NONCOOPERATIVE N-PERSON GAME THEORY (N ≥ 2) USING EXPECTED

RELATIVE DESIRABILITY CRITERION

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ABSTRACT

Considered is noncooperative discrete N-person game theory for $N \geq 2$ (where the players choose their strategies separately and independently). Payoffs can be of a very general nature and are not necessarily numbers. However, each player is able to quantitatively specify the relative desirability of the possible game outcomes (N-dimensional, a payoff to each player) according to his preferences. That is, he specifies a positive number for each outcome and these numbers are such that their ratios quantitatively represent the relatively desirability of the corresponding outcomes to this player. For each player, the criterion is the expected relative desirability to him of what occurs for the game, and an optimum mixed strategy maximizes the minimum value of this criterion over all possible mixed strategies that could be used by the other players. An optimum solution is obtained by use of classical minimax game theory. Practical implications of applying this N-person game theory are examined. Also, a possible approach for developing quantitative functions to provide the relative desirability numbers is outlined.

INTRODUCTION AND DISCUSSION

Considered is the case of N players (N ≥ 2) with finite numbers of strategies. A player chooses his strategy separately and independently of the strategy choice by the other player(s). The players use mixed strategies. That is, each player specifies selection probabilities (sum to unity, with a unit probability possible) for his possible strategies and randomly selects the strategy he uses according to these probabilities. Optimization of a mixed strategy involves optimum choice of the probability values in the mixed strategy. (See, for example, ref. 1 for a more thorough discussion of concepts in discrete game theory.)

A combination of N payoffs, one to each player, occurs for every possible combination of a choice of a strategy by the players. These combinations of N payoffs are possible game outcomes. The number of possible strategy combinations is

$$\prod_{i=1}^{N} r(i),$$

where $r(i) \geq 2$ is the number of strategies for player i, (i = 1,...,N).

The payoffs can be of an extremely general nature. In fact, some payoffs might not be numbers (could identify categories, etc.). However, the outcomes are such that each player, according to his preferences, is able to quantitatively state the relative desirability of the possible game outcomes. That is, he can specify a positive number for each outcome such that the ratios of these numbers quantitatively represent the relative desirability of the corresponding outcomes. These numbers, one

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for each possible outcome, provide the basis for the solution to the game.

Determination of the positive numbers that quantitatively establish the relative desirability of the outcomes to a player should often be achievable through use of paired comparisons. That is, for each two outcomes a player quantitatively states the ratio of how much he prefers one outcome compared to the other. This needs to be done so that there is consistency among ratios. The paired comparison approach can require huge effort, in the number of comparisons to be made and in establishing consistency among the ratios.

Substantial application effort can be avoided if a suitable numerical function of the N payoffs can be developed. That is, this function furnishes a positive number for each possible outcome and these numbers establish relative desirability for the player considered. However, development of a satisfactory function of this nature can be exceedingly difficult. One possible approach to such a development, which occurs as two steps, is outlined in the final section. The first step consists of developing a suitable numerical function for ranking the outcomes according to increasing desirability. An appropriate increasing function of the ranking function provides the function that yields the positive numbers being sought.

To be emphasized is the fact that quantitative relative evaluation of the desirability of outcomes not only considers the payoff to the player doing the evaluation but also considers the corresponding payoffs to the other players. Thus, for each player, his evaluation quantitatively measures

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the relative desirability of what can happen for the game, including what occurs for the other players.

A suitable criterion for each player would be the expected desirability of what occurs for the game (after a mixed strategy choice by every player). Only relative desirability is determined, but expected desirability equals some fixed (unknown) positive constant multiplied times expected relative desirability. Thus, a mixed strategy that is optimum with respect to expected relative desirability is also optimum for expected desirability. Consequently, the criterion used by each player is the expected value (over the random choice involved in using mixed strategies) of the relative desirability for what occurs in the game.

An optimum mixed strategy for a player is one that maximizes the minimum value of his expected relative desirability over the possible mixed strategies that could be used by the other players (or player). When cooperation does not occur, this is the best that can be assured by a player on the basis of the criterion used. A way of determining optimum mixed strategies with the required property is described in the next section.

The effort needed to apply the game theory of this paper can be very large. First, N payoffs are to be evaluated for each possible combination of strategies. The number of such combinations can be huge even when all of $N, r(1), \ldots, r(N)$ have moderate sizes. For example, let $N = 8$ and $r(1) = \ldots = r(N) = 10$. Then, there are 10^8 strategy combinations and 10^9 payoffs are to be evaluated. Second, determining the positive numbers for quantitatively establishing relative desirability of outcomes can also require huge effort. However, this effort is greatly reduced when

a suitable function is developed for providing these numbers, although development of such a function can require appreciable effort. Third, solution of a game can require much time and expense, due to the huge number of strategy combinations involved. In summary, much of the application difficulty is due to the massiveness of the number of possible outcomes, but substantial difficulty can arise in obtaining the positive numbers for establishing quantitative relative desirability.

A description of a procedure for determining optimum mixed strategies is given in the next section. A discussion of an approach toward development of relative desirability functions with quantitative properties occurs in the final section.

SOLUTION PROCEDURE

The same method of solution is used for all players and is stated for player i. Given for player i is a positive number for every possible game outcome, where the ratios of these numbers quantitatively represent the relative desirability (to player i) of the corresponding outcomes. That is, one of these positive numbers occurs for each possible strategy combination.

Expression of these numbers for player i in matrix form, according to the corresponding strategy combinations, is useful in stating the solution procedure. Here, the rows represent the strategies for player i and the columns represent the combinations of strategies for the other players.

For definiteness, in the combinations the strategy for the other player with lowest designation number occurs first,..., the strategy for the other player with the highest designation number occurs last. The resulting matrix is called the desirability matrix for player i.

Player i desires to maximize the expected value of the number in his desirability matrix that occurs when all players have chosen mixed strategies. More specifically, player i wishes to determine a mixed strategy for his use that maximizes the minimum of the expected values of this number over all possible mixed strategies for the other players.

Optimum mixed strategies with this property can be determined by direct use of results that have been developed for two-person zero-sum game theory with an expected value basis (for example, see the results for this case in ref. 1). This is accomplished by considering the desirability matrix of player i to be his payoff matrix in a two-person zero-sum game with an expected value basis. An optimum strategy for player i in this two-person game has the required property of maximizing the minimum value of the criterion used over all possible mixed strategies for the other players.

A potential application difficulty is that the number of columns in the desirability matrix for player i, which equals

$$\prod_{\substack{j=1 \\ j \neq i}}^{N} r(j),$$

may be too large for use of existing computer programs for solution of

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two-person zero-sum games.

DESIRABILITY FUNCTION DEVELOPMENT

Nearly complete freedom is available to player i in selecting the
positive numbers on which his quantitative relative desirability for the
game outcomes is based. However, this does not imply that any choice made
for these numbers is necessarily satisfactory. On the contrary, great
care is required to determine a suitable set of these numbers. To have
so much freedom is a valuable asset, but only if a careful and wise choice
is made.

Development of a quantitative relative desirability function can be
approached in two steps, the first of which is not so difficult. The first
step is to develop a suitable preference function for ordering the outcomes
according to increasing desirability. The values of this preference
function are real numbers expressed in the same unit. The second step
is to develop an increasing function of the values of the preference
function. This second function yields the positive numbers that are sought.
Of course, all increasing functions of a preference function are equivalent
preference functions, but the function sought has the additional property
that ratios of its values provide a quantitative measure of the relative
desirability of the outcomes furnishing these values.

The preference functions considered are of an elementary nature and
like those given in ref. 2 for noncooperative N-person game theory based on
a percentile criterion. Only a preference function is needed for applying

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N-person percentile game theory, which is a strong application advantage. However, more information would seem to be encompassed in use of a criterion based on expected relative desirability for the game outcomes.

Four examples of elementary preference functions are given to illustrate considerations in development of suitable preference functions. As a standardization, the preference function for player 1 is considered in all the examples. This function is written $D_1(p_1,\ldots,p_N)$, where $(p_1,\ldots,p_N)$ denotes a general outcome. Increasing value of this function represents increasing desirability to player 1 (equal value represents equal desirability).

Without great loss of generality, and for simplicity, values of the payoff p_i to player i are expressed as real numbers in the same unit. These numbers are such that increasing values of p_i represent nondecreasing (usually increasing) desirability to player i. The forms considered for $D_1(p_1,\ldots,p_N)$ are always such that, in their use, any differences in the types of units used for the payoffs cause no difficulties in the statement $D_1(p_1,\ldots,p_N)$.

The first example involves additive changes in the p_i and the situation is such that an addition of A to p_1 has the same desirability to player 1 as the combination of an addition of $e_i w_i a_i$ to p_i for $i=2,\ldots,N$. Here, a_i is positive, e_i is 1 or -1 (depending on whether an increase or decrease is to occur), $w_2 + \ldots + w_N = 1$ with all $w_i \geq 0$, and A can be positive or negative. The preference function

$$D_1^{(a)}(p_1,\ldots,p_N) = p_1 + A \sum_{i=2}^{N} e_i p_i / a_i$$

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should be suitable, since $D_1^{(a)}(p_1 + A, p_2, \ldots, p_N)$ equals

$$p_1 + A + A \sum_{i=2}^{N} e_i p_i / a_i \equiv p_1 + A \sum_{i=2}^{N} e_i (p_i + e_i w_i a_i) / a_i$$

equals $D_1^{(a)}(p_1, p_2 + e_2 w_2 a_2, \ldots, p_N + e_N w_N a_N)$ for all possible values

of $p_1, \ldots, p_N$.

The second example involves multiplicative changes in the p_i and requires that they all have positive values. The situation is such that multiplication of p_1 by the positive factor $(1 + B)$ has the same desirability to player 1 as the combination of multiplying p_i by the factor $(1 + e_i v_i)^{w_i}$ for $i = 2, \ldots, N$. Here, $0 < v_i < 1$, the value of B can be positive or negative, $e_i = 1$ or -1 (depending on whether an increase or decrease is to occur), and $w_2 + \ldots + w_N = 1$, with all $w_i \geq 0$. The preference function

$$D_1^{(m)}(p_1, \ldots, p_N) = \log_{10} p_1 + \sum_{i=2}^{N} \{[\log_{10}(1 + B)]/[\log_{10}(1 + e_i v_i)]\} \log_{10} p_i$$

should be suitable, since $D_1^{(m)}[(1 + B)p_1, p_2, \ldots, p_N]$ equals

$$\log_{10}(1 + B) p_1 + \sum_{i=2}^{N} \{[\log_{10}(1 + B)]/[\log_{10}(1 + e_i v_i)]\} \log_{10} p_i$$

$$\equiv \log_{10} p_1 + \sum_{i=2}^{N} \{[\log_{10}(1 + B)]/[\log_{10}(1 + e_i v_i)]\} \log_{10}(1 + e_i v_i)^{w_i} p_i$$

equals $D_1^{(m)}[p_1, (1 + e_2 v_2)^{w_2} p_2, \ldots, (1 + e_N v_N)^{w_N} p_N]$ for all positive values

of $p_1, \ldots, p_N$.

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The third example involves both addition and multiplication, where changes in $p_1, \ldots, p_J$ are by addition and changes in $p_{J+1}, \ldots, p_N$ are by multiplication (with $p_{J+1}, \ldots, p_N$ all positive). The situation is such that an addition of A to p_1 has the same desirability to player 1 as the combination of an addition of $e_j w_j a_j$ to p_j for $j = 2, \ldots, J$, and multiplication of p_j by $(1 + e_j v_j)^{w_j}$ for $j = J + 1, \ldots, N$. Here, $w_2 + \ldots + w_N = 1$ with all $w_j \geq 0$, the value of A can be positive or negative, and the e_j, a_j, v_j have the same properties as for the first and second examples. The preference function

$$D_1^{(am)}(p_1, \ldots, p_N) = p_1 + A \sum_{j=2}^{J} e_j p_j / a_j + A \sum_{j=J+1}^{N} [\log_{10}(1 + e_j v_j)]^{-1} \log_{10} p_j$$

should be suitable, since $D_1^{(am)}(p_1 + A, p_2, \ldots, p_N)$ equals

$$p_1 + A + A \sum_{j=2}^{J} e_j p_j / a_j + A \sum_{j=J+1}^{N} [\log_{10}(1 + e_j v_j)]^{-1} \log_{10} p_j$$

$$\equiv p_1 + A \sum_{j=2}^{J} e_j (p_j + e_j w_j a_j) / a_j + A \sum_{j=J+1}^{N} [\log_{10}(1 + e_j v_j)]^{-1} \log_{10}(1 + e_j v_j)^{w_j} p_j$$

equals $D_1^{(am)}[p_1, p_2 + e_2 w_2 a_2, \ldots, p_j + e_j w_j a_j, (1 + e_{J+1} v_{J+1})^{w_{J+1}} p_{J+1}, \ldots,$

$(1 + e_N v_N)^{w_N} p_N]$ for all permissible values of $p_1, \ldots, p_N$.

The final example also involves both addition and multiplication, but p_1 changes by multiplication. Again, as a standardization, the changes in $p_2, \ldots, p_J$ are by addition and the changes in $p_{J+1}, \ldots, p_N$ are by multiplication (with $p_1, p_{J+1}, \ldots, p_N$ all positive for this case). The situation is such that multiplication of p_1 by the positive factor $(1 + B)$ has the same desirability to player 1 as the combination of an addition of $e_j w_j a_j$ to p_j for $j = 2, \ldots, J$, and multiplication of p_j by $(1 + e_j v_j)^{w_j}$ for $j = J + 1, \ldots, N$. Here, $w_2 + \ldots + w_N = 1$ with all $w_j \geq 0$, the value of B can be positive or negative, and the e_j, a_j, v_j have the same properties as for the first two examples. The preference function

$$D_1^{(ma)} (p_1, \ldots, p_N) = \log_{10} p_1 + [\log_{10}(1 + B)] \sum_{j=2}^{J} e_j p_j / a_j$$

$$+ \sum_{j=J+1}^{N} \{ [\log_{10}(1 + B)] / [\log_{10}(1 + e_j v_j)] \} \log_{10} p_j$$

should be suitable, since $D_1^{(ma)} [(1 + B) p_1, p_2, \ldots, p_N]$ equals

$$\log_{10}(1 + B) p_1 + [\log_{10}(1 + B)] \sum_{j=2}^{J} e_j p_j / a_j$$

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$$+ \sum_{j=J+1}^{N} \{[\log_{10}(1 + B)]/[\log_{10}(1 + e_j v_j)]\}\log_{10}P_j$$

$$\equiv \log_{10}P_1 + [\log_{10}(1 + B)] \sum_{j=2}^{J} e_j(P_j + e_j w_j a_j)/a_j$$

$$+ \sum_{j=J+1}^{N} \{[\log_{10}(1 + B)]/\log_{10}(1 + e_j v_j)]\} \log_{10}(1 + e_j v_j)^{w_j}P_j$$

equals $D_1^{(ma)}[P_1, P_2 + e_2 w_2 a_2, \ldots, P_J + e_J w_J a_J, (1 + e_{J+1}v_{J+1})^{w_{J+1}}P_{J+1}, \ldots,$

$(1 + e_N v_N)^{w_N}P_N]$ for all permissible values of $P_1, \ldots, P_N$.

Development of a suitable preference function converts the original N-dimensional problem into a one-dimensional problem. A plot of the values of the preference function over the possible game outcomes, according to increasing function value, provides a graphical representation of the preference function values.

Development, on the same representation, of a graph that is increasing in the corresponding value for the preference function graph is a way of obtaining the positive values being sought. The values for this new graph should be positive and their ratios should quantitatively represent the relative value, to the player, of outcomes which yield these values. In principle, all game values for the preference function graph would be considered in development of the new graph. In practice, however, specific consideration might be limited to a relatively small fraction of the values

for the preference function graph, with isolated values that are repre-
sentative of the various parts being considered. This provides isolated
representative values for the new graph and interpolation among these
isolated values furnishes the overall new graph.

REFERENCES

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