Consultants were asked survey questions using a Likert-type scale, and open-ended questions in which they were asked to explain any of their lower ratings. Below are the questions asked:

- Q1: Does the progression of the subcomponents (from least complex to most complex) represent how children learn? [Please explain your answer in writing.]

- Q2: Are there any essential skills missing in this learning progression? If so, which skills might you add and where would they be placed?

- Q3: How logical is the progression from least complex to most complex? [Select one: Very logical, Mostly logical, Somewhat logical, Not logical]

    - Q3a: If you rated this question as "Somewhat logical" or "Not logical," please provide comments and suggestions for improvement.

- Q4: How mathematically precise and accurate are the subcomponents? [Select one: Very precise and accurate, Mostly precise and accurate, Somewhat precise and accurate, Not precise and accurate]

    - Q4a: If you rated this question as "Somewhat precise and accurate" or "Not precise and accurate," please provide comments and suggestions for improvement.

- Q5: How developmentally appropriate are the subcomponents in relation to the grade bands specified? [Select one: Very developmentally appropriate, Mostly developmentally appropriate, Somewhat developmentally appropriate, Not developmentally appropriate]

    - Q5a: If you rated this question as "Somewhat developmentally appropriate" or "Not developmentally appropriate," please provide comments and suggestions for improvement.

Results Part 1

<u>Relations:</u>

- Question 1: Both consultants agreed that the subcomponents of Relations represent how children learn. One consultant said that it "makes sense" and another consultant agreed that "In general, it moves from working with quantities (sets of objects) to symbolic representation (numerals)."

- Question 2: One consultant commented that there should be an additional skill about written/printed number lines to be introduced in Grade 1. The second consultant thought that all skills were included but wondered about including another way to assess transitivity by "comparing two containers at a time and then recording which container overflows when poured into another one." This consultant suggested that this was a skill recommended by NCTM to *compare* quantities, and offered it as another way to assess transitivity.

- Questions 3-5: Both consultants rated the subcomponents within Relations as having a very logical vertical progression, as very precise and accurate, and mostly developmentally appropriate.

- Question 6: One consultant commented that Kindergarten is too early to introduce "place value" language, suggesting instead that a prior skill might be about using tens and ones to make a comparison. While place value could begin in grade one, kindergartners might look at how many tens you could make out of each number, or making piles of tens and piles of ones.

Composition/Decomposition:

- Question 1: Both consultants noted issues with the vertical progression of the subcomponents within the Composition/Decomposition. Once again, the notion of place value was questioned as being introduced too early in kindergarten by one consultant. Additionally, the second consultant also highlighted the need to consider more carefully those statements that spanned Kindergarten through Grade 2 as not being appropriate for Kindergarten students; in particular, composing with three or more parts is not generally asked.

- Question 2: Both consultants agreed that there were no essential skills missing from Composition and Decomposition.

- Questions 3-5: Both consultants rated the subcomponents within Composition and Decomposition as having a very logical vertical progression, as mostly or very precise and accurate, and mostly developmentally appropriate.

- Question 6: Neither consultant had any additional comments.

Properties of Operations:

- Question 1: One consultant agreed that the vertical progression of the subcomponents progress from least complex to most complex, however, both consultants commented about the structure and specificity of the statements. The structure of the subcomponents within Properties of Operations was arranged so that all skills spanned all three grades. A consultant noted that this wasn't likely the case and that some researchers (c.f., Rittle-Johnson, et al.) placed mathematical equivalence as a skill in Grade 5.

- Question 2: Regarding missing skills, one consultant expressed the importance of including "true vs NOT true equations" suggesting that knowing what *isn't* true is necessary for having an understanding of what is true. The other consultant had many suggestions for additional skills, or modifications. Suggestions included having "property" be the element that varies in two statement: a) "Recognize numerical equivalence using at least one *property* of operations within contextual situations." And b) Recognize numerical equivalence using decomposition and at least one *property* of operations within contextual situations." This consultant also suggested appending with "amount in a true equation" for all of the statements within the subcomponent of Mathematical Equivalence: Known Starts.

- Questions 3-5: Both consultants selected the progression as "Mostly logical" from least complex to most complex. One consultant indicated that the subcomponents were "Very precise and accurate" and the other consultant selected "Somewhat precise and accurate" indicating that certain statements needed to be appended with a phrase ("amount in a true equation”) to specify that the use of the equal sign indicates a true equation. Both consultants were concerned about the developmental appropriateness of the statements especially since the skills spanned grades K-2. Kindergarten students may be introduced to some of the ideas of working with properties (i.e., associative and commutative properties of addition) but symbolically, they aren’t ready.

- Question 6: Both consultants provided extensive comments throughout answering questions 1-5.

Methods Part 2

In Part 2, consultants were asked to comment on the elements within each subcomponent that vary across and within grade levels. Consultants were asked three questions (7, 8, and 9) with questions 8 and 9 requesting further explanations if needed.

- Q7: Do the "number/quantity," "word problems/story situations", "benchmarks," and "unit" specifications represent an appropriate progression within and across grades? [Please explain your answer in writing.]

- Q8: How logical is the progression of "number/quantity," "word problems/story situations", "benchmarks," and "unit" specifications from least complex to most complex? [Select one: Very logical, Mostly logical, Somewhat logical, Not logical]

    - Q8a: If you rated this question as "Somewhat logical" or "Not logical," please provide comments and suggestions for improvement.

- Q9: Overall, how developmentally appropriate are the "number/quantity," "word problems/story situations", "benchmarks," and "unit" specifications in relation to the grade bands specified? [Select one: Very developmentally appropriate, Mostly developmentally appropriate, Somewhat developmentally appropriate, Not developmentally appropriate]

    - Q9a: If you rated this question as "Somewhat developmentally appropriate" or "Not developmentally appropriate," please provide comments and suggestions for improvement.

Results Part 2

<u>Relations</u>:

- Question 7: Consultants had differing opinions about appropriateness of progression for *three out of four* of the elements that vary.

    - For “number/quantity,” “word problems/story situations,” and “unit” one expert consultant expressed that the specifications were appropriate, and the other consultant explained how they were not appropriate. One consultant felt that to include place value in kindergarten, the number/quantity would have to be increased to greater than nineteen to at least twenty-five.
    - Both consultants expressed that “story situations” were possible for kindergartners and should be included as given verbally, with pictures, matching, or counting how many don’t have a match within the number five.
    - For “unit,” one consultant felt that including five in kindergarten was not appropriate and suggested using one and two as the possible units.
    - Both consultants were in agreement that the specifications within “benchmarks” were appropriate.

- Questions 8 and 9: Both consultants agreed that the elements that vary within Relations followed a mostly logical or very logical progression and they were mostly developmentally appropriate.

Composition and Decomposition:

- Question 7: Consultants had differing opinions about *two out of three* of the elements that vary.
    - For "number/quantity" and "unit" one expert consultant expressed that the specifications were appropriate, and the other consultant explained how they were not appropriate. Again, one consultant recommended increasing the number/quantity for kindergarten to twenty-five to "be able to do more with groups of tens and ones."
    - Additionally, the same consultant felt that the minimum number for "unit" would need to be increased to two to avoid the issue that "If the unit is 1, then the only part unknown you can have is 0."
- Questions 8 and 9: Both consultants agreed that the elements that vary within Composition and Decomposition followed a very logical progression and they were very developmentally appropriate or mostly developmentally appropriate.

Properties of Operations:

- Question 7: Consultants had differing opinions about *three out of four* of the elements that vary.
    - For "word problems/story situations," "benchmarks," and "unit" one expert consultant expressed that the specifications were appropriate, and the other consultant explained how they were not appropriate because the consultant did not see the variable in each of the subcomponents.
- Questions 8 and 9: One of the consultants thought that the elements that vary within Properties of Operations followed a mostly logical progression and that they were very developmentally appropriate. The other consultant rated the progressions not logical and not developmentally appropriate because this consultant wasn't able to connect the variables to the subcomponents and didn't agree with using one as a benchmark.

Methods Part 3

Consultants were asked to comment on the mathematical models or representations specified within each subcomponent. Examples of models or representations include bar diagrams, place value blocks, counters, linking cubes, patterned/unpatterned dots, 5-, and 10-frames, bar diagrams, area models, and other representations of tens and ones. Some of the models or representations repeat between "target" skills of one grade and "foundational" skills of the next grade. Consultants were asked two questions: did the researchers leave out any important representations? and are the models or representations developmentally appropriateness and/or represent a progression of the models or representations?

Results Part 3

Relations:

- Consultants noted that *six out of twenty* of the subcomponents within Relations were missing important models or representations. Suggestions for which representations should be added included: Unifix cubes, an open number line at the end of Grade 1-2, a traditional number line for the end of Grade 1 and throughout Grade 2, and counters for kindergartners.
- Additionally, consultants provided feedback about the developmental appropriateness of *eight out of twenty* of the subcomponents. Comments included clarifying questions about "number" being symbolic, and beakers as a unit of measure, and when precisely a skill is foundational, bridging, or target; to the importance of introducing open number lines only after a written number line has been introduced, and using 1-100 chart rather than 0-99 chart.
- Lastly, both consultants rated the models or representations an overall rating of "mostly developmentally appropriate" for the entire Relations progression.

Composition/Decomposition:

- Consultants noted that *one out of thirteen* of the subcomponents within Composition and Decomposition was missing Unifix cubes that fit on fingers to be added within Kindergarten foundational and bridging skills.
- Additionally, consultants provided feedback about the developmental appropriateness of *three out of thirteen* subcomponents. One consultant expressed discomfort with place value blocks being used in Kindergarten, favoring instead the use of bar diagrams and "tens and ones" at the end of Kindergarten (target) or perhaps only the beginning of grade one (foundational). The same consultant also suggested that place value blocks made sense in grade two at least in bridging and target skills up to 999.
- Lastly, both consultants gave the models or representations an overall rating of "mostly developmentally appropriate" for the entire Composition/Decomposition progression.

Properties of Operations:

- Consultants noted that none (*zero out of twenty-eight*) of the subcomponents within Properties of Operations were missing any important representations.
- Additionally, consultants provided feedback about the developmental appropriateness of all (*twenty-eight*) essentialized statements. One consultant commented about all but one of the subcomponents with regard to use of place value blocks. This consultant expressed discomfort with place value blocks being used in Kindergarten, favoring instead the use of bar diagrams and "tens and ones" at the end of Kindergarten (target) or perhaps only the beginning of grade one (foundational). The same consultant also suggested that place value blocks made sense in grade two at least in bridging and target skills up to 999. The second consultant also noticed that the differences between grade bands was largely due to using or not using symbolic representations in the progressions.

- Lastly, as an overall rating, one consultant rated the representations as “mostly developmentally appropriate” and the second consultant rated them as “somewhat developmentally appropriate”. This second consultant was primarily concerned about the grade band expectations for Kindergarten and grade one being too conceptual, seeking instead a more concrete illustration for younger students like a balance scale instead of having envisioning or mental expectations of the concepts.

Discussion

Relations:
No changes were made to the overall structure or vertical ordering of the subcomponents for Relations. However, in response to the suggestions regarding written number lines and place value, two new subcomponents were added and one subcomponent was revised.

Two skills were added in the first core concept of comparison. To account for finding how many tens in a number at the kindergarten level, a skill was added to capture skills using foundational place value concepts within the numbers 11-19. The skill spanned from kindergarten Target to Grade 2 Target. Also, an additional skill was added to capture the *written number line* strategy and was placed before *mental number lines* because one of the consultants suggested starting the skill in Grade 1. In this version, the *mental number lines* skill began at Grade 1 target so the added skill using *written number lines* was added to begin at Grade 1 foundational with both skills extended to Grade 2 Target.

No changes were documented for the core concepts of Ordinality and Transitivity. However, the core concept of Representations of Order in Comparison Situations included minor adjustments to boundaries. Subcomponents involving matching and counting strategies had an upper bound of Grade 1 Target rather than extending to Grade 2 Target. Also, the skill stating *find how much more/less between two numbers in a given context using tools such as an open number line or chart* was replaced with a statement to incorporate the ideas of word problems/story situations in kindergarten coupled with the matching and counting strategy.

Composition and Decomposition:
No changes were made to the overall structure of the learning progression for Composition and Decomposition. In response to the suggestions regarding place value extending into kindergarten, statements were added in both core concepts of Composition and Decomposition to address *foundations of place value* beginning at the bridging level of kindergarten. The additional skills were separate from the place value skill spanning from the bridging level of first grade to the target level of second grade.

After team deliberations, four skills were adjusted from the target level of first grade to the target level in kindergarten. While the consultants were concerned about skills regarding three parts in kindergarten, the team considered that the teaching timeframe provides opportunities for students to engage with composing numbers with three parts before engaging in concepts related to associativity. Also, the last skill regarding identifying missing parts in more than one way was moved up to the bridging level of first grade.

Properties of Operations:
Before revising the Properties of Operations progression, we requested additional information from the consultants to clarify their overall recommendations. Proposed revision options were sent to the consultants to facilitate a conversation that would help us reconcile their feedback. The purpose of the conversation was to come to consensus on when specific symbols and actions would be introduced. The following topics emerged as concerns.

- *Place value*. The consultants noted that place value was not developmentally appropriate in kindergarten and recommended moving place value to the end of first grade. However, by the end of kindergarten, students may begin using mathematical vocabulary, such as "10 ones is 1 ten," which is solidified at the end of first grade. The consultants also recommended using proportional representations with base-10 at the end of second grade, and blocks should not be placed on a place value mat since the place value mat already specifies units.
- *Numerical equivalence*. Initially *numerical equivalence* caused confusion as the name of the core concept did not clearly imply that symbols were not involved. The consultants recommended clarification regarding the equivalence of quantity without symbols and suggested expressions could be introduced at the end of kindergarten.
- *Equal sign and structure*. Emphasizing an instructional progression of basic equality using a balance towards a relational understanding of the equal sign, the consultants recommended starting with a physical balance in kindergarten up to the middle of first grade. Then, from mid-first grade to mid-second grade students may transition using both the balance and the symbol. Finally, by the end of second grade, students should work primarily within symbols. For kindergarten students, the equal sign is not the most important concept of equality. As part of the transition from concrete to symbolic, kindergarten students may begin to associate balance scales with an equal sign on the fulcrum. Additionally, to determine whether a student is using a relational definition of the equal sign, students must engage in equations that are true as well as equations that are not true. In regards to equation structure, the consultants recommended that some equations (e.g., a=a, a+b=c; c=a+b) may be possible at the end of kindergarten and should be introduced in numerical relationships first. Equation structure with operations on both side of the equal sign should be moved to first and second grade.
- *Mathematical equivalence*: *known and unknown starts*. The purpose of *mathematical equivalence* was for students to identify what makes the contextual situations maintain equality. All associated tasks should be contextualized with a story and visuals. Then, symbols could be introduced at the end. Skills within *mathematical equivalence* may develop in parallel with Equal Sign and Structure. The contextual situations associated with *mathematical equivalence* were not intended to be solved since equation solving is addressed in the next core concept.
- *Equation solving*. Both external consultants cautioned about implications for kindergarten students and solving equations from the symbolic representation. As such, concrete objects and manipulatives, including balances, must be associated with symbolic representations. Additionally, since this core concept is the last, skills should represent the culmination of all previous skills included in the Properties of Operations learning progression.

As a result of the discussion with the consultants, the core concepts were condensed from five core concepts to four. Since the original core concepts 10 and 11 both addressed mathematical equivalence and the only difference was whether or not the starting number was given, the two core concepts were combined into one. Table 6 outlines some of the major actions taken based on the external consultants' comments.

Table 6

Refinement of the Properties of Operations progression after the consultants' review

Original Core Concept	**After External Review**	**Subcomponent Refinements**
8. Numerical Equivalence	8. Equivalence of Quantity and Number	A new skill was added to the end of the core concept to address the equivalence of two expressions without involving the equal sign.
9. Equal Sign and Structure	9. Equal Sign as a Relational Symbol	All skills were rewritten to focus on the recognition of true and not true equations. Specific equation structures were separated into two different skills to associate with appropriate grade levels. A new skill was added to use quantities on a balance. Boundaries were adjusted.
10. Mathematical Equivalence: Known Starts 11. Mathematical Equivalence: Unknown Starts	10. Maintaining Equality	Quantities and contextual situations were specified within the skill statement for clarity. Grade-level boundaries were adjusted.
12. Equation Solving	11. Solving for Unknown Values	Specific statements for concrete objects, contextual situations, symbolic representations were listed for each of the two major content skills.

Balances were included but not to the extent as recommended by the consultants. The research team decided that the use of balances as a scaffolding tool was best suited instructionally rather than specifically stated in the learning progression. Meanwhile, place value was listed as an important topic for discussion before making any decisions on the boundaries for place value skills. Proportional representations with base-10 blocks were not included in the progression as the details are beyond the scope of the current project.

Final Revisions

Purpose. The purpose of the final internal review process was to ascertain whether or not the external consultant comments were appropriately addressed and to update the NRR learning progressions for the version that would be used in the empirical studies (e.g., cognitive interviews, teacher survey). The finalized NRR learning progression is presented in Appendix A.

Method. The MMaRS research team invited two colleagues from within the university who teach the elementary mathematics methods course for pre-service teachers to engage in the final review process. The chain of evidence was reviewed and discussed, starting with the literature review and including the consultants' meeting, internal development and review process, and external review by the consultants. The research team reviewed the adjustments and the rationale to support the revisions.

Results. Table 7 outlines the concerns, initial responses, and final responses that resulted from the last internal review.

Table 7

Concerns and Responses based on the Internal Review

Initial Issue or Concern	Initial Response	Final Response
Place value concepts in kindergarten	Add "foundations of place value statement beginning in kindergarten to address "tens and ones"	Instead of stating "foundations of place value" and having another skill stating "place value," include only one skill statement of "concepts of place value." The importance of place value to numeric relational reasoning is the foundational place value (i.e., tens and ones, one ten is ten ones).
Place value blocks in kindergarten	Remove place value blocks in kindergarten foundation and Grade 1 target. Emphasize tens and ones; do not use place value mat.	The place value mat will not be used in the cognitive interviews and place value blocks are introduced at Grade 1 bridging.
Measuring volume is not included	Volume statement was not added as a separate skill to support numeric relational reasoning. A	No additional refinements needed.

	skill statement using balances with unspecified weights.	
How are word problems addressed in transitivity?	Skills that stated "in a context" were changed to "in a word problem."	No additional refinements needed.
Skills for ordering three numbers were included in kindergarten, but a skill for comparing two numbers was not included in kindergarten. Kindergarten included only quantities.	A skill was added for comparing two numbers beginning in kindergarten.	No additional refinements needed.
When are numeric symbols introduced?	Kindergarten Bridging: Numeric symbols Kindergarten Target/Grade 1 Foundational: Equal Sign Grade 1 Bridging: Equation Solving	No additional refinements needed.
The way unit is used in relations is different than the way unit is used in comp/decomp. The unit of 5 is not appropriate in relations, but it is appropriate in comp/decomp.	Unit was defined differently for Relations and Comp/Decomp. Relations excluded the number 5.	No additional refinements needed.
Compare two quantities to a benchmark to find which is more/less. Compare two numbers to a benchmark to find which is more/less.	Questions associated with this skill were intended to ask "Which number is closer to 10?" The child would be given two numbers and a number line with a benchmark listed. Question posed to the internal review team: • Because this skill is still addressing two numbers and a benchmark as a reference, should these	Skill moved from Comparison core concept to the end of the Representations of Order in Comparison Situations since comparing to a benchmark relies on skills of "finding how much more/less than a number."

	skills be moved to transitivity after unspecified weights?	

Discussion. Grade span rules were put into place for consistency. For example, if a subcomponent ended at the target level for one grade, ideally, the skill should be appropriate for the foundation level of the next grade level. In other words, if a skill previously ended at Grade 1 Target, the skill was automatically bumped up to Grade 2 Foundational. Likewise, if a skill previously started at Grade 2 Foundational, it was bumped down to Grade 1 Target.

All of the development team's proposed changes were implemented with the exception of the concerns regarding place value and benchmarks. In lieu of the changes proposed in Table 7, the following revisions were implemented based on recommendations from the internal review team:

- Place value skills were combined into one skill statement and used the phrasing "concepts of place value" to incorporate both foundations of place value that focus on tens and one as well as the conceptual understanding of the meaning of each digit within a two- or three-digit number.

- The skill regarding comparisons to a benchmark was moved from Comparison to Representations of Order in Comparison Situations. The skill was placed as the last skill in Relations. The rationale was based on the fact that the skills regarding *find how much more/less* are prerequisite skills to determine which number is closer or further away from a benchmark.

The resulting version of the learning progression was used for the empirical studies, including the cognitive interviews and the teacher surveys.

Next Steps

Our next steps include the process of empirically recovering the learning progression through the use of cognitive interviews and a teacher survey.

The purpose of the cognitive interviews and teacher survey is to collect empirical evidence to support or refute the conceptualization of content, ordering, developmental appropriateness, and interconnectedness of the learning progressions. The cognitive interview protocols associate each skill within the learning progression with a question prompt. Thirty-two students in grades K-3 will be interviewed using one-on-one interview protocols that are aligned to the NRR Learning Progressions. The protocols will provide opportunities for children to engage in the content as well as provide their thinking to support answers given.

A teacher survey has been designed to assess teachers' knowledge of Numeric Relational Reasoning and their instructional planning focused on the core concepts. We ask teachers to rate their understanding of skill statements, the developmental appropriateness for students in the grade they primarily teach, and the time of year at which they teach the particular skills.

Technical reports for both the cognitive interviews and teacher surveys will document the methods, results, and implications for the NRR learning progression.

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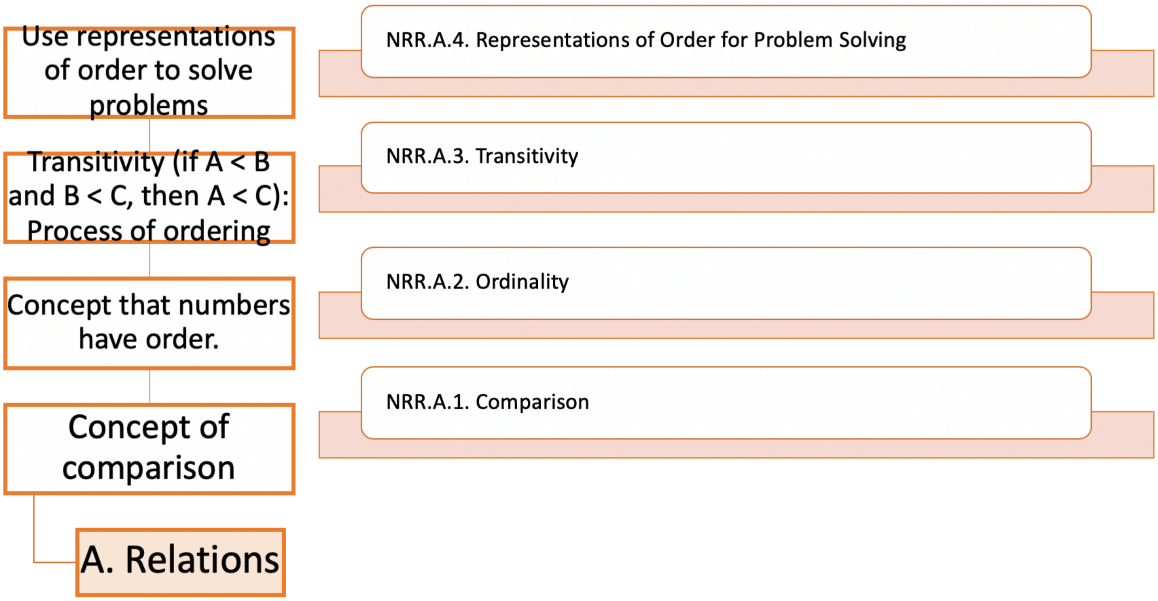
Appendix A – Finalized NRR Learning Progressions (as of Fall 2018)

Elements that Vary

Number Ranges	Kindergarten			Grade 1			Grade 2		
	F	**B**	**T**	**F**	**B**	**T**	**F**	**B**	**T**
Numerical/ Number/ Quantity	5 No symbolic	10	19	19	99 No symbolic	99	99	999 No symbolic	999
Word Problems / Story Situations	-	-	-	5 No symbolic	10	19	19	99 No symbolic	99
Benchmarks	-	5	5,10	5,10	Multiples of 10	Multiples of 10	Multiples of 10	Multiples of 100	Multiples of 100
Unit	1	1	1	1	10	10	10	10/100	10/100
Unspecified quantity ®	-	2	2	2	3	3	3	3	3
Number of Parts (C/D and Prop. Of Ops)	2	2	2	2	2	2/3	2/3	2	2/3
Comparisons	10	18	18	18	18	100	100	1000	100

Relations

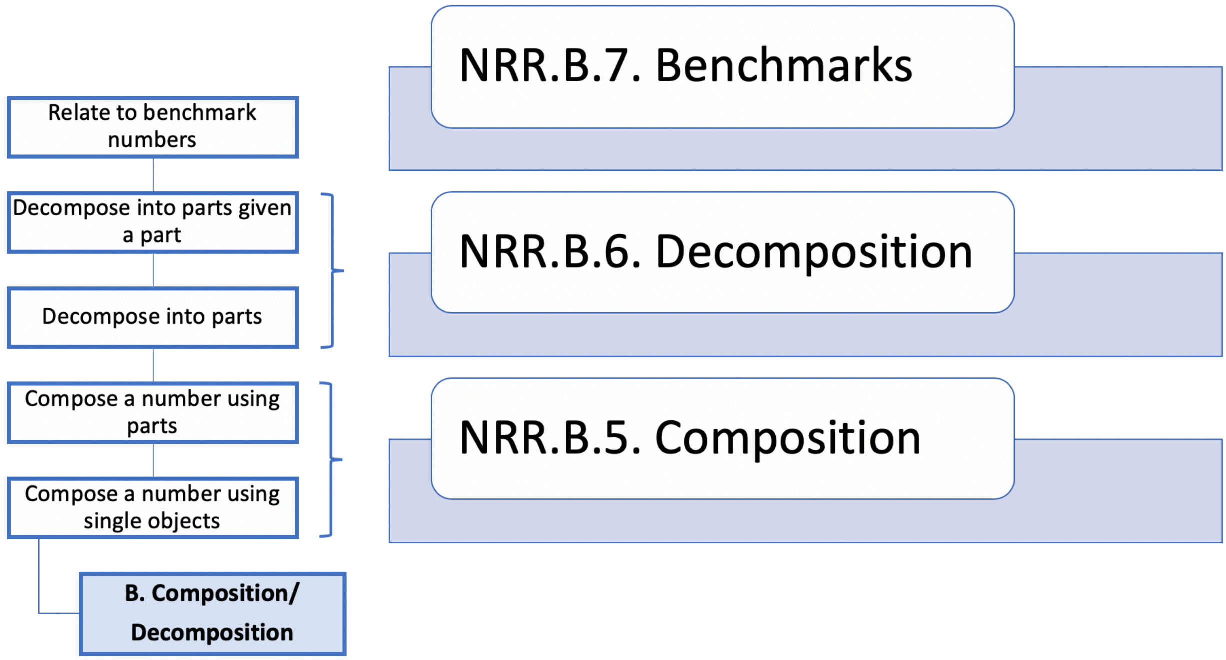


Relations	Initial Development	Final Version
NRR.A.1. Comparison	i. Use counting and matching strategies to find which is more/less for two quantities. ii. Use comparisons to benchmarks to find which is more/less for two quantities. (e.g., 5, 10, multiples of 10, multiples of 100) iii. Use place value and mental number lines to compare two numbers (progressing from one-	iv. Find which is more/less for two **quantities** i. Use counting and matching strategies (Grades K and 1 only) ii. Use mental images (subitize) v. Find which is more/less for two **quantities** by using comparisons to **benchmarks**

	digit → two-digit → three-digit) to determine which is more/less. i. Beginning in first grade, record the results of comparisons with the symbols >, <.	vi. Determine which is more/less for two **numbers** i. Use place value and mental number lines ii. Record results of comparisons with symbols >, < (Grades 1 and 2 only)
NRR.A.2. Ordinality	i. Given a number, mentally find 1/10/100 more or 1/10/100 less without having to count; explain **reasoning.**	a. Find a **unit** more/less than a given **number** i. Use mental math without counting
NRR.A.3. Transitivity	i. Using number relationships or tools (e.g., open number line) to order three numbers. ii. Order two/three unspecified quantities shown with visuals (bridging includes interval marks, target does not include interval marks).	a. Compare two unspecified **quantities** given visuals (i.e., beaker, balances/scales?) (interval marks bridging to no interval marks) b. Order three unspecified **quantities** given visuals (interval marks bridging to no interval marks) c. Order three **numbers** i. Use number relationships or tools (open number line bridging to mental number line)

NRR.A.4. Representations of Order for Problem Solving	i. Given two sets, show comparison situation with objects or drawing and find how much more/less using matching or counting strategies. ii. As a target for first grade, given a comparison word problem, find how much more/less.	a. Model/draw a comparison situation to find how much more/less between two sets of **numbers** i. Use matching or counting strategies ii. Use tools (open number line → 100s chart) b. Find how much more/less between two **quantities** in a comparison word problem i. Use tools (open number line → 100s chart)

Composition and Decomposition

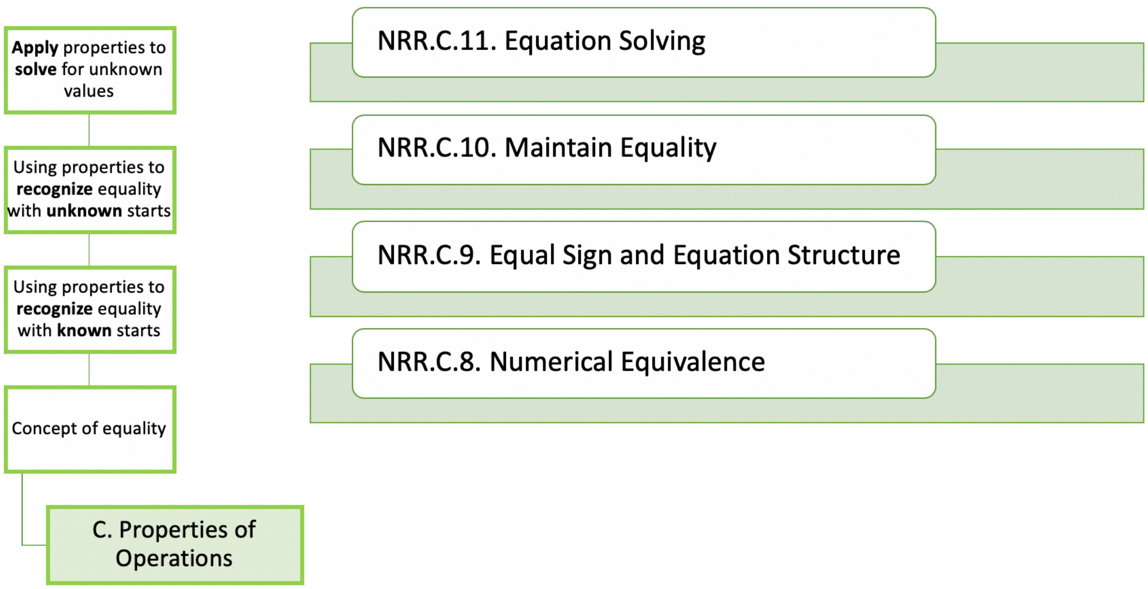


Composition and Decomposition of Number	Initial Development	Final Version

NRR.B.5.
 	
Composition
	i. Compose a number using single objects.
	
	ii. Compose a number using two parts.
	    i. Using place value
	
	iii. Compose a number using three parts. (first-grade target)
	    i. Using place value
		iv. Compose a **number**
		    i. Using single objects
		    ii. Using two parts
		    iii. Using place value
		    iv. Using three parts (Grade 1 target and Grade 2)

	ii. More than one way	
NRR.B.6. Decomposition	i. Decompose a number into two parts. i. Using place value ii. Decompose a number into three parts (first-grade target) i. Using place value ii. More than one way iii. Given one part, identify the missing part. i. Given ten, identify the missing part ii. Identify two additional parts (first-grade target)	
NRR.A.7. Benchmarks	i. Given ten(s), identify missing part	

Properties of Operations



Properties of Operations	Initial Development	Final Version
NRR.C.8. Numerical Equivalence	i. Given two equivalent sets, recognize that the quantity of each set remains the same regardless of size, color, or arrangement. ii. Given up to three equivalent sets, recognize that the quantity of each set remains the same regardless of	a. Given equivalent sets (two bridging to three), recognize that the quantity of each part remains the same regardless of size, color, or arrangement. i. Include array, area, and pre-grouped models b. Given a set broken into **parts**, recognize that order does not change the quantity of the set (commutative property).

	size, color, or arrangement. iii. Given a set broken into two parts, recognize that order does not change the quantity of the set (commutative property). i. Include array and area models (second-grade target as a precursor for distributive property). iv. Given a set broken into three parts (where two parts are positioned closer together), recognize that re-associating one part to make a 5/10/multiple of ten with the single part does not change the quantity of the set (associative property). v. Given problems in the form a + b – b or a – b + b, recognize the	i. Include array, area, and pre-grouped models ii. Use equal parts in array and area models as a precursor for distributive property (Grade 2) c. Given problems in the form a + b – b or a – b + b, recognize the initial quantity remains the same (undoing or additive inverse).

	initial quantity remains the same (undoing or additive inverse).	
NRR.C.9. Equal Sign **and Equation Structure**	i. Recognize a relational definition of the equal sign in various equation structures. i. No operations ii. Operations on right side iii. Operations on both sides iv. Multiple instances of a number	

NRR.C.10. Equation Structure **Maintain Equality**	i. Maintain equality with known starts without properties of operations (e.g., addition or subtraction property of equality) ii. Maintain equality with known starts with properties of operations iii. Maintain equality with unknown starts with properties of operations	
NRR.C.11. Equation Solving	i. Apply properties of operations to solve for an unknown value in any position.	

Appendix B: Consultant Primer Document

The Numerical Relational Reasoning construct is composed of three learning progressions: relations, composition/decomposition, and properties of operations. Each learning progression is subdivided into core concepts and skills are specified within essentialized statements from least to greatest levels of complexity. The foundation of each learning progression was developed at a joint meeting in March, 2018 with members of the MMaRS team and expert consultants. The meeting resulted in a general outline for each of the three learning progressions. The essentialized statements were developed from the initial skills outlined and supported by existing literature (see reference list).

Each learning progression includes three parts for review:

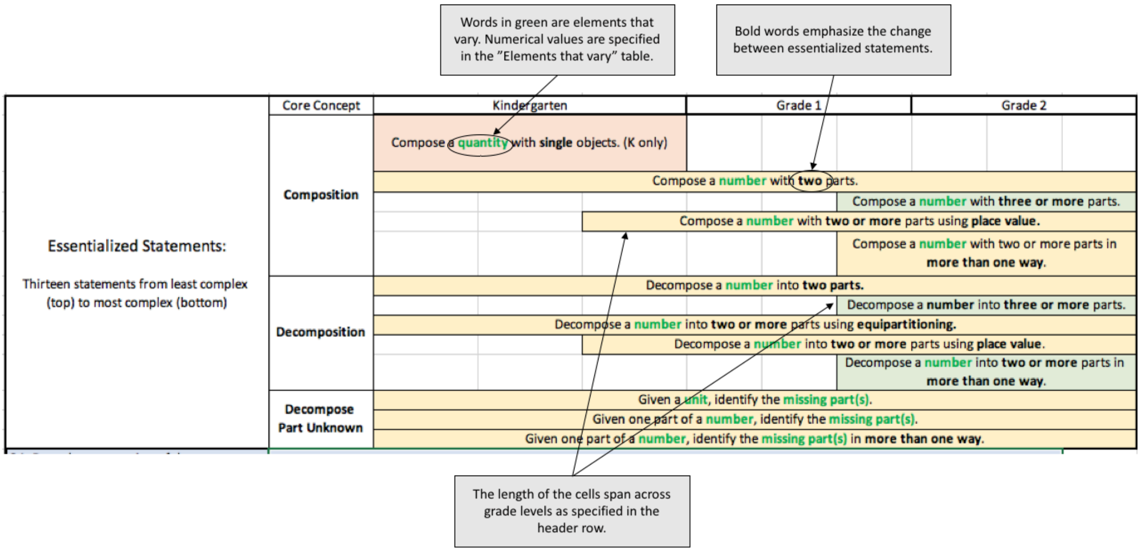
1. Vertical progression
2. Elements that vary
3. Representations

Part 1: Vertical progression

Respond to the vertical progression of essentialized statements. The figure below provides an example and explanation of the layout of part 1.

The essentialized statements are listed vertically from least complex to most complex. Also, the grade-level designations are specified by the cell lengths. For example, the first essentialized statement, *Compose a quantity with single objects* is constrained to Kindergarten only.

Please provide written explanations for questions that are rated either somewhat logical/developmentally appropriate or not logical/developmentally appropriate. Elements that vary across and within grade levels are in green. Part 2 explains these elements further. Changes from one essentialized statement to the next are bolded.



Part 2: Elements that vary

The figure below shows an example of numerical values that vary across grade levels. Each grade level is subdivided into three sections: Foundational (skills students should have entering a grade level), Bridging (skills students need to master to attain before target skill), and Target (skills students should master by the end of a grade level). The purpose of these delineations outline assessment goals within and across grade levels.

Please provide written explanations for questions that are rated either somewhat logical/developmentally appropriate or not logical/developmentally appropriate.

F: Foundational
B: Bridging
T: Target

Gray cells indicate an exact repeat from previous grade level. Moving from bridging to target may or may not repeat.

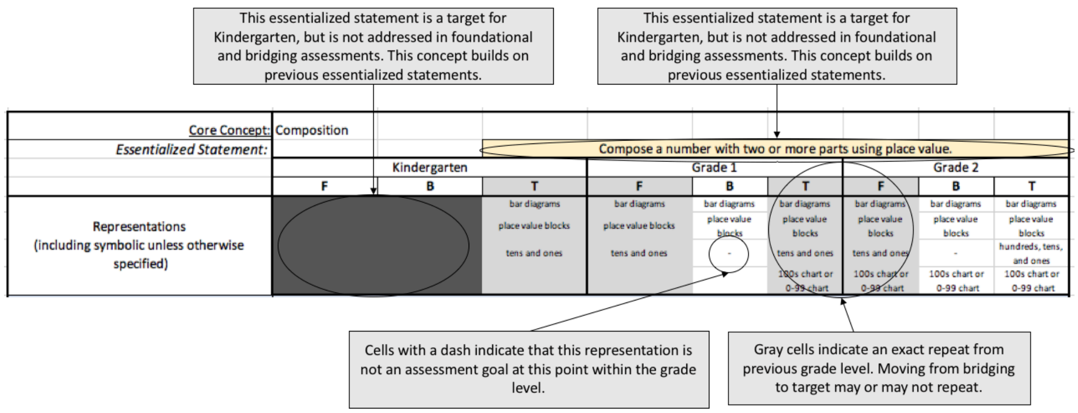
Elements that vary	Kindergarten			Grade 1			Grade 2		
	F	B	T	F	B	T	F	B	T
Number/ Quantity	5	10	19	19	99	99	99	999	999
Unit	1	1 or 5	1 or 10	1 or 10	10	10	10	10 or 100	10 or 100
Missing Parts	1	1	1	1	2	2	2	1	2

Definitions of terms are included after the Part 3 description.

Part 3: Representations

Each of the essentialized statements include associated representations. Please evaluate the representations used to elicit students' thinking for each essentialized statement within and across the grades. Consider the horizontal progression of the representations.

Essentialized statements span across grade level as specified by the placement and lengths of cells. Representations in adjacent light gray cells display the same representations since the target skill level at the end of one grade is the same as the foundational skill for the beginning of the next grade. Dark gray cells indicate that the skill is not addressed at those levels within a grade.



Similarly, cells with a dash indicate that a representation is not included within that assessment goal even when previously defined. This is typically the case when the numeric value increases. To illustrate the following figure displays a detailed progression of *Compose a number with two or more parts using place value.*

Kindergarten: Target	Compose a number within 19 with two or more parts using place value including bar diagrams, place value blocks, and representations of tens and ones.
Grade 1: Foundational	Compose a number within 19 with two or more parts using place value including bar diagrams, place value blocks, and representations of tens and ones.
Grade 1: Bridging	Compose a number within **99** with two or more parts using place value including bar diagrams and place value blocks.
Grade 1: Target	Compose a number within **99** with two or more parts using place value including bar diagrams, place value blocks, **and representations of tens and ones**.
Grade 2: Foundational	Compose a number within **99** with two or more parts using place value including bar diagrams, place value blocks, **and representations of tens and ones**.
Grade 2: Bridging	Compose a number within **999** with two or more parts using place value including bar diagrams and place value blocks.
Grade 2: Target	Compose a number within **999** with two or more parts using place value including bar diagrams, place value blocks, **and representations of tens and ones**.

Definition of terms

Benchmark	Typically used in make 10 strategies. 5, 10, and multiples of 10 are included as benchmarks.
Mathematical Equivalence	Distinct from numerical equivalence. Mathematical equivalence includes an understanding of the equal sign and equations (Rittle-Johnson, Matthews, Taylor, & McEldoon, 2011).
Numerical Equivalence	Distinct from mathematical equivalence. Students can use strategies such as matching and counting to determine the same amount given different representations of a quantity. Numerical equivalence does not include the use of the equal sign. Symbolic representations of number involve expressions but not equations.
Number	Distinct from quantity. Numbers refer to written numerals or symbolic representations.
Part	Part of a whole where a whole can be a quantity or a number. The phrase "parts of a quantity/whole" is used in the same manner as "addends of an expression."
Place value	Place value refers to the meaning of hundreds, tens, and ones within a given representation.
Quantity	Distinct from number. Quantities are sets of objects.
Sets	Include concrete objects without symbolic representations.
Unit	Include 1, 10, and 100. Generally used within the comparison skills when comparing more/less than a number.

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