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DISTRIBUTION OF QUADRATIC FORMS IN THE MULTIVARIATE SINGULAR NORMAL DISTRIBUTION

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Distribution of Quadratic Forms in the Multivariate Singular Normal Distribution

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Summary

The purpose of this paper is to investigate the distribution and independence properties of quadratic forms of the type $\underline{X}'A\underline{X}$ where $\underline{X}$, a px1 random vector, has the multivariate normal distribution with the corresponding mean vector $\underline{\mu}$ and the pxp variance covariance matrix V of rank $k < p$.

1. Introduction

The distribution and properties of quadratic forms of the type $\underline{X}'A\underline{X}$ where $\underline{X}$ has a p-dimensional nonsingular multivariate normal distribution with the corresponding mean vector $\underline{\mu}$ and a nonsingular variance covariance matrix V and A is a pxp symmetric matrix have been extensively studied by several authors [Graybill, 1961]. In this note, the similar results are derived when the variance covariance matrix is singular.

2. Notation and Preliminaries

Throughout this paper all matrices and vectors will consist of real elements. The notation D = [B, C] will be used to indicate that D has been partitioned into sub-matrices B and C and, similarly

$D = \begin{bmatrix} B, & C \\ E, & F \end{bmatrix}$ will denote the partitioning of D into the submatrices B, C, E, and F, of suitable dimensions. In general, upper and lower case Roman letters will denote matrices and scalars respectively.

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Lower case Greek letters will usually denote parameters, and underscored letters will be used to denote column vectors. The symbol $\emptyset$ will denote the null matrix. The rank of the matrix D will be denoted by r(D), the trace of D by tr(D), and the transpose of D by D'. The symbol $\sim$ will be used to mean "is distributed." $\underline{X} \sim N_p(\underline{\mu}, V)$ will indicate that the px1 random vector $\underline{X}$ has the multivariate normal distribution with mean $\underline{\mu}$ and variance covariance matrix V. The noncentral chi-square distribution with n degrees of freedom and noncentrality parameter λ will be denoted by $\chi^2(n, \lambda)$.

3. Theory

Theorem 3.1: Let the random vector $\underline{X}$ be distributed as $N_p(\underline{\mu}, V)$ with $r(V) = k \leq p$. A sufficient+ condition that $\underline{X}'A\underline{X}$ be distributed as a noncentral chi-square with k degrees of freedom and noncentrality parameter $\lambda = \frac{1}{2}(\underline{\mu}'A\underline{\mu})$ is that A be the unique generalized inverse of V; that is, $A = V^*$.

Proof: Suppose $A = V^*$. Since V is a p by p symmetric matrix of rank $k \leq p$, then there exists an orthogonal matrix P such that

$$P'VP = D_1 = \begin{bmatrix} D & \emptyset \\ \emptyset & \emptyset \end{bmatrix}$$

where D is a k by k diagonal matrix with the nonzero characteristic roots of V displayed on the diagonal. Let P be partitioned as $P = [P_1, P_2]$ where P_1 is p by k and P_2 is p by p-k, then the unique generalized inverse of V (Gateley, 1962) is given by

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+It can be shown by constructing a counter example that the condition given in the above theorem is not necessary.

$$V^* = P_1 D^{-1} P_1' \quad \text{and hence,}$$

$$A = V^* = P_1 D^{-1} P_1' \quad .$$

If $\underline{Y} = P'\underline{X}$,

$$\text{then} \quad \underline{X}'A\underline{X} = \underline{Y}' \begin{bmatrix} D^{-1} & \emptyset \\ \emptyset & \emptyset \end{bmatrix} \underline{Y} \quad .$$

Partition $\underline{Y}$ as

$$Y = \begin{bmatrix} \underline{Y}_1 \\ \underline{Y}_2 \end{bmatrix} \quad ,$$

where $\underline{Y}_1$ is k by one, $\underline{Y}_2$ is p-k by one. $\underline{Y}_1$ is distributed as $N_k(\underline{\alpha}_1, D)$ where $\underline{\alpha}_1$ is the vector consisting of the first k components of $P'\underline{\mu}$; That is, $\underline{\alpha}_1 = P_1'\underline{\mu}$. Then

$$\underline{X}'A\underline{X} = [\underline{Y}_1', \underline{Y}_2'] \begin{bmatrix} D^{-1} & \emptyset \\ \emptyset & \emptyset \end{bmatrix} \begin{bmatrix} \underline{Y}_1 \\ \underline{Y}_2 \end{bmatrix} \quad ,$$

$$\underline{X}'A\underline{X} = \underline{Y}_1' D^{-1} \underline{Y}_1 \quad .$$

$Y_1' D^{-1} \underline{Y}_1$ is distributed as a noncentral chi-square with k degrees of freedom and noncentrality $\lambda = \frac{1}{2}\underline{\alpha}_1 D^{-1} \underline{\alpha}_1$ (Graybill, 1961). Hence $\underline{X}'A\underline{X}$ is distributed as a noncentral chi-square with k degrees of freedom and noncentrality

$$\lambda = \frac{1}{2}(P_1'\underline{\mu})' \; D^{-1} \; (P_1'\underline{\mu})$$

$$= \frac{1}{2}\underline{\mu}' P_1 D^{-1} P_1' \underline{\mu}$$

$$= \frac{1}{2}\underline{\mu}' A \underline{\mu} \quad .$$

<u>Theorem 3.2</u>: Let the random vector $\underline{Y}$ be distributed as $N_p(\mu,V)$, with $r(V) = k \le p$. Then $\underline{Y}'A\underline{Y}$ is distributed as a noncentral chi-square with k degrees of freedom and $\lambda = \frac{1}{2}(\underline{\mu}'A\underline{\mu})$ if and only if A is a generalized inverse of V.

<u>Proof</u>: Suppose A is a generalized inverse of V. Then $k = r(V) = r(AV) = tr(AV)$, and hence, $VAV = V$ (Rao, 1962).

There exists an orthogonal matrix P such that

$$P'VP = \begin{bmatrix} D & \emptyset \\ \emptyset & \emptyset \end{bmatrix} ,$$

where D is the k by k diagonal matrix with the nonzero characteristic roots of V displayed on the diagonal. Letting P be partitioned as

$$P'VP = \begin{bmatrix} D & \emptyset \\ \emptyset & \emptyset \end{bmatrix} \quad \text{and } V = P_1DP_1' = PD^{1/2}D^{1/2}P_1' \quad ,$$

where $D^{1/2}$ is the k by k diagonal matrix of the positive square roots of the characteristic roots of V, such that $D^{1/2}D^{1/2} = D$. There exists a vector $\underline{X}$ such that $\underline{Y} = P_1D^{1/2}\underline{X}$, where $\underline{X}$ is distributed as $N_k(\underline{\mu}_x,I_k)$, and $\underline{\mu} = P_1D^{1/2}\underline{\mu}_x$. (For if such a vector did not exist, then there would exist no vector $\underline{X}$ such that $\underline{X} = D^{-1/2}P_1'\underline{Y}$, where $\underline{X}$ is distributed as $N_k(\underline{\mu}_x,I_k)$, which is a contradiction, since the existence and distribution of $\underline{Y}$ determin the existence and distribution of $\underline{X}$. Then

$$\underline{Y}'A\underline{Y} = \underline{X}'D^{1/2}P_1'AP_1D^{1/2}\underline{X} \quad ,$$

and since $D^{1/2}P_1'AP_1D^{1/2}$ can be shown to be an independent matrix, it follows that $\underline{X}'D^{1/2}P_1'AP_1D^{1/2}\underline{X}$ is distributed as a noncentral chi-square with k degrees of freedom and noncentrality parameter λ where $k = tr(AV) = tr(AP_1D^{1/2}D^{1/2}P_1') = tr(D^{1/2}P_1'AP_1D^{1/2})$ and $tr(D^{1/2}P_1'AP_1D^{1/2}) = r(D^{1/2}P_1'AP_1D^{1/2})$ (Searle, 1966). The noncentrality parameter λ is given by

$$\begin{aligned}\lambda &= \frac{1}{2}\underline{\mu}_x'D^{1/2}P_1'AP_1D^{1/2}\underline{\mu}_x \\ &= \frac{1}{2}(P_1D^{1/2}\underline{\mu}_x)'A(P_1D^{1/2}\underline{\mu}_x) \\ &= \frac{1}{2}\underline{\mu}'A\underline{\mu} \quad .\end{aligned}$$

Hence $\underline{Y}'A\underline{Y}$ is distributed as a noncentral chi-square with k degrees of freedom and $\lambda = \frac{1}{2}\underline{\mu}'A\underline{\mu}$.

Conversely, suppose $\underline{Y}'A\underline{Y}$ is distributed as a noncentral chi-square with k degrees of freedom and $\lambda = \frac{1}{2}\underline{\mu}'A\underline{\mu}$. Then, as above, $\underline{X}'D^{1/2}P_1'AP_1D^{1/2}\underline{X}$ is distributed as a noncentral chi-square with k degrees of freedom and $\lambda = \frac{1}{2}\underline{\mu}_x'D^{1/2}P_1'AP_1D^{1/2}\underline{\mu}_x$. Hence, $D^{1/2}P_1'AP_1D^{1/2}$ is idempotent of rank k, so that

$$D^{1/2}P_1'AP_1D^{1/2} = I_k \quad , \text{ or}$$

$$P_1D^{1/2}D^{1/2}P_1'AP_1D^{1/2}D^{1/2}P_1' = P_1D^{1/2}D^{1/2}P_1' \quad ,$$

and VAV = V, implying that A is a generalized inverse of V (Rao, 1962).

<u>Theorem 3.3</u>: Let the random vector $\underline{Y}$ be distributed as $N_p(\underline{\mu},V)$ with $r(V) = k \le p$. Then $\underline{Y}'A\underline{Y}$ is distributed as a noncentral chi-square with

r_1 degrees of freedom and noncentrality $\lambda = \frac{1}{2}\underline{\mu}'A\underline{\mu}$ if AV is idempotent of rank r_1 .

<u>Proof</u>: As in the proof of Theorem 3.2, let P be an orthogonal matrix such that

$$P'VP = \begin{bmatrix} D & \emptyset \\ \emptyset & \emptyset \end{bmatrix} , \quad V = P_1 D^{1/2} D^{1/2} P_1' , \text{ and } \underline{Y} = P_1 D^{1/2} \underline{X} ,$$

where $\underline{X}$ is distributed as $N_k(\underline{\mu}_x, I_k)$.

Since $\underline{Y}'A\underline{Y} = X'D^{1/2}P_1'AP_1D^{1/2}X$ and assuming AV an idempotent matrix of rank r, it can be shown that $D^{1/2}P_1'AP_1D^{1/2}$ is an idempotent matrix of rank and hence $\underline{X}'D^{1/2}P_1'AP_1\underline{X}$ is distributed as a noncentral chi-square with r_1 degrees of freedom and noncentrality λ , where $r_1 = r(AV) =$ $tr(AP_1D^{1/2}D^{1/2}P_1') = tr(D^{1/2}P_1'AP_1D^{1/2})$, and $tr(D^{1/2}P_1'AP_1D^{1/2}) =$ $r(D^{1/2}P_1'AP_1D^{1/2}) = r(D^{1/2}P_1'AP_1D^{1/2}) = r_1$, the noncentrality parameter is given by

$$\lambda = \frac{1}{2}\underline{\mu}_x D^{1/2}P_1'AP_1D^{1/2}\underline{\mu}_x$$

$$= \frac{1}{2}\underline{\mu}'A\underline{\mu} \text{ , as derived above.}$$

Thus $\underline{Y}'A\underline{Y}$ is distributed as a noncentral chi-square with r_1 degrees of freedom and $\lambda = \frac{1}{2}\underline{\mu}'A\underline{\mu}$.

If in addition it is known that AV is symmetric, the converse of Theorem 3.3 is true: if $\underline{Y}'A\underline{Y}$ is distributed as a noncentral chi-square with r_1 degrees of freedom and noncentrality parameter λ , where

$\underline{Y}$ is $N_p(\underline{\mu},V)$ as in Theorem 3.3 , then $\underline{X}'D^{1/2}P_1'AP_1D^{1/2}\underline{X}$ as a noncentral chi-square with r_1 degrees of freedom and noncentrality $\lambda = \frac{1}{2}\underline{\mu}'A\underline{\mu}$, $r_1 = r(AV)$ as in Theorem 3.3. $D^{1/2}P_1'AP_1D^{1/2}$ is idempotent or rank r_1 . But $AP_1D^{1/2}D^{1/2}P_1'$ has the same nonzero characteristic roots as $D^{1/2}P_1'AP_1D^{1/2}$ (Scheffé, 1959). Hence $AP_1D^{1/2}D^{1/2}P_1' = AV$ is idempotent of rank r_1 .

Theorem 3.4: Let the random vector $\underline{Y}$ be distributed as $N_p(\underline{\mu},V)$, where $r(V) = k \le p$. Then $\underline{Y}'A\underline{Y}$ is distributed as a noncentral chi-square with with r_1 degrees of freedom and $\lambda = \frac{1}{2}\underline{\mu}'A\underline{\mu}$ if and only if $V(AVA - A)V = \emptyset$, where $r_1 = tr(AV)$.

Proof: As in the proof of Theorem 3.2, let P be an orthogonal matrix such that

$$P'VP = \begin{bmatrix} D & \emptyset \\ \emptyset & \emptyset \end{bmatrix} , \quad V = P_1D^{1/2}D^{1/2}P_1' \quad , \text{ and } \underline{Y} = P_1D^{1/2}\underline{X} ,$$

where $\underline{X}$ is distributed as $N_k(\underline{\mu}_x, I_k)$.

Suppose $V(AVA - A)V = \emptyset$. Then $D^{1/2}P_1'AP_1D^{1/2}$ is an idempotent matrix, and $tr(D^{1/2}P_1'AP_1D^{1/2}) = tr(AP_1D^{1/2}D^{1/2}P_1') = tr(AV) = r_1$, so that $r(D^{1/2}P_1'AP_1D^{1/2}) = r_1$. Thus $\underline{Y}'A\underline{Y} = \underline{X}'D^{1/2}P_1'AP_1D^{1/2}\underline{X}$ is distributed as a noncentral chi-square with r_1 degrees of freedom and noncentrality

$\lambda = \frac{1}{2}\underline{\mu}_x' D^{1/2} P_1' A P_1 D^{1/2} \underline{\mu}_x = \frac{1}{2}\underline{\mu}' A \underline{\mu}$.

Conversely, suppose that $\underline{Y}' A \underline{Y} = \underline{X}' D^{1/2} P_1' A P_1 D^{1/2} \underline{X}$ is distributed as a noncentral chi-square with r_1 degrees of freedom and $\lambda = \frac{1}{2}\underline{\mu}' A \underline{\mu}$.

It follows that $D^{1/2} P_1' A P_1 D^{1/2}$ is idempotent of rank r_1. Hence

$$D^{1/2} P_1' A P_1 D^{1/2} D^{1/2} P_1' A P_1 D^{1/2} = D^{1/2} P_1' A P_1 D^{1/2}$$

$$(P_1 D^{1/2}) D^{1/2} P_1' A P_1 D^{1/2} D^{1/2} P_1' A P_1 D^{1/2} (D^{1/2} P_1') = (P_1 D^{1/2}) D^{1/2} P_1' A P_1 D^{1/2} (D^{1/2} P_1')$$

$$P_1 D^{1/2} D^{1/2} P_1' A P_1 D^{1/2} D^{1/2} P_1' A P_1 D^{1/2} D^{1/2} P_1' = P_1 D^{1/2} D^{1/2} P_1' A P_1 D^{1/2} D^{1/2} P_1' \quad , \text{ or}$$

$$VAVAV = VAV \quad ,$$

$$V(AVA - A)V = \emptyset \quad ,$$

and $\mathrm{tr}(AV) = \mathrm{tr}(D^{1/2} P_1' A P_1 D^{1/2}) = r_1$, as outlined above.

[Note: Rao (1965, page 443) proved the result stated in Theorem 3.4 for the case when $\underline{\mu} = \underline{0}$ and Rao (1966) proved the result for $(Y-\mu)A(Y-\mu)$].

It should be noted that the maximum number of degrees of freedom of a noncentral chi-square variate of the type $\underline{X}' A \underline{X}$, where $\underline{X}$ is distributed as $N_p(\underline{\mu}, V)$, $r(V) = k \le p$, is k, based on the proofs of Theorems 3.1, 3.2, 3.3 and 3.4.

The proofs of the following Theorems can be derived from the method given in Theorem 3.2.

<u>Theorem 3.5</u>: Let the random vector $\underline{Y}$ be distributed as $N_p(\underline{\mu}, V)$, where $r(V) = k \le p$. Then $\underline{Y}' A \underline{Y}$ and $\underline{Y}' B \underline{Y}$ are independent quadratic

forms if and only if $VAVBV = \emptyset$.

Theorem 3.6: If B is a q by p matrix, A is a p by p symmetric matrix, and $\underline{Y}$ is distributed as $N_p(\underline{\mu}, V)$ with $r(V) = k \leq p$, then the quadratic $\underline{Y}'A\underline{Y}$ is independent of the linear forms $B\underline{Y}$ if $BVA = \emptyset$.

Theorem 3.7: Let the random vector $\underline{Y}$ be distributed as $N_p(\underline{\mu}, V)$ with $r(V) = k \leq p$. If $\underline{Y}'A\underline{Y} = \sum_{i=1}^{m} \underline{Y}'A_i\underline{Y}$, where $tr(A_iV) = p_i$ and $tr(AV) = p$, then any one of the following conditions is necessary and sufficient that the $\underline{Y}'A_i\underline{Y}$ be independently distributed as noncentral chi-squares with p_i degrees of freedom and $\lambda_i = \frac{1}{2}\underline{\mu}'A_i\underline{\mu}$:

(1) $V(AVA - A)V = \emptyset$ and $\sum_{i=1}^{m} p_i = p$.

(2) $V(AVA - A)V = \emptyset$ and $V(A_iVA_i - A_i)V = \emptyset$, for $i = 1,2,\cdots, m$.

(3) $V(AVA - A)V = \emptyset$ and $VA_iVA_jV = \emptyset$, $i \neq j$.

4. Conclusions

It has been demonstrated that the distributional properties of quadratic forms in the multivariate singular normal distribution are essentially those of quadratic forms in the nonsingular normal distribution. Thus, with a few exceptions, the singular case may be treated as the nonsingular case, with care being exercised to meet the exact conditions of the theorems. The main conditions on distribution and independence as given in Theorems 3.4 and 3.5 reduce at once to those given in Graybill (1961), if the variance covariance matrix is nonsingular. Further, the method of proof of the theorems suggests that other distributional properties of quadratic forms in nonsingular multivariate normal variates extend also to the singular case.

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