OPTIMAL SPACING PROBLEMS

by

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Let $X_{1:1},\ldots,X_{n:n}$ denote the order statistics for a random sample from a distribution of the form $F(\frac{x-\mu}{\sigma})$, where F is a known distributional form and μ and σ are, respectively, location and scale parameters. Estimates of μ and/or σ are often obtained by using linear functions of $k < n$ order statistics. Such estimators have received considerable attention in the statistical literature primarily due to their computational simplicity, high efficiency and frequent robust behavior under departures from distributional assumptions. The loss in efficiency from using a subset of the order statistics is compensated, in many instances, by the decrease in time spent computing estimators and analyzing data. Moreover, these types of estimators are easy to use in censored samples where most other estimation techniques, such as maximum likelihood, are of a less computationally tractable nature. One particularly simple estimator that is a linear combination of k sample quantiles (and, hence, of k sample order statistics) is the asymptotically best linear unbiased estimator (ABLUE) developed by Ogawa (1951). We now discuss this estimator and the associated problem of optimal quantile (spacing) selection.

THE ABLUE

Let $Q(u) = \inf\{x : F(x) \geq u\}$, $0 < u < 1$, denote the quantile function for F and, assuming that F admits a continuous density $f = F'$, define the density-quantile function $fQ(u) = f(Q(u))$, $0 \leq u \leq 1$. Also define the sample quantile function by

$$\tilde{Q}(u) = X_{j:n}, \quad \frac{j-1}{n} < u \leq \frac{j}{n}, \quad j = 1,\ldots,n.$$

Given a spacing $U = \{u_1,\ldots,u_k\}$ (k real numbers satisfying $0 < u_1 < \cdots < u_k < 1$) it was shown by Mosteller (1946) that, provided fQ is continuous and positive at each of the u_i, the corresponding sample quantiles, $\tilde{Q}(u_1),\ldots,\tilde{Q}(u_k)$, have a k-variate normal limiting distribution with means $\mu + \sigma Q(u_i)$, $i = 1,\ldots,k$, and variance-covariance matrix composed of the elements $\sigma^2 u_i(1-u_j)/[nfQ(u_i)fQ(u_j)]$, $i \leq j$, $i,j = 1,\ldots,k$. Thus, asymptotically, the $\tilde{Q}(u_i)$ follow a linear model with known covariance structure so that asymptotically best linear unbiased estimators (ABLUE's) of μ and σ may be constructed using generalized least squares. Motivated by the work of Mosteller (1946) and Yamanouchi (1949), Ogawa (1951) developed explicit formulas for these estimators and their asymptotic relative Fisher efficiencies (ARE's) (see also Sarhan and Greenberg (1962, pgs. 47-55)). These formulae, for the various estimation situations, can be summarized as follows:

Case 1: σ known. The ABLUE of μ when σ is known is

$$\mu^*(U) = [X(U) - \sigma K_3(U)]/K_1(U) \tag{1}$$

where, taking $u_0 = 0$, $u_{k+1} = 1$ and assuming $fQ(0)=fQ(1)=fQ(0)Q(0)=fQ(1)Q(1) = 0$,

$$K_1(U) = \sum_{i=1}^{k+1} \frac{[fQ(u_i)-fQ(u_{i-1})]^2}{u_i-u_{i-1}} \tag{2}$$

$$K_3(U) = \sum_{i=1}^{k+1} \frac{[fQ(u_i)-fQ(u_{i-1})][fQ(u_i)Q(u_i)-fQ(u_{i-1})Q(u_{i-1})]}{u_i-u_{i-1}} \tag{3}$$

and

$$X(U) = \sum_{i=1}^{k+1} \frac{[fQ(u_i)-fQ(u_{i-1})][fQ(u_i)\tilde{Q}(u_i)-fQ(u_{i-1})\tilde{Q}(u_{i-1})]}{u_i-u_{i-1}} . \tag{4}$$

The ARE of $\mu^*(U)$ is given by

$$ARE(\mu^*(U)) = K_1(U)/I_{\mu\mu} , \tag{5}$$

with $I_{\mu\mu}$ denoting the Fisher information for location parameter estimation, i.e.,

$$I_{\mu\mu} = E\left[\left(\frac{f'(X)}{f(X)}\right)^2\right] = \int_0^1 [(fQ)'(u)]^2 du.$$

Case 2: μ known. The ABLUE of σ when μ is known is

$$\sigma^*(U) = [Y(U)-\mu K_3(U)]/K_2(U) \tag{6}$$

where

$$K_2(U) = \sum_{i=1}^{k+1} \frac{[fQ(u_i)Q(u_i)-fQ(u_{i-1})Q(u_{i-1})]^2}{u_i-u_{i-1}} \tag{7}$$

and

$$Y(U) = \sum_{i=1}^{k+1} [fQ(u_i)Q(u_i)-fQ(u_{i-1})Q(u_{i-1})] \cdot [fQ(u_i)\tilde{Q}(u_i)-fQ(u_{i-1})\tilde{Q}(u_{i-1})]/(u_i-u_{i-1}). \tag{8}$$

The ARE of this estimator is given by

$$ARE(\sigma^*(U)) = K_2(U)/I_{\sigma\sigma} , \tag{9}$$

with $I_{\sigma\sigma}$, the Fisher information for scale parameter estimation, defined by

$$I_{\sigma\sigma} = E\left[\left(\frac{Xf'(X)}{f(X)}\right)^2\right] - 1 = \int_0^1 [(fQ\cdot Q)'(u)]^2 \, du$$

and $fQ\cdot Q$ denotes the product of fQ and Q.

<u>Case 3: Both μ and σ unknown</u>. The ABLUE's for μ and σ are

$$\mu^*(U) = [K_2(U)X(U)-K_3(U)Y(U)]/\Delta(U) \tag{10}$$

$$\sigma^*(U) = [-K_3(U)X(U)+K_1(U)Y(U)]/\Delta(U) \tag{11}$$

where

$$\Delta(U) = K_1(U)K_2(U)-K_3(U)^2 . \tag{12}$$

The joint ARE for these estimators is

$$ARE(\mu^*(U),\sigma^*(U)) = \Delta(U)/|I(\mu,\sigma)| \tag{13}$$

with $|I(\mu,\sigma)|$ denoting the determinant of the Fisher information matrix for location and scale parameter estimation, i.e. $I(\mu,\sigma)$ is the 2×2 symmetric matrix with diagonal elements $I_{\mu\mu}$ and $I_{\sigma\sigma}$ and offdiagonal element

$$I_{\mu\sigma} = E\left[X\left(\frac{f'(X)}{f(X)}\right)^2\right] = \int_0^1 (fQ)'(u)(fQ\cdot Q)'(u)du.$$

Upon examination of the ARE expressions (5), (9) and (13) it is seen that they are all functions of the spacing U. Thus, an estimator based on optimal quantiles can be obtained by choosing U to maximize the ARE for the estimator of interest. A spacing that maximizes one of (5), (9) or (13) (or, equivalently, (2), (7) or (12)) will be termed an <u>optimal spacing</u>.

The determination of optimal spacings under various choices for F, e.g., the normal, Cauchy, etc. has been a popular area of statistical research and is the subject of the next section.

Optimal Spacing Selection

Early work on optimal spacing selection (although not explicitly discussed as such) dates back at least to papers by Sheppard (1899) and Pearson (1920) who considered the use of certain simplified estimators (that were, in fact, ABLUE's) for the mean and standard deviation of a normal distribution. Both Sheppard and Pearson examined the problem of estimating μ or σ using estimators of the form $\tilde{\mu} = \frac{1}{2}[\tilde{Q}(1-u_1)+\tilde{Q}(u_1)]$ and $\tilde{\sigma} = b[\tilde{Q}(1-u_1)-\tilde{Q}(u_1)]$ for $u_1 \leq .5$ and $b > 0$. They found the optimal value of u_1 to be approximately .27, for estimation of μ, and took $u_1 \approx .069$ with $b \approx .34$ for the estimation of σ. These values are in agreement with those later obtained by Kulldorff (1963) for optimal spacings for the normal distribution. Pearson (1920) also considered the estimation of μ and σ using ABLUE's based on four quantiles. The approximations he gave for the optimal spacings are also quite close to the exact values given by Kulldorff.

The majority of progress on the optimal spacing problem has, however, been made since the development of the ARE expressions (5), (9) and (13), that are applicable to

general F, by Ogawa (1951). As these expressions are non-linear functions of the u_i's the computation of optimal spacings has also been facilitated, in recent years, by the advent of high speed computers.

Spacings that satisfy a necessary condition for optimality may be obtained by differentiating expression (5), (9) or (13), with respect to the u_i's, and equating the resulting system of simultaneous equations to zero. The usual approach to this problem has been to examine the solutions to these equations (there may be many solutions) for a particular probability law of interest. The solution that provides the highest ARE is then taken as the optimal spacing. For location parameter estimation this procedure, for most distributions of practical interest, reduces to solving

$$2(fQ)'(u_i) - \frac{fQ(u_i)-fQ(u_{i-1})}{u_i-u_{i-1}} - \frac{fQ(u_{i+1})-fQ(u_i)}{u_{i+1}-u_i} = 0, \quad i=1,\ldots,k, \quad (14)$$

and, for scale parameter estimation, a similar necessary condition is

$$2(fQ\cdot Q)'(u_i) - \frac{fQ(u_i)Q(u_i)-fQ(u_{i-1})Q(u_{i-1})}{u_i-u_{i-1}}$$

$$- \frac{fQ(u_{i+1})Q(u_{i+1})-fQ(u_i)Q(u_i)}{u_{i+1}-u_i} = 0, \quad i=1,\ldots,k, \quad (15)$$

where $fQ\cdot Q$ denotes the product of fQ and Q. In some cases it is possible to show that (14) and (15) have unique solutions. However, even when this is not the case, their solutions provide a set of optimal spacing candidates that

may be examined to locate a spacing providing high ARE. Important early references that utilize (14) and (15) for spacing selection are Higuchi (1954), Saleh and Ali (1966), and Chan and Kabir (1969). General methods for determining when (14) and (15) are necessary conditions for an optimal spacing can be found in Cheng (1975) and Eubank, Smith and Smith (1982). In particular, if $(fQ)''$ and $(fQ\cdot Q)''$ are continuous and positive on $[0,1]$, (14) and (15) are satisfied by optimal spacings and if, in addition, $\log(fQ)''$ and $\log(fQ\cdot Q)''$ are concave on $(0,1)$ these equation systems have unique solutions for each k (see Eubank, Smith and Smith (1982)). For simultaneous estimation of μ and σ the maximization of (13) is usually mathematically, and frequently even numerically, intractable. A notable exception is the Cauchy distribution for which the optimal spacing consists of the uniformly spaced points $i/k+1$, $i=1,\ldots,k$.

For a discussion of some of the early work on optimal spacing selection see Harter (1971) and Johnson and Kotz (1970a,b). A bibliography containing many of the more recent references is provided in Eubank (1982).

Approximate solutions to the problem of optimal spacing selection that are based on spacings generated by density functions on $[0,1]$ have been considered by Särndal (1962) and Eubank (1981). Assuming that $(fQ)''$ and $(fQ\cdot Q)''$ are continuous let $\psi(u) = ((fQ)''(u),(fQ\cdot Q)''(u))^t$; then, Eubank (1981) showed that asymptotically (as $k \to \infty$) optimal spacings are

provided by the (k+1)-tiles of the densities

$$h(u)=\begin{cases} |(fQ)''(u)|^{2/3}/\int_0^1 |(fQ)''(s)|^{2/3}ds, & \sigma \text{ known,} \\ |(fQ\cdot Q)''(u)|^{2/3}/\int_0^1 |(fQ\cdot Q)''(s)|^{2/3}ds, & \mu \text{ known,} \\ [\psi(u)^t I(\mu,\sigma)^{-1}\psi(u)]^{1/3}/\int_0^1 [\psi(s)^t I(\mu,\sigma)^{-1}\psi(s)]^{1/3}ds, & \text{both } \mu \text{ and } \sigma \text{ unknown.} \end{cases} \tag{16}$$

Letting H^{-1} denote the quantile function for h in the parameter estimation problem of interest, an approximate solution is then provided by the spacing $U_k=\{H^{-1}(\frac{1}{k+1}),\ldots,H^{-1}(\frac{k}{k+1})\}$. Examples of these solutions are given in Table 1.

This approach provides spacings that are optimal in an asymptotic (as $k \to \infty$) sense. For instance, if $\{U_k\}$ denotes the sequence of spacings chosen from the density proportional to $|(fQ)''(u)|^{2/3}$, by successively increasing k, and $\{U_k^*\}$ is a corresponding sequence of optimal spacings, then

$$\lim_{k\to\infty} \frac{1-\mathrm{ARE}(\mu^*(U_k))}{1-\mathrm{ARE}(\mu^*(U_k^*))} = 1.$$

This has the interpretation that the sequence $\{U_k\}$ and $\{U_k^*\}$ have identical asymptotic properties, in terms of their ARE behavior, and suggests that, for large k, U_k might be used as a computationally expedient alternative to the optimal spacing U_k^*. Similar results hold for the other estimation situations. The approximate solutions provided by the densities in (16) have been found to work surprisingly well

Table 1. Asymptotically optimal spacings for various distributions

Distribution	Unknown Parameter(s)	$H^{-1}(i/k+1)$
Cauchy	μ and σ	$i/k+1$
Exponential	σ	$1-(1-i/k+1)^3$
Extreme Value (largest value)	μ	$(i/k+1)^3$
Logistic	μ	$i/k+1$
Normal ($F=\Phi$)	μ	$\Phi(\sqrt{3}\ \Phi^{-1}(i/k+1))$
Pareto ($F(x) = 1-(1+x)^{-\nu}$, $x,\nu > 0$)	σ	$1-(1-i/k+1)^{3\nu/2+\nu}$

even for k as small as 7 or 9. For example, when k=7 the optimal spacing for the estimation of the scale parameter of the exponential distribution is given by Sarhan, Greenberg, and Ogawa (1963) as $U_7^* = \{.3121, .5513, .7277, .8506, .9297, .9746, .9948\}$ with a corresponding ARE of .969. In contrast, the approximate solution U_7, obtained from Table 1, consists of the points $1-(1-i/8)^3$ and provides an ARE of .958. Consequently, the loss in ARE from using the approximate solution (16) is, in this case, only .011. Such comparisons for the other laws in Table 1 provide analogous results. See Eubank (1981) for further comparisons and details regarding the computation and computational savings available from the use of the densities in (16).

The spacing densities in (16), or equivalently their H^{-1} functions, may be viewed as describing (asymptotically) the areas of concentration of the optimal spacings for a distribution and, hence, are useful for the purpose of comparison between distributions. There are at least three common shapes for the H^{-1} functions, namely i) uniform ($H^{-1}(u)=u$), such as for the logistic (σ known) and Pareto (μ known) when $\nu=1$, ii) skewed ($H^{-1}(u)$ frequently behaves like u^3) e.g., the exponential (μ known) and extreme value (σ known) and iii) symmetric ($H^{-1}(u)=H^{-1}(1-u)$) as illustrated by the normal distribution. In contrast to uniform or symmetric shaped spacing densities, those which are skewed

indicate that, for large k, estimators will be composed of quantiles corresponding to predominantly large or small percentage values. Thus, for instance, since $H^{-1}(u)=u^3$ for the extreme value, it follows that estimators of μ for the extreme value distribution will be based predominantly on data values that are below the median, for k sufficiently large.

For many problems, such as those arising from the study of survival data, the object of interest is the p-th population percentile for some $0 < p < 1$. Under the assumed location and scale parameter model the p-th percentile is $\mu+\sigma Q(p)$ for which an asymptotically best linear unbiased estimator is $\mu^*(U)+\sigma^*(U)Q(p)$. This estimator has asymptotic (as $n \to \infty$) variance

$$\sigma^2[K_2(U)+Q(p)^2K_1(U)-2Q(p)K_3(U)]/n\Delta(U)$$

which may be minimized as a function of U to obtain spacings that are optimal for percentile estimation. Optimal spacing selection, in this setting, is discussed by Ali, Umbach Saleh, and Hassanein (1983). A general approximate solution, similar to those discussed previously, is given in Eubank (1981).

Tests regarding certain hypotheses about μ and σ that are based on the ABLUE's have also been developed. When both

μ and σ are unknown a test of $H_0:\mu=\mu_0$ against $H_a:\mu\neq\mu_0$ can be conducted using the statistic

$$\sqrt{K_1(U)}(\mu^*(U)-\mu_0)/\sqrt{\Omega_1(U)/(k-2)} \tag{17}$$

where, for a symmetric spacing (i.e., $u_{k-i+1}=1-u_i$),

$$\Omega_1(U)=S(U)-[K_1(U)+K_2(U)]\mu^*(U)^2$$

and

$$S(U) = \sum_{i=1}^{k+1} \frac{[fQ(u_i)\tilde{Q}(u_i)-fQ(u_{i-1})\tilde{Q}(u_{i-1})]^2}{u_i-u_{i-1}}$$

(c.f. Ogawa (1951) or Sarhan and Greenberg (1962, pgs. 291-299)). This statistics has a Student's t limiting distribution with k-2 degrees of freedom, when H_0 is true, and an asymptotic noncentral t-distribution, under H_a, with noncentrality $\sqrt{K_1(U)}(\mu-\mu_0)/\sigma$. Thus, the power of this test is an increasing function of $K_1(U)$ and, consequently, optimal spacings, for testing purposes, may be obtained by maximizing $K_1(U)$ over all symmetric spacings. In some (although not all) cases this is equivalent to optimal spacing selection for $\mu^*(U)$. A test for $H_0:\sigma=\sigma_0$ against $H_a:\sigma\neq\sigma_0$, when μ is known and, therefore, taken as zero, can be obtained from

$$\sqrt{K_2(U)}(\sigma^*(U)-\sigma_0)/\sqrt{\Omega_2(U)/(k-1)} \tag{18}$$

where $\Omega_2(U)=S(U)-K_2(U)\sigma^*(U)^2$ (c.f. Sarhan and Greenberg (1962, pgs. 380-382)). This statistic also has an asymptotic t-distribution under H_0, except with k-1 degrees of freedom, and has (asymptotically) a noncentral t-distribution under H_a with noncentrality $\sqrt{K_2(U)}(\sigma-\sigma_0)/\sigma$. As the power of the

test increases with $K_2(U)$, it follows that selecting optimal spacings for testing hypotheses about σ and for the estimation of σ are equivalent problems. The use of (17) and (18) with the Cauchy, logistic and normal distributions has been considered by Chan and Cheng (1971) and Chan, Cheng, Mead and Panjer (1973) who have also investigated the problem of spacing selection and some small sample behavior. Similar results for (18), are given by Ogawa (1951) (see also Sarhan and Greenberg (1962, pgs. 380-382)), Chan, Cheng and Mead (1972), and Cheng (1980) for the exponential, extreme value and Rayleigh distributions respectively.

In previous discussions it has been assumed that the quantiles utilized in estimation were selected from complete samples. However, the ABLUE is also readily adapted for use with censored samples. For instance, if the sample is censored from the left, with censoring proportion α, then this may be viewed as observing $\tilde{Q}(u)$ only on $[\alpha,1]$. Consequently, the ABLUE may be computed as before except now the spacing must satisfy $\alpha \leq u_1 < u_2 < \cdots < u_k < 1$. Optimal spacings are then obtained by maximizing (5), (9) or (13) over spacings of this form. Similar comments hold for right and both left and right censored samples. Most of the previous results, such as the necessary conditions (14) and (15) and the approximate solutions (16) are found to also hold for censored samples after appropriate modifications (c.f. Cheng (1975) and Eubank (1981)). References

containing the optimal spacings for censored samples from various probability distributions can also be found in Eubank (1982). A closely related problem where the ABLUE is used to predict future observations in what may be viewed as right censored samples is discussed by Kaminsky and Nelson (1975).

The distributional form, F, is not always known in practice. Frequently, our knowledge is only sufficient to restrict attention to several possible candidates for the underlying probability law. In such instances it may be advantageous to use spacings that are robust relative to the various models that are being considered. One measure of robustness for a spacing is its guaranteed ARE(GARE), i.e., its minimum ARE over all the candidates for F. An approach to robust spacing selection for location parameter estimation that is based on GARE has been developed by Chan and Rhodin (1980) (see also Eubank (1983)).

Related Problems

The problem of optimal spacing selection for the ABLUE's is closely related to a variety of other statistical problems. In the one parameter case, the problem of optimal spacing selection is equivalent to i) optimal grouping for maximum likelihood estimation of μ or σ from grouped data (Kulldorff (1961)), ii) optimal grouping for the asymptotically most-powerful group rank tests for the two sample location and scale

problems (Gastwirth (1966)), and iii) regression design selection for time series models with regression function fQ or fQ•Q and Brownian bridge error (Eubank (1981)). In addition, there is a structural similarity between these problems and i) optimal strata selection with proportional allocation (Dalenius (1950)), ii) problems of optimal grouping considered by Cox (1957), Rade (1963) and Ekman (1969) and iii) certain problems of optimal grouping for chi-squared tests of homogeneity and for multivariate distributions (Bofinger (1970, 1975) and Hung and Kshirsagar (1984)).

The connections and relationships between these problems have been examined by Adatia and Chan (1982) and Eubank (1983). Under appropriate restrictions all these problems are equivalent for normal and gamma distributions.

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This article is an overview of the problem of optimal quantile (spacing) selection for estimation of location and/or scale parameters by the asymptotically best linear unbiased estimators. The problems of optimal spacing selection for tests of hypotheses and quantile estimation are also discussed. Connections with other related problems are indicated.

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