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Technical Report No. SMU-DS-TR-192
Department of Statistics NASA Contract

April 1985

Research sponoroed by
National Aeronautics and Space Administration
Contract No. NCC 9-9

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Summary

Maximum likelihood estimators for model parameters in linear structural relationships having correlated measurement errors are presented. Comparisons are made between asymptotic mean squared errors of least squares and maximum likelihood estimators of the slope parameter. These comparisons indicate that the least squares estimator is only preferable to the maximum likelihood estimator when the slope parameter is sufficiently small relative to the ratio of error standard deviations; however, unlike structural relationships having independent measurement errors, the region of preference is not simply given by an upper bound on this scaled slope parameter.

Some key words: Errors in variables; Maximum likelihood; Regression

1. Introduction

Linear structural relationships are linear models $Y = \alpha + \beta X$ between two stochastic variates (Y,X) in which both variates are measured with error:

$$y_i = Y_i + v_i, \quad x_i = X_i + u_i \ , \qquad i = 1,2,\ldots,n \ . \tag{1.1}$$

If one assumes that X, u, and v are mutually independent with $X \sim N(\mu_X,\sigma_X^2)$, $u \sim N(0,\sigma_u^2)$, $v \sim N(0,\sigma_v^2)$ and that $\lambda = \sigma_v^2/\sigma_u^2$, the error variance ratio, is known then the maximum likelihood estimators of the intercept and slope parameters are (Kendall and Stuart 1977, Chapter 29) $\tilde{\alpha} = \bar{y} - \tilde{\beta}\bar{x}$ and

$$\begin{aligned} \tilde{\beta} &= s(\lambda) + \mathrm{sign}(s_{xy})\{s^2(\lambda) + \lambda\}^{\frac{1}{2}} \ , \\ s(\lambda) &= (s_{yy} - \lambda s_{xx})/(2s_{xy}) \ , \end{aligned} \tag{1.2}$$

where s_{yy}, s_{xx}, and s_{xy} are the sample variances and covariance, respectively. These estimators of α and β are consistent and asymptotically normal.

If β^* is any estimator of β which is only a function of the sample variances and covariance, the variance of an intercept estimator of the form $\alpha^* = \bar{y} - \beta^*\bar{x}$ is given by

$$\mathrm{var}(\alpha^*) = n^{-1}\sigma_u^2(\beta^2 + \lambda) + \{n^{-1}(\sigma_X^2 + \sigma_u^2) + \mu_X^2\} \ \mathrm{var}(\beta^*) \ . \tag{1.3}$$

Since equation (1.3) is a monotonic function of $\mathrm{var}(\beta^*)$, discussion of the asymptotic mean squared error properties of least squares and structural model estimators is usually confined to discussion of those properties for the slope estimators.

Lakshminarayanan and Gunst (1984) derive the asymptotic variance of the structural model slope estimator (1.2):

$$\mathrm{var}(\tilde{\beta}) = n^{-1}\{(\beta^2 + \lambda)\gamma + \lambda\gamma^2\} \ , \tag{1.4}$$

where $\gamma = \sigma_u^2/\sigma_X^2$ is the noise-to-signal ratio for the observable predictor variable x. They compare the asymptotic mean squared errors of the least squares and the maximum likelihood estimators and show that for large sample sizes the maximum likelihood estimator has a smaller mean squared error than least squares unless the variance ratio is very small. The magnitude of reduction in mean squared error using the maximum likelihood estimator increases with the magnitude of β for a fixed variance ratio.

Simultaneous measurement of two or more physical quantities using a single measuring device can lead to structural relationships in which the measurement errors are correlated. For example, an engine analyzer may measure several automobile exhaust emissions simultaneously. The purpose of this paper is to investigate asymptotic properties of the maximum likelihood estimator of the slope parameter in structural models in which measurement errors are correlated.

2. Correlated Measurement Errors

Assuming that $\mathrm{corr}(u_i, v_i) = \rho$, $i = 1,2,\ldots,n$ and that $\mathrm{corr}(u_i, v_j) = 0$ for $i \neq j$, the maximum likelihood estimator of the slope parameter is

$$\tilde{\beta} = s(\lambda,\theta) + \mathrm{sign}\{u(\theta)\}\{s^2(\lambda,\theta) + t(\lambda,\theta)\}^{\frac{1}{2}} , \tag{2.1}$$

where $\theta = \rho\lambda^{\frac{1}{2}}$,

$$s(\lambda,\theta) = (s_{yy} - \lambda s_{xx})/\{2u(\theta)\} ,$$

$$t(\lambda,\theta) = (\lambda s_{xy} - \theta s_{yy})/u(\theta) ,$$

$$u(\theta) = s_{xy} - \theta s_{xx} .$$

When the measurement errors on y and x are correlated the variance of an intercept estimator $\alpha^* = \bar{y} - \beta^*\bar{x}$ is still a monotonic function of $\mathrm{var}(\beta^*)$, the only difference in equation (1.3) is the addition of the term $-2\beta\theta$ to the expression in the first set of parentheses. Using statistical differentials (Serfling 1980, Chapter 6), the asymptotic variance (to $O(n^{-2})$) of a first-order approximation to $\tilde{\beta}$ is

$$n^{-1}\gamma\{(\beta - \theta)^2 + (1 + \gamma)(\lambda - \theta^2)\} \ . \tag{2.2}$$

The estimator (2.1) is consistent so equation (2.2) also represents an approximate asymptotic mean squared error of $\tilde{\beta}$.

Using the same technique, the asymptotic mean squared error of the least squares estimator $\hat{\beta} = s_{xy}/s_{xx}$ of β can be shown to equal

$$n^{-1}\gamma(1 + \gamma)^{-2}\{(1 + n\gamma)(\beta - \theta)^2 + (1 + \gamma)(\lambda - \theta^2)\} \ . \tag{2.3}$$

Let $R = \mathrm{var}(\tilde{\beta})/\mathrm{mse}(\hat{\beta})$. Then

$$R = (1 + \gamma)^2\{\Psi + (1 + \gamma)\}/\{(1 + n\gamma)\Psi + (1 + \gamma)\} \ , \tag{2.4}$$

where $\Psi = (\beta - \theta)^2/(\lambda - \theta^2)$. Note that the true slope parameter and the variance ratio only affect R through the parameter Ψ. One can rewrite Ψ as

$$\Psi = (\phi - \rho)^2/(1 - \rho^2) \ , \tag{2.5}$$

where $\phi = \beta\lambda^{-\frac{1}{2}}$ is the slope parameter scaled by a ratio of the error standard deviations. The inequality $R > 1$ leads to the following condition for which the asymptotic variance of the maximum likelihood estimator exceeds the asymptotic mean squared error of the least squares estimator:

$$\Psi < (1 + \gamma)(2 + \gamma)/(n - 2 - \gamma). \tag{2.6}$$

3. Asymptotic Comparisons and Simulation Results

Figure 1 contains graphs of the mean squared error ratio (2.4) for a structural model having independent measurement errors. The mean squared error ratio is plotted as a function of the noise-to-signal ratio γ for several (positive) values of the scaled slope parameter ϕ and for two sample sizes. Since $\rho = 0$, $\Psi = \phi^2$ for these graphs. The mean squared error ratio is seen to be a monotonically decreasing function of ϕ, leading to preference for the maximum likelihood estimator for large ϕ, especially when the sample size is not small.

Mandel (1984) describes conditions under which the least squares estimator should be preferred to an alternative estimator defined by the joint solution of his equations (10a) and (10b). The solution of these equations is the maximum likelihood estimator (2.1). One set of conditions that he cites for preference of the least squares estimator is: (i) the true model is linear, (ii) $\rho = 0$, and (iii) $|\phi| << 1$. Inequality (2.6) of the previous section quantifies Mandel's condition (iii) and it also shows that the upper bound on ϕ is a function of both γ and n. Thus preference for the least squares estimator depends on the values of the scaled slope parameter, the noise-to-signal ratio, and the sample size. In application this result suggests that the least squares estimator would only be preferable to the maximum likelihood estimator when ϕ is close to zero and the sample size is not too large.

Figure 2 contains graphs of the mean squared error ratio (2.4) for a structural model having correlated errors ($\rho = .5$). The mean squared error ratio is again plotted as a function of the noise-to-signal ratio γ for several values of ϕ; however, since $\rho \neq 0$, $\Psi \neq \phi^2$. Note that for these models the mean squared error ratio initially increases with ϕ and then decreases for ϕ sufficiently large. The region of preference for the maximum likelihood estimator is not a simple bounded function on ϕ; it is a bounded function $c(\gamma, n)$ on Ψ, where the bound is given in inequality (2.6).

A simulation similar to that reported in Lakshminarayanan and Gunst (1984) was conducted to assess the effects of correlation on the agreement between simulated mean squared errors and the asymptotic variance formula (2.2). Essentially the same model configurations were used in this simulation with the addition of several values of the correlation between the errors:

$$\alpha = 0, \quad \beta = 3, \quad \mu_X = 0, \quad \sigma_X^2 = 5, \quad \sigma_v^2 = 5;$$

$$\lambda = .1, .2, .5, 1, 2, 5, 10; \quad \gamma = .1, 1;$$

$$\rho = 0, .1, .3, .5, .7, .9.$$

Two hundred replications of each model configuration were simulated for samples of size 20, 50, 100, and 200. For samples of size 50 or greater there were only two instances where the relative error between the average $\tilde{\beta}$ and the true value of β exceeded 4%: $n = 50$, $\lambda = 10$, $\gamma = 1$, and $\rho = 0, .1$ (relative error = 7%). Thus, as in the previous study, samples of size 50 or greater provide good agreement between estimates from equation (2.1) and the true value of β. This close an agreement was observed for all values of the correlation coefficient included in the simulation.

Ratios of simulated mean squared errors ($\sum(\tilde{\beta}_i - \beta)^2/200$) to the asymptotic variance (2.2) indicate that much larger sample sizes are needed for the asymptotic variance formula to adequately represent the observed sample variability. With sample sizes of $n = 200$ the sample to asymptotic mean squared error ratios ranged between .83 and 1.15 for the smaller noise-to-signal ratio and between .83 and 1.31 for the larger one. Approximately half of the sample mean squared errors were within 5% of the asymptotic values for the smaller noise-to-signal ratio, compared to only about one-fourth for the larger noise-to-signal ratio. No consistent pattern with the correlation coefficient was observed for the ratios of sample to asymptotic mean squared error.

This research was supported in part by the National Aeronautics and Space Administration and by the Office of Naval Research.

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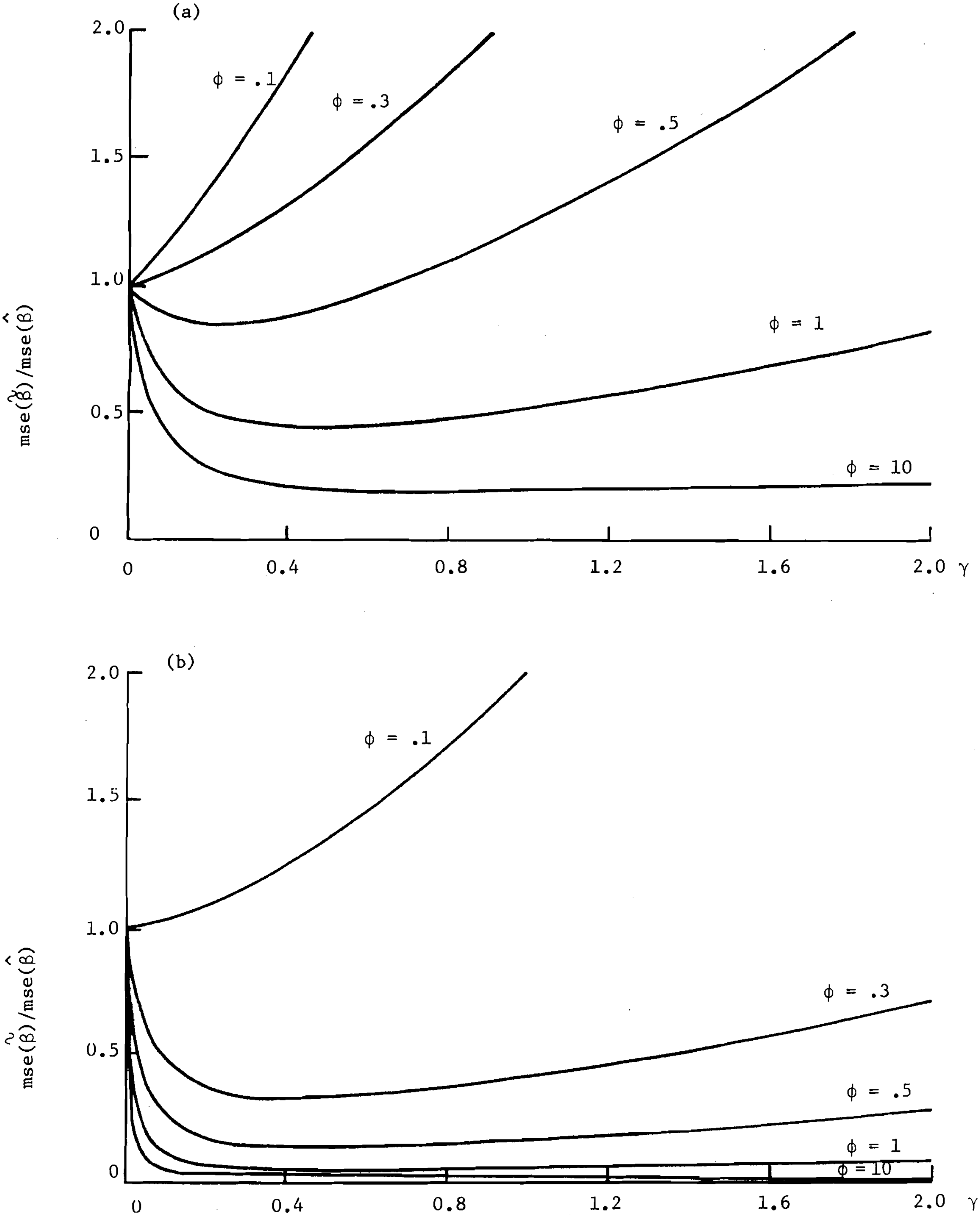

Fig. 1. Ratio of asymptotic mean squared errors, $\rho = 0$, (a) $n = 50$, (b) $n = 200$.

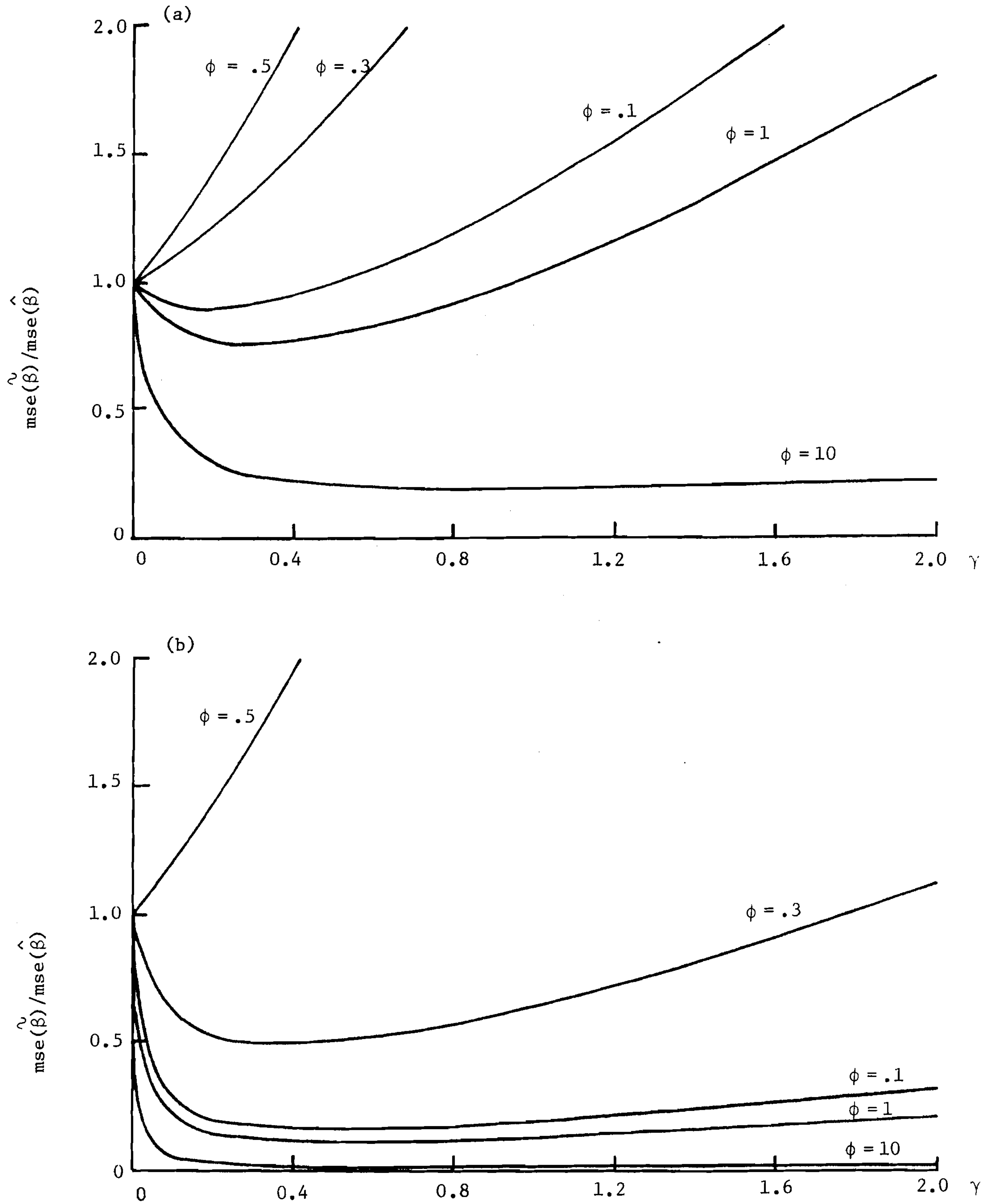

Fig. 2. Ratio of asymptotic mean squared errors, ρ = .5, (a) n = 50, (b) n = 200.