A NOTE ON THE RANK TRANSFORMATION FOR INTERACTION

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Abstract

For a two-way layout with interactions, necessary and sufficient conditions are given for the asymptotic distribution of rank transform statistic to be chi-squared under the null hypothesis of no interaction. When both main effects are present, it is shown that as the number of replications approaches infinity, the expected value of the rank transform test for interactions also approaches infinity under the null hypothesis of no interaction. However, when there is at most one main effect, the test is asymptoically chi-squared.

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1. Introduction. The rank transform procedure was proposed by Conover and Iman (1981) as a bridge between nonparametrics and classical analysis of variance procedures. In the rank transform procedure, all of the observations are ranked together without regard to row or column membership, and the classical normal theory tests are applied to the ranks, instead of to the observations. This procedure has gained much popularity because it is very easy to implement with most common statistical packages (whereas other nonparametric methods are often unavailable to the data analyst) and because it often yields significance where other tests do not.

The asymptotic properties of the rank transform statistic for testing for interaction in a two-way layout are studied in this article. Confusion about the performance of the rank transform statistic for interaction stems from several seemingly contradictory simulation studies. Using only small sample sizes, the simulation studies of Iman (1974) and Conover and Iman (1976) show that this rank transform statistic performs well at detecting interaction in a two-way layout. On the other hand, simulations by Blair, Sawilowsky, and Higgins (1987) and Sawilowsky, Blair, and Higgins (1989) show that the rank transform procedure is very poorly behaved for detecting interaction in two and three-way layouts with large numbers of replications. The fact that this particular rank transform statistic performs well for small numbers, but not large numbers of replications, points to the critical need to study the asymptotic properties of the test. In contrast to the asymptotic properties of the rank transform test for interaction, the asymptotic properties of rank transform statistics to detect column effects and row effects (with and without dependent data) have been widely studied, (cf. Hora and Conover (1984), Iman, Hora, and Conover (1984), Hora and Iman (1988), Kepner and Robinson (1988), Thompson and Ammann (1989), Thompson (1990a), Thompson (1990b), Thompson and Ammann (1990)).

This discussion of the asymptotic properties of the rank transform statistic for testing for interaction in a two-way layout with replication is motivated by the fact that the critical points for the rank transform test are identical to the critical points for the normal theory test and are obtained from the F-distribution. The asymptotic null distribution of the normal theory test for interaction is chi-squared with $(I-1)(J-1)$ degrees of freedom. Hence, for the rank transform test to have acceptable behavior, it is necessary that its asymptotic null distribution also be chi-squared with $(I-1)(J-1)$ degrees of freedom. It is shown that the asymptotic null distribution of the rank transform statistic for interaction is not chi-squared if both main effects are present and unknown because the expected value of the test statistic appoaches infinity (under the null hypothesis of no interaction) as the number of replications approaches infinity. As a result, this particular rank statistic is grossly liberal for testing the null hypothesis of main effects with no interaction effects versus the alternative of main effects with interaction effects. However, it is shown that this rank test is a suitable asymptotic test for testing the null hypothesis of no nested or interaction effect against the alternative of exactly one main effect and a nested or interaction effect.

These are the first theoretical results proving that a commonly used rank transform statistic has unacceptable properties. They are important results because despite the limited theoretical results and the contradictory simulation results, the rank transform procedure has become popular with social scientists, business professionals, and other researchers in both academic and industrial fields. This is not particularly surprising – easily implemented tests that detect alternatives are attractive. Further contributing to the inappropriate use of the rank transform are two widely used manuals for statistical procedures that endorse the procedure without reservation. The 1985 release of SAS states:

> "Many nonparametric Statistical methods use ranks rather than the original values of the variable. For example, a set of data may be passed through PROC RANK to obtain the ranks for a response variable that could then be fit to an analysis-of variance model using the ANOVA or GLM procedures." (SAS,1985,p.647)

The 1987 IMSL User's Manual Stat/Library also suggests applying analysis of variance tests to ranked data. These endorsements of the rank transform are misleading. In general, a test statistic based on the rank transform does not behave like its normal theory counterpart. Frequently, the two tests detect entirely different alternatives. As shown in this paper, extreme care must be taken when using a rank transform statistic to assure that the null hypothesis is correctly specified and that the test has the correct asymptotic null distribution.

In Section 2 the model and the rank transform statistic are defined. In Section 3 the asymptotic properties of the rank transform are discussed. The proofs are in Section 4

2. Definitions and Preliminary Notation. Consider the model for a two-way layout with interaction: $X_{ijn}=\mu+\alpha_i+\beta_j+\gamma_{ij}+\epsilon_{ijn}$, $1\le i\le I$, $1\le j\le J$, and $1\le n\le N$, where the ϵ_{ijn} are iid random variables with an absolutely continuous cdf $F(x)$ such that $F(0)=\frac{1}{2}$. The parameters for the main effects, α_i and β_j are considered unknown and completely arbitrary. Because the main thrust of this paper is to show when the rank transform has undesirable properties, we will restrict ourselves to the balanced case. Without loss of generality assume that $\sum_{i=1}^{I}\alpha_i=0$, $\sum_{j=1}^{J}\beta_j=0$, $\sum_{i=1}^{I}\gamma_{ij}=0$, and $\sum_{j=1}^{J}\gamma_{ij}=0$. The null hypothesis of no interaction effect is $H_o:\gamma_{ij}=0$ for all $1\le i\le I$, $1\le j\le J$; the alternative of an interaction effect is $H_a:\gamma_{ij}\neq 0$ for some i,j. Let $F_{ij}(x)=F(x-\alpha_i-\beta_j)$ denote the cdf of X_{ijn} under the null hypothesis, and let the "average" cdf be $H(x)=(IJ)^{-1}\sum_{j=1}^{J}\sum_{i=1}^{I}F_{ij}(x)$. Let X_{ij}, $1\le i\le I$, $1\le j\le J$, denote IJ independent random varible with cdfs $F_{ij}(x)$. Because $H(x)$ is increasing on the support of X_{ij}, it follows that $\mathrm{var}(H(X_{ij}))>0$. We will also make the assumption that $\mathrm{var}(H(X_{ij}))<\infty$ for all i and j.

To define the rank transform statistic for this model, first define the function $u(x)=1$ or 0 as to whether $x\ge 0$ or $x<0$, so that the rank of X_{ijn} is $R_{ijn}=\sum_{c=1}^{N}\sum_{b=1}^{J}\sum_{a=1}^{I}u(X_{ijn}-X_{abc})$. Note that R_{ijn} is the rank of X_{ijn} among all of the $M=IJN$ observations. Because the primary intent of this paper is to discuss the validity of the rank transform statistic for interaction under commonly used conditions, we will restrict our attention to approximate scores generated by $\phi(u)=u$. For notational convenience let $a_{ijn}=a_M(R_{ijn})=R_{ijn}/(M+1)$. Then the rank transform statistic is

$$T_N=\frac{\frac{1}{N}\sum_{j=1}^{J}\sum_{i=1}^{I}\left(a_{ij\cdot}-\frac{1}{J}a_{i\cdot\cdot}-\frac{1}{I}a_{\cdot j\cdot}+\frac{1}{IJ}a_{\cdot\cdot\cdot}\right)^2}{\frac{(I-1)(J-1)}{IJ(N-1)}\sum_{n=1}^{N}\sum_{j=1}^{J}\sum_{i=1}^{I}\left(a_{ijn}-\frac{1}{N}a_{ij\cdot}\right)^2},$$

where a dot in the subscript denotes summation over that index. Note that the rank transform statistic, T_N, is exactly the classical normal theory test with the scored ranks, a_{ijn}, substituted in place of the observations, X_{ijn}. Define

$$Q_N = \frac{1}{N}\sum_{j=1}^{J}\sum_{i=1}^{I}\left(a_{ij\cdot} - \frac{1}{J}a_{i\cdot\cdot} - \frac{1}{I}a_{\cdot j\cdot} + \frac{1}{IJ}a_{\cdot\cdot\cdot}\right)^2$$

to be the quadratic form in the numerator of T_N, and define

$$D_N = \frac{1}{IJ(N-1)}\sum_{n=1}^{N}\sum_{j=1}^{J}\sum_{i=1}^{I}\left(a_{ijn} - \frac{1}{N}a_{ij\cdot}\right)^2$$

to be the denominator of T_N, divided by $(I-1)(J-1)$.

3. Asymptotic Properties of T_N Under H_0. To show that the asymptotic distribution of T_N is not chi-squared and that $E(T_N)\to\infty$ under the null hypothesis when both main effects are present, we will first show that $D_N \xrightarrow{P} c$ where $0 \le c < \infty$ and that $E(Q_N)\to\infty$. Because D_N and Q_N are both functions of the linear rank statistic $a_{ij\cdot} = \sum_{n=1}^{N} a_{ijn}$, the following lemma concerning the asymptotic normality of $a_{ij\cdot}$ is useful. First, however, define

$$\mu_{ij} = N\int H(x)dF_{ij}(x) = NE[H(X_{ij})]\ ,$$

$$\sigma_{ij}^2 = \lim_{N\to\infty} M^2(M+1)^{-2}\mathrm{var}\left(H(X_{ij}) - (IJ)^{-1}\sum_{b=1}^{J}\sum_{a=1}^{I}F_{ij}(X_{ab})\right),$$

$$\mathbf{a} = (a_{11\cdot}\ ,\ a_{12\cdot}\ ,\ \dots\ ,\ a_{1J\cdot}\ ,\ a_{21\cdot}\ ,\ \dots\ ,\ a_{IJ\cdot})'\ ,\ \text{and}$$

$$\mu = (\mu_{11\cdot}\ ,\ \mu_{12\cdot}\ ,\ \dots\ ,\ \mu_{1J\cdot}\ ,\ \mu_{21\cdot}\ ,\ \dots\ ,\ \mu_{IJ\cdot})'.$$

Let Σ be an $IJ\times IJ$ dimensional matrix whose rows are indexed by the ordered pairs (i,j), $1\le i\le I$, $1\le j\le J$, with the second index running faster than the first. Similarly, index the columns by the ordered pairs (r,s), $1\le r\le I$, $1\le s\le J$. Let the elements of Σ be given by

$$\sigma_{(i,j),(r,s)} = \mathrm{cov}\left(H(X_{ij}) - \frac{1}{IJ}\sum_{b=1}^{J}\sum_{a=1}^{I}F_{ij}(X_{ab}),\ \ H(X_{rs}) - \frac{1}{IJ}\sum_{b=1}^{J}\sum_{a=1}^{I}F_{rs}(X_{ab})\right).$$

Note that $\sigma_{(i,j),(i,j)} = \sigma_{ij}^2$. Also note that $0<\mathrm{var}(H(X_{ij}))<\infty$ implies that $0<\sigma_{(i,j),(r,s)}<\infty$ for all $1\le i,r\le I$, $1\le j,s\le J$.

Lemma 3.1. Under the null hypothesis, $N^{-1/2}(\mathbf{a}-\mu)\xrightarrow{d} N_{IJ}(0,\Sigma)$; in particular, $N^{-1/2}(a_{ij\cdot}-\mu_{ij})/\sigma_{ij} \xrightarrow{d} N(0,\ 1)$.

By using Lemma 3.1 and decomposing D_N as

$$D_N = \frac{1}{IJ(N-1)} \sum_{n=1}^{N} \sum_{j=1}^{J} \sum_{i=1}^{I} a_{ijn}^2 - \frac{1}{IJ} \sum_{j=1}^{J} \sum_{i=1}^{I} \left(\left(N(N-1)\right)^{-1/2} a_{ij\cdot} \right)^2 ,$$

Theorem 3.3 shows that under the null hypothesis $D_N \xrightarrow{P} c$ where $0 \leq c < \infty$.

Theorem 3.2. Under the null hypothesis, D_N converges in probability to the nonnegative, finite constant $c = \frac{1}{3} - \frac{1}{IJ} \sum_{j=1}^{J} \sum_{i=1}^{I} \left(E(H(X_{ij})) \right)^2$.

Next, to show that $E(Q_N) \rightarrow \infty$ under the null hypothesis of no interaction when both main effects are present, define an IJxIJ matrix A. The IJ rows of A are indexed by the ordered pair (a,b) , $1 \leq a \leq I$, $1 \leq b \leq J$, where the second element in the pair runs faster; the IJ columns of A are similarly indexed; and the elements of A are defined as

$$[A]_{(a,b)(i,j)} = \delta(a,i)\delta(b,j) - I^{-1}\delta(b,j) - J^{-1}\delta(a,i) + (IJ)^{-1}$$

where $\delta(a,i) = 1$ or 0 as to whether $a = i$ or $a \neq i$. Note that A is idempotent and symmetric, and that Q_N can be written as the quadratic form $Q_N = N^{-1} a' A\, a = (N^{-1/2} a)' A (N^{-1/2} a)$. The expected value of the quadratic form Q_N is $E(Q_N) = tr(A\Sigma) + N^{-1} e' A e$ where $e_{ij} = E(a_{ij\cdot})$ and $e = (e_{11}, e_{12}, \dots, e_{1J}, e_{21}, \dots, e_{IJ})'$. The proof of Theorem 3.3 shows that $tr(A\Sigma)$ is finite and gives conditions under which $N^{-1} e' A e$ is also finite.

Theorem 3.3. Under the null hypothesis of no interaction effect, $E(Q_N) \rightarrow tr(A\Sigma) < \infty$ only if $F_{ij}(x) - \frac{1}{J} F_{i\cdot}(x) - \frac{1}{I} F_{\cdot j}(x) + \frac{1}{IJ} F_{\cdot\cdot}(x) = 0$. Otherwise, $E(Q_N) \rightarrow \infty$.

When both main effects α_i and β_j are regarded as unrestricted and unknown parameters, the solution to the functional equation

$$F(x - a_i - \beta_j) - \frac{1}{J} \sum_{r=1}^{J} F(x - a_i - \beta_r) - \frac{1}{I} \sum_{r=1}^{J} F(x - a_s - \beta_j) + \frac{1}{IJ} \sum_{r=1}^{J} \sum_{s=1}^{I} F(x - a_s - \beta_r) = 0$$

is $F(x) = mx + b$ for some constants m and b. This is impossible because the cdf $F(x)$ is by definition a non-linear function. Hence, when the model contains both main effects and the null hypothesis of no interaction effect holds, it follows from Theorems 3.2 and 3.3 and Slutsky's theorem that T_N is not asymptotically chi-squared because $E(T_N) \to \infty$. This means that as the number of replications increases, the test becomes more and more liberal. This result is entirely in keeping with the simulation results of Blair, Sawilowsky, and Higgens (1987) that show, when both main effects are the present, that the nominal α-levels approach 1 as the number of replications increases. Thus, T_N, should never be used as a rank transform test for interaction in the model defined in Section 2.

However, for any fixed value of x, $F_{ij}(x) - \frac{1}{J}F_{i\cdot}(x) - \frac{1}{I}F_{\cdot j}(x) + \frac{1}{IJ}F_{\cdot\cdot}(x) = 0$ for all i and j defines a system of IJ equations in terms of the IJ variables $F_{ij}(x)$, $1 \le i \le I$, $1 \le j \le J$. One obvious solution to this system of equations is $F_{ij}(x) = F_i(x)$ or $F_{ij}(x) = F_j(x)$ which is equivalent to $\alpha_i = 0$ for all $1 \le i \le I$ or $\beta_j = 0$ for all $1 \le j \le J$. Hence, $E(Q_N) \to tr(A\Sigma) < \infty$ when the model has at most one main effect, and T_N is a valid test statistic with acceptable asymptotic properties for testing the null hypothesis $H_{o1}: \gamma_{ij} = 0$ in the model $X_{ijn} = \mu + \alpha_i + \gamma_{ij} + \epsilon_{ijn}$ where $\sum_{i=1}^{I} \alpha_i = 0$ and $\sum_{j=1}^{J} \gamma_{ij} = 0$. This can be interpreted either as testing the null hypothesis that there is exactly one main effect with a nested effect, or equivalently, as testing the null hypothesis $H_{o2}: \beta_j + \gamma_{ij} = 0$ for all i and j in the model $X_{ijn} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijn}$ (as defined in Section 2). With either of these interpretations, T_N is no longer a rank transform statistic: it looks like the classical F-test for interaction, but it tests an entirely different null hypothesis. Theorem 3.3 shows that $E(Q_N) < \infty$ under these conditions. It follows immediately from Theorems 4.3 and 6.1 of Thompson (1990b) that under H_{o1} and H_{o2}, the statistic T_N converges in distribution to $\chi^2_{(I-1)(J-1)}/(I-1)(J-1)$ and that under a sequence of Pitman alternatives T_N converges in distribution to a noncentral chi-square .

It is interesting to compare the results of this paper with those of Akritas (1990). In the above test for a nested factor, the data are ranked all together and the ranks are substituted into one of the standard analysis of variance procedures, but not the procedure classically used to detect a nested effect. In the tests that Akritas proposes, all of the data are ranked together. Then, a function of the ranks, namely, the ranks divided by an estimate of the standard deviation, are substituted into the classical F-test for detecting a nested effect. Hence, the two tests are fundamentally different. Even though Akritas claims that his test is a rank transform test, it clearly is not. The tests that Akritas proposes are sophisticated rank tests that adjust for heteroscedasticity. Furthermore, in the same paper Akritas claims that the rank transform procedure cannot be used to test for a nested effect in the above nested design, or to test for an interaction effect in a two-way layout. While these may be true, the arguments (which lend insight into the complications introduced by ranking non-identically distributed random variables) only show that his type of tests (in which the ranks are first adjusted for heteroscedasticity) can not be derived via his methods for these cases. This paper actually proves when the rank transform test is not chi-squared for detecting interaction.

4. Proofs.

Proof of Lemma 3.1. The univariate result follows by writing $a_{ij\cdot}$ as the linear rank statistic $a_{ij\cdot}=\sum_{c=1}^{N}\sum_{b=1}^{J}\sum_{a=1}^{I} d_{abc}a_{abc}$ where $d_{abc}=\delta(a,i)\delta(b,j)$, applying Theorem 3.3 of Thompson and Ammann (1989), and simplifying the expression for the variance. Note that the condition $\operatorname{var}(H(X_{ij}))>0$ ensures that $\lim_{N\to\infty}\sigma_{ij}^2>0$. The multivariate results follows by noting that $\operatorname{var}(H(X_{ij}))>0$ implies that diagonal elements of Σ are non-zero. Let λ be any vector such that $\lambda'\Sigma\lambda>0$. Then, $N^{-1/2}\lambda'\mathbf{a}$ is a linear rank statistic with regression constants $d_{ijn}=N^{-1/2}\lambda_{ij}$. From Theorem 3.3 of Thompson and Ammann (1989), the linear rank statistic $N^{-1/2}\lambda'\mathbf{a}$ is $AN(\lambda'\mu,\ \sigma^2(\lambda))$ where $\sigma^2(\lambda)=\sum_{n=1}^{N}\sum_{j=1}^{J}\sum_{i=1}^{I} Z_{ijn}(\lambda)$ and $Z_{ijn}(\lambda)$ corresponds to $N^{-1/2}\lambda'\mathbf{a}$ as defined by equation (3.3) of Thompson and Ammann (1989). Straightforward computations show that $\sigma^2(\lambda)=\lambda'\Sigma\lambda$. Q. E. D.

Proof of Theorem 3.2. As in Thompson and Ammann (1989), the first term in the expanded expression for D_N converges to $\int x^2dx=\frac{1}{3}$. From Lemma 3.1, it follows that $\left(N(N-1)\right)^{-1/2}a_{ij\cdot}$ is $AN\left(\int H(x)dF_{ij}(x),\ \ \sigma_{ij}^2/(N-1)\right)$. Because $\sigma_{ij}^2<\infty$ it follows that $\sigma_{ij}^2/(N-1)\to 0$. Hence, $\left(N(N-1)\right)^{-1/2}a_{ij\cdot}$ converges in probability to $\int H(x)dF_{ij}(x)=E(H(X_{ij}))$, which in turn implies that $D_N\to c$. To show that $0\le c<\infty$, note that $D_N\ge 0$ for all N implies that $\frac{1}{3}\ge c\ge 0$. Q. E. D.

Proof of Theorem 3.3. The elements of A do not depend on N and the elements of Σ converge to finite values so that $\mathrm{tr}(A\Sigma)$ is finite. Hence, $\lim_{N\to\infty} E(Q_N) < \infty$ iff

$$e'Ae = \sum_{j=1}^{J}\sum_{i=1}^{I}\left(e_{ij} - \tfrac{1}{J}e_{i\cdot} - \tfrac{1}{I}e_{\cdot j} + \tfrac{1}{IJ}e_{\cdot\cdot}\right)^2 = O(N) .$$

This holds iff $e_{ij} - \frac{1}{J}e_{i\cdot} - \frac{1}{I}e_{\cdot j} + \frac{1}{IJ}e_{\cdot\cdot} = O(N^{1/2})$ for every pair (i,j). Theorem 3.3 of Thompson and Ammann (1989) and Lemma 1.5.5.A of Serfling (1980) imply that $\lim_{N\to\infty}(e_{ij} - \mu_{ij})/\sigma_{ij} = 0$. Because $0 < \sigma_{ij} < \infty$, both e_{ij} and μ_{ij} converge to the same limit as N approaches infinity. Therefore, we have that $e_{ij} - \frac{1}{J}e_{i\cdot} - \frac{1}{I}e_{\cdot j} + \frac{1}{IJ}e_{\cdot\cdot}$ is $O(N^{1/2})$ iff $E(\mu_{ij} - \frac{1}{J}\mu_{i\cdot} - \frac{1}{I}\mu_{\cdot j} + \frac{1}{IJ}\mu_{\cdot\cdot})$ is also $O(N^{1/2})$. This happens iff $\int H(x)dG(x) = 0$ where $G(x) = F_{ij}(x) - \frac{1}{J}F_{i\cdot}(x) - \frac{1}{I}F_{\cdot j}(x) + \frac{1}{IJ}F_{\cdot\cdot}(x)$. The function $H(x)$ is positive on the support of X_{ij} for all $1 \le i \le I$, $1 \le j \le J$, so $\int H(x)dG(x) = 0$ iff $G(x)$ is a constant function almost everywhere. Because $\lim_{x\to\infty} G(x) = \lim_{x\to-\infty} G(x) = 0$, it follows that $G(x)$ is equal to a constant almost everywhere iff $G(x) = 0$ almost everywhere. Q. E. D.

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