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LIFE ESTIMATION AND RENEWAL THEORY

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1. Introduction: In an ordinary renewal process, an item is replaced, on failure, by an identical item. The life distributions of items successively used are, therefore, the same. If such a renewal process is, therefore, observed for a long time, we can obtain a sample of "life-times" from this life distribution and from this, we can estimate the parameters of the life distribution or the distribution itself. But this is, in many practical situations, not feasible because the life of the item under consideration may be large and one has to wait very very long before one gets a reasonably large sample. It is therefore essential to look for some alternative method of estimation.

In practice one comes across situations, where not one but several identical renewal processes are simultaneously occurring. For example, consider an Army depot of spare parts which caters to the needs of several vehicles or tanks operating under similar conditions. This depot has to supply spare parts like gear boxes to these vehicles for replacements. Each vehicle or tank thus corresponds to a renewal process and all these renewal processes are going on simultaneously. Let t be the time elapsed since the beginning of a renewal process and v_t denote the time up to the next renewal measured from t. Then v_t is called the Forward Recurrence Time. Similarly, the time since the last renewal to the present

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time t is called the Backward Recurrence Time v_t. If all the identical simultaneous renewal processes are observed from a given instant of time to their next renewals, we shall get a large sample from the distribution of v_t. The depot can easily get this information from the vehicles and then making use of the relationship between the distributions of v_t and $F(x)$, the distribution function (d.f.) of the life of the item, one can easily estimate $F(x)$ or its parameters. The explicit relation and the procedure are well-known but for the sake of completeness are described briefly in this paper. The main aim of this paper is, however, to extend this idea to the estimation of a whole matrix of life distributions associated with a Markov Renewal Process (M.R.P.)

2. Markov Renewal Processes (M. R. P.) A Markov renewal process (see Pyke, 1961) is one in which there are m, a finite number of states and the process makes transitions from one state to another according to a transition probability matrix $P = p_{ij}$ $(i,j=1,2,\ldots,m)\left(\sum p_{ij} = 1\right)$, but unlike a Markov chain, the holding time in any state is not fixed but is a random variable. If the process makes a transition from state i to state j, the d.f. of the time spent in state i, before it makes a transition to state j, may depend on both i and j. We shall denote this by $F_{ij}(x)$. We thus have $F = \left[F_{ij}(x)\right]$, the matrix of all such life distributions. Such a situation can occur in practice, if an item is not replaced by an identical item but by a "different" one. Items can be repaired and then used again. But the life distribution of a repaired item may not be the same as the original. In fact, for certain types of items, they

can be repaired a number of times and used repeatedly. We have thus different types of items, with different life distributions but all these items serve the same purpose on the vehicle. The types of items are then the states of the MRP and the lives of these different types will be the holding times in the different states. The supply of the items to the vehicle is governed by the matrix P and the Markov Chain, imbedded, in the process, could be of zero-order also. We shall define subsequently, forward recurrence times for the different transitions of an M.R.P. and indicate their use in the estimation of the moments of the life-distributions of the matrix F.

3. Ordinary Renewal Process. If μ_1' is the mean life of an item in an ordinary renewal process and $F(x)$ is the d.f. of the life, then it has been proved (see Cox 1962) that the limiting distribution of v_t, for large t, has

$$\int_0^x \{1-F(x)\} \,/\, \mu_1' \quad dx \tag{3.1}$$

as its d.f. It is therefore easy to see that the r^{th} raw moments M_r' of this distribution is connected to the moments μ_r' of F(x), by the relation

$$M_r' = \mu_{r+1}' \,/\, (r+1)\mu_1' \tag{3.2}$$

Also, the ordinate of the probability dendity function(p.d.f.) of the distribution of v_t, at x=0 is $^1/\mu_1'$. Thus, from a large sample from the distribution of v_t (which can be obtained by observing a large number of simultaneously occurring renewal processes, from a given instant of time till a renewal occurs on each of them), one can construct a histogram and fit a smooth curve to obtain the p.d.f. of v_t. μ_1' then can be estimated from its ordinate at zero and the higher moments μ_{r+1}' can then be estimated from the sample moments, of this distribution of Vt, using (3.2).

The limiting distribution of v_t and of u_t, the backward recurrence time is the same. The procedure is thus the same, whether are uses v_t or u_t to estimate the life distribution F9x).

4. Forward Recurrence time for an M.R.P.

We set $Q_{ij}(x) = p_{ij}F_{ij}(x)$. We define the Laplace-Steiltjes transform (L.-S.T.) of $Q_{ij}(x)$ by

$$q_{ij}(s) = \int_0^\infty e^{-sx}\, dQ_{ij}(x) \tag{4.1}$$

and denote by q(s), the mxm matrix of the quantities q_{ij},(i,j=1,2,...,m). Let z_t denote the state of a M.R.P. at time t. We now define

$R_{jk}^{i}(t_o,x)$ = Prob.(that the first transition, after time t_o form the beginning of the process, is from state j to state k and that the holding time in state j, measured form t_o onwards is $\leq x \mid Z_o = i$) ....(4.2)

i,j,k=1,2,...,m.

We further define the following L.-S.T.'s.

$$r_{jk}^{i}(t_o,s) = \int_0^\infty e^{sx}\, d_x\ R_{jk}^{i}(t_o,x) \qquad ....(4.3)$$

and

$$*r^{i}_{jk}(s_o,s) = \int_0^\infty e^{-s_o t_o} \, r^{i}_{jk}(t_o,s)dt_o \qquad \text{............(4.4)}$$

It has been proved, by the author (197oa), elsewhere that

$$*r^{i}_{jk}(s_o,s) = \left\{I-q(s_o)\right\}^{-1}_{ij} \quad (q_{jk}(s) - q_{jk}(s_o) \,/\, (s_o-s), \text{......(4.5)}$$

where

$\{A\}_{ij}$ denotes the element in the i^{th} row and j^{th} column of the matrix A. $R^{i}_{jk}(t_o,x)$ is the d.f. of x, the forward recurrence time of the M.R.P., measured from t_o and corresponds to the transition from j to k.

Its limiting behavior as $t_o \to \infty$, can be studied by letting $s_o \to o$ in the L.-S.T. For this one needs the expansion of the matrix $(I-q(s_o))^{-1}$ in powers of s. This was obtained by the author in collaboration with Y. P. Gupta, in an earlier paper (1967). An easier method of expanding this by using generalized inverse of a matrix, in a paper sent for publication (1970b). The final result is that

$$*r^{i}_{jk}(s_o,s) \to \frac{d_j}{\sum_{j=1}^{m} d_j n_j} \left\{ \frac{p_{jk} - q_{jk}(s)}{S} \right\} = r_{jk}(s) \qquad (4.6)$$

as $s_o \to o$, where $d' = \{d_1,d_2,\ldots,d_m\}$ is the vector of the stationary s state probabilities of the imbedded Markov Chain iè

$$\underline{d}' \, \underline{P} = d' \qquad (4.7)$$

and η_j is the mean of the d.f. $\sum_{j=1}^{m} Q_{jk}(x)$. As is obvious, the limiting distribution does not depend on i, the initial state of the M.R.P. (4.6) is the L.-S.T. of

$$R_{jk}(x) = \frac{d_j}{\sum_{j=1}^{m} d_j \eta_j} \int_0^x \left\{ p_{jk} - Q_{jk}(x) \right\} dx \tag{4.8}$$

and this represents the d.f. of the forward recurrence time. If $\mu_{jk}^{(r)}$ denote the raw moments of the d.f. $F_{jk}(x)$, (r=1,2,....)

$$q_{jk}(s) = \left\{ 1 - \frac{s}{1!} \mu_{jk}^{(1)} + \frac{s^2}{2!} \mu_{jk}^{(2)} - \ldots\ldots\ldots \right\} \cdot p_{jk} \tag{4.9}$$

and so, from (4.6).

$$r_{jk}(s) = \frac{d_j p_{jk}}{\sum_{j=1}^{m} d_j \eta_j} \left\{ \mu_{jk}^{(1)} - \frac{s}{2!} \mu_{jk}^{(2)} + \frac{s^2}{3!} \mu_{jk}^{(3)} - \ldots \right\} \ldots (4.10)$$

The constant term in (4.10) is not 1, showing that $R_{jk}(x)$ is not a d.f., in the strict sense of the term. We shall however continue to call, the coefficient of $(-1)^r s^r / r!$ in (4.10) as the r^{th} moment of the d.f. $R_{jk}(x)$ and shall denote it by $M_{jk}^{(r)}$. Then from (4.10)

$$m_{jk}^{(r)} = \frac{d_j p_{jk}}{\sum_{j=1}^{m} d_j \eta_j} \cdot \frac{\mu_{jk}^{(r+1)}}{(r+1)} \tag{4.11}$$

(r=1,2,....)

This is an analogous expression to (3.2) of the ordinary renewal process. When r=0, we get

$$\frac{d_j p_{jk}}{\sum_{j=1}^{m} d_j \eta_j} \quad \mu_{jk}^{(1)} ,$$

showing that the total probability $R_{jk}(\infty)$ is not one but

$d_j p_{jk} \mu_{jk}^{(1)} / \sum_{j=1}^{m} d_j \eta_j$. Also observe that

$$\eta_j = \sum_{j=1}^{m} p_{jk} \mu_{jk}^{(1)} \tag{4.12}$$

The p.d.f. corresponding to the d.f. $R_{jk}(x)$ is

$$\frac{d_j p_{jk}}{\sum_{j=1}^{m} d_j \eta_j} \quad (1-F_{jk}(x)) \tag{4.13}$$

and its ordinate at x=0, is $d_j p_{jk} / \sum_{j=1}^{m} d_j \eta_j$.................... (4.14)

The estimation procedure for the matrix F is now outlined in the next section

5. Estimation of the matrix F of life distributions:

Suppose that a sufficiently large amount of time has elasped since the beginning of a number of M.R.P.'s which are simulatneously in progress.

Then begin to observe the M.R.P.'s from a given instant of time till the next transition occurs on each of them. Record the initial states and the final states of each of these transitions and the time from the instant of observation to the occurrence of the transition for each of the M.R.P. Suppose N is the number of the M.R.P's. Let N_{jk} be the number of processes for which the transition was from state j to state k ($j,k=1,2,\ldots m$) and the forward recurrence times for these be $x_1(j,k)$, $x_2(j,k),\ldots,$ $x_{N_{jk}}(j,k)$. Then it is obvious that N_{jk} / N estimates $d_j p_{jk} \mu_{jk}^{(1)} / \sum_{j=1}^{m} d_j n_j$. If a histogram is constructed, from these $x(j,k)$'s and asmooth curve is fitted, the ordinate at $x=0$ of the curve will estimate $d_j p_{jk} / \sum_{j=1}^{m} d_j n_j$. From these two, $\mu_{jk}^{(1)}$ can easily be estimated, and the higher moments $\mu_{jk}^{(r)}$ of $F_{jk}(x)$ can then be estimated from the

sample moments

$$\sum_{\alpha=1}^{N_{jk}} x_\alpha^r (j,k) / N_{jk}$$

and the relation (4.11). This procedure must be repeated for every (j,k) to estimate the moments of the life-distributions in the matrix F. It is assumed here that the matrix P of transition probabilities is known, and that all the states are positive recurrent.

6. Backward Recurrence time of an M.R.P.

In the case of an ordinary renewal process, the limiting distributions of the forward and backward recurrence times are the same and hence the life distribution can be estimated from either. But this symmetry is obviously not present for an M.R.P., because not only the time but also the type of the transition enter into our consideration.

Let us consider an M.R.P. at time t_o, from its beginning. The time elapsed since the last transition is then the backward recurrence time and if this last transition was into state j, we define

$$S_{ij}(t_o,x) = \text{Prob}(Z_{t_o}=j;\ \begin{matrix}\text{time elapsed since the last}\\ \text{transition } x\end{matrix} \mid Z_o=i) \tag{6.1}$$

It can be shown, from the theory of Markovian renewals that the above distribution function of the backward recurrence time (corresponding to state j) tends, as $t_o \to \infty$, to

$$\frac{d_j}{\sum_{j=1}^{m} d_j n_j} \int_o^x \left\{1 - \sum_{k=1}^{m} Q_{jk}(x)\right\} dx \tag{6.2}$$

It does not depend on the initial state i of the M.R.P. The L.-S.T. of the above limiting d.f. is

$$\frac{d_j}{\sum_{j=1}^{m} d_j n_j} \left(1 - \sum_{k=1}^{m} q_{jk}(s)\right) \tag{6.3}$$

Since the d.f. (6.2) involves $\sum_{k=1}^{m} Q_{jk}(x)$ and not the Q^1_{jk} S alone, it is obvious that the backward recurrence times will not be useful in estimating the parameters of the individual life distributions, but only of the sum

$\sum_{k=1}^{m} Q_{jk}(x)$. But, in practice, the d.f.'s of the holding times rarely depend on both the initial and final states of a transition. They depend on one of them only. Thus in the case of the spare parts illustration cited earlier, the life distribution of a component will depend only its type and not on the type of component to be used immediately after it. So F_{jk} will depend on j alone and so even the backward recurrence time can be used for estimation purposes, in exactly a similar way as the forward recurrence time.

7. Remarks: It is assumed throughout this paper that the p_{ij} are known. If they are also unknown, the method described in this paper is of no use. The only alternative in that case is to observe one M.R.P. for a very long time and count the different transitions and the different holding times. Moore and Pyke (1968) have described the estimation procedure in this case.

By the method described in this paper, it is possible to estimate the moments of the different F_{jk}. only if all the N_{jk} are non-null and sufficiently large. This will be so, only if one observes a very large number of simultaneous Markov renewal processes.

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