

COMPARISONS IN A CLASS OF APPROXIMATE F-TESTS

by

James M. Davenport

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DEPARTMENT OF STATISTICS
Southern Methodist University

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CHAPTER I

INTRODUCTION

In statistical methods, it is often desirable to test the general null hypothesis that the relation

$$C_{\theta}^1 + \dots + C_{\theta}^m = C_{\theta}^{m+1} + \dots + C_{\theta}^s \quad (1)$$

holds among the variances θ_i , where the C_i are known real numbers. An

assumption which is essential in deriving the results in the sequel for testing relation (1), is that there exist a number of independent estimates V_i of each of the variances θ_i , respectively, ($i = 1, 2, \dots, s$). We assume that each estimate V_i is based on n_i degrees of freedom, and

$n_i V_i / \theta_i$ follows the chi-square distribution with n_i degrees of freedom. Additional assumptions are that both $\sum_{i=1}^m C_i \theta_i > 0$ and $\sum_{i=m+1}^s C_i \theta_i > 0$; when the null hypothesis is true and also when the alternative hypothesis is true.

This problem has been encountered several times in the application of statistical methods, especially in the analysis of variance. Several methods are well known for testing some particular cases of relation (1). These are as follows:

1. The F-test. This tests the relation $\theta_1 = \theta_2$.
2. Bartlett's (1937) test of homogeneity of variances. This tests the series of $k-1$ relations $\theta_1 = \theta_2 = \dots = \theta_k$.

3. The two-tailed Behrens-Fisher problem.

This tests the relation that

$$\theta_3 = \theta_1 + \theta_2 \quad (2)$$

holds among the variances, as is shown

by Cochran (1951).

4. Satterthwaite's (1941 and 1946) F' -statistic,

which is an approximate F -test.

An estimate of a general linear combination of variances such as

$$\sum_{j=1}^k C_j \theta_j \quad \text{is } \sum_{j=1}^k C_j V_j, \quad \text{where } V_j \text{ is the independent mean square estimate of } \theta_j \quad (j = 1, 2, \dots, k).$$

The exact distributions of such linear combinations of independent

χ^2 distributed random variables have been derived by several authors,

e.g., see Hsu (1938), Robbins and Pittman (1949), Box (1954), Ruben (1963)

and Wang (1967). Such distributions are too complicated for tabulation

and practical use. Therefore, approximations to these distributions

are used.

Smith (1936) has proposed using $F'_1 = V_3 / (V_1 + V_2)$ which follows

approximately an F -distribution with n_3 degrees of freedom in the numer-

ator and the denominator degrees of freedom estimated by

$$(V_1 + V_2)^2 / (V_1^2/n_1 + V_2^2/n_2), \quad \text{for testing the relation (2) versus the}$$

alternative that $\theta_3 > \theta_1 + \theta_2$.

Satterthwaite (1941 and 1946) later generalized the technique to

the general linear relation among variances, such as relation (1). The

test criterion is

$$F' = \frac{C_{1V1} + C_{2V2} + \dots + C_{mVm}}{C_{m+1V} + \dots + C_{sVs}} \quad (3)$$

When relation (1) is true, the null distribution of F' follows an F -

distribution, approximately. The degrees of freedom of F' , F_1 and F_2 ,

are found by the moment fitting method and are given by

$$F_1 = \frac{(C_{\theta}^1 + \dots + C_{\theta}^m) \frac{n_1}{2}}{(C_{\theta}^{m+1} + \dots + C_{\theta}^s) \frac{n}{2}}; \quad F_2 = \frac{(C_{\theta}^{m+1} + \dots + C_{\theta}^s) \frac{n}{2}}{(C_{\theta}^1 + \dots + C_{\theta}^m) \frac{n}{2}} \quad (4)$$

Since F_1 and F_2 involve the unknown parameters θ_i ($i = 1, 2, \dots, s$),

they must be estimated, and their estimates are given by

$$\hat{F}_1 = \frac{(C_V^1 + \dots + C_V^m) \frac{n_1}{2}}{(C_V^{m+1} + \dots + C_V^s) \frac{n}{2}}; \quad \hat{F}_2 = \frac{(C_V^{m+1} + \dots + C_V^s) \frac{n}{2}}{(C_V^1 + \dots + C_V^m) \frac{n_1}{2}} \quad (5)$$

The test procedure is to compute the degrees of freedom using the data

and then find the corresponding upper α -level critical point from an

F -distribution with v_1 and v_2 degrees of freedom, i.e., $F_{\alpha}(v_1, v_2)$. The null hypothesis is then rejected if $F' > F_{\alpha}(v_1, v_2)$ and is not rejected otherwise. The probability of the type I error is assumed to be α .

In general, the alternative hypothesis will be that the left side

of relation (1) will be greater than the right side of the relation.

However, if the alternative hypothesis is two-sided, the numerator of F'

will be whichever of the estimates of the variance is larger, so that

the alternative will again be one-sided, as above. The resulting prob-

ability will be doubled.

Note that v_1 and v_2 are not dependent on all of the v_i 's. That

is, v_1 and v_2 are invariant if the numerator and denominator of each

$$\frac{\frac{s_u}{s_{(C)}} + \dots + \frac{\tau_u^{w_u}}{s_{\Delta/\tau_{\Delta}^{w_u}} \tau_{(C)}^{w_u}}}{\frac{s_{(C)}}{z} + \dots + \frac{s_{\Delta/\tau_{\Delta}^{w_u}} \tau_{(C)}^{w_u}}{z}} = z_{\Delta} \quad ; \quad \frac{\frac{w_u}{w_{(C)}} + \dots + \frac{\tau_u}{w_{\Delta/\tau_{\Delta}^{w_u}} \tau_{(C)}}}{\frac{w_{(C)}}{z} + \dots + \frac{w_{\Delta/\tau_{\Delta}^{w_u}} \tau_{(C)}}{z}} = \tau_{\Delta}$$

or, letting $x_1 = 1/v^m$, $x_2 = 1/v^{m-1}$, ..., $x_m = 1/v$, and $x_{m+1} = 1/v^m$, we have

(1) $i_{m+1} = i, \dots, i_{s-1}, i_s = i$, it follows that

$$\frac{\frac{s_u}{z^{(C)}} + \dots + \frac{\tau_u}{z^{(C)} X^w}}{\frac{s_u}{z^{(C)}} + \dots + \frac{\tau_u}{z^{(C)} X^w}} = z_u, \quad \frac{\frac{w_u}{z^{(C)}} + \dots + \frac{\tau_u}{z^{(C)} X^w}}{\frac{w_u}{z^{(C)}} + \dots + \frac{\tau_u}{z^{(C)} X^w}} = \tau_u$$

Therefore, $F_{\alpha}(\nu^{I_2})$ is a random variable depending on the $(s-2)$ stochastic variables $x_1, \dots, x_{s-2}$. Let $g(x_1, \dots, x_{s-2})$ be the joint density function of these random variables and $G(x_1, \dots, x_{s-2})$ be the distribution function. Then Z_{α} , where

$$(9) \quad \left| \begin{matrix} \mathbb{I}_X & \dots & X^{-2} \end{matrix} \right|_{\mathbb{F}} \leq \left| \begin{matrix} \mathbb{I}_X & \dots & X^{-2} \end{matrix} \right|_{\mathbb{F}'} \leq \left| \begin{matrix} \mathbb{I}_X & \dots & X^{-2} \end{matrix} \right|_{\mathbb{F}''} = \alpha_Z$$

It is a random variable, with its non-central moments given by

$$(L) \quad \cdot \quad (z^{-s_x}, \dots, 1_x) \otimes_{\mathbb{Z}} \int \dots \int = \int_{\mathbb{Z}} \mathbb{E}$$

Methods of performing the integration of equation (7) are developed in Chapter II. These methods are generalizations of a method developed by Cochran (1951).

• (1951) Cochrane

If $r = 1$ in equation (7), then $E(Z^0)$ can be considered the mean or

average performance of the F' approximate test, i.e., the true probability

of the type I error of the test procedure. $E(Z^{\alpha}_T)$ gives the higher moments

of the test procedure

Cochran (1951) investigated the accuracy of F^*_1 for testing relation (2) and the power of the test, while Hudson and Krutchkoff (1968) used a Monte Carlo procedure to investigate the approximation of Satterthwaite's approximate F-statistic for testing $\theta_1 + \theta_2 = \theta_3 + \theta_4$ and $\theta_1 = \theta_2 + \theta_3 - \theta_4$. Davenport (1969) and Davenport and Webster (1971), investigate F^*_1 thoroughly. These articles deal only with $E(Z^{\alpha})$.

An alternative approach to the derivation of equation (7) when

$n = 1$ is as follows. Let A represent the event that $F' > F^{\alpha}_{(1, \nu_1, \nu_2)}$. The conditional probability of A, given $X_1, \dots, X_{s-2}$, is given by

$$P(A|X_1, \dots, X_{s-2}) \quad (8)$$

The unconditional probability of the event A is then

$$P(A) = \int \dots \int P(A|X_1, \dots, X_{s-2}) d(x_1, \dots, x_{s-2}) \quad (9)$$

which is the same as equation (7) with $r = 1$. Hence, $E(Z^{\alpha})$ can also be considered the unconditional probability that F' will exceed the critical point attributed to it by the test procedure.

The most common occurrences of relation (1) are when the $C^*_i = 1$ ($i = 1, 2, \dots, s$). Such is the case for relation (2), and such is the case when testing the main effect variance component in a three factor factorial random effect model (see Chapter IV).

Of special interest in this paper are the following four relations together with their respective alternatives:

H_a

(10)

$$\theta_1 + \theta_2 > \theta_3 + \theta_4$$

(11)

$$\theta_1 - \theta_2 > \theta_3 - \theta_4$$

(12)

$$\theta_1 > \theta_2 + \theta_3 - \theta_4$$

(13)

$$\theta_2 + \theta_3 - \theta_4 > \theta_1$$

 H_o

$$\theta_1 + \theta_2 = \theta_3 + \theta_4$$

$$\theta_1 - \theta_2 = \theta_3 - \theta_4$$

$$\theta_1 = \theta_2 + \theta_3 - \theta_4$$

$$\theta_2 + \theta_3 - \theta_4 = \theta_1$$

Associated with each of the above relations are the following respective approximate test statistics, which are special cases of equation (3):

(14)

$$F'_2 = \frac{\frac{1}{V_1} + \frac{1}{V_2}}{\frac{1}{V_3} + \frac{1}{V_4}}$$

(15)

$$F'_3 = \frac{\frac{1}{V_1} - \frac{1}{V_2}}{\frac{1}{V_3} - \frac{1}{V_4}}$$

(16)

$$F'_4 = \frac{\frac{1}{V_1}}{\frac{1}{V_2} + \frac{1}{V_3} - \frac{1}{V_4}}$$

(17)

$$F'_5 = \frac{\frac{1}{V_2} + \frac{1}{V_3} - \frac{1}{V_4}}{\frac{1}{V_1}}$$

In addition, associated with each approximate test statistic above are, respectively, the random variables $Z_{\alpha,2}$, $Z_{\alpha,3}$, $Z_{\alpha,4}$, and $Z_{\alpha,5}$ where each is the appropriate case of equation (6).

In Chapter II, methods of obtaining $E(Z_{\alpha,i}^2)$, ($i = 2, 3, 4, 5$), are derived. Chapter III is devoted to the investigation of the accuracy of the integration technique for $E(Z_{\alpha,2}^2)$. The equations for $E(Z_{\alpha,i}^2)$, differ little for different values of i ; therefore, it is felt that the accuracy

of only one of them warrants investigation. In Chapter IV, the methods of Chapter II are then applied to the testing of main effect variance components in the analysis of variance for certain three factor factorial models. The four tests given by equations (14), (15), (16), and (17) are then compared and the "best" is chosen on the basis of which is closest to the nominal α and the power function. Chapter V is devoted to the further investigations into the behavior of the test procedure using F_2^2 .

GENERALIZATIONS OF COCHRAN'S METHOD

CHAPTER II

The purpose of this chapter is to derive methods to evaluate $E(Z_{r,1}^2)$ ($r = 2, 3, 4, 5$). Consider relation (10) with its associated test statistic

$$F'_2 = (v_1 + v_2)/(v_3 + v_4) \quad . \quad \text{The degrees of freedom are given by}$$

$$v_1 = \frac{(v_1 + v_2)^2}{\frac{n_1}{v_2} + \frac{n_2}{v_2} + \frac{n_1}{n_2} + \frac{n_2}{1}} \quad (18)$$

and

$$v_2 = \frac{(v_3 + v_4)^2}{(1 + w)^2} = \frac{\frac{v_3}{n_4} + \frac{v_4}{n_4}}{\frac{w}{n_4} + \frac{1}{n_4}} \quad (19)$$

where $u = v_1/v_2$ and $w = v_3/v_4$. Hence, equation (7) can be written in the following form, when $i = 2$:

$$E(Z_{r,2}^2) = \int_{-\infty}^0 \int_{-\infty}^0 P[F'_2 > F_{\alpha(v_1, v_2)} | u, w] g(u, w) du dw, \quad (20)$$

where $g(u, w)$ is the joint p.d.f. of the random variables u and w . Note that u and w are independent random variables.

Equation (20) can be made more tractable by the following method.

It is well known that

$$\bar{Q} = \frac{(n_3 + n_4) [n_1 v_1 / \theta_1 + n_2 v_2 / \theta_2]}{(n_1 + n_2) [n_3 v_3 / \theta_3 + n_4 v_4 / \theta_4]} \quad (21)$$

is distributed as a central F-distribution with $(n_1 + n_2)$ and $(n_3 + n_4)$

degrees of freedom, and that $\bar{Q}$ is independent of u and w . Then the conditional distribution of F'_1 given u and w is related to the distribution

of $\bar{Q}$ as follows:

$$F'_1 = \bar{Q} \cdot \frac{(v_1 + v_2) (n_1 + n_2) [n_3 v_3 / \theta_3 + n_4 v_4 / \theta_4] (\theta_1 + \theta_2) (\theta_3 + \theta_4)}{(v_3 + v_4) (n_3 + n_4) [n_1 v_1 / \theta_1 + n_2 v_2 / \theta_2] (\theta_1 + \theta_2) (\theta_3 + \theta_4)}$$

$$= \bar{Q} \cdot \frac{(1 + w) (n_3 + n_4) [n_1 u / \theta_1 + n_2 u / \theta_2] (\theta_1 + \theta_2) (\theta_3 + \theta_4)}{(1 + u) (n_1 + n_2) [n_3 w / \theta_3 + n_4 w / \theta_4] (\theta_1 + \theta_2) (\theta_3 + \theta_4)}$$

$$= \bar{Q} \cdot \phi_2 / K_2, \quad (22)$$

where $\phi_2 = (\theta_1 + \theta_2) / (\theta_3 + \theta_4)$, $u = \theta_1 / \theta_2$, $w = \theta_3 / \theta_4$ and

$$K_2 = \frac{(1 + w) (n_3 + n_4) [n_1 u / \theta_1 + n_2 u / \theta_2]}{(1 + u) (n_1 + n_2) [n_3 w / \theta_3 + n_4 w / \theta_4]} \quad (23)$$

Hence, substituting equation (22) into equation (20) yields

$$E(Z_{\alpha, 2}^2) = \int_0^\infty \int_{-\infty}^\infty P\left[\bar{Q} > \frac{\phi_2}{K_2 F_{\alpha}(v_1 v_2)} \mid u, w\right] g(u, w) du dw \quad (24)$$

A change of variable will be applied to equation (24) since the range of integration is unbounded. Consider the following transformation:

$$(25) \quad s = \frac{n_1 u / n_2}{1 + n_1 u / n_2} \quad \text{and} \quad t = \frac{n_3 w / n_4}{1 + n_3 w / n_4}$$

Then equation (24) becomes

$$(26) \quad E(Z_{\alpha, 2}^r) = \int_0^1 \int_0^1 P \left[\tilde{Q} > \frac{K_{EF}^{2\alpha} (v_1^{*2})}{\phi_2} \mid s, t \right] h(s, t) ds dt$$

where

$$(27) \quad v_1^* = \frac{\left(1 + \frac{n_2 s}{n_1 (1-s)} \right)^2}{\frac{(n_2 s)^2}{2} + \frac{n_3 (1-s)}{2} + \frac{n_2}{1}}$$

$$(28) \quad v_2^* = \frac{\left(1 + \frac{n_4 t}{n_3 (1-t)} \right)^2}{\frac{(n_4 t)^2}{2} + \frac{n_1 (1-t)}{2} + \frac{n_4}{1}}$$

$$(29) \quad K_2^* = \frac{(n_3 + n_4) \left(1 + \frac{n_4 t}{n_3 (1-t)} \right) \left[\frac{n_2 s}{2} + \frac{n_3 (1-s)}{2} + n_2 \right]}{(n_1 + n_2) \left(1 + \frac{n_2 s}{n_1 (1-s)} \right) \left[\frac{n_4 t}{2} + \frac{n_1 (1-t)}{2} + n_4 \right]}$$

and where $h(s, t)$ is the joint density of the random variables s and t . Note that u, w , and ϕ_2 are unchanged. The density functions $g(u, w)$ and $h(s, t)$ are derived in Appendix I.

Relation (10) can be equivalently stated as $H_0: \phi_2 = 1$ versus

$H_a: \phi_2 > 1$, where $\phi_2 = (\theta_1 + \theta_2)/(\theta_3 + \theta_4)$. Therefore, letting $\phi_2 = 1$

in equation (26) gives a method to evaluate $E(Z_{\alpha,2}^T)$ for all values of

r , when the null hypothesis is true. Likewise, for values of $\phi_2 > 1$,

equation (26) yields the power function of the test.

Equation (26) is the form of the equation for $E(Z_{\alpha,2}^T)$ that is

numerically integrated in Chapter III and Chapter IV.

When testing relation (11) using $F'_3 = (V_1 - V_2)/(V_3 - V_4)$, the

derivation of an equation to evaluate $E(Z_{\alpha,3}^T)$ exactly parallels that of

$E(Z_{\alpha,2}^T)$, with the following changes.

Equations (18) and (19) become

$$V_1 = \frac{(V_1 - V_2)^2}{(u - 1)^2} = \frac{\frac{u_1}{V_1^2} + \frac{u_2}{V_2^2}}{\frac{u_1}{V_1^2} + \frac{u_2}{V_2^2} + \frac{1}{V_2^2}} \quad (30)$$

$$V_2 = \frac{(V_3 - V_4)^2}{(w - 1)^2} = \frac{\frac{u_3}{V_3^2} + \frac{u_4}{V_4^2}}{\frac{u_3}{V_3^2} + \frac{u_4}{V_4^2} + \frac{1}{V_4^2}} \quad (31)$$

where, again, $u = V_1/V_2$ and $w = V_3/V_4$. Equations (22) and (23) are

changed as follows:

$$F'_3 = \phi_3 \cdot \phi_3/K_3 \quad ,$$

where $\phi_3 = (\theta_1 - \theta_2)/(\theta_3 - \theta_4)$ and

$$K_3 = \frac{(u_3 + u_4)(w - 1)(u - 1)[n_1 u/u + n_2]}{(u_1 + u_2)(u - 1)(w - 1)[n_3 w/w + n_4]} .$$

The quantities U , W , and $\tilde{O}$ are unchanged. Note that K_3 may assume negative values. Hence, equation (24) becomes

$$E(Z_r^{\alpha, \beta}) = \int \int_{[K_3 > 0]} \left(P \left[\tilde{O} > \frac{K_{3F}^{\alpha} (v_1 v_2)}{\phi_3} \mid u, w \right] g(u, w) dudw \right. \\ \left. + \int \int_{[K_3 < 0]} \left(P \left[\tilde{O} > \frac{K_{3F}^{\alpha} (v_1 v_2)}{\phi_3} \mid u, w \right] g(u, w) dudw \right) \right) \quad (32)$$

where v_1 and v_2 are now given by equations (30) and (31). The second term in the sum in equation (32) is zero, and this term corresponds to F_3 being negative.

Now the transformation given in (25) is applied to (32) with the

following results:

$$E(Z_r^{\alpha, \beta}) = \int \int_{[K_3 > 0]} \left(P \left[\tilde{O} > \frac{K_{3F}^{\alpha} (v_1 v_2)}{\phi_3} \mid s, t \right] h(s, t) dsdt \right) \quad (33)$$

where

$$v_1^* = \frac{(n_2^2 s) \frac{n_1^1 (1-s)}{2} + \frac{n_3^1 (1-s)}{2}}{\left(\frac{n_2^2 s}{n_1^1 (1-s)} - 1 \right) \frac{n_3^1 (1-s)}{2} + \frac{n_4^1 (1-s)}{2}},$$

$$v_2^* = \frac{(n_4^2 t) \frac{n_3^1 (1-t)}{2} + \frac{n_4^1 (1-t)}{2}}{\left(\frac{n_4^2 t}{n_3^1 (1-t)} - 1 \right) \frac{n_3^1 (1-t)}{2} + \frac{n_4^1 (1-t)}{2}},$$

and

random variables. When $i = 4$, equation (7) can be written

where $x = v_2/v_4$ and $y = v_3/v_4$. Note that x and y are not independent

$$n_2 = \frac{\frac{n_2}{v_2} + \frac{n_3}{v_3} + \frac{n_4}{v_4}}{(v_2 + v_3 - v_4)^2} = \frac{\frac{n_2}{x} + \frac{n_3}{y} + \frac{n_4}{1}}{(x + y - 1)^2} \quad (36)$$

$$n_1 = n_1 \quad (35)$$

by

formulae. For $F'_4 = v_1/(v_2 + v_3 - v_4)$, the degrees of freedom are given

relation (12) using F'_4 is slightly different from the two previous

The derivation of a formula to calculate $E(Z_{\alpha,4}^2)$ for testing

hypotheses by numerical integration.

a method to calculate values for $E(Z_{\alpha,3}^2)$ under the null and alternative

$t = 1$ occurs with probability zero. Consequently, equation (33) offers

(34). Also s and t are continuous random variables; hence, $s = 1$ and/or

and $(\theta_3 - \theta_4) > 0$. Therefore neither u nor w can equal one in equation

such as that in (11), are strictly greater than zero, i.e., $(\theta_1 - \theta_2) > 0$

possible since it was assumed in Chapter I that both sides of relations,

$H_a: \phi_3 > 1$, where $\phi_3 = (\theta_1 - \theta_2)/(\theta_3 - \theta_4)$. Division by zero is not

Observe that relation (11) can be stated as $H_0: \phi_3 = 1$ versus

$$K_3^* = \frac{(n_1 + n_2) \left(\frac{n_1}{n_2 s} - 1 \right) (w - 1) \left[\frac{n_1}{n_2 t} \frac{w(1-t)}{w(1-t)} + n_4 \right]}{(n_3 + n_4) \left(\frac{n_3}{n_4 t} - 1 \right) (u - 1) \left[\frac{n_3}{n_4 s} \frac{u(1-s)}{u(1-s)} + n_2 \right]} \quad (34)$$

$$E(Z^{\alpha, q}_r) = \int_{-\infty}^0 \int_{-\infty}^0 P\left[P_{F_1^q}^{\alpha} > P_{F_1^q}^{\alpha}(n_1 n_2) \mid x, y\right] k(x, y) dx dy, \quad (37)$$

where $k(x, y)$ is the joint p.d.f. of x and y .

The integration can be simplified by the following method. It is

well known that

$$H = \frac{(n_2 + n_3 + n_4^q)^{n_1^q} \left[\frac{n_2^q}{n_3^q} + \frac{n_3^q}{n_4^q} + \frac{n_4^q}{n_1^q} \right]}{(n_2 + n_3 + n_4^q)^{n_1^q}}$$

is distributed as a central F-distribution with n_1 and $(n_2 + n_3 + n_4^q)$

degrees of freedom, and that H is independent of x and y . Hence, it

follows that

$$F_1^q = H \cdot \frac{(n_2 + n_3 + n_4^q)^{n_1^q} (x + y - 1)}{(x + y - 1) \left[\frac{n_2^q}{n_3^q} + \frac{n_3^q}{n_4^q} + \frac{n_4^q}{n_1^q} \right]}$$

$$= H \cdot \phi^q / K^q, \quad (38)$$

where $\phi^q = \theta_1 / (\theta_2 + \theta_3 - \theta_4^q)$, $x = \theta_2 / \theta_4^q$, $y = \theta_3 / \theta_4^q$, and

$$K^q = \frac{(n_2 + n_3 + n_4^q)^{n_1^q} (x + y - 1)}{(n_2^q + n_3^q + n_4^q) \left[\frac{n_2^q}{n_3^q} + \frac{n_3^q}{n_4^q} + \frac{n_4^q}{n_1^q} \right]} \quad (39)$$

Again, note that K^q may be negative; hence, the integral over the region where $K^q \leq 0$, is zero. Substituting equation (38) into equation (37)

yields

$$E(Z_r^{\alpha, \bar{q}}) = \int_{[0, \infty]} \int_{[0, \infty]} P \left[H > \frac{\phi_{\bar{q}}}{K_F^{\bar{q}} \alpha(n_1 n_2)} \mid x, y \right] k(x, y) dx dy \quad (40)$$

The range of integration can be changed to the unit square by the trans-

formation

$$p = \frac{\frac{n_2}{n_4} x}{1 + \frac{n_2}{n_4} x}, \quad q = \frac{\frac{n_3}{n_4} y}{1 + \frac{n_3}{n_4} y} \quad (41)$$

Equation (40) then becomes

$$E(Z_r^{\alpha, \bar{q}}) = \int_{[0, \infty]} \int_{[0, \infty]} P \left[H > \frac{\phi_{\bar{q}}}{K_F^{\bar{q}} \alpha(n_1 n_2)} \mid p, q \right] f(p, q) dp dq \quad (42)$$

where $f(p, q)$ is the joint p.d.f. of p and q , and

$$n_1^* = n_1$$

$$n_2^* = \frac{\frac{(n_2^{\bar{q}} p)^2}{2} + \frac{(n_3^{\bar{q}} (1-p))^2}{2}}{\frac{(n_2^{\bar{q}} p)^2}{2} + \frac{(n_3^{\bar{q}} (1-p))^2}{2} + \frac{(n_2^{\bar{q}} p)^2}{2} + \frac{(n_3^{\bar{q}} (1-p))^2}{2}}$$

$$K_{\bar{q}}^* = \frac{(n_2^{\bar{q}} p)^2 + (n_3^{\bar{q}} (1-p))^2}{\frac{(n_2^{\bar{q}} p)^2}{2} + \frac{(n_3^{\bar{q}} (1-p))^2}{2} + \frac{(n_2^{\bar{q}} p)^2}{2} + \frac{(n_3^{\bar{q}} (1-p))^2}{2}}$$

The quantities $X, Y, \phi_{\bar{q}}$, and H are unchanged. The density functions

$k(x, y)$ and $f(p, q)$ are derived in Appendix II.

Relation (12) can equivalently be stated as $H_0: \phi_4 = 1$ versus $H_a: \phi_4 > 1$, where $\phi_4 = \theta_1/(\theta_2 + \theta_3 - \theta_4)$. Again division by zero is not possible, for it is assumed that $(\theta_2 + \theta_3 - \theta_4) > 0$; therefore, it follows that $X + Y > 1$. So the quantity K_4^* will be undefined only on sets of probability zero. Equation (42) provides an effective way to calculate $E(Z_{\alpha,4}^r)$ under the null and alternative hypotheses.

The quantity $E(Z_{\alpha,5}^r)$ can be shown to be related to $E(Z_{\alpha,4}^r)$ by the following procedure. Testing relation (13) using $F_5' = (V_2 + V_3 - V_4)/V_1$ gives degrees of freedom

$$n_1' = \frac{(V_2 + V_3 - V_4)^2}{(X + Y - 1)^2} = \frac{\frac{n_2}{V_2^2} + \frac{n_3}{V_3^2} + \frac{n_4}{V_4^2}}{\frac{n_2}{X^2} + \frac{n_3}{Y^2} + \frac{n_4}{1}}$$

$$n_2' = n_1,$$

where, again, $x = V_2/V_4$ and $y = V_3/V_4$. It is easily shown that

$$F_5' = H_1' \cdot \phi_5/K_4^* \quad (43)$$

where

$$(44) \quad \left\{ \begin{array}{l} H_1' = 1/H \sim F(n_2 + n_3 + n_4, n_1) \\ K_4^* = 1/K_4 \\ \phi_5 = (\theta_2 + \theta_3 - \theta_4)/\theta_1 = 1/\phi_4 \end{array} \right.$$

When $i = 5$, equation (7) becomes

$$E(Z_r^{\alpha, 5}) = \int_{-\infty}^0 \int_{-\infty}^0 P[F_5^1 > F_\alpha(n_1^1 n_2^1)] \left| x, y \right| \left[\int_r k(x, y) dx dy \right]$$

which, by equation (43) is

$$E(Z_r^{\alpha, 5}) = \int_{-\infty}^0 \int_{-\infty}^0 P\left[F_5^1 \cdot \frac{K_1^4}{\phi_5} > F_\alpha(n_1^1 n_2^1)\right] \left| x, y \right| \left[\int_r k(x, y) dx dy \right],$$

and by (44), the above reduces to

$$E(Z_r^{\alpha, 5}) = \int_{-\infty}^0 \int_{-\infty}^0 P\left[\frac{1}{H} \cdot \frac{\phi_4}{K_4} > F_\alpha(n_1^1 n_2^1)\right] \left| x, y \right| \left[\int_r k(x, y) dx dy \right] \quad (45)$$

By the well known reciprocal relations of the F-distribution, equation

(45) reduces to

$$E(Z_r^{\alpha, 5}) = \int_{-\infty}^0 \int_{-\infty}^0 P\left[H \cdot \frac{K_4}{\phi_4} > F_{(1-\alpha)}(n_1^1, n_2^1)\right] \left| x, y \right| \left[\int_r k(x, y) dx dy \right]$$

Since $n_1^1 = n_1$ and $n_2^1 = n_2$, this becomes

$$E(Z_r^{\alpha, 5}) = \int_{-\infty}^0 \int_{-\infty}^0 \left(1 - P\left[H \cdot \frac{K_4}{\phi_4} > F_{(1-\alpha)}(n_1 n_2)\right]\right) \left| x, y \right| \left[\int_r k(x, y) dx dy \right]$$

and if $r = 1$

$$E(Z_{\alpha, 5}) = 1 - \int_{-\infty}^0 \int_{-\infty}^0 P\left[H \cdot \frac{K_4}{\phi_4} > F_{(1-\alpha)}(n_1 n_2)\right] \left| x, y \right| \left[\int k(x, y) dx dy \right] \quad (46)$$

The second term in equation (46) is the same as equation (40), with the exception that α is changed to $(1-\alpha)$. Hence, it follows that

$$E(Z_{\alpha,5}) = 1 - E(Z_{(1-\alpha),4}) \quad (47)$$

Relation (13) can equivalently be stated as $H_0: \phi_5 = 1$ versus

$H_a: \phi_5 > 1$. Also it is equivalent to $H_0: \phi_4 = 1$ versus $H_a: \phi_4 < 1$,

since it is assumed that $(\theta_2 + \theta_3 - \theta_4) > 0$. Therefore, equation (47)

gives a method to evaluate $E(Z_{\alpha,5})$, under the null and alternative

hypotheses given in (13).

In summary, equations (26), (33), (42) and (47) provide methods to

calculate the quantities $E(Z_{\alpha,1}^i)$ ($i = 2, 3, 4, 5$) related to testing

relations (10), (11), (12), and (13) using the approximate statistics

$$F_1^i \quad (i = 2, 3, 4, 5) \quad .$$

CHAPTER III

ACCURACY INVESTIGATIONS OF THE INTEGRATION TECHNIQUE

As stated in Chapter I, the accuracy investigation of the integration techniques will be limited to an investigation of the accuracy of $E(Z^{.05,2})$, i.e., equation (26) with $r = 1$ and $\alpha = .05$.

The method used to calculate the percentile points of an F-distribution is the Cornish-Fisher (1960) method. The upper-tail probabilities of an F-distribution are calculated by the well known relations between the incomplete F, the incomplete Beta, and the cumulative binomial probabilities. The actual integration technique is the Gauss-Legendre quadrature.

The first matter to investigate is to determine the number of points to be used in the Gauss-Legendre quadrature. Several cases were chosen to represent the various extreme and reasonable situations for the values of n_1, n_2, n_3, n_4, U and W . Then the integration for these cases of $E(Z^{.05,2})$ was performed using the Gauss-Legendre quadrature with 20, 32, 40, and 48 points. The results of this investigation are summarized in Table I. It is felt that using a 32-point quadrature is sufficient to obtain answers accurate to $\pm .001$.

The next matter to investigate is to determine if a transformation different from that used in (25) would have a more "smoothing" effect on the function being integrated. The alternative transformation is the probability integral transformation, and it is applied to equation (24).

Comparisons of Values of $E(Z_{.05,2})$ for Different Gauss-Legendre Quadratures

TABLE 1

U	W	n_1	n_2	n_3	n_4	ϕ	Quad 20	Quad 32	Quad 40	Quad 48
7	1	2	12	4	4	1	.0395551588	.0395551568	**	**
9	1	3	35	9	9	1	.0435991124	.0435991135	**	**
9	1	3	37	9	9	1	.0435960086	.0435960094	**	**
5	1	4	6	4	4	1	.0378475380	.0378475373	**	**
11	3	4	36	4	28	1	.0707096530	.0707096299	**	**
250	1	4	64	16	16	1	.0538111489	.0491849055	.0488888819	.0487817820
250	1	4	64	16	16	15	.9446595740	.9502690810	.9502527600	.9502623960

** Value is unchanged from the next lower point quadrature.

Davenport (1969) gives the results of this transformation. This additional method to calculate $E(Z^{.05,2})$ was accomplished again with 20, 32, and 40 point Gauss-Legendre quadratures for several cases in order to make comparisons with the method given in Chapter II. The results are given in Table 2. Again, the 32-point quadrature with the original method of Chapter II seems adequate.

The third and final investigation into accuracy is to compare the results for $E(Z^{.05,2})$ given by Hudson and Krutchkoff (1968), [who will be referred to as H & K henceforth]. Only one case was considered due to the nature of a requirement of the program used; that $n_1 + n_2$ must be even and $n_3 + n_4$ must be even. That case is $u = 7, w = 1, n_1 = 2, n_2 = 12, n_3 = 4, n_4 = 4$, and $\phi = 1$.

H & K's value for the above case based on 2000 randomly observed values of the event $\left(F'_2 > F^{.05(V_1^2 V_2^2)} \right)$ is .0425. That is 85 out of 2000 times, the calculated F'_2 exceeded its calculated critical point,

$$F^{.05(V_1^2 V_2^2)}.$$

The answer for this case given by the numerical integration technique discussed earlier in this chapter is .03955. Assuming that this is the true value of p in a binomial population with $n = 2000$ and variance

$(2000)p(1-p)$, then an estimate of p obtained by the Monte Carlo results is approximately normally distributed with mean p and variance $\sigma^2 = \frac{p(1-p)}{2000}$.

Therefore, the interval $u \pm 2\sigma$ contains about 95% of all possible values for $\hat{p}$. This interval is (.03085, .04825). H & K's value falls in this interval.

A Monte Carlo study for this case was also accomplished. The results are given in Table 3.

Comparisons of the (s,t) Transformation Method and the Probability
Integral Transformation Method for Calculating $E(Z_{.05,2})$

TABLE 2

U	W	n_1	n_2	n_3	n_4	ϕ	(s,t) Transformation			Probability Integral Transformation		
							Quad 20	Quad 32	Quad 40	Quad 20	Quad 32	Quad 40
1	1	6	6	6	6	1	.0398267699	**	**	.0398307331	.0398281646	.0398277022
1	80	6	6	6	6	1	.0521569576	.0521592053	**	.0521799610	.0521816967	.0521800464
4	4	12	12	12	12	1	.0519825979	.0519825926	**	.0519378263	.0519697506	.0519759133
5	16	2	30	4	20	1	.0623891684	.0623891666	**	.0618426110	.0618629996	.0618663309
16	16	12	12	12	12	1	.0533240378	+	+	.0533735207	.0534602242	.0534761608

** Value is unchanged from the next lower point quadrature.

+ Value not calculated.

TABLE 3

Monte Carlo Results for $E(Z^{.05,2})$

With $U = 7$, $W = 1$, $n_1 = 2$, $n_2 = 12$, $n_3 = 4$, $n_4 = 4$ and $\phi = 1$

Sample Size	Number of Times $F' > F^{.05}(v_1, v_2)$	Estimate of $E(Z^{.05,2})$	Cumulative Sample Size	Estimate of $E(Z^{.05,2})$
2,000	82	.0410	2,000	.0410
2,000	79	.0395	4,000	.04025
4,000	148	.0370	8,000	.0386
4,000	153	.03825	12,000	.0385
4,000	140	.0350	16,000	.037625
4,000	173	.04325	20,000	.03875

In conclusion, the (s, t) transformation technique [equation (26)] with a 32-point quadrature is thought to be sufficiently accurate, and these results agree with H & K's and the present Monte Carlo results. All of the preceding investigations were performed for $E(Z^{.05,2})$. Since the formulae $E(Z^{.05,1})$ ($i = 3, 4, 5$) are very little different from the formulae for $E(Z^{.05,2})$, this investigation is felt to be adequate for all four cases. In addition, since the quadrature is on the (u, v) variables, the same relative accuracy applies to $E(Z^{.05,1})$ because the only change is to raise the probability statement to a power.

APPLICATION TO RANDOM EFFECT MODELS

Consider the usual three factor factorial design with all random effects. The model is

$$x_k = 1 + \alpha_k + \beta_k + \gamma_k + \delta_k + \epsilon_k + \zeta_k + \eta_k + \theta_k + \iota_k + \kappa_k + \lambda_k + \mu_k + \nu_k + \xi_k + \omicron_k + \pi_k + \rho_k + \sigma_k + \tau_k + \upsilon_k + \phi_k + \chi_k + \psi_k + \omega_k$$

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i = 1, 2, ... , a	(the number of levels of factor A)
j = 1, 2, ... , b	" " " " " "
k = 1, 2, ... , c	" " " " " "
l = 1, 2, ... , n	(the number of replications) .

The usual assumptions of independence, zero expectation, and normality are made concerning each of the components in the model, except for μ . The analysis of variance table for this model with the expected mean squares is given in Table 4.

It is well known that no exact tests exist for any of the following hypotheses:

It is well known that no exact tests exist for any of the following

$$\begin{array}{llll}
\text{H}_O: \sigma^2_\gamma = 0 & \text{versus} & \text{H}_a: \sigma^2_\gamma > 0 & (50) \\
\text{H}_O: \sigma^2_\beta = 0 & \text{versus} & \text{H}_a: \sigma^2_\beta > 0 & (49) \\
\text{H}_O: \sigma^2_\alpha = 0 & \text{versus} & \text{H}_a: \sigma^2_\alpha > 0 & (48)
\end{array}$$

TABLE 4

ANOVA For Three-Factor Factorial Random Effects Model

Factor	Mean Square	Degrees of Freedom	Expected Mean Square
A	MSA	(a-1)	$\Sigma_1 = \sigma^2 + n\sigma_{\alpha\beta\gamma}^2 + n\sigma_{\alpha\beta}^2 + n\sigma_{\alpha\gamma}^2 + n\sigma_{\alpha}^2 + nb\sigma_{\alpha}^2 + nb\sigma_{\alpha\beta}^2 + nb\sigma_{\alpha\gamma}^2 + nb\sigma_{\alpha}^2 + na\sigma_{\alpha}^2 + na\sigma_{\alpha\beta}^2 + na\sigma_{\alpha\gamma}^2 + na\sigma_{\alpha}^2 + nabc\sigma_{\alpha}^2 + nabc\sigma_{\alpha\beta}^2 + nabc\sigma_{\alpha\gamma}^2 + nabc\sigma_{\alpha}^2$
B	MSB	(b-1)	$\Sigma_2 = \sigma^2 + n\sigma_{\alpha\beta\gamma}^2 + n\sigma_{\alpha\beta}^2 + n\sigma_{\alpha\gamma}^2 + n\sigma_{\beta}^2 + na\sigma_{\beta}^2 + na\sigma_{\alpha\beta}^2 + na\sigma_{\beta\gamma}^2 + na\sigma_{\beta}^2 + nb\sigma_{\beta}^2 + nb\sigma_{\alpha\beta}^2 + nb\sigma_{\beta\gamma}^2 + nb\sigma_{\beta}^2 + nabc\sigma_{\beta}^2 + nabc\sigma_{\alpha\beta}^2 + nabc\sigma_{\beta\gamma}^2 + nabc\sigma_{\beta}^2$
C	MSC	(c-1)	$\Sigma_3 = \sigma^2 + n\sigma_{\alpha\beta\gamma}^2 + n\sigma_{\alpha\beta}^2 + n\sigma_{\alpha\gamma}^2 + n\sigma_{\beta}^2 + na\sigma_{\gamma}^2 + na\sigma_{\alpha\gamma}^2 + na\sigma_{\beta\gamma}^2 + na\sigma_{\gamma}^2 + nb\sigma_{\gamma}^2 + nb\sigma_{\alpha\gamma}^2 + nb\sigma_{\beta\gamma}^2 + nb\sigma_{\gamma}^2 + nabc\sigma_{\gamma}^2 + nabc\sigma_{\alpha\gamma}^2 + nabc\sigma_{\beta\gamma}^2 + nabc\sigma_{\gamma}^2$
AB	MSAB	(a-1)(b-1)	$\Sigma_4 = \sigma^2 + n\sigma_{\alpha\beta\gamma}^2 + n\sigma_{\alpha\beta}^2 + n\sigma_{\alpha\gamma}^2 + n\sigma_{\beta}^2 + na\sigma_{\alpha\beta}^2 + na\sigma_{\alpha\gamma}^2 + na\sigma_{\beta\alpha}^2 + na\sigma_{\beta\gamma}^2 + na\sigma_{\alpha\beta}^2 + nb\sigma_{\alpha\beta}^2 + nb\sigma_{\alpha\gamma}^2 + nb\sigma_{\beta\alpha}^2 + nb\sigma_{\beta\gamma}^2 + nabc\sigma_{\alpha\beta}^2 + nabc\sigma_{\alpha\gamma}^2 + nabc\sigma_{\beta\alpha}^2 + nabc\sigma_{\beta\gamma}^2$
AC	MSAC	(a-1)(c-1)	$\Sigma_5 = \sigma^2 + n\sigma_{\alpha\beta\gamma}^2 + n\sigma_{\alpha\beta}^2 + n\sigma_{\alpha\gamma}^2 + n\sigma_{\beta}^2 + na\sigma_{\alpha}^2 + na\sigma_{\alpha\beta}^2 + na\sigma_{\alpha\gamma}^2 + na\sigma_{\alpha}^2 + nb\sigma_{\alpha}^2 + nb\sigma_{\alpha\beta}^2 + nb\sigma_{\alpha\gamma}^2 + nb\sigma_{\alpha}^2 + nabc\sigma_{\alpha}^2 + nabc\sigma_{\alpha\beta}^2 + nabc\sigma_{\alpha\gamma}^2 + nabc\sigma_{\alpha}^2$
BC	MSBC	(b-1)(c-1)	$\Sigma_6 = \sigma^2 + n\sigma_{\alpha\beta\gamma}^2 + n\sigma_{\alpha\beta}^2 + n\sigma_{\alpha\gamma}^2 + n\sigma_{\beta}^2 + na\sigma_{\beta}^2 + na\sigma_{\alpha\beta}^2 + na\sigma_{\beta\gamma}^2 + na\sigma_{\beta}^2 + nb\sigma_{\beta}^2 + nb\sigma_{\alpha\beta}^2 + nb\sigma_{\beta\gamma}^2 + nb\sigma_{\beta}^2 + nabc\sigma_{\beta}^2 + nabc\sigma_{\alpha\beta}^2 + nabc\sigma_{\beta\gamma}^2 + nabc\sigma_{\beta}^2$
ABC	MSABC	(a-1)(b-1)(c-1)	$\Sigma_7 = \sigma^2 + n\sigma_{\alpha\beta\gamma}^2 + n\sigma_{\alpha\beta}^2 + n\sigma_{\alpha\gamma}^2 + n\sigma_{\beta}^2 + na\sigma_{\alpha\beta\gamma}^2 + na\sigma_{\alpha\beta}^2 + na\sigma_{\alpha\gamma}^2 + na\sigma_{\beta}^2 + nb\sigma_{\alpha\beta\gamma}^2 + nb\sigma_{\alpha\beta}^2 + nb\sigma_{\alpha\gamma}^2 + nb\sigma_{\beta}^2 + nabc\sigma_{\alpha\beta\gamma}^2 + nabc\sigma_{\alpha\beta}^2 + nabc\sigma_{\alpha\gamma}^2 + nabc\sigma_{\beta}^2$
Error	MSE	abc(n-1)	$\Sigma_8 = \sigma^2$
TOTAL	---	abcn-1	-----

However, an approximate testing procedure due to Satterthwaite (1941 and 1946), is as follows: Form two linear combinations of expected mean

squares such that these two quantities are equal when the null hypothesis is true. Estimates of these expected mean squares are used to form a ratio which is approximately distributed as an F. The degrees of freedom are estimated as is explained in Chapter I. An additional requirement, not stated by Satterthwaite, but which is felt to be quite reasonable and is not too restrictive, is that each of these two linear combinations of expected mean squares should be greater than zero [i.e., linear combinations of expected mean squares that are negative are excluded].

Now consider the test in (48). When $\sigma^2_{\alpha} = 0$, then all of the

following null hypotheses are true. Also the alternatives are given

when $\sigma^2_{\alpha} > 0$.

H_0 :

$$\Sigma_1 = \Sigma_4 + \Sigma_5 - \Sigma_7$$

$$\Sigma_1 + \Sigma_7 = \Sigma_4 + \Sigma_5$$

$$\Sigma_1 - \Sigma_4 = \Sigma_5 - \Sigma_7$$

$$\Sigma_1 - \Sigma_5 = \Sigma_4 - \Sigma_7$$

$$\Sigma_1 + \Sigma_7 - \Sigma_4 = \Sigma_5$$

$$\Sigma_1 + \Sigma_7 - \Sigma_5 = \Sigma_4$$

H_a :

$$\Sigma_1 > \Sigma_4 + \Sigma_5 - \Sigma_7$$

$$\Sigma_1 + \Sigma_7 > \Sigma_4 + \Sigma_5$$

$$\Sigma_1 - \Sigma_4 > \Sigma_5 - \Sigma_7$$

$$\Sigma_1 - \Sigma_5 > \Sigma_4 - \Sigma_7$$

$$\Sigma_1 + \Sigma_7 - \Sigma_4 > \Sigma_5$$

$$\Sigma_1 + \Sigma_7 - \Sigma_5 > \Sigma_4$$

(51)

(52)

(53)

(54)

(55)

(56)

Note that each one of the above hypotheses is one of the relations (10),

(11), (12), or (13); hence, can be tested by its appropriate F_1^1 statistic.

Also, $E(Z_{\alpha,1}^2)$ can be computed for each of the relations (51) through (56).

It should be pointed out here that the test for $\sigma^2_{\beta} = 0$ and $\sigma^2_{\gamma} = 0$

can be accomplished by the same tests by simply relabeling the main effects (i.e., switch effect B and A, etc.). Therefore, testing $\sigma^2_{\alpha} = 0$ is the

only case that it is necessary to investigate.

The mixed model where A is a fixed effect component, and effects

B and C are random can also be included as equivalent to the above test.

The only difference is that the power function here is now a non-central

F and must be computed by a different method than that given in Chapter II.

This was not studied in this work.

The natural question arises: "Which one of the respective approximate tests for relations (51) through (56) is 'best'?" The purpose of this paper is to partially answer this question by computing the true probability of type I error $E(Z_{\alpha,1}^r)$ under the null hypothesis] and the power function $E(Z_{\alpha,1}^r)$ under the alternative hypothesis] for a few particular three factor factorial designs. This is done for all six cases [(51) through (56)]. These results are then used to make comparisons among the six approximate tests to determine which test is closest in its approximation to the nominal α -level of the test, and also to determine which test has the most power.

Since the formulae for $E(Z_{\alpha,1}^r)$ are functions of the unknown variances σ_1^2 , a range of values must be used for each "nuisance" variance. The F -ratios are invariant under a change in scale; hence, the error variance, σ_2^2 , is taken to be equal to one in all cases. The values for the variance components are given in Table 5 and Table 11. Two designs are used to make the initial comparisons. They are the symmetric 3^3 -design, and the asymmetric design with the number of levels given as: $a = 7$, $b = 7$, and $c = 3$. Three values for the nominal α are used, and they are .05, .025, and .01. The results are given in Tables 5 through 10. Also for convenience, the number of replications is taken to be equal to one.

From the results of Tables 5 through 10, it is apparent that the approximate test statistics F_3' and F_5' can be removed from consideration as a method of testing (48). Both of these tests are in considerable error in their approximation to the nominal α , and both exhibit the disturbing property that, in some cases, $P(\text{Rejecting } H_0 | H_a)$ can be smaller than the true type I error. Therefore, testing relations (53), (54), (55), and (56) are not satisfactory ways to carry out Satterthwaite's procedure for

TABLE 5

Type-I Error & Power of Approximate Tests; $\sigma^2=1$, Nominal $\alpha=.05$, 3^3 -design

$\sigma^2_{\alpha\beta\gamma}$	$\sigma^2_{\alpha\beta}$	$\sigma^2_{\alpha\gamma}$	σ^2_{α}	Testing (51) with F'_4	Testing (52) with F'_2	Testing (53) with F'_3	Testing (54) with F'_3	Testing (55) with F'_5	Testing (56) with F'_5
.01	.01	.01	0	.000095	.0348	.1435	(same as testing (53))	.2646	(same as testing (55))
.01	.01	.01	1	.0976	.4741	.0598		.5693	
.01	.01	.01	4	.3116	.8001	.0790		.8360	
.01	1	1	0	.0151	.0396	.2095		.3745	
.01	1	1	1	.1091	.1649	.1931		.3818	
.01	1	1	4	.3875	.4618	.2778		.5807	
.01	1	1	10	.6322	.6946	.4230		.7623	
.01	4	4	0	.0370	.0422	.2695		.4055	
.01	4	4	1	.0715	.0731	.2890		.3820	
.01	4	4	4	.1885	.1990	.3585		.4135	
.01	4	4	10	.3794	.4290	.4879		.5349	
1	1	1	0	.0058	.0378	.1877		.3450	
1	1	1	1	.0525	.1410	.1559		.3501	
1	1	1	4	.2536	.4068	.1562		.5338	
1	1	1	10	.4921	.6480	.2363		.7220	
4	4	4	0	.0116	.0390	.2023		.3653	
4	4	4	1	.0259	.0702	.1940		.3485	
4	4	4	4	.0905	.1650	.1777		.3732	
4	4	4	10	.2341	.3227	.1941		.4806	

TABLE 6

Type-I Error & Power of Approximate Tests; $\sigma^2=1$, Nominal $\alpha=.025$, 3^3 -design

$\sigma_{\alpha\beta\gamma}^2$	$\sigma_{\alpha\beta}^2$	$\sigma_{\alpha\gamma}^2$	σ_{α}^2	Testing (51) with F'_4	Testing (52) with F'_2	Testing (53) with F'_3	Testing (54) with F'_3	Testing (55) with F'_5	Testing (56) with F'_5
.01	.01	.01	0	.0000098	.0156	.1435	(same as testing (53))	.2339	(same as testing (55))
.01	.01	.01	1	.0444	.3585	.0511		.4414	
.01	.01	.01	4	.2180	.7295	.0462		.7601	
.01	1	1	0	.0040	.0186	.2129		.3420	
.01	1	1	1	.0485	.0985	.1710		.2971	
.01	1	1	4	.2630	.3556	.1763		.4608	
.01	1	1	10	.5191	.6083	.2903		.6678	
.01	4	4	0	.0155	.0203	.2568		.3722	
.01	4	4	1	.0341	.0380	.2562		.3305	
.01	4	4	4	.1124	.1250	.2736		.3226	
.01	4	4	10	.2731	.3357	.3609		.4179	
1	1	1	0	.0011	.0174	.1898		.3130	
1	1	1	1	.0179	.0807	.1520		.2723	
1	1	1	4	.1442	.3003	.1082		.4127	
1	1	1	10	.3626	.5527	.1432		.6176	
4	4	4	0	.0028	.0182	.2055		.3329	
4	4	4	1	.0075	.0357	.1953		.3021	
4	4	4	4	.0373	.0982	.1635		.2902	
4	4	4	10	.1311	.2248	.1395		.3658	

TABLE 7

Type-I Error & Power of Approximate Tests; $\sigma^2=1$, Nominal $\alpha=.01$, 3^3 -design

$\sigma_{\alpha\beta\gamma}^2$	$\sigma_{\alpha\beta}^2$	$\sigma_{\alpha\gamma}^2$	σ_{α}^2	Testing (51) with F'_4	Testing (52) with F'_2	Testing (53) with F'_3	Testing (54) with F'_3	Testing (55) with F'_5	Testing (56) with F'_5
.01	.01	.01	0	.0000006	.0055	.1524	(same as testing (53))	.2148	(same as testing (55))
.01	.01	.01	1	.0141	.2367	.0536		.2892	
.01	.01	.01	4	.1267	.6301	.0265		.6333	
.01	1	1	0	.0006	.0071	.2423		.3213	
.01	1	1	1	.0146	.0489	.1865		.2260	
.01	1	1	4	.1421	.2426	.1208		.3160	
.01	1	1	10	.3745	.4964	.1604		.5232	
.01	4	4	0	.0046	.0080	.2781		.3509	
.01	4	4	1	.0120	.0163	.2641		.2938	
.01	4	4	4	.0532	.0661	.2298		.2421	
.01	4	4	10	.1669	.2393	.2409		.2865	
1	1	1	0	.0001	.0065	.2143		.2928	
1	1	1	1	.0039	.0380	.1726		.2101	
1	1	1	4	.0604	.1934	.0995		.2754	
1	1	1	10	.2213	.4339	.0812		.4655	
4	4	4	0	.0004	.0069	.2339		.3125	
4	4	4	1	.0013	.0148	.2227		.2687	
4	4	4	4	.0101	.0485	.1821		.2214	
4	4	4	10	.0538	.1344	.1310		.2465	

TABLE 8
Type-I Error & Power of Approximate Tests; $\sigma^2=1$, Nominal $\alpha=.05$
Number of Levels of Main Effects; $a=7$, $b=7$, $c=3$

$\sigma^2_{\alpha\beta\gamma}$	$\sigma^2_{\alpha\beta}$	$\sigma^2_{\alpha\gamma}$	σ^2_{α}	Testing (51) with F'_4	Testing (52) with F'_2	Testing (53) with F'_3	Testing (54) with F'_3	Testing (55) with F'_5	Testing (56) with F'_5
.01	.01	.01	0	.0039	.0395	.1654	.1522	.0909	.1602
.01	.01	.01	1	.9302	.9840	.0985	.1523	.9862	.9929
.01	.01	.01	4	.9599	.9996	.2143	.1970	.9997	.9998
.01	1	1	0	.0519	.0537	.1285	.3992	.1200	.4490
.01	1	1	1	.5228	.5184	.5230	.8235	.5363	.7595
.01	1	1	4	.9288	.9286	.9262	.9831	.9303	.9751
.01	1	1	10	.9910	.9905	.9898	.9981	.9915	.9974
.01	4	4	0	.0530	.0526	.1239	.3893	.1247	.4526
.01	4	4	1	.1770	.1765	.2165	.5584	.2134	.5135
.01	4	4	4	.5579	.5561	.5693	.8346	.5706	.7883
.01	4	4	10	.8588	.8581	.8611	.9612	.8622	.9474
1	1	1	0	.0498	.0434	.1198	.3831	.1149	.4100
1	1	1	1	.4797	.4767	.4232	.7796	.4977	.7191
1	1	1	4	.9122	.9116	.8810	.9576	.9160	.9666
1	1	1	10	.9882	.9881	.9717	.9796	.9886	.9962
4	4	4	0	.0514	.0498	.1268	.3989	.1209	.4421
4	4	4	1	.1604	.1508	.1876	.5530	.1971	.4871
4	4	4	4	.5123	.5133	.5035	.8179	.5266	.7499
4	4	4	10	.8295	.8281	.8206	.9526	.8352	.9318

TABLE 9
Type-I Error & Power of Approximate Tests; $\sigma^2=1$, Nominal $\alpha=.025$
Number of Levels of Main Effects; $a=7$, $b=7$, $c=3$

$\sigma^2_{\alpha\beta\gamma}$	$\sigma^2_{\alpha\beta}$	$\sigma^2_{\alpha\gamma}$	σ^2_{α}	Testing (51) with F'_4	Testing (52) with F'_2	Testing (53) with F'_3	Testing (54) with F'_3	Testing (55) with F'_5	Testing (56) with F'_5
.01	.01	.01	0	.0005	.0178	.1654	.1542	.0631	.1249
.01	.01	.01	1	.8866	.9734	.0550	.1471	.9764	.9894
.01	.01	.01	4	.9561	.9993	.1360	.1553	.9994	.9998
.01	1	1	0	.0266	.0257	.0992	.4030	.0917	.4078
.01	1	1	1	.4167	.4129	.3978	.7894	.4258	.7037
.01	1	1	4	.8939	.8940	.8833	.9772	.8935	.9666
.01	1	1	10	.9854	.9849	.9826	.9972	.9858	.9964
.01	4	4	0	.0277	.0274	.0949	.3991	.0961	.4109
.01	4	4	1	.1123	.1119	.1460	.5405	.1440	.4484
.01	4	4	4	.4542	.4524	.4584	.8066	.4617	.7391
.01	4	4	10	.8009	.7999	.7984	.9507	.8010	.9314
1	1	1	0	.0247	.0242	.1017	.3771	.0868	.3694
1	1	1	1	.3714	.3702	.2751	.7271	.3866	.6561
1	1	1	4	.8699	.8699	.8056	.9475	.8730	.9551
1	1	1	10	.9807	.9808	.9506	.9751	.9809	.9947
4	4	4	0	.0262	.0253	.0992	.4000	.0926	.4010
4	4	4	1	.0988	.0921	.1150	.5258	.1317	.4211
4	4	4	4	.4055	.4074	.3719	.7807	.4159	.6922
4	4	4	10	.7620	.7609	.7341	.9377	.7653	.9105

TABLE 10
Type-I Error & Power of Approximate Tests; $\sigma^2=1$, Nominal $\alpha=.01$
Number of Levels of Main Effects; $a=7$, $b=7$, $c=3$

$\sigma^2_{\alpha\beta\gamma}$	$\sigma^2_{\alpha\beta}$	$\sigma^2_{\alpha\gamma}$	σ^2_{α}	Testing (51) with F'_4	Testing (52) with F'_2	Testing (53) with F'_3	Testing (54) with F'_3	Testing (55) with F'_5	Testing (56) with F'_5
.01	.01	.01	0	.000032	.0061	.1654	.1660	.0457	.0995
.01	.01	.01	1	.8149	.9542	.0217	.1299	.9569	.9834
.01	.01	.01	4	.9325	.9986	.0823	.1545	.9988	.9996
.01	1	1	0	.0111	.0107	.0841	.3877	.0734	.3719
.01	1	1	1	.2999	.2980	.2518	.7267	.3003	.6334
.01	1	1	4	.8377	.8391	.8072	.9663	.8310	.9541
.01	1	1	10	.9748	.9746	.9664	.9959	.9743	.9949
.01	4	4	0	.0119	.0118	.0763	.3924	.0776	.3737
.01	4	4	1	.0611	.0609	.0888	.4929	.0875	.3799
.01	4	4	4	.3374	.3358	.3282	.7542	.3341	.6767
.01	4	4	10	.7164	.7154	.7012	.9316	.7064	.9089
1	1	1	0	.0097	.0098	.0971	.3581	.0688	.3349
1	1	1	1	.2559	.2582	.1266	.6312	.2649	.5782
1	1	1	4	.8029	.8054	.6799	.9262	.8015	.9380
1	1	1	10	.9669	.9675	.9118	.9722	.9657	.9924
4	4	4	0	.0108	.0105	.0870	.3823	.0744	.3655
4	4	4	1	.0513	.0476	.0651	.4666	.0798	.3537
4	4	4	4	.2890	.2925	.2203	.7123	.2912	.6200
4	4	4	10	.6655	.6662	.5983	.9099	.6605	.8806

testing (48). Also it is apparent that there is no change in the relative behavior of any of the tests for different values of the nominal- α . Hence, all additional investigations are at the .05 level for the nominal- α .

In addition, the results of Tables 5 through 10 appear to indicate that F'_1 is slightly better than F'_4 . To further confirm this indication, some additional designs were chosen to be used in making further comparisons. The same values as before for the variance components are used. The nominal- α is equal to .05, $\sigma^2 = 1$ (the error variance), and again the number of replications is taken to be equal to one. The three designs are: 5₃-design; the number of levels -- $a = 3$, $b = 7$, and $c = 3$; and the number levels -- $a = 3$, $b = 7$, and $c = 7$. These results are given in Table 11.

Two things now seem clear: (1) F'_1 is in no case appreciably better than F'_2 ; and, (2) when the degrees of freedom of the numerator of F'_1 are small, this test is very conservative in its approximation to the nominal- α , and hence, there is a corresponding reduction in power. For this reason, it is felt that F'_1 is the "best" of the approximate test statistics considered here. It shows considerably more stability than any of the others. These conclusions are in complete agreement with those of Hudson and Krutchoff (1968).

Therefore, testing relation (52) by using the F'_2 approximate F-statistic is the "best" way to test (48), among the approximate tests considered here. Further investigations into the behavior and properties of $E(Z_{\tau}^{\alpha, 2})$ are considered in Chapter V.

Some general recommendations concerning the use of Satterthwaite approximate F-statistics are as follows:

TABLE 11
Type-I Error & Power of Approximate Tests; $\sigma^2=1$, Nominal $\alpha=.05$

$\sigma^2_{\alpha\beta\gamma}$	$\sigma^2_{\alpha\beta}$	$\sigma^2_{\alpha\gamma}$	σ^2_{α}	5 ³ -Design		Number of Levels a=3, b=7, c=3		Number of Levels a=3, b=7, c=7	
				Testing (51) with F' ₄	Testing (52) with F' ₂	Testing (51) with F' ₄	Testing (52) with F' ₂	Testing (51) with F' ₄	Testing (52) with F' ₂
.01	.01	.01	0	.0031	.0330	.0005	.0382	.0046	.0281
.01	.01	.01	1	.8616	.9574	.3815	.7420	.7901	.8960
.01	.01	.01	4	.9272	.9966	.6063	.9225	.8820	.9716
.01	1	1	0	.0477	.0452	.0501	.0530	.0492	.0466
.01	1	1	1	.5104	.5073	.2654	.2721	.4559	.4504
.01	1	1	4	.8932	.8907	.5960	.6031	.7840	.7798
.01	1	1	10	.9757	.9753	.7908	.7947	.9023	.9012
.01	4	4	0	.0487	.0479	.0559	.0561	.0499	.0491
.01	4	4	1	.1803	.1772	.1182	.1187	.1814	.1791
.01	4	4	4	.5434	.5412	.2926	.2930	.4744	.4727
.01	4	4	10	.8208	.8200	.5080	.5079	.7039	.7031
1	1	1	0	.0458	.0426	.0381	.0504	.0481	.0440
1	1	1	1	.4700	.4616	.2239	.2478	.4335	.4236
1	1	1	4	.8740	.8702	.5504	.5732	.7692	.7631
1	1	1	10	.9704	.9696	.7593	.7751	.8947	.8920
4	4	4	0	.0473	.0445	.0474	.0524	.0490	.0459
4	4	4	1	.1626	.1555	.1004	.1042	.1686	.1627
4	4	4	4	.5002	.4928	.2560	.2677	.4504	.4437
4	4	4	10	.7913	.7867	.4652	.4753	.6844	.6793

1. The subtraction of mean squares should be avoided when they are to be used in constructing approximate F-statistics. That is, if possible, the mean squares should be arranged so that they are added.
2. If subtraction of mean squares is necessary, then this approximated chi-square should be used in the denominator of the approximate F, never in the numerator.
3. If an exact chi-square is available, then it should be used in the numerator of the approximate F, and the approximated chi-square (whether added or subtracted) should be used in the denominator.
4. It would be best, if the degrees of freedom associated with the mean squares that are used in constructing the approximated chi-square, are not extremely different.

CHAPTER V

PROPERTIES OF THE F'_2 TEST

$E(Z_r^{\alpha,2})$ given by equation (26) can be considered as a function of

the arguments U and W , with parameters ϕ, r, n_1, n_2, n_3 , and n_4 . Hence,

let

$$E(Z_r^{\alpha,2}) = F_{\alpha,r}(U, W; \phi, n_1, n_2, n_3, n_4) \quad (57)$$

When it is clear what values the arguments and parameters assume, equation (57) will be written $F_{\alpha,r}$ for short. The purpose of this chapter is to

determine some properties of the function $F_{\alpha,r}$.

Recall that F'_2 is used to test $H_0: \theta_1 + \theta_2 = \theta_3 + \theta_4$ versus

$H_a: \theta_1 + \theta_2 > \theta_3 + \theta_4$, and that $U = \theta_1/\theta_2$, $W = \theta_3/\theta_4$, and

$$\phi = (\theta_1 + \theta_2)/(\theta_3 + \theta_4).$$

The asymptotic behavior of $F_{\alpha,1}$ (i.e., $r = 1$) as U and W approach

extreme values can be determined by observing the relation $\theta_1 + \theta_2 = \theta_3 + \theta_4$.

Ignoring infinite variances, then the following hold:

$$W \rightarrow \infty \Rightarrow \theta_4 \rightarrow 0$$

$$W \rightarrow 0 \Rightarrow \theta_3 \rightarrow 0$$

$$U \rightarrow \infty \Rightarrow \theta_2 \rightarrow 0$$

$$U \rightarrow 0 \Rightarrow \theta_1 \rightarrow 0$$

Therefore, it follows that (when $\phi = 1$)

$$\lim_{\substack{U \rightarrow 0 \text{ or } \infty \\ W \rightarrow 0 \text{ or } \infty}} F_{\alpha, 1} = \alpha.$$

This property can be seen in Figures 1 through 4. The 4-tuple (n_1, n_2, n_3, n_4) gives the respective values of the degrees of freedom.

If $U \rightarrow 0$ or ∞ as W remains constant, then relation (10) reduces to

$$H_0: \theta_1 = \theta_3 + \theta_4 \quad \text{versus} \quad H_a: \theta_1 > \theta_3 + \theta_4.$$

$$(1 = 1 \text{ or } 2)$$

This can be tested using the F_1^1 statistic mentioned in Chapter I. The

properties of this test statistic are thoroughly investigated by

Davenport (1969) and Davenport and Webster (1971). A similar result

holds as $W \rightarrow 0$ or ∞ as U remains constant (i.e., it can be related to

$$F_1^1).$$

The function $F_{\alpha, 1}$ has a type of symmetry that reduces the possible

cases that need investigation. It can be shown algebraically that

$$F_{\alpha, 1}^1(U, W; \phi, n_1, n_2, n_3, n_4) = F_{\alpha, 1}^1\left(\frac{1}{U}, \frac{1}{W}; \phi, n_2, n_1, n_4, n_3\right) \quad (58)$$

by using the well known reciprocal relation of the F distribution and

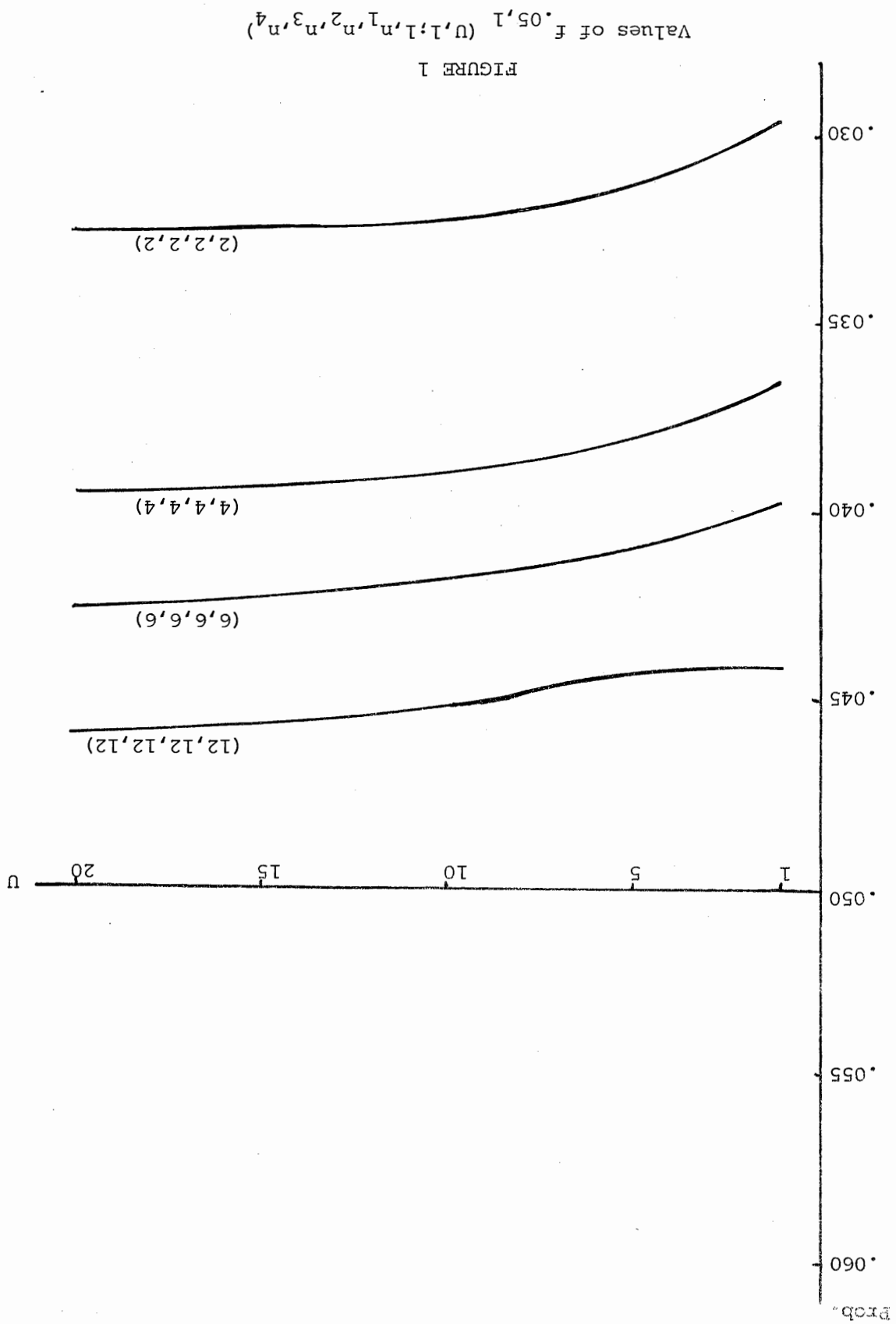
following the same type of derivation as used for equation (26) in

Chapter II. Hence, the particular cases of $F_{\alpha, 1}$ that needed investigation

are when $U > 1$ and $W > 1$.

As an illustrative example of this symmetry, consider the case

where $n_1 = 30$, $n_2 = 2$, $n_3 = 2$, and $n_4 = 30$ in Figure 4. If you wished



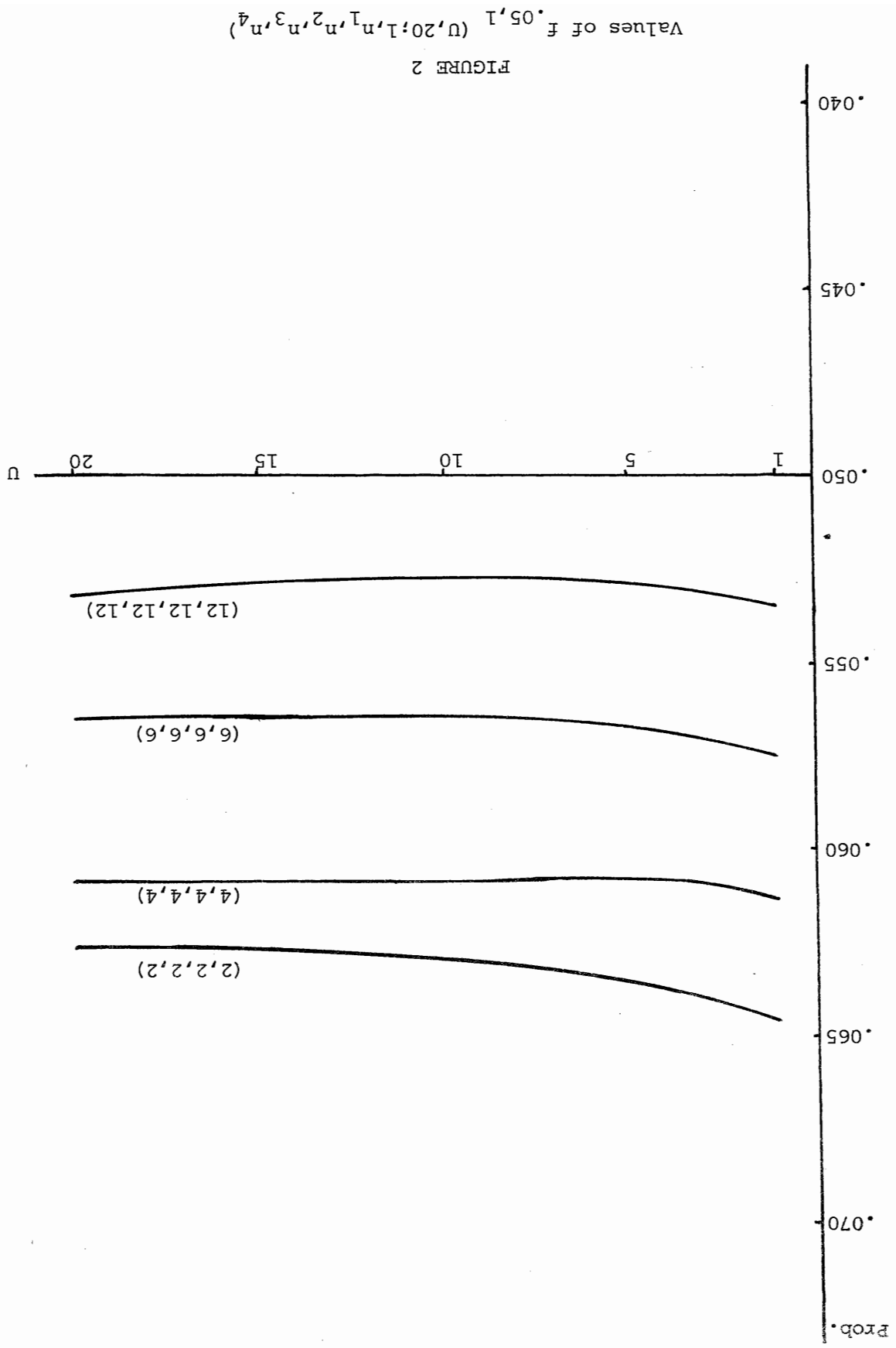
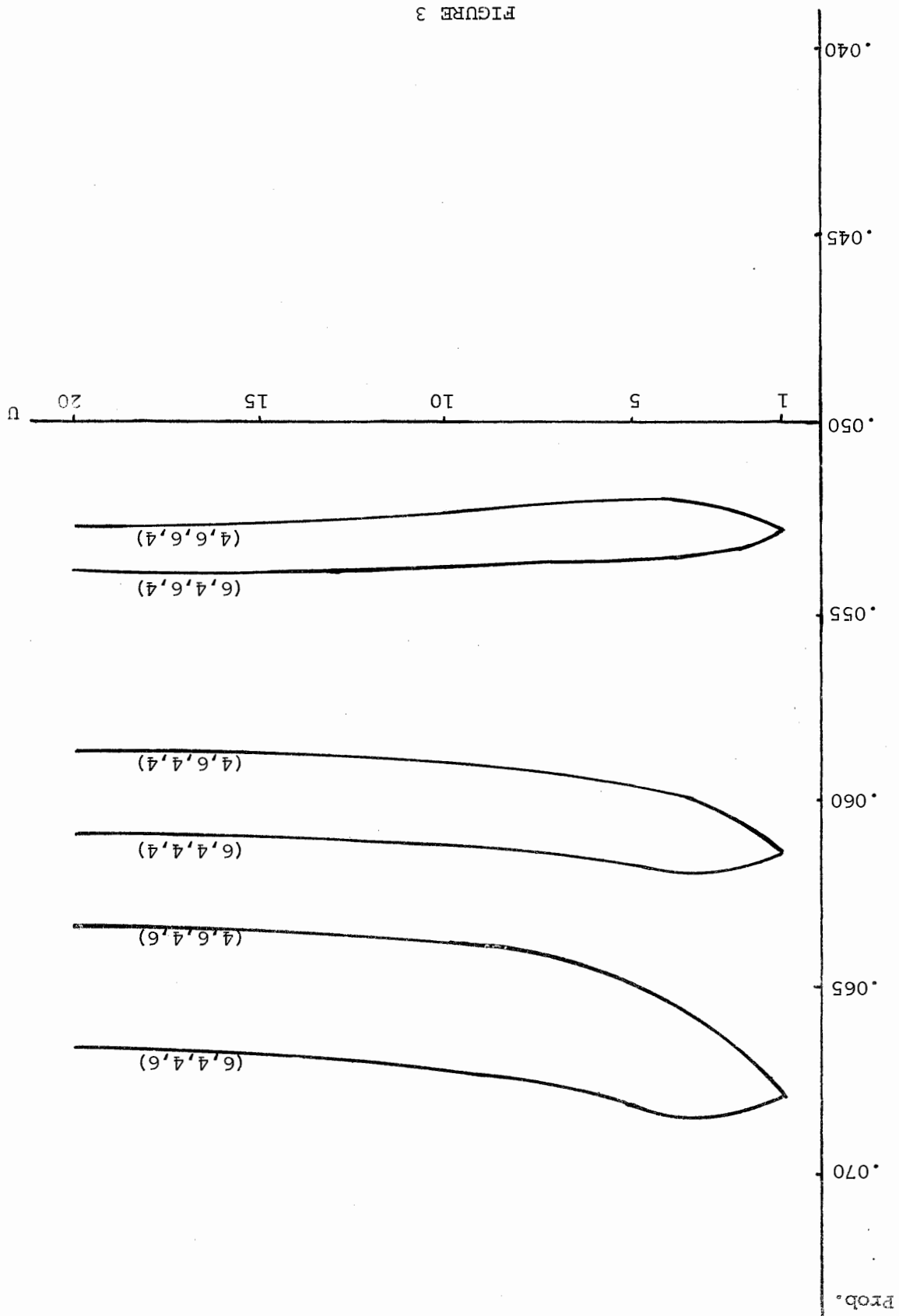
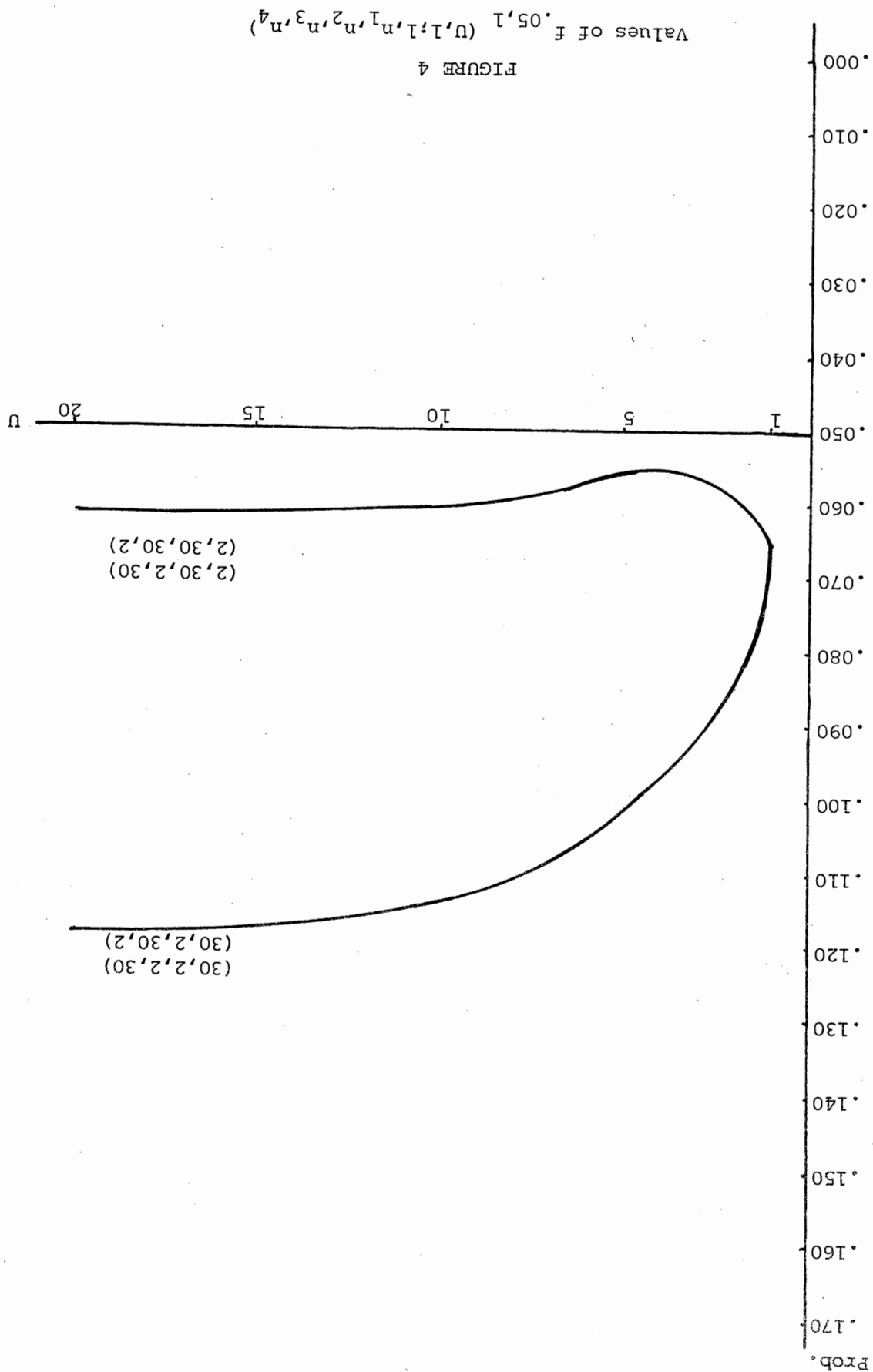


FIGURE 2
Values of $F_{.05,1}(u, 20; 1, n_1, n_2, n_3, n_4)$



Values of $F_{.05, 1}(u, 10; 1, n_1, n_2, n_3, n_4)$

FIGURE 3



to construct the graph for the case $n_1 = 2, n_2 = 30, n_3 = 2, n_4 = 30$ for values of U between zero and one (i.e., $0 < U \leq 1$), you could easily do so by plotting the same function values as the graph in Figure 4 (mentioned above) at $1/U$ instead of U . Note that if $n_3 = 30$ and $n_4 = 2$, the function values are unchanged in this example since $W = 1$. The same is true for the power curves.

Additional comprehensive properties of $F_{\alpha,1}$ (when $\phi = 1$) are difficult to assess without extensive investigation, which is beyond the limits of this paper. However, some general properties were observed and are as follows:

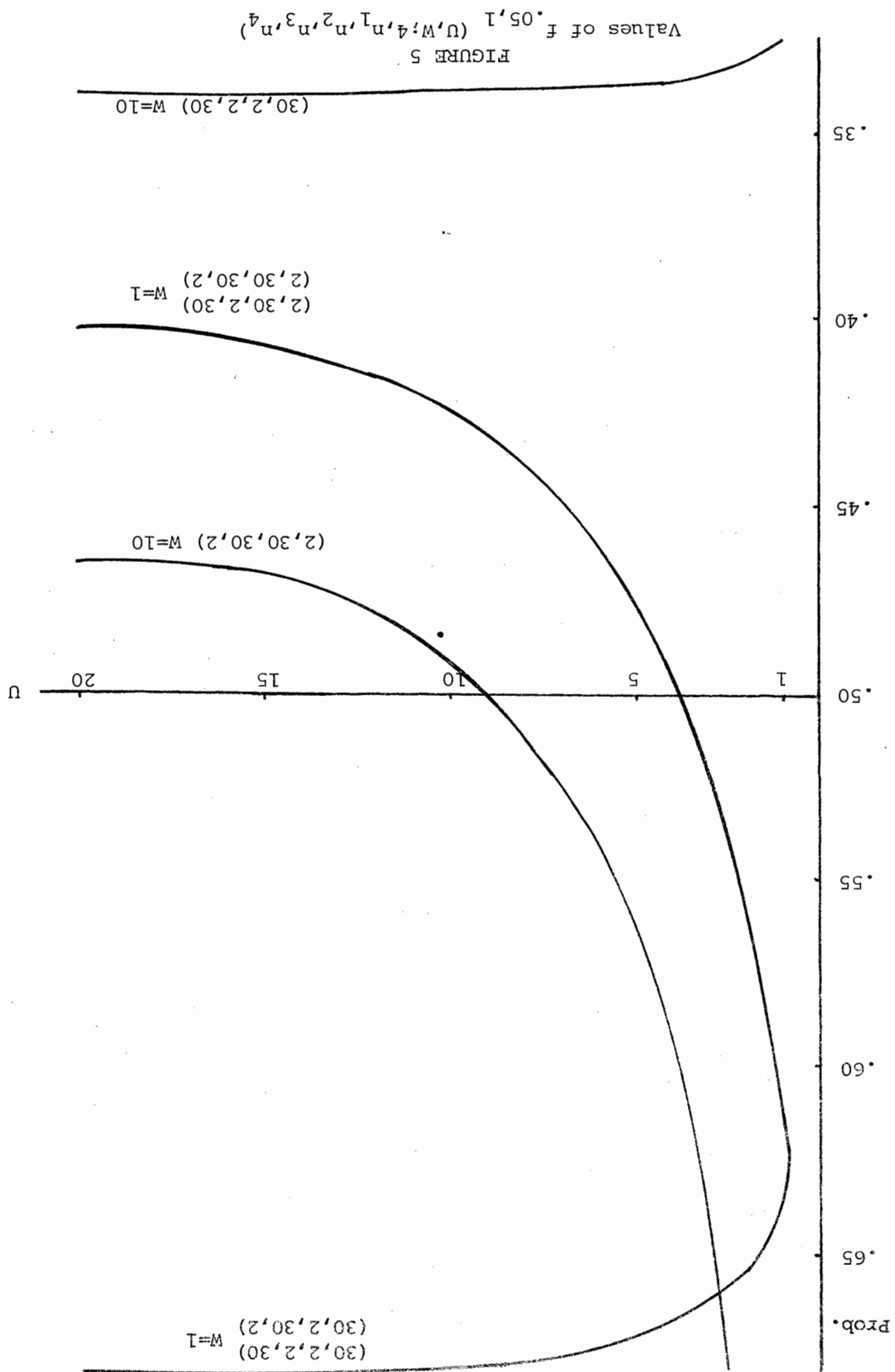
1. As the sample sizes increase together, $F_{\alpha,1}$ approaches α -- Figures 1 and 2.

2. If $U > 1$, then $F_{\alpha,1}$ is closer to α when $n_1 < n_2$. Likewise, if $U < 1$, then $F_{\alpha,1}$ is closer to α when $n_1 > n_2$ -- Figure 3.

3. If $W > 1$, then $F_{\alpha,1}$ is closer to α when $n_3 > n_4$. Likewise, if $W < 1$, then $F_{\alpha,1}$ is closer to α when $n_3 < n_4$ -- Figure 3.

4. If $|n_1 - n_2|$ and/or $|n_3 - n_4|$ are large, then $|F_{\alpha,1} - \alpha|$ can also be large -- Figure 4.

The power function for $F_{\alpha,1}$ is easily computed by letting the parameter ϕ assume values greater than one. First it is necessary to determine the effects of the nuisance parameters U, W on the power of the test. As is shown in Figure V, the power either increases slightly or decreases with increasing values of U . However, the power may either increase or decrease by a considerable amount with increasing values of



W. Therefore, the power curves are given in Figures 6 and 7 for both small and large values of U and W and for some special cases of the degrees of freedom. The 4-tuple (n_1, n_2, n_3, n_4) gives the values of the respective degrees of freedom in each case.

Hald (1952) has proven that an approximated chi-square of the type $(v_1 + v_2)$ has degrees of freedom that is bounded by $\min(n_1, n_2)$ and $(n_1 + n_2)$. Hence, the power of the test F_1^2 will always be between the power of an F with $(n_1 + n_2, n_3 + n_4)$ degrees of freedom and an F with $[\min(n_1, n_2), \min(n_3, n_4)]$ degrees of freedom. This gives some insight into the reason many properties for $F_{\alpha, 1}$ are so closely related to $|n_1 - n_2|$ and $|n_3 - n_4|$.

As was pointed out in Chapter I, $E(Z_{\alpha, 1}^r)$ can be used to calculate

the higher non-central moments of the variable Z. This is accomplished here for $i = 2$ and 4 , $\alpha = .05$, and $r = 2$ and 3 . The standard formulae are then used to calculate the standard deviation and the skewness. The cases that were considered are some of the special cases of those used from the 3^3 and 5^3 designs given in Tables 5 and 11. Those cases used are given in Table 12, along with the calculated values of the mean,

the standard deviation, and the skewness. It was hoped that the standard deviation would offer additional information to be used in discriminating between F_1^2 and F_1^4 ; however, such is not the case. But, this investigation further confirms that the F_1^4 approximate test is, in general, more

"conservative" than it needs to be. As was expected, the distribution of $Z_{\alpha, 1}$ is highly skewed when the mean is near zero or one.

The distribution of a random variable like Z_{α} [given in equation (6)] is next to impossible to obtain, even though all of its moments can be

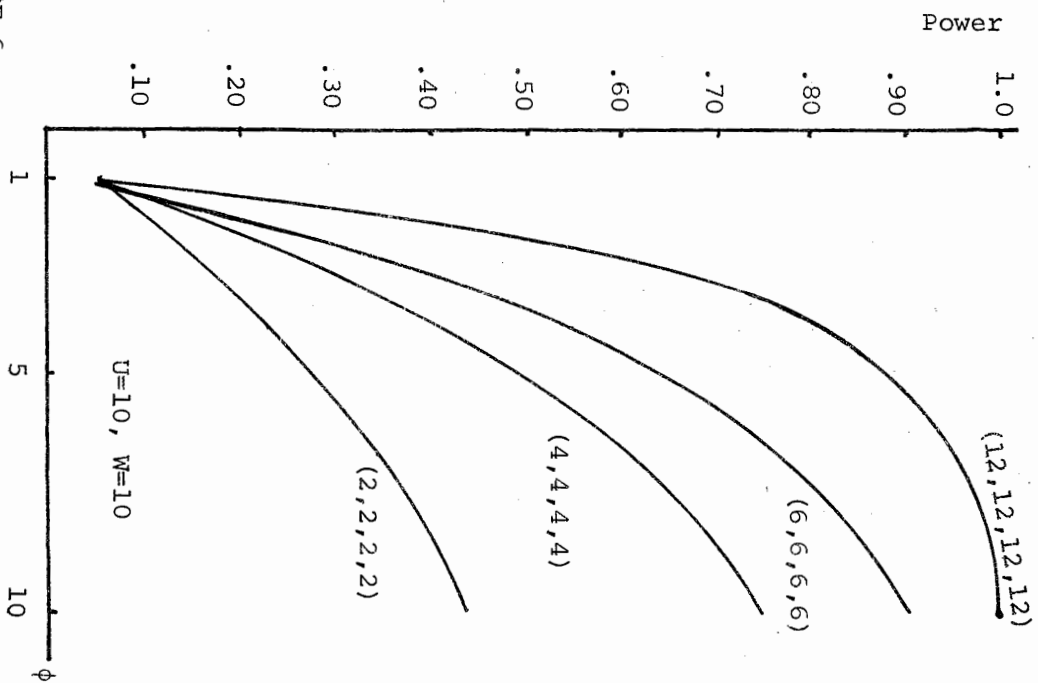
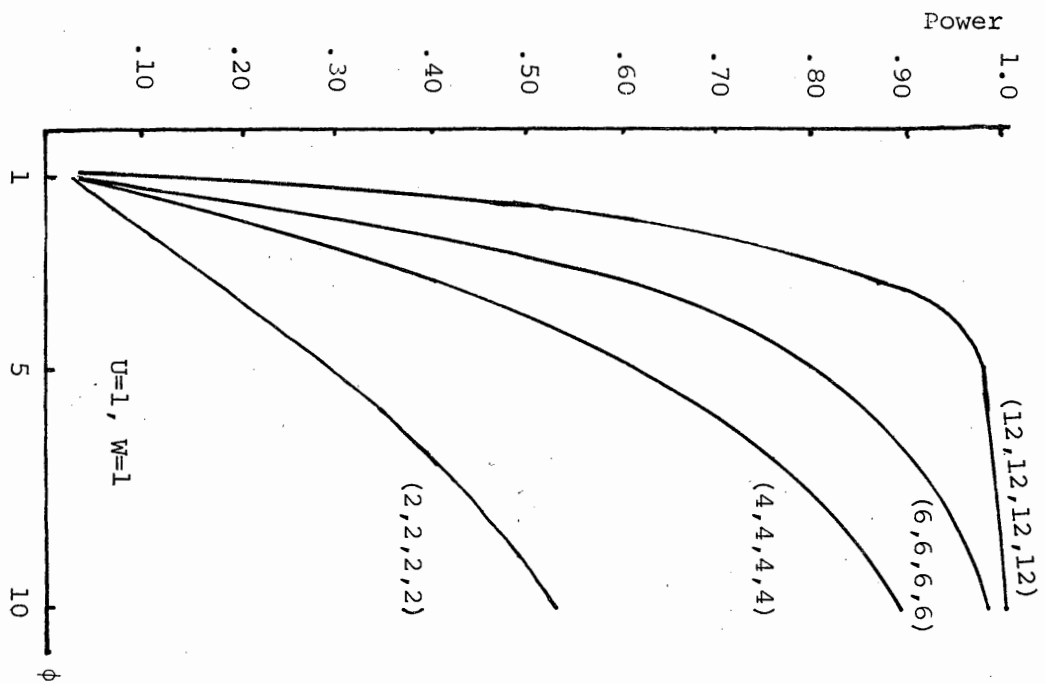


FIGURE 6
Power of the F'_2 -Test; Nominal $\alpha=.05$

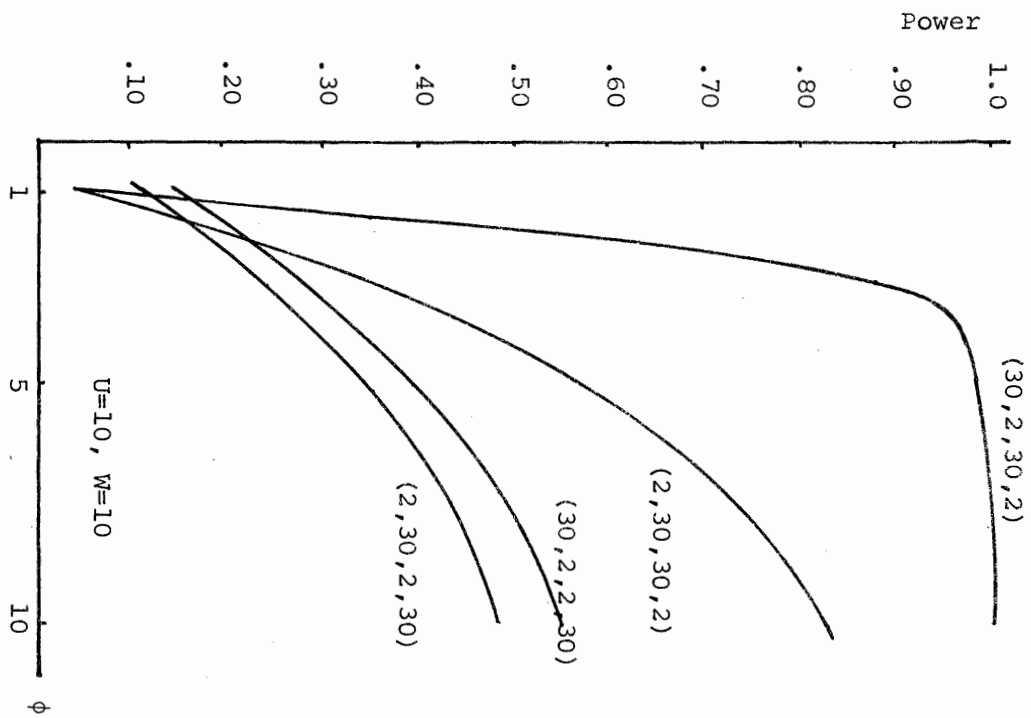
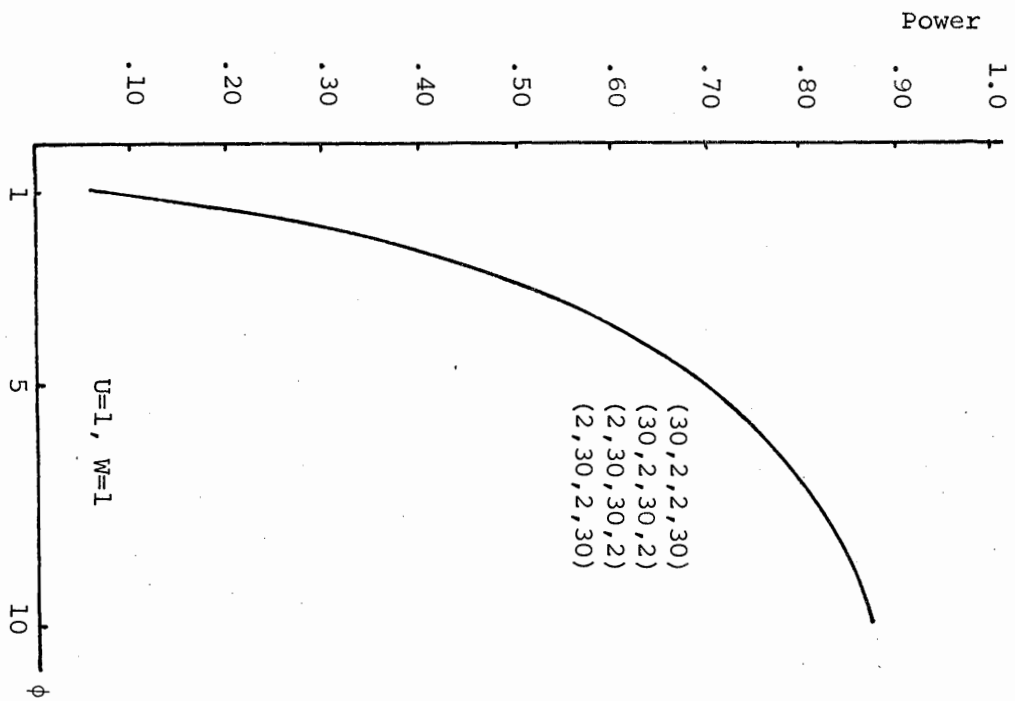


FIGURE 7
Power of the F_2' -Test; Nominal $\alpha=.05$

TABLE 12

Mean, Standard Deviation, and Skewness of the Approximate Test Procedures F'_2 and F'_4
 $\sigma^2_{\alpha\beta\gamma}=1$, $n(\text{replications})=1$, Nominal $\alpha=.05$

				F' ₂			F' ₄		
				Mean	Standard Deviation	Skewness	Mean	Standard Deviation	Skewness
$\sigma^2_{\alpha\beta\gamma}$	$\sigma^2_{\alpha\beta}$	$\sigma^2_{\alpha\gamma}$	σ^2_{α}						
3 ³ -Design									
.01	.01	.01	0	.0348	.0276	1.6430	.000095	.00015	1.4780
.01	.01	.01	1	.4741	.3087	.0069	.0976	.1110	.5164
.01	4	4	0	.0422	.0692	2.4160	.0370	.0443	2.5830
.01	4	4	1	.0731	.1080	1.9820	.0715	.0662	1.9850
.01	4	4	4	.1990	.2226	1.0640	.1885	.1066	1.1570
.01	4	4	10	.4290	.3115	-.0009	.3794	.1235	.5731
1	1	1	0	.0378	.0535	2.2990	.0058	.0047	.6258
1	1	1	1	.1410	.1637	1.3670	.0525	.0323	.0008
1	1	1	10	.6480	.3397	-.6880	.4921	.1809	-1.8600
5 ³ -Design									
.01	.01	.01	0	.0330	.0632	4.2210	.0031	.0024	.2750
.01	.01	.01	1	.9574	.1537	-4.4190	.8616	.2578	-2.9800
.01	4	4	0	.0479	.1295	3.8700	.0487	.0428	2.2630
.01	4	4	1	.1772	.2769	1.6030	.1803	.0883	1.0320
.01	4	4	4	.5412	.4030	-.1816	.5434	.1023	.1141
.01	4	4	10	.8200	.3183	-1.6700	.8208	.0552	-.3233
1	1	1	0	.0426	.1131	4.0660	.0458	.0473	2.5220
1	1	1	1	.4616	.3897	.1356	.4700	.1203	.2579
1	1	1	10	.9696	.1396	-5.4590	.9704	.0124	-.9506

computed. However, some general properties of the distribution of Z^α are as follows:

1. Z^α has a continuous density function on the interval $(0, 1)$.
2. The density of Z^α is uni-nodal.
3. If $E(Z^\alpha)$ is near zero or one, the density of Z^α is highly

skewed.

The distribution of Z^α can be approximated by a Beta (α^*, β^*) by fitting the first two moments. For convenience, let the random variable $X \sim \text{Beta}(\alpha^*, \beta^*)$. Then X has density function

$$f(x) = \frac{\Gamma(\alpha^* + \beta^*)}{\Gamma(\alpha^*)\Gamma(\beta^*)} x^{\alpha^*-1} (1-x)^{\beta^*-1} \quad 0 < x < 1.$$

The mean and variance of X are given as

$$E(X) = \frac{\alpha^*}{\alpha^* + \beta^*}, \quad \text{Var}(X) = \frac{\alpha^* \beta^*}{(\alpha^* + \beta^*)^2 (\alpha^* + \beta^* + 1)} \quad (59)$$

Then the parameters α^* and β^* can be approximated by equating $E(Z^\alpha) = E(X)$ and $\text{Var}(Z^\alpha) = \text{Var}(X)$, where $E(Z^\alpha)$ and $\text{Var}(Z^\alpha)$ are computed by means of

equation (26).

As a numerical example, this was accomplished for the case that is the third line of Table 12 ($\alpha = .05$) for both F_1^2 and F_1^4 . The results

are as follows:

F_1^2	F_1^4
$\alpha^* = .314$	$\alpha^* = .637$
$\beta^* = 7.13$	$\beta^* = 16.6$

(60)

In conclusion, the distribution of $Z_{\alpha,2}$ when the null hypothesis [relation (10)] is true, is such that the probability that $Z_{\alpha,2}$ will be

α^*	β^*	$P(0 < X < .05)$	$P(.05 < X < .10)$	$P(.10 < X < .50)$	$P(.50 < X < 1.00)$
.5	.5	.5949	.1723	.2307	.0021
.5	.17	.8101	.1296	.0603	.0000

TABLE 13
Probability Intervals for a Beta Distributed Random
Variable (X) with Parameters α^* and β^*

Probability intervals for the two cases are given in Table 13. The nearest tabulated cases are for $(\alpha^* = .5, \beta^* = 7)$ and $(\alpha^* = .5, \beta^* = 17)$. except that the tables for the Incomplete Beta are not that extensive. Probability intervals could be calculated for these two cases, of Z_{α} is very bad.

out that there are cases where the Beta approximation to the distribution While the fit in these two cases seems to be adequate, it should be pointed

$$E\left(Z_3^{.05,2}\right) = .00148$$

$$E\left(Z_3^{.05,4}\right) = .000493$$

central moments in each case are: respectively, .0016 and .000455. The true values for the third non- were calculated using the results in (60). Those values found are, As a check to see how "good" this fit is, the third non-central moments

"close" to its mean is "large". In fact it can be stated, in general, that (for $\alpha = .05$) $P(Z_{\alpha, 2} < .05) > .5$. Therefore, it is felt that F'_2 is a useful test statistic and can be used with some degree of "confidence". To reiterate, F'_2 is preferred to F'_4 because $Z_{\alpha, 4}$ has, in general, a more highly skewed distribution than $Z_{\alpha, 2}$. Hence, when the null hypothesis is true, the test based on F'_4 is, in general, more conservative than the test based on F'_2 , and also there is a corresponding reduction in power.

APPENDIX I

Let v_1, v_2, v_3, v_4 be independent mean square estimates of the

respective variances $\theta_1, \theta_2, \theta_3, \theta_4$, each based on n_1 degrees of free-

dom such that $n_1 v_1 / \theta_1 \sim \chi^2$ -square with n_1 degrees of freedom. It is

well known that

$$x = \frac{v_1}{\theta_1} \frac{1}{2} \sim \frac{u}{n} \sim F(n_1, n_2)$$

and

$$y = \frac{v_3}{\theta_3} \frac{3}{4} = \frac{w}{m} \sim F(n_3, n_4)$$

are independent central F-statistics, where $u = v_1/v_2$, $w = v_3/v_4$, $m = \theta_3/\theta_4$, and $u = \theta_1/\theta_2$. The joint density of x and y is

$$f(x, y) = \frac{C_1 C_2}{\frac{n_1}{n_3} \frac{1}{2} \frac{n_3}{n_1-2} \frac{x}{2} \frac{n_3}{n_3-2} \frac{y}{2}} \frac{\left[1 + \frac{n_1}{n_3} x \right] \frac{1}{2} \frac{n_1}{n_1+n_2}}{\left[1 + \frac{n_3}{n_4} y \right] \frac{1}{2} \frac{n_3}{n_3+n_4}} \quad (A.1)$$

$$y > 0, x > 0$$

where

$$C_1 = \frac{\Gamma\left(\frac{n_1}{2} + \frac{n_2}{2}\right) \Gamma\left(\frac{n_1}{2}\right) \Gamma\left(\frac{n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right) \Gamma\left(\frac{n_2}{2}\right) \Gamma\left(\frac{n_1}{2}\right)} \quad \text{and} \quad C_2 = \frac{\Gamma\left(\frac{n_3}{2} + \frac{n_4}{2}\right) \Gamma\left(\frac{n_3}{2}\right) \Gamma\left(\frac{n_4}{2}\right)}{\Gamma\left(\frac{n_3}{2}\right) \Gamma\left(\frac{n_4}{2}\right) \Gamma\left(\frac{n_3}{2}\right)}.$$

Let $u = x \cdot v$ and $w = y \cdot w$. The Jacobian of transformation is $\frac{1}{u \cdot w}$.

Then the joint density of u and w is

$$g(u, w) = \frac{c_1 c_2 \left(\frac{u}{n_1}\right)^2 \left(\frac{w}{n_3}\right)^2 \left(\frac{u}{n_2}\right)^2 \left(\frac{w}{n_4}\right)^2 \left[1 + \frac{u}{n_1}\right] \left[1 + \frac{w}{n_3}\right]}{\frac{u}{n_1} \frac{w}{n_3} \frac{u}{n_2} \frac{w}{n_4} \left[1 + \frac{u}{n_1}\right] \left[1 + \frac{w}{n_3}\right]}$$

$$u > 0, w > 0.$$

Let

$$s = \frac{\frac{u}{n_1}}{\frac{u}{n_2} + 1} \quad \text{and} \quad t = \frac{\frac{w}{n_3}}{\frac{w}{n_4} + 1}.$$

The Jacobian of this transformation is

$$\left(\frac{u}{n_2}\right) \left(\frac{w}{n_4}\right) \frac{1}{(1-s)(1-t)}.$$

Therefore the joint density of s and t is

$$h(s, t) = \frac{c_1 c_2 s^2 (1-s) \frac{u}{n_1} \frac{w}{n_3} \left[1 + \frac{u}{n_1}\right] \left[1 + \frac{w}{n_3}\right]}{\frac{u}{n_1} \frac{w}{n_3} \frac{u}{n_2} \frac{w}{n_4} \left[1 + \frac{u}{n_1}\right] \left[1 + \frac{w}{n_3}\right]}.$$

$$0 < s < 1, 0 < t < 1.$$

APPENDIX II

Let v_1, v_2, v_3, v_4 be independent mean square estimates of the

respective variances $\theta_1, \theta_2, \theta_3, \theta_4$, each based on n_i degrees of

freedom such that $n_i v_i / \theta_i \sim \chi^2$ -square with n_i degrees of freedom. Let

$$u = \frac{v_2 \theta_4}{v_4 \theta_2} \sim \frac{x}{k} \sim F(n_2, n_4)$$

and

$$w = \frac{v_3 \theta_4}{v_4 \theta_3} \sim \frac{y}{k} \sim F(n_3, n_4)$$

where $x = v_2/v_4, y = v_3/v_4, x = \theta_2/\theta_4, y = \theta_3/\theta_4$. It is well known

that u and w are central F -statistics but they are not independent.

Therefore, the joint density of x and y must be derived from the original

χ^2 -square variates $n_i v_i / \theta_i$ ($i = 2, 3, 4$). Let $z_i = n_i v_i / \theta_i$ ($i = 2, 3, 4$).

Then it is well known that the joint density of the z_i is

$$g(z_2, z_3, z_4) = \frac{1}{n_2^{n_2-2} n_3^{n_3-2} n_4^{n_4-2}} e^{-\frac{z_2}{n_2} - \frac{z_3}{n_3} - \frac{z_4}{n_4}}$$

$$z_2 > 0, z_3 > 0, z_4 > 0,$$

where

$$c_2 = \Gamma\left(\frac{n_2}{2}\right) \Gamma\left(\frac{n_3}{2}\right) \Gamma\left(\frac{n_4}{2}\right)$$

Now consider the transformation

$$\left(\frac{2}{n_2 n_3 n_4} \right)^T = {}^T C_1$$

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$$C_1 \left(\frac{u^4}{z^2} \right) \left(\frac{u^4}{z^3} \right) \frac{z}{u^3} \frac{z}{u^2} \frac{z}{u^2} \frac{z}{u^2} = \frac{C_2 \frac{z}{u^3} \frac{z}{u^2} \frac{z}{u^2} \frac{z}{u^2} \left[1 + \frac{u^4}{z^3} + \frac{u^4}{z^2} \right]}{\frac{z}{u^4} \frac{z}{u^3} \frac{z}{u^2}}$$

variable z , yields as the joint density of x and y

Therefore, making the transformation and integrating out the undesired

$$\begin{array}{r} \lambda \quad x \quad u \\ \lambda \quad x \quad u \\ \hline \lambda \quad x \quad u \end{array}$$

The Jacobian transformation is

$$\frac{x^{\bar{v}}_u}{x^{\bar{z}}_u} = z_{\bar{z}} \quad \bar{v}_{\bar{z}} = z$$

$$\frac{\lambda^{\bar{v}} u}{\lambda^{\varepsilon} z^{\varepsilon} u} = \varepsilon_z \quad ; \quad \frac{\bar{v}_z^{\varepsilon} u}{\varepsilon_z^{\bar{v}} u} \cdot \lambda = \frac{\bar{v}_\Lambda}{\varepsilon_\Lambda} = \Lambda$$

$$Z = \frac{\partial}{\partial Z} \frac{\partial^2 Z_u}{\partial Z_u^2} \cdot X = \frac{\partial}{\partial \Lambda} = X$$

Let

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13. ABSTRACT

In statistical methods, it is often desirable to test the relation

$$\sum_{k=1}^m c_{\theta}^m = \sum_{j=k+1}^p c_{\theta}^j$$

holds among the variances θ_i . An assumption made in this paper is that there exists independent mean square estimates V_i of the variances θ_i such that $n_i V_i / \theta_i$ follows the chi-square distribution for each $i = 1, 2, \dots, p$. Also each side of the above equation is assumed to be strictly greater than zero. An approximate test of the above relation is Satterthwaite's approximate F-test.

A method by Cochran is generalized to obtain true type I error and power of the approximate F-tests for the following relations:

$$\begin{aligned} \theta_1 &= \theta_2 + \theta_3 - \theta_4 \\ \theta_1 + \theta_2 &= \theta_3 + \theta_4 \\ \theta_1 - \theta_2 &= \theta_3 - \theta_4 \\ \theta_2 + \theta_3 - \theta_4 &= \theta_1 \end{aligned}$$

continued on next sheet.

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continued from first sheet.

The alternatives are that the left side of the relation is greater than the right side.

These results are then applied to testing the main effects in a three factor factorial design with the random effects. The different methods that are available to carry out this test are all special cases of the approximate F-test. The various methods are compared, and the "best" testing procedure is chosen. In addition, the properties of this "best" test are investigated.