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CORRELATION BETWEEN TWO HOTELLING'S T^2

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1. Introduction:

Let B be a p×p symmetric matrix having the Wishart distribution

(1.1) $$W_p(B|I|f)dB = C_{pf}|B|^{(f-p-1)/2} e^{-1/2 \operatorname{tr}B} dB ,$$

where

(1.2) $$C_{pf}^{-1} = 2^{fp/2} \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma\left(\frac{f+1-i}{2}\right) ,$$

and dB stands for the product of the differentials of the $p(p+1)/2$ distinct elements of B. Let $\underline{x}$ and $\underline{y}$ be two vector variables of p components, distributed independently of B, and also independently of each other, as

(1.3) $$\frac{1}{(2\pi)^{p/2}} e^{-1/2\, \underline{x}'\underline{x}} d\underline{x} ,$$

and

(1.4) $$\frac{1}{(2\pi)^{p/2}} e^{-1/2\, \underline{y}'\underline{y}} d\underline{y}$$

respectively. While considering the problem of multivariate statistical outliers, Wilks (1963) used statistics of the type,

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(1.5) $\quad r = |B + \underline{y}\underline{y}'|/|B + \underline{x}\underline{x}' + \underline{y}\underline{y}'| \quad , \quad s = |B + \underline{x}\underline{x}'|/|B + \underline{x}\underline{x}' + \underline{y}\underline{y}'|$.

He has remarked that the exact distribution (joint) of r and s is complicated and has given the expected values, variances and covariance of r and s. Unfortunately, his expressions for the variance and covariance are in error. The purpose of this note is to derive the exact joint distribution of r and s and to give correct expressions for the moments.

2. Joint distribution:

In the joint distribution of B, $\underline{x}$ and $\underline{y}$, make the following transformation

$$(2.1) \qquad \begin{aligned} A &= B + \underline{x}\,\underline{x}' + \underline{y}\,\underline{y}' , \\ \underline{u} &= A^{-1/2}\,\underline{x} , \\ \underline{\nu} &= A^{-1/2}\,\underline{y} , \end{aligned}$$

where $A^{-1/2}$ is any matrix such that $A^{-1/2} \cdot A^{-1/2} = A^{-1}$. The Jacobian of transformation from B to A is 1 and that from $\underline{x}$ to $\underline{u}$ or $\underline{y}$ to $\underline{\nu}$ is $|A|^{1/2}$ and hence, the joint distribution of A, $\underline{u}$ and $\underline{\nu}$ comes out as

$$(2.2) \qquad \frac{C_{pf}}{(2\pi)^p} \, |A|^{\frac{(f+2)-p-1}{2}} \, e^{-1/2\ \mathrm{tr}A} \cdot |I - \underline{u}\underline{u}' - \underline{\nu}\underline{\nu}'|^{\frac{f-p-1}{2}} \, dA\,d\underline{u}\,d\underline{\nu} \quad ,$$

as $\quad |B| = |A - A^{1/2}\underline{u}\underline{u}'\,A^{1/2} - A^{1/2}\underline{\nu}\underline{\nu}'\,A^{1/2}| = |A|\,|I - \underline{u}\underline{u}' - \underline{\nu}\underline{\nu}'|$.

This shows that A has a Wishart distribution of f+2 degrees of freedom and is independent of $\underline{u}$ and $\underline{\nu}$. Splitting the constant suitably, the joint distribution of $\underline{u}$ and $\underline{\nu}$ is

$$(2.3) \qquad \frac{\Gamma(f+1)}{(2\pi)^p\,\Gamma(f-p+1)} \, |I - \underline{u}\underline{u}' - \underline{\nu}\underline{\nu}'|^{\frac{f-p-1}{2}} \, d\underline{u}\,d\underline{\nu} \quad .$$

Observe that the statistics r, s of Wilks are given by

$$(2.4) \qquad r = \frac{|B+\underline{y}\underline{y}'|}{|B+\underline{x}\underline{x}'+\underline{y}\underline{y}'|} = \frac{|A-\underline{x}\underline{x}'|}{|A|} = |I - \underline{u}\underline{u}'| = 1 - \underline{u}'\underline{u} \quad ,$$

and

$$(2.5) \qquad s = \frac{|B+xx'|}{|B+xx'+yy'|} = 1 - \underline{\nu}'\underline{\nu} \quad .$$

Also observe that in (2.3)

$$(2.6) \qquad \begin{aligned} |I - \underline{uu}' - \underline{\nu\nu}'| &= (1 - \underline{u}'\underline{u})(1 - \underline{\nu}'\underline{\nu}) - (\underline{u}'\underline{\nu})^2 \\ &= rs - (\underline{u}'\underline{\nu})^2 \quad . \end{aligned}$$

In (2.3), transform from $\underline{\nu}$ to $\underline{w} = [w_1 , w_2 , \ldots, w_p]'$, by an orthogonal transformation

$$(2.7) \qquad \underline{w} = L\underline{\nu} \ ,$$

where

L is a p×p orthogonal matrix, whose last row is $\underline{u}'/\sqrt{\underline{u}'\underline{u}}$. The Jacobian of this transformation is $|L| = 1$ and $\underline{\nu}'\underline{\nu} = \underline{w}'\underline{w} = 1-s$. Also

$$(2.8) \qquad u'\nu = \underline{u}'L'L\underline{\nu} = [0 \ \ldots \ 0, \sqrt{u'u}\,]\underline{w} = \sqrt{1-r} \cdot w_p \quad .$$

The joint distribution of $\underline{u}$ and $\underline{w}$ is, therefore,

$$(2.9) \qquad \frac{\Gamma(f+1)}{(2\pi)^p \Gamma(f-p+1)} \left\{ rs - (1-r)w_p^2 \right\}^{\frac{f-p-1}{2}} d\underline{u}d\underline{w} \quad .$$

From $\underline{u}$, transform to $r = 1 - u'u$ and p-1 other variables $\phi_1, \phi_2, \ldots, \phi_{p-1}$ by

$$(2.10) \qquad \begin{aligned} u_1 &= (1-r)^{1/2}\cos\phi_1 \cos\phi_2 \ldots \cos\phi_{p-1} \ , \\ u_j &= (1-r)^{1/2}\cos\phi_1 \cos\phi_2 \ldots \cos\phi_{p-j} \sin\phi_{p-j+1} \\ & \qquad (j=2,\ 3,\ \ldots\ p) \end{aligned}$$

Similarly, transform from $\underline{w}$ to $s = 1 - \underline{w}'\underline{w}$ and p-1 other variables $\theta_1, \theta_2, \ldots, \theta_{p-1}$ by

$$(2.11) \qquad \begin{aligned} \nu_1 &= (1-s)^{1/2}\cos\theta_1 \cos\theta_2 \ldots \cos\theta_{p-1} \ , \\ \nu_j &= (1-s)^{1/2}\cos\theta_1 \cos\theta_2 \ldots \cos\theta_{p-j} \sin\theta_{p-j+1} \quad . \\ & \qquad (j=2,\ 3,\ \ldots,\ p) \end{aligned}$$

The Jacobian of transformation from $\underline{u}$ to $r, \phi_1, \dots \phi_{p-1}$ is

$$\frac{1}{2}(1-r)^{\frac{1}{2}p-1} \prod_{i=1}^{p-2} \cos^{p-i-1}\phi_i$$

and a similar expression in s and θ_i for the Jacobian of transformation from $\underline{w}$ to s and the θ's. Now θ_{p-1} and ϕ_{p-1} vary from 0 to 2π, the other θ's and ϕ's vary from $-\pi/2$ to $\pi/2$ while r and s vary from 0 to 1. Integrating out all the ϕ's and all θ's except θ_1, we obtain the joint distribution of r, s and θ_1 as

$$(2.12) \qquad \frac{\Gamma(f+1)}{4\pi\Gamma(p-1)\Gamma(f-p+1)} \left\{rs - (1-r)(1-s)\sin^2\theta\right\}^{\frac{f-p-1}{2}} \cos^{p-2}\theta \; drdsd\theta$$

where θ_1 is replaced by θ .

The joint distribution of r,s alone can now be obtained by integrating out θ but this does not seem to yield a manageable expression, as the bracket in (2.12) will have to be expanded in a series.

3. Moments of r,s .

Only the product moment of r and s is difficult to obtain. The mean and variance of r (or s) can be very easily obtained from the marginal distribution of r, which is related to the well-known Hotelling's T^2 by $r = \frac{1}{1+\left(\frac{T^2}{f+1}\right)}$. In the joint distribution of $\underline{u}$ and $\underline{\nu}$, given by (2.3), if we transform to $\underline{z} = [z_1, \dots, z_p]'$ from $\underline{\nu}$ by

$$(3.1) \qquad \underline{\nu} = (I - \underline{uu}')^{1/2}\underline{z} \quad ,$$

we shall find that $\underline{u}$ and $\underline{z}$ are independently distributed as

$$(3.2) \qquad K(\underline{u}|f)d\underline{u} = \frac{f}{\pi^{p/2}(f-p)} \cdot \frac{\Gamma(f/2)}{\frac{\Pi}{2}(f-p)} \left|I - \underline{uu}'\right|^{\frac{f-p}{2}} d\underline{u}$$

(3.3) and $K(\underline{z}|f-1)d\underline{z}$, respectively.

From (3.2), one can easily show that

$$E(r^h) = E(1-\underline{u}'\underline{u})^h = E|I - uu'|^h \tag{3.4}$$

$$= \frac{f(f+2h-p)}{(f-p)(f+2h)} \cdot \frac{\Gamma\left(\frac{f-p}{2}+h\right)\Gamma\left(\frac{f}{2}\right)}{\Gamma\left(\frac{f}{2}+h\right)\Gamma\left(\frac{f-p}{2}\right)}$$

This will also be the h^{th} moment of s by symmetry. This leads to

$$E(r) = \frac{f-p+2}{f+2}, \quad v(r) = \frac{2p(f-p+2)}{(f+2)^2(f+4)} \quad . \tag{3.5}$$

$$\begin{aligned} \text{Now } Cov(r,s) &= E\{(1-\underline{u}'\underline{u})(1-\underline{\nu}'\underline{\nu})\} - E(r)E(s) \\ &= E\{(1-\underline{u}'u)[1-\underline{z}(I-\underline{uu}')\underline{z}]\} - \{E(r)\}^2 \quad \text{by (3.1)} \\ &= E(r) - E\{r[\underline{z}'\underline{z}-(\underline{z}'\underline{u})^2]\} - \{E(r)\}^2 \end{aligned}$$

$$= E(r) - E(r)E(z'z) + E\{r(z'u)^2\} - \{E(r)\}^2 \quad , \tag{3.6}$$

as $\underline{z}$ and r are independent. Since $\underline{z}$ has the same distribution as $\underline{u}$ with f changed f-1 ,

$$E(z'z) = 1 - E(1 - \underline{u}'\underline{u}) \text{ with f replaced by f-1}$$

$$= \frac{p}{f+1} \tag{3.7}$$

Hence (3.6) reduces to

$$Cov(r,s) = \frac{-p(f-p+2)}{(f+1)(f+2)^2} + E\{r(z'u)^2\} \quad . \tag{3.8}$$

Now

$$E\{r(\underline{z}'\underline{u})^2\} = \int (1-u'u)(\underline{z}'\underline{u})^2 K(\underline{u}|f)K(\underline{z}|f-1)d\underline{u}d\underline{z} \tag{3.9}$$

where the integration is over the range of values of $\underline{u}$ and $\underline{z}$ such that $\underline{u}'\underline{u} \le 1$, $\underline{z}'\underline{z} \le 1$. Transform from $\underline{z}$ to $\underline{\xi} = [\xi_1, \ldots, \xi_p]$ by the transformation

$$\xi = L\underline{z} \quad ,$$

where L is already defined to be a p×p orthogonal matrix, whose last row is $\underline{u}'/\sqrt{\underline{u}'\underline{u}}$. Then,

$$\underline{z}'\underline{u} = \underline{z}'L'L\underline{u} = \xi'L\underline{u} = \xi_p\sqrt{\underline{u}'\underline{u}} = (1-r)^{1/2}\xi_p \quad .$$

Hence (3.9) reduces to

$$(3.10) \qquad \int r(1-r)K(\underline{u}|f)d\underline{u} \cdot \int \xi_p^2 \, K(\underline{\xi}|f-1)d\underline{\xi} = E(r-r^2) \cdot \frac{1}{p} E(\underline{\xi}'\underline{\xi})$$, due

to symmetry of the distribution of $\underline{\xi}$. Now $\underline{\xi}$ has the same distribution as $\underline{u}$ with f replaced by f-1 and hence finally, (3.10) reduces to

$$\frac{p(f-p+2)}{(f+4)(f+2)} \cdot \frac{1}{f+1} \quad .$$

The covariance between r and s, therefore, is (from (3.5))

$$(3.11) \qquad \frac{-2p(f-p+2)}{(f+1)(f+2)^2(f+4)} \quad .$$

Remarks:

Wilks considers a sample of size n and has a Wishart matrix based on n-1 degrees of freedom as deviations are from the sample means. He then removes two observations as outliers and thus his (n-1)-2 corresponds to our f . His E(r) agrees with our result, with this correspondence but the other moments are in error.

Reference

Wilks, S. S. (1963). "Multivariate Statistical Outliers," Sankhyā, Vol. 25, p. 407-426.