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# TESTING EQUALITY OF MEANS IN A p-VARIATE NORMAL DISTRIBUTION HAVING EQUAL VARIANCES AND EQUAL CORRELATION COEFFICIENTS

by

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DEPARTMENT OF STATISTICS
Southern Methodist University

# TESTING EQUALITY OF MEANS IN A p-VARIATE NORMAL DISTRIBUTION HAVING EQUAL VARIANCES AND EQUAL CORRELATION COEFFICIENTS

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Testing Equality of Means in a p-Variate Normal Distribution Having Equal Variances and Equal Correlation Coefficients

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The null hypothesis $H_0$: $\underline{\mu} = \underline{\mu}^*$ , where $\underline{\mu}$ and $\underline{\mu}^*$ are $p \times 1$ vectors such that

$$\underline{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \cdot \\ \cdot \\ \cdot \\ \mu_p \end{pmatrix} \quad \text{and} \quad \underline{\mu}^* = \begin{pmatrix} \mu \\ \mu \\ \cdot \\ \cdot \\ \cdot \\ \mu \end{pmatrix} ,$$

is to be tested on the basis of a sample of size n, $\underline{Y}_1$ , $\underline{Y}_2$ , $\cdots$ , $\underline{Y}_n$ , from a p-variate normal population having equal variances and equal correlation coefficients. Since $\mu$ can be either specified or unspecified, and $\sigma^2$ and $\rho$ can be either known or unknown, there are eight cases of the null hypothesis to consider. One of these cases has been considered by S. S. Wilks [6].

In this paper four of the remaining seven cases will be considered. The test criterion will be derived by the likelihood-ratio method for each case.

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CHAPTER I

Testing of statistical hypothesis about the parameters of normal distributions is an important part of the theory of mathematical statistics and its applications. In 1946 an important paper was published by S. S. Wilks [6], in which he derived criteria for testing hypotheses about the unknown parameters of a p-variate normal distribution. It is of interest to derive test criteria, by the same method used by Wilks [6], for the cases where certain parameters are considered to be known quantities.

This paper is concerned with testing equality of means on the basis of a sample of size n, $\underline{Y}_1, \underline{Y}_2, \cdots, \underline{Y}_n$, from a p-variate normal distribution having equal variances and equal correlation coefficients. Thus, the normal distribution which we will be considering has p + 2 parameters, namely, $\mu_1, \mu_2, \cdots, \mu_p, \sigma^2$, and $\rho$. The null hypothesis is

$$H_0\colon \ \underline{\mu} = \underline{\mu}^* \quad ,$$

where $\underline{\mu}$ and $\underline{\mu}^*$ are $p \times 1$ vectors such that

$$\underline{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \cdot \\ \cdot \\ \cdot \\ \mu_p \end{pmatrix} \quad \text{and} \quad \underline{\mu}^* = \begin{pmatrix} \mu \\ \mu \\ \cdot \\ \cdot \\ \cdot \\ \mu \end{pmatrix} \quad .$$

But, since $\mu$ can be either specified or unspecified, $\sigma^2$ is either known or unknown, and $\rho$ is either known or unknown, there are eight cases of the null hypothesis to consider. Treating each case as a separate hypothesis and letting a subscript of zero denote that a parameter is known or specified, the eight hypotheses are as follows:

$H_{0_1}$: $\underline{\mu} = \underline{\mu}_0^*$ , where $\rho = \rho_0$ and $\sigma^2 = \sigma_0^2$ and $\underline{\mu}_0$ is a $p \times 1$ vector such that

$$\underline{\mu}_0^* = \begin{pmatrix} \mu_0 \\ \mu_0 \\ \cdot \\ \cdot \\ \cdot \\ \mu_0 \end{pmatrix} ;$$

$H_{0_2}$: $\underline{\mu} = \underline{\mu}_0^*$ , where $\rho = \rho_0$ and $\sigma^2$ is unknown;

$H_{0_3}$: $\underline{\mu} = \underline{\mu}_0^*$ , where $\rho$ is unknown and $\sigma^2 = \sigma_0^2$ ;

$H_{0_4}$: $\underline{\mu} = \underline{\mu}_0^*$ , where both $\rho$ and $\sigma^2$ are unknown;

$H_{0_5}$: $\underline{\mu} = \underline{\mu}^*$ , where $\rho = \rho_0$ and $\sigma^2 = \sigma_0^2$ ;

$H_{0_6}$: $\underline{\mu} = \underline{\mu}^*$ , where $\rho = \rho_0$ and $\sigma^2$ is unknown;

$H_{0_7}$: $\underline{\mu} = \underline{\mu}^*$ , where $\rho$ is unknown and $\sigma^2 = \sigma_0^2$ ;

$H_{0_8}$: $\underline{\mu} = \underline{\mu}^*$ , where both $\rho$ and $\sigma^2$ are unknown.

With this paper, test criteria is now available for testing five of the above eight hypotheses. Test criteria are derived in this paper for testing hypotheses $H_{0_1}$ , $H_{0_2}$ , $H_{0_5}$ , and $H_{0_6}$ . S. S. Wilks [6] derived the test criterion for testing the hypothesis $H_{0_8}$ . In each case the test criterion is derived by the likelihood-ratio method. Time did not allow derivation of test criteria for testing hypotheses $H_{0_3}$ , $H_{0_4}$ , and $H_{0_7}$ .

Wilks [6] derived the test criterion for testing the null hypothesis $H_{0_8}$ based on a sample of size n, $\underline{Y}_1$ , $\underline{Y}_2$ , $\cdots$ , $\underline{Y}_n$ where each $\underline{Y}_i$ is a $p \times 1$ vector for $i = 1, 2, \cdots, n$ , from a p-variate normal distribution having mean vector $\underline{\mu}$ , equal variances and equal correlation coefficients, all of which are unknown. He found the likelihood ratio to be

$$\lambda_m = \left[\frac{(s^2)^p(1 - r)^{p-1}(1 + [p - 1]r)}{(s_0^2)^p(1 - r_0)^{p-1}(1 + [p - 1]r_0)}\right]^{\frac{n}{2}} , \tag{1}$$

where

$$\bar{Y} = \frac{1}{np} \sum_{j=1}^{n} \sum_{i=1}^{p} Y_{ij} ,$$

$$s_{ik} = \frac{1}{n} \sum_{j=1}^{n} (Y_{ij} - \bar{Y}_i)(Y_{kj} - \bar{Y}_k) ,$$

$$\bar{Y}_i = \frac{1}{n} \sum_{j=1}^{n} Y_{ij} ,$$

$$\bar{Y}_j = \frac{1}{p} \sum_{i=1}^{p} Y_{ij} ,$$

$$s^2 = \frac{1}{p} \sum_{i=1}^{p} s_{ii} ,$$

$$r = \frac{\sum_{i \neq j=1}^{p} s_{ij}}{(p - 1) \sum_{i=1}^{p} s_{ii}} ,$$

$$s_{0_{ik}} = \frac{1}{n} \sum_{j=1}^{n} (Y_{ij} - \bar{Y})(Y_{kj} - \bar{Y}) ,$$

$$s_0^2 = \frac{1}{p} \sum_{i=1}^{p} s_{0_{ii}} ,$$

and

$$r_0 = \frac{\sum_{i \neq j=1}^{p} s_{0_{ij}}}{(p - 1) \sum_{i=1}^{p} s_{0_{ii}}} .$$

The distribution of $L_m = \lambda_m^{\frac{2}{n(p-1)}} = \frac{s^2(1-r)}{s_0^2(1-r_0)}$ , under $H_{0_8}$ , was found to be

$$dF(L_m) = \frac{\Gamma\left(\frac{1}{2}n(p-1)\right)}{\Gamma\left(\frac{1}{2}n(n-1)(p-1)\right)\Gamma\left(\frac{1}{2}(p-1)\right)} L_m^{\frac{1}{2}(n-1)(p-1)-1} (1-L_m)^{\frac{1}{2}(p-1)-1} \tag{2}$$

for $0 \leq L_m \leq 1$ . Then an exact test of $H_{0_8}$ can be made on the basis of $dF(L_m)$ by use of tables for the Incomplete Beta Function, where

$$\int_0^{L_\alpha} dF(L_m) = \alpha ,$$

and $L_\alpha$ is the 100α% point. The null hypothesis $H_{0_8}$ is rejected at the 100α% significance level if

$$L_m < L_\alpha \tag{3}$$

Wilks also stated, but did not prove, that $H_{0_8}$ could be tested by using

$$F = \frac{n(n-1)\sum_{i=1}^{p}(\bar{Y}_i - \bar{Y})^2}{\sum_{j=1}^{n}\sum_{i=1}^{p}(Y_{ij} - \bar{Y}'_j - \bar{Y}_i + \bar{Y})^2} \tag{4}$$

where $\bar{Y}'_j = \frac{1}{p}\sum_{i=1}^{n} Y_{ij}$ for $j = 1, 2, \cdots , p$ and under $H_{0_8}$ , F has an F distribution with p-1 and (n-1)(p-1) degrees of freedom. The null hypothesis $H_{0_8}$ is rejected if

$$F > F_{\alpha,p-1,(n-1)(p-1)} \tag{5}$$

where

$$\int_{F_{\alpha,p-1,(n-1)(p-1)}}^{\infty} g(F)dF = \alpha$$

and g(F) is the density function of a F distribution with p-1 and (n-1(p-1) degrees of freedom.

Anderson [1] stated, but did not prove, that the test criterion for testing the null hypothesis

$$H_0: \ \underline{\mu} = \underline{\mu}_0$$

where $\underline{\mu}_0$ is a specified vector such that all components are not necessarily equal. The null hypothesis is rejected if

$$n(\bar{\underline{Y}} - \underline{\mu}_0)'V^{-1}(\bar{\underline{Y}} - \underline{\mu}_0) \geqq \chi^2_{\alpha,p} \quad , \tag{6}$$

where

$$\int_{\chi^2_{\alpha,p}}^{\infty} f(x)dx = \alpha$$

and f(x) is the density function of a chi-square distribution with p degrees of freedom, and $\bar{\underline{Y}}$ is the mean of a sample of size n, $\underline{Y}_1$ , $\underline{Y}_2$ , $\cdots$ , $\underline{Y}_n$ , from a p variate normal distribution with unknown mean vector $\underline{\mu}$ and known covariance matrix V. The null hypothesis $H_{0_1}$ is a special case of this hypothesis, namely, the case where the p components of $\underline{\mu}_0$ are all equal.

The test criterion for testing a special case of the null hypothesis $H_{0_6}$ was derived by Hogg and Craig [4]. For $\rho = 0$ they found the likelihood ratio to be

$$\lambda = \left[ \frac{\sum_{j=1}^{p} \sum_{i=1}^{n} (Y_{ij} - \bar{Y}_{ij})^2}{\sum_{j=1}^{p} \sum_{i=1}^{n} (Y_{ij} - \bar{Y})^2} \right]^{\frac{np}{2}} \tag{7}$$

where

$$\bar{Y}_{ij} = \frac{1}{n} \sum_{i=1}^{n} Y_{ij} \qquad \text{for } j = 1, 2, \cdots, p$$

and

$$\bar{Y} = \frac{1}{np} \sum_{j=1}^{p} \sum_{i=1}^{n} Y_{ij} \quad .$$

The null hypothesis is rejected if

$$F = \frac{p(n-1)}{p-1} \frac{\sum_{j=1}^{p} \sum_{i=1}^{n} (Y_{ij} - \bar{Y}_{ij})^2}{\sum_{j=1}^{p} \sum_{i=1}^{n} (Y_{ij} - \bar{Y})^2} \geqq F_{\alpha,p-1,p(n-1)} \tag{8}$$

where

$$\int_{F_{\alpha,p-1,p(n-1)}}^{\infty} g(F)dF = \alpha$$

and $g(F)$ is the density function of an F distribution with $p-1$ and $p(n-1)$ degrees of freedom.

The test criteria for testing two other hypotheses will be given though they are not exactly the same as any of the eight hypotheses. Fix [2] states the test criterion for testing the null hypothesis

$$H_0: \ \mu_i = \mu_{0_i} \qquad i = 1, 2, \cdots, p$$

against

$$H_A: \ \mu_i \neq \mu_{0_i}$$

for at least one value of $i$, where $Y_i$ has a normal distribution with mean $\mu_i$, variance 1, and $\text{cov}(Y_i, Y_j) = 0$ for $i \neq j = 1, 2, \cdots, p$. The null hypothesis is rejected if

$$\sum_{i=1}^{p} (Y_i - \mu_{0_i})^2 > \chi^2_{\alpha,p} \tag{9}$$

where

$$\int_{\chi^2_{\alpha,p}}^{\infty} h(x)\,dx = \alpha$$

and $h(x)$ is the density function of a chi-square distribution with p degrees of freedom. This is a very special case of the test of the hypothesis above by Anderson [1], namely, for $n = 1$, $\sigma^2 = 1$, and $\rho = 0$ .

Stuart [5] derived a test criterion for testing the null hypothesis

$$H_0:\ \mu_i = \mu_0 \ , \quad i = 1, 2, \cdots , p \ ,$$

on the basis of a sample of size $n = 1$, $\underline{Y}_1$ , from a p-variate normal distribution having equal but unknown variances $\sigma^2$ and correlation coefficients all equal to a known value, $\rho_0$ . The test statistic is given by

$$t = \frac{(\bar{Y} - \mu_0)[p(p-1)]^{\frac{1}{2}}}{\left[\sum_{i=1}^{p} (Y_i - \bar{Y})^2\right]^{\frac{1}{2}}} \left[\frac{1 - \rho_0}{1 + (p-1)\rho_0}\right]^{\frac{1}{2}} . \tag{10}$$

It is of interest to note that the likelihood-ratio method was not used.

In this chapter, we shall see how a p-variate normal distribution having equal variances and equal correlation coefficients can arise.

Let $X_0 , X_1 , \cdots , X_p$ be p + 1 normally distributed random variables such that for $i \neq j = 1, 2, \cdots , p$ ,

$$E(X_0) = 0 \quad \text{and} \quad Var(X_0) = \sigma_x^2 \quad , \tag{11}$$

$$E(X_i) = \mu_i \quad \text{and} \quad Var(X_i) = \sigma_x^2 \quad , \tag{12}$$

$$cov(X_0 , X_i) = b\sigma_x^2 \quad , \tag{13}$$

$$cov(X_i , X_j) = 0 \quad , \tag{14}$$

where b is a real constant.

We define a new set of random variables

$$Y_i = X_i + aX_0 \quad \text{for} \quad i = 1, 2, \cdots , p \quad , \tag{15}$$

where a is a real constant. Now, for $i \neq j = 1, 2, \cdots , p$ ,

$$E(Y_i) = \mu_i \quad , \tag{16}$$

$$Var(Y_i) = (1 + 2ab + a^2)\sigma_x^2 \quad , \tag{17}$$

$$cov(Y_i , Y_j) = (2ab + a^2)\sigma_x^2 \quad , \tag{18}$$

so that
$$\rho(Y_i , Y_j) = \frac{2ab + a^2}{1 + 2ab + a^2} \quad . \tag{19}$$

Hence, we have a p-variate normal distribution with equal variances and equal correlation coefficients.

Let $\rho = \rho(Y_i , Y_j)$ and $\sigma^2 = \mathrm{Var}(Y_i) = (1 + 2ab + a^2)\sigma_x^2$ for $i \neq j = 1 , 2, \cdots , p$ . Now $\underline{Y}$ has a p-variate normal distribution with mean vector $\underline{\mu}$ and covariance matrix V , denoted by $N(\underline{\mu} , V)$, where

$$\underline{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \cdot \\ \cdot \\ \cdot \\ Y_p \end{pmatrix} \qquad \underline{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \cdot \\ \cdot \\ \cdot \\ \mu_p \end{pmatrix} , \text{ and } V = \begin{pmatrix} \sigma^2 & \rho\sigma^2 & \cdots & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 & \cdots & \rho\sigma^2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \rho\sigma^2 & \rho\sigma^2 & \cdots & \sigma^2 \end{pmatrix}$$

Let

$$K = \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \rho & \rho & \cdots & 1 \end{pmatrix} \tag{20}$$

Now

$$V = \sigma^2 K \tag{21}$$

and the density function of $\underline{Y}$ is given by

$$f(\underline{Y}') = \frac{1}{\sigma(2\pi)^{\frac{p}{2}}|K|^{\frac{1}{2}}} \exp\left[ - \frac{1}{2\sigma^2}(\underline{Y} - \underline{\mu})'K^{-1}(\underline{Y} - \underline{\mu}) \right] \tag{22}$$

where

$$K^{-1} = \frac{1}{1 - \rho}\left( I - \frac{\rho J}{1 + (p - 1)\rho} \right) , \tag{23}$$

I is the $p \times p$ identity matrix and J is a $p \times p$ matrix of 1's .

# CHAPTER III

The likelihood ratio method will be used to derive the test criteria for testing equality of means in the p-variate normal distribution with covariance matrix $\sigma^2 K$ where K is defined in (20). The likelihood function will be denoted by L . Now

$$L = \prod_{i=1}^{n} f(\underline{Y}_i' , \underline{\mu} , \sigma^2 , \rho)$$

$$= \frac{1}{(\sigma^2)^{\frac{n}{2}}(2\pi)^{\frac{np}{2}}|K|^{\frac{n}{2}}} \exp\left[ - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (\underline{Y}_i - \underline{\mu})' K^{-1} (\underline{Y}_i - \underline{\mu}) \right] \tag{24}$$

and the natural logarithm of L is given by

$$\ln L = - \frac{n}{2} \ln \sigma^2 - \frac{np}{2} \ln(2\pi) - \frac{n}{2} \ln|K| - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (\underline{Y}_i - \underline{\mu})' K^{-1} (\underline{Y}_i - \underline{\mu}) . \tag{25}$$

The parameter space will be denoted by $\Omega$ and the subspace specified by the null hypothesis will be denoted by $\omega$ .

Now the test criterion will be derived for testing the null hypothesis

$$H_{0_1} : \quad \underline{\mu} = \underline{\mu}_0^*$$

against the alternative

$$H_{A_1} : \quad \underline{\mu} \neq \underline{\mu}_0^* .$$

Here

$$\Omega = \{(\mu_1, \mu_2, \cdots, \mu_p, \sigma^2, \rho): -\infty < \mu_i < \infty$$

$$\text{for } i = 1, 2, \cdots, p, \sigma^2 = \sigma_0^2 > 0, -1 < \rho = \rho_0 < 1\}$$

and

$$\omega = \{(\mu_1, \mu_2, \cdots, \mu_p, \sigma^2, \rho): -\infty < \mu_i = \mu_0 < \infty$$

$$\text{for } i = 1, 2, \cdots, p, \sigma^2 = \sigma_0^2 > 0, -1 < \rho = \rho_0 < 1\}$$

Since all of the parameters are specified in $\omega$, the maximum of the likelihood function in $\omega$ is given by

$$L(\hat{\omega}) = L(\omega) = \frac{1}{(\sigma_0^2)^{\frac{n}{2}}(2\pi)^{\frac{np}{2}}|K_0|^{\frac{n}{2}}} \exp\left[-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(\underline{Y}_i - \underline{\mu}_0^*)'K_0^{-1}(\underline{Y}_i - \underline{\mu}_0^*)\right], \quad (26)$$

where $K_0^{-1}$ is defined in (23) but with $\rho$ replaced by $\rho_0$. Now L is maximized in $\Omega$ by

$$\hat{\underline{\mu}} = \bar{\underline{Y}} = \begin{pmatrix} \bar{Y}_1 \\ \bar{Y}_2 \\ \cdot \\ \cdot \\ \cdot \\ \bar{Y}_p \end{pmatrix},$$

where $\bar{Y}_i = \frac{1}{n}\sum_{j=1}^{n} Y_{ij}$ for $i = 1, 2, \cdots, p$, and the maximum of the likelihood function in $\Omega$ is given by

$$L(\hat{\Omega}) = \frac{1}{(\sigma_0^2)^{\frac{n}{2}}(2\pi)^{\frac{np}{2}}|K_0|^{\frac{n}{2}}} \exp\left[-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(\underline{Y}_i - \bar{\underline{Y}})'K_0^{-1}(\underline{Y}_i - \bar{\underline{Y}})\right]. \quad (27)$$

The ratio of (25) to (26) is the likelihood ratio:

$$\lambda = \frac{\exp\left[-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(\underline{Y}_i - \underline{\mu}_0^*)'K_0^{-1}(\underline{Y}_i - \underline{\mu}_0^*)\right]}{\exp\left[-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(\underline{Y}_i - \bar{\underline{Y}})'K_0^{-1}(\underline{Y}_i - \bar{\underline{Y}})\right]}$$

$$= \exp\left(-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}[(\underline{Y}_i - \underline{\mu}_0^*)'K_0^{-1}(\underline{Y}_i - \underline{\mu}_0^*) - (\underline{Y}_i - \bar{\underline{Y}})'K_0^{-1}(\underline{Y}_i - \bar{\underline{Y}})]\right) \tag{28}$$

But

$$\sum_{i=1}^{n}[(\underline{Y}_i - \underline{\mu}_0^*)'K_0^{-1}(\underline{Y}_i - \underline{\mu}_0^*) - (\underline{Y}_i - \bar{\underline{Y}})'K_0^{-1}(\underline{Y}_i - \bar{\underline{Y}})] = n(\bar{\underline{Y}} - \underline{\mu}_0^*)'K_0^{-1}(\bar{\underline{Y}} - \underline{\mu}_0^*) \; . \tag{29}$$

Now

$$\lambda = \exp\left[-\frac{n}{2\sigma_0^2}(\bar{\underline{Y}} - \underline{\mu}_0^*)'K_0^{-1}(\bar{\underline{Y}} - \underline{\mu}_0^*)\right] \; . \tag{30}$$

Next we need to determine the distribution of $\lambda$ . Anderson [1] showed that

$$Q = \frac{n}{\sigma_0^2}(\bar{\underline{Y}} - \underline{\mu}_0^*)'K_0^{-1}(\bar{\underline{Y}} - \underline{\mu}_0^*) \tag{31}$$

has a $\chi^2$-distribution with p degrees of freedom under the null hypothesis. Now the distribution of $\lambda$ can be determined, but the test may be done using Q as a criterion. Since

$$\lambda = \exp\left(-\frac{Q}{2}\right) \; ,$$

$$Q = -2 \ln \lambda \; . \tag{32}$$

Then

$$\alpha = \int_0^A g(\lambda | H_{0_1}) d\lambda$$

$$= \int_{-2 \ln A}^{\infty} h(Q | H_{0_1}) dQ$$

$$= \int_{\chi^2_{\alpha,p}}^{\infty} h(Q | H_{0_1}) dQ \quad , \tag{33}$$

where $h(Q|H_{0_1})$ is the density function of a chi-square distribution with p degrees of freedom. The null hypothesis $H_{0_1}$ is rejected if

$$Q = \frac{n}{\sigma_0^2} (\bar{\underline{Y}} - \underline{\mu}_0^*)' K_0^{-1} (\bar{\underline{Y}} - \underline{\mu}_0^*) \geqq \chi^2_{\alpha,p} \quad . \tag{34}$$

This is the same critical region that Anderson [1] gives in (6).

the

$$H_{0_2}: \quad \underline{\mu} = \underline{\mu}_0^*$$

against the alternative hypothesis

$$H_{A_2}: \quad \underline{\mu} \neq \underline{\mu}_0^* \quad .$$

Here the total parameter space is

$$\Omega = \{(\mu_1 , \mu_2 , \cdots , \mu_p , \sigma^2 , \rho): -\infty < \mu_i < \infty$$
$$\text{for } i = 1, 2, \cdots , p , \sigma^2 > 0 , -1 < \rho = \rho_0 < 1\}$$

and

$$\omega = \{(\mu_1 , \mu_2 , \cdots , \mu_p , \sigma^2 , \rho): -\infty < \mu_i = \mu_0 < \infty$$
$$\text{for } i = 1, 2, \cdots , p , \sigma^2 > 0 , -1 < \rho = \rho_0 < 1\}$$

The likelihood function is given by

and

$$L(\hat{\omega}) = \frac{1}{(2\pi)^{\frac{np}{2}} |K_0|^{\frac{n}{2}} \left[\frac{1}{n} \sum_{i=1}^{n} (\underline{Y}_i - \underline{\mu}_0^*)' K_0^{-1} (\underline{Y}_i - \underline{\mu}_0^*)\right]^{\frac{n}{2}}} \exp\left(-\frac{n}{2}\right) \tag{37}$$

The likelihood function is maximized in $\Omega$ by

$$\hat{\underline{\mu}} = \bar{\underline{Y}} \tag{38}$$

and

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (\underline{Y}_i - \bar{\underline{Y}})' K_0^{-1} (\underline{Y}_i - \bar{\underline{Y}}) \quad , \tag{39}$$

so that

$$L(\hat{\Omega}) = \frac{1}{(2\pi)^{\frac{np}{2}} |K_0|^{\frac{n}{2}} \left[\frac{1}{n} \sum_{i=1}^{n} (\underline{Y}_i - \bar{\underline{Y}})' K_0^{-1} (\underline{Y}_i - \bar{\underline{Y}})\right]^{\frac{n}{2}}} \exp\left(-\frac{n}{2}\right) \quad . \tag{40}$$

Now the likelihood ratio is

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \left[\frac{\sum_{i=1}^{n} (\underline{Y}_i - \bar{\underline{Y}})' K_0^{-1} (\underline{Y}_i - \bar{\underline{Y}})}{\sum_{i=1}^{n} (\underline{Y}_i - \underline{\mu}_0^*)' K_0^{-1} (\underline{Y}_i - \underline{\mu}_0^*)}\right]^{\frac{n}{2}} \quad . \tag{41}$$

The next step is to obtain the distribution of $\lambda$ under $H_{0_2}$ .

The sum in the denominator of (41) may be put in the form

$$\sum_{i=1}^{n} (\underline{Y}_i - \underline{\mu}_0^*)' K_0^{-1} (\underline{Y}_i - \underline{\mu}_0^*) = \sum_{i=1}^{n} (\underline{Y}_i - \bar{\underline{Y}})' K_0^{-1} (\underline{Y}_i - \bar{\underline{Y}}) + n(\bar{\underline{Y}} - \underline{\mu}_0^*)' K_0^{-1} (\bar{\underline{Y}} - \underline{\mu}_0^*) \tag{42}$$

so that $\lambda$ may be written

$$\lambda = \left[ \frac{1}{1 + \frac{n(\bar{\underline{Y}} - \underline{\mu}_0^*)' K_0^{-1} (\bar{\underline{Y}} - \underline{\mu}_0^*)}{\sum_{i=1}^{n} (\underline{Y}_i - \bar{\underline{Y}})' K_0^{-1} (\underline{Y}_i - \bar{\underline{Y}})}} \right]^{\frac{n}{2}} . \tag{43}$$

Anderson [1] showed that $\frac{Q_1}{\sigma^2} = \frac{n}{\sigma^2}(\bar{\underline{Y}} - \underline{\mu}_0^*)' K^{-1} (\bar{\underline{Y}} - \underline{\mu}_0^*)$ has a chi-square distribution with p degrees of freedom where $H_{0_2}$ is true. Now we need to find the distribution of

$$\frac{Q_2}{\sigma^2} = \sum_{i=1}^{n} (\underline{Y}_i - \bar{\underline{Y}})' \frac{K_0^{-1}}{\sigma^2} (\underline{Y}_i - \bar{\underline{Y}}) .$$

The np × 1 vector $\underline{Y}_*$ , where

$$\underline{Y}_*' = (Y_{11}, Y_{12}, \cdots, Y_{1p}, Y_{21}, Y_{22}, \cdots, Y_{2p}, \cdots,$$
$$Y_{n1}, Y_{n2}, \cdots, Y_{np}) ,$$

has a multivariate normal distribution with mean vector $\underline{\mu}_*$ and covariance matrix $\sigma^2 V_0$ , where

$$\underline{\mu}_*' = (\mu_1, \mu_2, \cdots, \mu_p, \mu_1, \mu_2, \cdots, \mu_p, \cdots, \mu_1, \mu_2, \cdots, \mu_p)$$

and

$$V_0 = \begin{pmatrix} K_0 & O & \cdots & O \\ O & K_0 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & K_0 \end{pmatrix} ,$$

$$K_0 = (1 - \rho_0)\left(I + \frac{\rho_0}{1 - \rho_0} J\right) ,$$

and O is a p × p matrix of zeros.

Now

$$Q_2 = (\underline{Y}_* - \bar{\underline{Y}}_*)'V_0^{-1}(\underline{Y}_* - \bar{\underline{Y}}) \tag{44}$$

where $\bar{\underline{Y}}_*$ is a np × 1 vector and $V_0^{-1}$ is an np × np matrix,

$$\bar{\underline{Y}}_*' = (\bar{Y}_1 , \bar{Y}_2 , \cdots , \bar{Y}_p , \bar{Y}_1 , \bar{Y}_2 , \cdots , \bar{Y}_p , \cdots , \bar{Y}_1 , \bar{Y}_2 , \cdots , \bar{Y}_p),$$

$$V_0^{-1} = \begin{pmatrix} K_0^{-1} & O & \cdots & O \\ O & K_0^{-1} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & K_0^{-1} \end{pmatrix} ,$$

and where O is a p × p matrix of zeros. The np × 1 vector $(\underline{Y}_* - \bar{\underline{Y}}_*)$ can be written as

$$(\underline{Y}_* - \bar{\underline{Y}}_*) = \begin{pmatrix} \left(\frac{n-1}{n}\right)I & -\frac{1}{n}I & \cdots & -\frac{1}{n}I \\ -\frac{1}{n}I & \left(\frac{n-1}{n}\right)I & \cdots & -\frac{1}{n}I \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n}I & -\frac{1}{n}I & \cdots & \left(\frac{n-1}{n}\right)I \end{pmatrix} = G\underline{Y}_* \tag{45}$$

Now

$$Q_2 = \underline{Y}_*'B\underline{Y}_* \tag{46}$$

where

$$B = G'V_0^{-1}G = \begin{pmatrix} \left(\frac{n-1}{n}\right)K_0^{-1} & -\frac{1}{n}K_0^{-1} & \cdots & -\frac{1}{n}K_0^{-1} \\ -\frac{1}{n}K_0^{-1} & \left(\frac{n-1}{n}\right)K_0^{-1} & \cdots & -\frac{1}{n}K_0^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n}K_0^{-1} & -\frac{1}{n}K_0^{-1} & \cdots & \left(\frac{n-1}{n}\right)K_0^{-1} \end{pmatrix} .$$

Next we determine if $BV_0$ is idempotent. Now

$$BV_0 = \frac{1}{n}\begin{pmatrix} (n-1)I & -I & \cdots & -I \\ -I & (n-1)I & \cdots & -I \\ \vdots & \vdots & \ddots & \vdots \\ -I & -I & \cdots & (n-1)I \end{pmatrix} \tag{47}$$

where rank $(BV_0) = (n-1)p$ . And it is easy to verify that

$$(BV_0)(BV_0) = BV_0 \quad , \tag{48}$$

so that $BV_0$ is idempotent of rank $(n-1)p$ . By Graybill [3], $\frac{Q_2}{\sigma^2}$ has a non-central chi-square distribution with $(n-1)p$ degrees of freedom and noncentrality parameter

$$\lambda_p = \frac{1}{2\sigma^2}\,\underline{\mu}_*'B\underline{\mu}_* \quad . \tag{49}$$

But it is easily verified that

$$\lambda_p = 0 \tag{50}$$

for any vector $\underline{\mu}_*'$ . Hence

$$\frac{Q_2}{\sigma^2} = \sum_{i=1}^{n} (\underline{Y}_i - \bar{\underline{Y}})' \frac{K_0^{-1}}{\sigma^2} (\underline{Y}_i - \bar{\underline{Y}}) \tag{51}$$

has a chi-square distribution with (n-1)p degrees of freedom regardless of whether or not $H_{0_2}$ is true.

Now we determine if $Q_1$ and $Q_2$ are independent. Since

$$Q_1 = \underline{Y}_*' A \underline{Y}_* \quad ,$$

where

$$A = \begin{pmatrix} K_0^{-1} & K_0^{-1} & \cdots & K_0^{-1} \\ K_0^{-1} & K_0^{-1} & \cdots & K_0^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ K_0^{-1} & K_0^{-1} & \cdots & K_0^{-1} \end{pmatrix} \quad ,$$

it is easily shown that

$$BV_0A = 0 \quad .$$

By Graybill [3], the quadratic forms $Q_1$ and $Q_2$ are independent. Hence, when $H_{0_2}$ is true,

$$F = (n-1) \frac{Q_1}{Q_2} \tag{53}$$

has an F distribution with p and (n-1)p degrees of freedom.

The likelihood ratio can be written as

$$\lambda = \left[ \frac{1}{1 + \frac{F}{n-1}} \right]^{\frac{n}{2}} \quad . \tag{54}$$

Since F is a monotonic decreasing function of $\lambda$ , the critical region of

## CHAPTER V

The criterion will be derived in this chapter by the likelihood ratio method for testing the null hypothesis

$$H_{0_5}: \ \underline{\mu} = \underline{\mu}^*$$



size $\alpha$ for testing $H_{0_2}$ based on F is

$$F_{\alpha,p,(n-1)p} < F < \infty \quad , \tag{55}$$

where

$$\int_{F_{\alpha,p,(n-1)p}}^{\infty} g(F)\,dF = \alpha \tag{56}$$

and g(F) is the density function of the F distribution with p and (n-1)p degrees of freedom. The test of $H_{0_2}$ may be performed as follows: We compute the quantity

$$F = \frac{n(\bar{\underline{Y}} - \underline{\mu}_0^*)'K_0^{-1}(\bar{\underline{Y}} - \underline{\mu}_0^*)}{\sum_{i=1}^{n} (\underline{Y}_i - \bar{\underline{Y}})'K_0^{-1}(\underline{Y}_i - \bar{\underline{Y}})} \tag{57}$$

and reject $H_{0_2}$ if $F > F_{\alpha,p,(n-1)p}$ ; otherwise accept $H_{0_2}$ .

$$\hat{\underline{\mu}}^* = \bar{\bar{\underline{Y}}} = \begin{pmatrix} \frac{1}{p} \sum_{i=1}^{n} \bar{Y}_i \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{p} \sum_{i=1}^{n} \bar{Y}_i \end{pmatrix} \tag{59}$$

and the maximum of the likelihood function is given by

$$L(\hat{\omega}) = \frac{1}{(\sigma_0^2)^{\frac{n}{2}} (2\pi)^{\frac{np}{2}} |K_0|^{\frac{n}{2}}} \exp\left[ -\frac{1}{2\sigma_0^2} \sum_{i=1}^{n} (\underline{Y}_i - \bar{\bar{\underline{Y}}})' K_0^{-1} (\underline{Y}_i - \bar{\bar{\underline{Y}}}) \right] \tag{60}$$

In $\Omega$ the likelihood function is maximized by

$$\hat{\underline{\mu}} = \bar{\underline{Y}} \tag{61}$$

so that

$$L(\hat{\Omega}) = \frac{1}{(\sigma_0^2)^{\frac{n}{2}} (2\pi)^{\frac{np}{2}} |K_0|^{\frac{n}{2}}} \exp\left[ -\frac{1}{2\sigma_0^2} \sum_{i=1}^{n} (\underline{Y}_i - \bar{\underline{Y}})' K_0^{-1} (\underline{Y}_i - \bar{\underline{Y}}) \right] \tag{62}$$

The ratio of (60) to (62) is the likelihood ratio, i.e.,

$$\lambda = \exp\left( -\frac{1}{2\sigma_0^2} \sum_{i=1}^{n} [(\underline{Y}_i - \bar{\bar{\underline{Y}}})' K_0^{-1} (\underline{Y}_i - \bar{\bar{\underline{Y}}}) - (\underline{Y}_i - \bar{\underline{Y}})' K_0^{-1} (\underline{Y}_i - \bar{\underline{Y}})] \right) \tag{63}$$

The likelihood ratio can be written as

$$\lambda = \exp\left( -\frac{n}{2\sigma_0^2} (\bar{\underline{Y}} - \bar{\bar{\underline{Y}}})' K_0^{-1} (\bar{\underline{Y}} - \bar{\bar{\underline{Y}}}) \right) \tag{64}$$

Now we need to find the distribution of $\lambda$ under $H_{05}$ . To do this we first find the distribution of

$$Q = \frac{n}{\sigma_0^2} (\bar{\underline{Y}} - \bar{\bar{\underline{Y}}})' K_0^{-1} (\bar{\underline{Y}} - \bar{\bar{\underline{Y}}}) \quad .$$

Now

$$(\bar{\underline{Y}} - \bar{\bar{\underline{Y}}}) = C\underline{Y}_* \quad ,$$

where

$$C = \frac{1}{n}\left(I - \frac{1}{p}J \;,\; I - \frac{1}{p}J \;,\; \cdots \;,\; I - \frac{1}{p}J\right)$$

is a $p \times np$ matrix, $I$ is the $p \times p$ identity matrix, $J$ is a $p \times p$ matrix of one's, and $\underline{Y}_*$ is a $np \times 1$ vector such that

$$\underline{Y}_*' = (Y_{11} , Y_{21} , \cdots , Y_{p1} , Y_{12} , Y_{22} , \cdots , Y_{p2} , \cdots ,$$

$$Y_{1n} , Y_{2n} , \cdots , Y_{pn}) \quad ,$$

so that

$$Q = \frac{\underline{Y}_*' A \underline{Y}_*}{\sigma_0^2} \quad , \tag{65}$$

where

$$A = nC'K_0^{-1}C \quad .$$

But

$$A = \frac{1}{n}\begin{pmatrix} A_1 & A_1 & \cdots & A_1 \\ A_1 & A_1 & \cdots & A_1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ A_1 & A_1 & \cdots & A_1 \end{pmatrix} \tag{66}$$

where $A_1 = \left(I - \frac{1}{p}J\right)K_0^{-1}\left(I - \frac{1}{p}J\right)$ is a $p \times p$ matrix. Since, by (23),

$$K_0^{-1} = \frac{1}{1 - \rho_0}\left(I - \frac{\rho_0 J}{1 + (p - 1)\rho_0}\right) \quad ,$$

$$A_1 = \frac{1}{1 - \rho_0}\left(I - \frac{1}{p}J\right) \quad . \tag{67}$$

The np × 1 vector $\underline{Y}_*$ has a multivariate normal distribution with mean vector $\underline{\mu}_*$ and covariance matrix $\sigma^2 V_0$ , where

$$\underline{\mu}_*' = (\mu_1 , \mu_2 , \cdots , \mu_p , \mu_1 , \mu_2 , \cdots , \mu_p , \cdots , \mu_1 , \mu_2 , \cdots , \mu_p) ,$$

and

$$V_0 = \begin{pmatrix} K_0 & O & \cdots & O \\ O & K_0 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & K_0 \end{pmatrix} ,$$

where O is a p × p matrix of zeros.

We now determine whether or not $AV_0$ is an idempotent matrix. It is easy to show that

$$AV_0 = \begin{pmatrix} A_2 & A_2 & \cdots & A_2 \\ A_2 & A_2 & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_2 & \cdots & A_2 \end{pmatrix} , \tag{68}$$

where $A_2 = I - \frac{J}{p}$ is a p × p matrix. Now

$$(AV_0)(AV_0) = \frac{1}{n} \begin{pmatrix} A_2^2 & A_2^2 & \cdots & A_2^2 \\ A_2^2 & A_2^2 & \cdots & A_2^2 \\ \vdots & \vdots & \ddots & \vdots \\ A_2^2 & A_2^2 & \cdots & A_2^2 \end{pmatrix} . \tag{69}$$

But

$$A_2^2 = \left(I - \frac{1}{p} J\right)^2$$

$$= \left(I - \frac{1}{p} J\right)$$

$$= A_2 \quad . \tag{70}$$

Therefore

$$(AV_0)(AV_0) = AV_0$$

and $AV_o$ is an idempotent matrix of rank p-1 .

By Graybill [3], Q has a non-central chi-square distribution with p-1 degrees of freedom and noncentrality parameter

$$\lambda_p = \frac{1}{2\sigma_0^2} \underline{\mu}_*' A \underline{\mu}_* \quad . \tag{71}$$

But, under $H_{05}$ , it is easy to verify that

$$\lambda_p = 0 \quad . \tag{72}$$

Therefore, under the null hypothesis,

$$Q = \frac{n}{\sigma_0^2} (\bar{\underline{Y}} - \bar{\bar{\underline{Y}}})' K_0^{-1} (\bar{\underline{Y}} - \bar{\bar{\underline{Y}}})$$

has a chi-square distribution with p-1 degrees of freedom.

Now the distribution of $\lambda = \exp(-\frac{1}{2} Q)$ could be determined and $H_{05}$ could be tested using $\lambda$ . But, since Q is a monotonic decreasing function of $\lambda$ , $H_{05}$ can be tested on the basis of Q . The critical region of size

$\alpha$ for testing $H_{0_5}$ based on $Q$ is

$$\chi^2_{\alpha,p-1} < Q < \infty \quad , \tag{73}$$

where

$$\int_{\chi^2_{\alpha,p-1}}^{\infty} h(Q)dQ = \alpha$$

and $g(Q)$ is the density of a central chi-square distribution with p-1 degrees of freedom. Then from a sample of size n, $\underline{Y}_1$ , $\underline{Y}_2$ , $\cdots$ , $\underline{Y}_n$ , from a normal distribution with mean vector $\underline{\mu}$ and covariance matrix $\sigma_0^2 K_0$ , the quantity

$$Q = \frac{n}{\sigma_0^2}(\bar{\underline{Y}} - \bar{\bar{\underline{Y}}})'K_0^{-1}(\bar{\underline{Y}} - \bar{\bar{\underline{Y}}}) \tag{74}$$

is computed. The null hypothesis, $H_{0_5}$: $\underline{\mu} = \underline{\mu}^*$ , is rejected if

$$Q > \chi^2_{\alpha,p-1} \quad ;$$

otherwise $H_{0_5}$ is accepted.

# CHAPTER VI

In this chapter the criterion for testing the null hypothesis

$$H_{0_6}: \quad \underline{\mu} = \underline{\mu}^*$$

against the alternative hypothesis

$$H_{A_6}: \quad \underline{\mu} \neq \underline{\mu}^*$$

will be derived by the likelihood ratio method.

Here

$$\Omega = \{(\mu_1, \mu_2, \cdots, \mu_p, \sigma^2, \rho): -\infty < \mu_i < \infty$$

$$\text{for } i = 1, 2, \cdots, p, \sigma^2 > 0, -1 < \rho = \rho_0 < 1\}$$

and

$$\omega = \{(\mu_1, \mu_2, \cdots, \mu_p, \sigma^2, \rho): -\infty < \mu_i = \mu < \infty$$

$$\text{for } i = 1, 2, \cdots, p, \sigma^2 > 0, -1 < \rho = \rho_0 < 1\} \quad .$$

The likelihood function is given by

$$L = \frac{1}{(\sigma^2)^{\frac{n}{2}} (2\pi)^{\frac{np}{2}} |K_0|^{\frac{n}{2}}} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (\underline{Y}_i - \underline{\mu})' K_0^{-1} (\underline{Y}_i - \underline{\mu})\right] \tag{75}$$

where

$$K_0^{-1} = \frac{1}{1-\rho_0} \left(I - \frac{\rho_0}{1+(p-1)\rho_0} J\right) \quad .$$

Now L is maximized in $\omega$ by

$$\underline{\hat{\mu}}^* = \underline{\bar{\bar{Y}}} = \begin{pmatrix} \frac{1}{p} \sum_{i=1}^{n} \bar{Y}_i \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{p} \sum_{i=1}^{p} \bar{Y}_i \end{pmatrix} \tag{76}$$

and

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (\underline{Y}_i - \underline{\bar{\bar{Y}}})' K_0^{-1} (\underline{Y}_i - \underline{\bar{\bar{Y}}}) \quad , \tag{77}$$

so that the maximum of L in $\omega$ is

$$L(\hat{\omega}) = \frac{1}{(2\pi)^{\frac{np}{2}} |K_0|^{\frac{n}{2}} \left[\frac{1}{n} \sum_{i=1}^{n} (\underline{Y}_i - \underline{\bar{\bar{Y}}})' K_0^{-1} (\underline{Y}_i - \underline{\bar{\bar{Y}}})\right]^{\frac{n}{2}}} \exp\left(-\frac{n}{2}\right) \tag{78}$$

In $\Omega$ , L is maximized by

$$\underline{\hat{\mu}} = \underline{\bar{Y}} \tag{79}$$

and

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (\underline{Y}_i - \underline{\bar{Y}})' K_0^{-1} (\underline{Y}_i - \underline{\bar{Y}}) \tag{80}$$

so that

$$L(\hat{\Omega}) = \frac{1}{(2\pi)^{\frac{np}{2}} |K_0|^{\frac{n}{2}} \left[\frac{1}{n} \sum_{i=1}^{n} (\underline{Y}_i - \underline{\bar{Y}})' K_0^{-1} (\underline{Y}_i - \underline{\bar{Y}})\right]^{\frac{n}{2}}} \exp\left(-\frac{n}{2}\right) \tag{81}$$

Now the likelihood ratio is

$$\lambda = \left[\frac{\sum_{i=1}^{n} (\underline{Y}_i - \underline{\bar{Y}})' K_0^{-1} (\underline{Y}_i - \underline{\bar{Y}})}{\sum_{i=1}^{n} (\underline{Y}_i - \underline{\bar{\bar{Y}}})' K_0^{-1} (\underline{Y}_i - \underline{\bar{\bar{Y}}})}\right]^{\frac{n}{2}} \tag{82}$$

But

$$\sum_{i=1}^{n} (\underline{Y}_i - \bar{\bar{\underline{Y}}})' K_0^{-1} (\underline{Y}_i - \bar{\bar{\underline{Y}}}) = \sum_{i=1}^{n} (\underline{Y}_i - \bar{\underline{Y}})' K_0^{-1} (\underline{Y}_i - \bar{\underline{Y}}) + n(\bar{\underline{Y}} - \bar{\bar{\underline{Y}}})' K_0^{-1} (\bar{\underline{Y}} - \bar{\bar{\underline{Y}}}) \tag{83}$$

so that the likelihood ratio can be written as

$$\lambda = \left[ \frac{1}{1 + \dfrac{n(\bar{\underline{Y}} - \bar{\bar{\underline{Y}}})' K_0^{-1} (\bar{\underline{Y}} - \bar{\bar{\underline{Y}}})}{\sum_{i=1}^{n} (\underline{Y}_i - \bar{\underline{Y}})' K_0^{-1} (\underline{Y}_i - \bar{\underline{Y}})}} \right]^{\frac{n}{2}} . \tag{84}$$

To find the distribution of the likelihood ratio under the null hypothesis we first find the distribution of

$$\frac{n(\bar{\underline{Y}} - \bar{\bar{\underline{Y}}})' K_0^{-1} (\bar{\underline{Y}} - \bar{\bar{\underline{Y}}})}{\sum_{i=1}^{n} (\underline{Y}_i - \bar{\underline{Y}})' K_0^{-1} (\underline{Y}_i - \bar{\underline{Y}})} .$$

It was proved in Chapter V that

$$Q = \frac{n}{\sigma^2} (\bar{\underline{Y}} - \bar{\bar{\underline{Y}}})' K_0^{-1} (\bar{\underline{Y}} - \bar{\bar{\underline{Y}}})$$

has a chi-square distribution with p-1 degrees of freedom under $H_{06}$ , and in Chapter IV it was proved that

$$\frac{Q_2}{\sigma^2} = \sum_{i=1}^{n} (\underline{Y}_i - \bar{\underline{Y}})' \frac{K_0^{-1}}{\sigma^2} (\underline{Y}_i - \bar{\underline{Y}})$$

has a chi-square distribution with (n-1)p degrees of freedom regardless of whether or not $H_{06}$ is true. Since by (65)

$$Q = \frac{\underline{Y}_*' A \underline{Y}_*}{\sigma^2}$$

where

$$A = \frac{1}{n}\begin{pmatrix} A_1 & A_1 & \cdots & A_1 \\ A_1 & A_1 & \cdots & A_1 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_1 & \cdots & A_1 \end{pmatrix}, \quad A_1 = \frac{1}{1 - \rho_0}\left(I - \frac{1}{p} J\right)$$

and by (46)

$$Q_2 = \underline{Y}_*'B\underline{Y}_*$$

where

$$B = \frac{1}{n}\begin{pmatrix} (n-1)K_0^{-1} & -K_0^{-1} & \cdots & -K_0^{-1} \\ -K_0^{-1} & (n-1)K_0^{-1} & \cdots & -K_0^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ -K_0^{-1} & -K_0^{-1} & \cdots & (n-1)K_0^{-1} \end{pmatrix},$$

it is easy to show that

$$AV_0B = 0 \tag{85}$$

Hence $Q$ and $Q_2$ are independent, and

$$F = \frac{p(n - 1)}{(p - 1)\sigma^2} \frac{Q}{Q_2} \tag{86}$$

has an F distribution with p-1 and (n-1)p degrees of freedom under $H_{0_6}$ .

Now the likelihood ratio can be written as

$$\lambda = \left[ \frac{1}{1 + \frac{p-1}{p(n-1)} F} \right]^{\frac{n}{2}} \quad . \tag{87}$$

Since F is a monotonic decreasing function of $\lambda$ , the test can be done just as well with F as a criterion as with $\lambda$ . The critical region for testing $H_{06}$ by a likelihood-ratio test based on F is

$$F_{\alpha,p-1,(n-1)p} < F < \infty \quad , \tag{88}$$

where

$$\int_{F_{\alpha,p-1,(n-1)p}}^{\infty} g(F)dF = \alpha$$

and g(F) is the density function of an F distribution with p-1 and (n-1)p degrees of freedom. To test $H_{06}$: $\underline{\mu} = \underline{\mu}^*$ we compute the quantity

$$F = \frac{n(\bar{\underline{Y}} - \bar{\bar{\underline{Y}}})'K_0^{-1}(\bar{\underline{Y}} - \bar{\bar{\underline{Y}}})}{\sum_{i=1}^{n} (\underline{Y}_i - \bar{\underline{Y}})'K_0^{-1}(\underline{Y}_i - \bar{\underline{Y}})} \tag{89}$$

and reject $H_{06}$ if $F > F_{\alpha,p-1,(n-1)p}$ ; otherwise accept $H_{06}$ .

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