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AN ALTERNATIVE TO AN F-TEST ON VARIANCES

BY

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TABLE OF CONTENTS

LIST OF TABLES

CHAPTER I

INTRODUCTION

Let Π_1 and Π_2 be two populations with variances σ_1^2 and σ_2^2 respectively. The variance of these two populations can be compared by testing the null hypothesis, $\sigma_1^2 = c\sigma_2^2$, where c is some specified constant, against an alternative hypothesis which may be either two-sided, such as $\sigma_1^2 \neq c\sigma_2^2$, or one-sided, such as $\sigma_1^2/\sigma_2^2 > c$. The tests considered throughout this paper will be for one-sided alternative hypotheses of the form above, and therefore will be upper-tail tests. Furthermore, since in the F-test and in the alternative test that shall be considered the value of the constant, c, does not affect the procedure, the null hypothesis will be that of equal variances and the alternative will be $\sigma_1^2 > \sigma_2^2$.

Now if the two populations are normally distributed, sufficiency and the monotone likelihood ratio property combine to make the F-test for comparing variances uniformly most powerful for testing $\sigma_1^2/\sigma_2^2 = 1$ versus $\sigma_1^2/\sigma_2^2 > 1$. This property, along with the efficiency of the statistics involved in forming the F-statistic have caused this test to be widely used and favored.

Because it is so favored and because the normality of the populations is questionable in many situations, there have been several papers directed to the problem of finding out what happens to the F-test when the populations involved are not normal distributions. Work of this nature has led to the notion of robustness for a test. The term robust will be used to denote a

The second kind of error occurs when the statistician assumes that the distributions are non-normal (but not specified) when in fact they are normal. Having assumed that the F-test is not appropriate and not assuming some other specific distribution leads the statistician to a non-parametric approach. In doing this the statistician is denying himself the power of the F-test when it would be properly useful. Because this second kind of error is more obvious in its defects and because the procedures involved in the non-parametric approach are more difficult than those for the F-test the first kind of error is probably more common.

Since both kinds of error need to be considered in developing an alternative to the F-test, both the robustness and the power of the alternative criterion for comparing variances will be considered.

The test statistic for the alternative test will be developed in Chapter II. Next, in Chapter III, the critical points will be obtained and an approximation to the power function for the alternative test under normality will be developed. This approximation will then be compared with the power function for the F-test. The effects of non-normality upon both the alternative test and the F-test will be considered in Chapter IV and Chapter V will be a summary of the results contained in the first four chapters. In the appendix some material used in the course of this paper is included, along with the unabridged data from the sampling experiments.

CHAPTER II

THE V-STATISTIC

The use of order statistics in estimates of location and scale parameters was developed extensively by Lloyd (1952). Lloyd used the theory of least squares to obtain his estimators and unfortunately these estimators were, in general, rather difficult to use. Downton (1966) decided that mathematical tractability was a more desirable feature for estimators based on order statistics than least squares optimality and so considered estimators with polynomial coefficients. One of the estimators that Downton (1966) gives is for the standard deviation of a normal population, namely

$$\sigma^* = \left[2\sqrt{\pi} \sum_{k=1}^{n} \left(k - \frac{n+1}{2}\right) z_k\right] / [n(n-1)] \qquad (1)$$

where n is the sample size and z_k denotes the k^{th} order statistic from the sample. When the underlying population is normal Downton shows that σ^* is an unbiased estimator of the standard deviation, is uncorrelated with the mean and is highly efficient even for small sample sizes.

Barnett, Mullen and Saw (1967) note that σ^* is a constant multiple of one of a general class of statistics which are linear combinations of order statistics with coefficients of ordered Tchebycheff-Hermite polynomials as weights. Thus, based on a paper by Saw and Chow (1966), Barnett, Mullen and Saw give a table of coefficients and the formulas necessary to obtain the first four moments about the origin for σ^*. This leads to a formula for the variance of σ^*, namely,

$$\text{var}(\sigma^*) = \sigma^2\{n(\pi/3 + 2\sqrt{3} - 4) + (6 - 4\sqrt{3} + \pi/3)\}\big/[n(n-1)] \tag{2}$$

which agrees with the results given by Downton (1966) even though it is expressed differently.

Barnett, Mullen and Saw (1967) also suggest using σ^* in a test of the hypotheses $\mu = \mu_0$ versus $\mu > \mu_0$ in a normal population. Their test statistic is

$$Y = (\bar{X} - \mu_0)\sqrt{n}/\sigma^* \ . \tag{3}$$

This test is only slightly less powerful than the usual t-test and the power of the Y test is easily obtained using a cumulative normal density.

Thus σ^* recommends itself as a candidate for use in test criteria proposed as alternatives to the uniformly most powerful tests.

Returning to the original problem, let Π_1 and Π_2 both be normally distributed, let n be the size of the sample drawn from population one and m be the size of the sample drawn from population two, and let the ordered sample from Π_1 be denoted by $z_1 < z_2 < \cdots < z_n$, while the ordered sample from Π_2 is denoted by $w_1 < w_2 < \cdots < w_m$. Then

$$\sigma_1^* = \left\{2\sqrt{\pi}\sum_{k=1}^{n}\left(k - \frac{n+1}{2}\right)z_k\right\}\Big/n(n-1) \tag{4}$$

and

$$\sigma_2^* = \left\{2\sqrt{\pi}\sum_{k=1}^{m}\left(k - \frac{m+1}{2}\right)w_k\right\}\Big/m(m-1) \tag{5}$$

are unbiased estimators of σ_1 and σ_2 respectively.

Thus when Π_1 and Π_2 are independent, the ratio

$$v = \sigma_1^*/\sigma_2^* \tag{6}$$

is the ratio of the u.b.e. of σ_1 and σ_2 respectively which are independent. Therefore v may be used as a test statistic for the hypothesis $\sigma_1 = \sigma_2$ versus $\sigma_1 > \sigma_2$. This is equivalent to testing equality of variances with the hypothesis $\sigma_1^2 = \sigma_2^2$ versus $\sigma_1^2 > \sigma_2^2$. The linear nature of the σ^*'s leads one to feel that v may be affected less by non-normality than the usual F-statistic. In order to verify this some properties of this alternative test for equality of variances need to be considered. But first an alternative form for the V-statistic will be given.

David (1968) discusses the estimator σ^* in terms of a statistic called "Gini's mean difference" which was discussed by Helmert in 1876 and was not new at that time according to the history included in David's article. This results in σ_1^* and σ_2^* being rewritten as

$$\sigma_1^* = \left[\sqrt{\pi} \sum_{i,j=1}^{n} |x_i - x_j|\right] \Big/ [2n(n-1)] \tag{7}$$

and

$$\sigma_2^* = \left[\sqrt{\pi} \sum_{i,j=1}^{m} |y_i - y_j|\right] \Big/ [2m(m-1)] \tag{8}$$

where $\{x_i\}$ is the unordered sample from Π_1 and $\{y_i\}$ is the unordered sample from Π_2. That (7) and (8) are equivalent to (4) and (5) is apparent from the following. Let $\{x_i\}$ be an unordered sample of size n and let $z_1 < \cdots < z_n$ be the corresponding ordered sample, then

$$\frac{1}{2} \sum_{i,j=1}^{n} |x_i - x_j| = \sum_{i=2}^{n} (z_i - z_1 + z_i - z_2 + \cdots + z_i - z_{i-1})$$

$$= \sum_{i=1}^{n} \left[(i-1)z_i - \sum_{j=1}^{i-1} z_j\right]$$

$$= \sum_{i=1}^{n} (i-1)z_i - \sum_{i=1}^{n} (n-i)z_i$$

$$= 2 \sum_{i=1}^{n} \left(i - \frac{n+1}{2}\right) z_i \ . \qquad (9)$$

Thus using (7) and (8) and cancelling common terms, an alternate expression for v is

$$v = \frac{m(m-1) \sum_{i,j=1}^{n} |x_i - x_j|}{n(n-1) \sum_{i,j=1}^{m} |y_i - y_j|} \ . \qquad (10)$$

This expression is more convenient for use with computers than is expression (6) when the sample data is not in an ordered form.

In both (10) and (6) v is a realization of a random variable which shall be denoted by V. The distributional properties of V are difficult to obtain directly as may be seen from a consideration of expressions (7) and (8). In particular both σ_1^* and σ_2^* are sums of dependent random variables each of which has a marginal half-normal distribution when Π_1 and Π_2 are normally distributed. The dependency makes finding the distributions of σ_1^* and σ_2^* difficult and since non-normality for Π_1 and Π_2 will also be considered an alternate approach to the distribution of V was used.

If c_α denotes the critical point for an upper tail test where the probability of a type I error is α, that is, c_α is that point at which the cumulative distribution function of V has value $1-\alpha$, then using an approach similar to that of Barnett, Mullen and Saw (1967), it is found that

$$\alpha = \Pr\{V > c_\alpha\} = \Pr\{\sigma_1^*/\sigma_2^* > c_\alpha\} = \Pr\{\sigma_1^* > c_\alpha \sigma_2^*\}$$

$$= \Pr\{\sigma_1^* - c_\alpha \sigma_2^* > 0\} = \Pr\{Z > 0\} \qquad (11)$$

where $Z = \sigma_1^* - c_\alpha \sigma_2^*$. Since Barnett, Mullen and Saw (1967) give the formulas necessary to compute the first four moments about the origin of σ_1^* and σ_2^* the first four moments (about the origin or the mean) of z are easily obtained as a function of c_α. The method by which the values of c_α are obtained will be outlined in the following chapter.

CHAPTER III

THE CRITICAL POINTS AND POWER OF THE

v-TEST UNDER NORMALITY

The σ^* statistic, when considered as a random variable, has a distribution which will be treated as approximately normal when Π_1 and Π_2 are normal. Moore (1968) has shown that under this condition of normality for the population involved the distribution of σ^* is asymptotically normal, and Barnett, Mullen and Saw (1967) have given values for Pearson's measures of skewness and kurtosis, β_1 and β_2, for σ^* for a few sample sizes. For example, when the sample size is ten, σ^* is distributed with skewness $\beta_1 = .07$ and kurtosis $\beta_2 = 3.02$.

Since z is a linear combination of asymptotically normal variables it is itself asymptotically normal and shall be considered to be approximatley normal when Π_1 and Π_2 are normal because of the approximate normality of σ_1^* and σ_2^* for relatively small sample sizes. Under this assumption an approximation to c_α may be obtained as follows.

$$\Pr\{V > c_\alpha\} = \Pr\{\mathbf{z} > 0\} \doteq [\sqrt{2\pi}\ \sigma_z]^{-1} \int_0^\infty \exp\left\{-\frac{1}{2}\left(\frac{z-\mu_z}{\sigma_z}\right)^2\right\} dz$$

where μ_z and σ_z^2 are respectively the mean and variance of z. If the transformation $U = (z - \mu_z)/\sigma_z$ is made, then

$$\Pr\{V > c_\alpha\} = \alpha \doteq \int_\delta^\infty G'(u)\,du = 1 - G(\delta) \tag{12}$$

where $G'(x)$ and $G(x)$ are respectively the normal (0, 1) p.d.f. and c.d.f., and where $\delta = -\mu_z/\sigma_z$. If d_α denotes the critical point of a standard normal distribution which corresponds to an upper tail area of α, then $d_\alpha \doteq \delta$. Since the exact moments of Z are known as functions of c_α, the expression $d_\alpha = -\mu_z/\sigma_z$ can be solved in terms of c_α to obtain an approximation to c_α, say $\hat{c}_\alpha$. In particular, the mean of Z is

$$\mu_z = \sigma_1 - c_\alpha \sigma_2 \tag{13}$$

and the variance of Z can be written as

$$\sigma_z^2 = c_2 \sigma_1^2 + c_3 c_\alpha^2 \sigma_2^2 \tag{14}$$

where c_2 and c_3 are constants given by Barnett, Mullen and Saw (1967) to be

$$c_2 = [n(\pi/3 + 2\sqrt{3} - 4) + (6 - 4\sqrt{3} + \pi/3)]/n(n-1) \tag{15}$$

and

$$c_3 = [m(\pi/3 + 2\sqrt{3} - 4) + (6 - 4\sqrt{3} + \pi/3)]/m(m-1) \ . \tag{16}$$

So under the null hypothesis, $\sigma_1^2 = \sigma_2^2$, the expression $\delta = d_\alpha$ becomes

$$\begin{aligned} d_\alpha &= \frac{(\hat{c}_\alpha - 1)\sigma_2}{\left[c_2\sigma_2^2 + \hat{c}_\alpha^2 c_3 \sigma_2^2\right]^{1/2}} \\ &= \frac{(\hat{c}_\alpha - 1)}{[c_2 + \hat{c}_\alpha^2 c_3]^{1/2}} \end{aligned} \tag{17}$$

thus

$$(c_2 + c_3 \hat{c}_\alpha^2) d_\alpha^2 = 1 - 2\hat{c}_\alpha + \hat{c}_\alpha^2$$

which becomes

$$(1 - c_3 d_\alpha^2)\hat{c}_\alpha^2 - 2\hat{c}_\alpha + (1 - c_2 d_\alpha^2) = 0$$

so by the quadratic formula

$$\hat{c}_\alpha = \left[2(1 - c_3 d_\alpha^2)\right]^{-1}\left\{2 \pm \sqrt{4 - 4(1 - c_3 d_\alpha^2)(1 - c_2 d_\alpha^2)}\right\} \tag{18}$$

the larger value of (18) becomes, upon simplification

$$\hat{c}_\alpha = \left[(1 - c_3 d_\alpha^2)^{-1}\right]\left[1 + d_\alpha \sqrt{c_3 + c_2(1 - c_3 d_\alpha^2)}\right] \tag{19}$$

which is the desired approximation to the critical point c_α. As the sample size increases, Z will asymptotically approach a normal distribution and the approximation in (19) will improve in accuracy. But a method of improving the accuracy of (19) for a given sample size is needed. This will be considered next.

Since the moments of Z are known functions of c_α these moments can now be approximated by using $\hat{c}_\alpha$. If these approximate moments of Z could be used to find interval probabilities for Z, then perhaps $\hat{c}_\alpha$ could be improved to any desired level. In order to do this a close approximation for the distribution of Z will be needed.

In order to approximate a distribution closely when only four moments are known it is desirable to match each of those four moments as closely as possible. A family of distributions which allow this to be done is given by Burr (1942). This family is characterized by the cumulative distribution function

$$\begin{aligned} F(x) &= 1 - (1 + x^c)^{-k} \qquad x \geq 0 \\ &= 0 \qquad\qquad\qquad\quad x < 0 \end{aligned} \tag{20}$$

where c and k are positive. Let λ_3 and λ_4 be measures of skewness and kurtosis defined by

$$\lambda_3 = \sqrt{\beta_1} \qquad \lambda_4 = \beta_2 - 3. \tag{21}$$

The family of Burr distributions includes distributions with λ_3 and λ_4 values covering a large area surrounding the values for a normal distribution, and because the parameters, c and k, together uniquely determine a pair of values of λ_3 and λ_4 this family of distributions is well suited for use in approximating the distribution of Z. For information about using a Burr distribution as an approximation for a normal distribution see Burr (1967).

The procedure for improving the approximation to c_α consists of the following. First use $\hat{c}_\alpha$ to obtain approximate values for the mean, variance, skewness and kurtosis of Z, denoted by $\hat{\mu}_z$, $\hat{\sigma}_2^2$, $\hat{\lambda}_{3z}$, $\hat{\lambda}_{4z}$ respectively. Next find the values of c and k which determine a Burr distribution of the form of (20) with a skewness, λ_{3x}, equal to $\hat{\lambda}_{3z}$ and a kurtosis, λ_{4x}, equal to $\hat{\lambda}_{4z}$ (see Burr (1942)). Now since skewness and kurtosis are invariant under any change in location or scale the Burr distribution above may be transformed to have a mean equal to $\hat{\mu}_z$ and a variance equal to $\hat{\sigma}_2^2$. If μ_x and σ_x^2 denote the original mean and variance of the Burr distribution, then this transformed Burr distribution has the form

$$F_Y(y) = 1 - \left\{1 + [(\sigma_x/\hat{\sigma}_z)(y - \hat{\mu}_z) + \mu_x]^c\right\}^{-k} \qquad y \geq \hat{\mu}_z - (\hat{\sigma}_z/\sigma_x)\mu_x$$
$$= 0 \qquad \text{otherwise.} \qquad (22)$$

Actually when $\hat{\lambda}_{3z}$ is negative it is easier to find a Burr distribution with shape parameters equal to $-\hat{\lambda}_{3z}$ and $\hat{\lambda}_{4z}$ and by flipping this distribution about the origin a distribution with the proper skewness and kurtosis is obtained. When this procedure is used the transformed Burr distribution which has approximately the same first four moments as z is of the form

$$F_Y(y) = \left\{1 + [(\sigma_x/\hat{\sigma}_z)(\hat{\mu}_z - y) + \mu_x]^c\right\}^{-k} \quad , \quad y \leq \hat{\mu}_z + (\hat{\sigma}_z/\sigma_x)\mu_x$$
$$= 0 \qquad \text{otherwise.} \qquad (23)$$

Having found a transformed Burr distribution of the form of (22) or (23) the next step is to use this distribution to improve the approximation to c_α. If Y denotes a random variable distributed according to this transformed Burr distribution, then

$$\Pr\{V > c_\alpha\} = \Pr\{Z < 0\} = \alpha. \tag{24}$$

Now

$$\Pr\{Z > 0\} \doteq \Pr\{Y > 0\} \tag{25}$$

and if the c.d.f. of Y is of the form of (22) then

$$\Pr\{V > c_\alpha\} \doteq \left\{1 + [\mu_x - (\sigma_x/\hat{\sigma}_z)\hat{\mu}_z]^c\right\}^{-k} \tag{26}$$

where the right hand side is not zero. The right hand side of (26) is an approximation to the known value of α. If the c.d.f. of Y is of the form (23) then the approximation for α is given by

$$1 - \left\{1 + [(\sigma_x/\hat{\sigma}_z)\hat{\mu}_z + \mu_x]^c\right\}^{-k} \quad \text{if} \quad \hat{\mu}_z + (\hat{\sigma}_z/\sigma_x)\mu_x \geq 0$$

$$\text{or} \quad 0 \quad \text{otherwise.} \tag{27}$$

The accuracy of this approximation is directly dependent upon the accuracy of $\hat{c}_\alpha$, and $\hat{c}_\alpha$ can be improved by comparing this approximation of α with α. Using the improved value of $\hat{c}_\alpha$, say $\hat{\hat{c}}_\alpha$, to calculate new approximations to the moments of z the above procedure is repeated. When the approximation for α is "close enough" to α then the present value of $\hat{\hat{c}}_\alpha$ is taken as the actual value of c_α.

The above process for finding a value for c_α is rather complicated, therefore the accuracy of the results are of interest. In the work done the approximations to α were considered close enough when they were within

0.0001 of α. This led to values of $\hat{\hat{c}}_\alpha$ that were accurate to three decimal places and thus to moments of Z that were also accurate to three decimal places. So the moments of the transformed Burr distribution were made to match the moments of Z to three decimal places, obtaining thereby a fit which was as close as possible without fitting error. The discrepancies between the cumulative distribution for Z, $F_Z(z)$ and its approximation, the c.d.f. for Y, $F_Y(y)$, are composed basically of the differences between the two distributions related to the differences between moments of order greater than four. Considering the work of Burr (1967) these discrepancies appear to be no greater than 0.004, and generally are less than 0.001, at least in the neighborhood of the normal distribution which is the area of concern. The only other factor affecting the discrepancies between $F_Z(z)$ and $F_Y(y)$ is the accuracy of $\hat{\hat{c}}_\alpha$. Based on the work done it seems that discrepancies between $\hat{\hat{c}}_\alpha$ and c_α should not affect the cumulative distribution functions by more than 0.0005.

Table 1 contains critical points for the v-test for three different levels and for nine different pairs of sample sizes. The sample sizes were obtained by letting the sample sizes for the two populations, n and m respectively, take on the values 5, 10 and 21 independently. For each of these nine pairs of sample sizes the critical points are tabled for α levels of .05, .10 and .25.

The critical points given in columns labeled c_α are the final values of $\hat{\hat{c}}_\alpha$.

If critical points for the v-test which are not given in Table 1 are desired, then they may be approximated using expression (19) for $\hat{c}_\alpha$. Furthermore, if the accuracy of $\hat{c}_\alpha$ and the direction in which $\hat{c}_\alpha$ errs from c_α were known for the desired combination of α-level and sample sizes then an even

TABLE 1

CRITICAL POINTS, c_α, FOR THE v-TEST AND RATIOS OF $\hat{c}_\alpha$ TO c_α

α	m \ n	5		10		21	
		c_α	$\hat{c}_\alpha/c_\alpha$	c_α	$\hat{c}_\alpha/c_\alpha$	c_α	$\hat{c}_\alpha/c_\alpha$
.05	5	2.5750	1.077	2.4089	1.092	2.3270	1.103
	10	2.0024	0.996	1.8090	1.009	1.7100	1.017
	21	1.7778	0.990	1.5714	1.002	1.4712	0.999
.10	5	2.0661	1.008	1.9487	1.014	1.8930	1.017
	10	1.6960	1.005	1.5795	0.998	1.5091	1.001
	21	1.5750	0.997	1.4240	1.002	1.3489	0.997
.25	5	1.4406	0.993	1.4002	0.983	1.3803	0.978
	10	1.3229	1.004	1.2636	0.999	1.2336	0.995
	21	1.2755	1.006	1.2069	1.002	1.1666	1.000

closer approximation to $\hat{c}_\alpha$ would be possible. In order to give some idea of the accuracy of $\hat{c}_\alpha$ and its position with respect to c_α the ratio $\hat{c}_\alpha/c_\alpha$ was included in Table 1.

Determining a general systematic pattern **for** accuracy and direction of deviation of $\hat{c}_\alpha$ would require the evaluation of the ratio $\hat{c}_\alpha/c_\alpha$ at more points than are available, and would be equivalent **to** preparing a more extensive table of critical values. However some observations based on the tabled values of $\hat{c}_\alpha/c_\alpha$ are possible. First it seems that $\hat{c}_\alpha$ improves in accuracy with increasing denominator sample size and total sample size and with larger α-levels. Secondly it will be noticed that two-thirds of the ratios differ from the value one by less than one percent. In these situations at least it appears that $\hat{c}_\alpha$ is a fairly accurate approximation to c_α.

With the critical point known it is natural to consider the power of the v-test when the two populations are both normally distributed. The power is a function of simple alternative hypotheses which are included in the compound alternative hypothesis $\sigma_1^2 > \sigma_2^2$.

Rewrite the alternative hypothesis as

$$\sigma_1^2 = c_1\sigma_2^2 \quad , \quad c_1 > 1 . \tag{28}$$

For any particular value of $c_1 > 1$, (28) represents a particular alternative hypothesis at which the power can be calculated. Also if $c = 1$, then (28) becomes the null hypothesis, $\sigma_1^2 = \sigma_2^2$. In order to find c_α, the moments of Z were computed, in effect, with $c_1 = 1$, and in order to obtain a point on the power curve of the v-test the moments of Z will have to be recomputed using both the value of c_α found earlier and a value of $c_1 > 1$. Having computed these moments, a transformed Burr distribution is found to match them and is used to evaluate (25)

$$\Pr\{Z > 0\} \doteq \Pr\{Y > 0\} \tag{25}$$

as before. When $c_1 > 1$ the left hand side of (25) is the power of the v-test for the alternative hypothesis $\sigma_1 = \sqrt{c_1}\,\sigma_2$. Thus an approximation to the power of the v-test is obtained. The accuracy of this approximation is the same as the approximation discussed above, namely the right hand side should not differ from the left hand side by more than 0.0045 at any point.

Table 2 compares this approximation to the power function of the v-test of the hypotheses $\sigma_1 = \sigma_2$ versus $\sigma_1 > \sigma_2$ with the power function of the F-test of the hypotheses $\sigma_1^2 = \sigma_2^2$ versus $\sigma_1^2 > \sigma_2^2$. The probabilities of type I error (α-levels) for which points on the power curves are tabled are .05, .10 and .25. The sample sizes used in Table 2 are the same as in the critical point tables.

Table 2 is divided into three sections according to the level of the tests. The first column of each section gives the values of c_1 which were used in obtaining the points on the power curves given in the following columns. Next, according to sample sizes for m and n, points corresponding to the same alternative hypothesis from the power curves of the F-test and the v-test are given. The values for the F-test were obtained from Pearson's <u>Tables of the Incomplete Beta Function</u> and the values for the v-test are approximations based upon the transformed Burr distribution fitted to the moments of Z.

There are several points in Table 2 where the approximation to the power value of the v-test exceeds the corresponding power value of the F-test. However, these points do not exceed the F-test values by more than

0.005, which is, based on earlier considerations, the size of the maximum discrepancy between the transformed Burr distribution and the cumulative distribution of Z which it approximates. Therefore these points where the v-test values are larger than the F-test values can be interpreted as points where the power of the v-test is almost exactly the same as the power of the F-test.

By the same criterion all of the approximations to the power curve points of the v-test can be considered to be accurate to two decimal places. Thus Table 2 shows that the v-test is almost as powerful as the F-test and that the use of the v-test entails the loss of an almost negligible amount of power when normality holds.

Box and Anderson (1955) presented the approximate power function for the modified F-test for one case ($\alpha = .05$, $m = n = 20$). For this case the modified F-test appears to be slightly less powerful than the v-test. In particular, the power lost in this example when using the modified F-test instead of the usual F-test when the underlying populations are normal is shown by Box and Anderson to be as great as .05, whereas the power loss due to using the v-test when normality holds is generally less than .02 (see Table 2).

In this chapter the v-test has been considered as an alternative to the F-test when the two populations sampled are normally distributed. An estimate for critical points of the v-test was developed, a method of obtaining more exact values of the critical points was outlined and a short table of critical points was given. Finally a procedure for approximating the power of the v-test was outlined and this approximation to the power function was compared pointwise with the power function for the corresponding F-test. Comparison of the power functions under normality is advocated by

TABLE 2

POWER OF F AND v-TESTS WHEN α LEVEL IS .05

c_1	n=5 m=5 F test	n=5 m=5 v test	n=10 m=5 F test	n=10 m=5 v test	n=21 m=5 F test	n=21 m=5 v test
1.0	.050	.050	.050	.050	.050	.050
1.21	.068	.069	.069	.069	.069	.069
1.44	.089	.089	.092	.091	.093	.092
1.69	.113	.113	.118	.116	.120	.119
1.96	.140	.138	.147	.146	.151	.149
2.25	.168	.165	.179	.175	.185	.182
3.24	.264	.254	.289	.279	.305	.295
4.0	.331	.317	.369	.354	.392	.379
5.76	.461	.444	.526	.506	.565	.548
9.0	.626	.616	.719	.702	.774	.760
16.0	.802	.790	.898	.891	.944	.939

c_1	n=5 m=10 F test	n=5 m=10 v test	n=10 m=10 F test	n=10 m=10 v test	n=21 m=10 F test	n=21 m=10 v test
1.0	.050	.050	.050	.050	.050	.050
1.21	.079	.078	.083	.082	.087	.085
1.44	.115	.107	.127	.124	.136	.133
1.69	.157	.148	.180	.174	.199	.191
1.96	.203	.194	.241	.231	.272	.260
2.25	.253	.243	.308	.293	.352	.332
3.24	.405	.396	.511	.485	.596	.574
4.0	.499	.489	.631	.622	.731	.713
5.76	.653	.642	.805	.791	.899	.891
9.0	.802	.789	.932	.921	.982	.978
16.0	.916	.907	.988	.986	.999	.999

c_1	n=5 m=21 F test	n=5 m=21 v test	n=10 m=21 F test	n=10 m=21 v test	n=21 m=21 F test	n=21 m=21 v test
1.0	.050	.050	.050	.050	.050	.050
1.21	.087	.087	.098	.098	.086	.086
1.44	.135	.136	.165	.165	.196	.186
1.69	.191	.192	.246	.248	.307	.289
1.96	.251	.254	.337	.342	.429	.405
2.25	.313	.317	.429	.432	.550	.544
3.24	.491	.494	.670	.670	.823	.813
4.0	.591	.592	.784	.780	.917	.908
5.76	.738	.735	.912	.906	.985	.982
9.0	.862	.858	.977	.975	.999	.999
16.0	.946	.944	.997	.997	1.000	1.000

(TABLE 2)

POWER OF F AND v-TESTS WHEN α LEVEL IS .10

c_1	n=5 m=5 F test	n=5 m=5 v test	n=10 m=5 F test	n=10 m=5 v test	n=21 m=5 F test	n=21 m=5 v test
1.0	.100	.100	.100	.100	.100	.100
1.21	.132	.130	.134	.133	.135	.134
1.44	.167	.164	.173	.170	.176	.173
1.69	.205	.199	.216	.210	.221	.217
1.96	.246	.237	.262	.254	.271	.264
2.25	.287	.276	.310	.299	.323	.313
3.24	.412	.395	.458	.441	.485	.470
4.0	.490	.466	.551	.533	,589	.571
5.76	.624	.614	.709	.693	.760	.745
9.0	.767	.754	.861	.852	.910	.903
16.0	.892	.880	.962	.956	.987	.985

c_1	n=5 m=10 F test	n=5 m=10 v test	n=10 m=10 F test	n=10 m=10 v test	n=21 m=10 F test	n=21 m=10 v test
1.0	.100	.100	.100	.100	.100	.100
1.21	.147	.147	.155	.153	.161	.159
1.44	.200	.202	.222	.215	.239	.231
1.69	.258	.261	.297	.286	.327	.315
1.96	.317	.321	.375	.361	.421	.405
2.25	.376	.380	.453	.431	.514	.496
3.24	.538	.542	.660	.655	.751	.736
4.0	.627	.629	.763	.752	.855	.846
5.76	.759	.756	.892	.881	.958	.955
9.0	.871	.866	.967	.962	.995	.994
16.0	.949	.946	.995	.995	.999	.999

c_1	n=5 m=21 F test	n=5 m=21 v test	n=10 m=21 F test	n=10 m=21 v test	n=21 m=21 F test	n=21 m=21 v test
1.0	.100	.100	.100	.100	.100	.100
1.21	.157	.158	.175	.176	.193	.185
1.44	.221	.225	.268	.270	.314	.298
1.69	.293	.296	.369	.372	.448	.426
1.96	.363	.367	.469	.473	.578	.583
2.25	.431	.435	.563	.568	.691	.686
3.24	.605	.606	.777	.774	.902	.894
4.0	.693	.692	.863	.858	.960	.953
5.76	.813	.809	.950	.946	.994	.994
9.0	.906	.902	.988	.988	.999	.999
16.0	.965	.964	.999	.999	1.000	1.000

(TABLE 2)

POWER OF F AND v-TESTS WHEN α LEVEL IS .25

c_1	n=5 m=5 F test	n=5 m=5 v test	n=10 m=5 F test	n=10 m=5 v test	n=21 m=5 F test	n=21 m=5 v test
1.0	.250	.250	.250	.250	.250	.250
1.21	.309	.306	.316	.313	.320	.318
1.44	.368	.364	.384	.379	.393	.389
1.69	.425	.420	.452	.445	.467	.461
1.96	.481	.475	.517	.509	.539	.531
2.25	.532	.532	.579	.571	.607	.599
3.24	.664	.663	.733	.728	.775	.769
4.0	.731	.729	.809	.805	.854	.850
5.76	.828	.822	.906	.905	.944	.942
9.0	.909	.902	.968	.965	.989	.988
16.0	.964	.961	.994	.994	.999	.999

c_1	n=5 m=10 F test	n=5 m=10 v test	n=10 m=10 F test	n=10 m=10 v test	n=21 m=10 F test	n=21 m=10 v test
1.0	.250	.250	.250	.250	.250	.250
1.21	.326	.327	.345	.337	.328	.332
1.44	.402	.403	.442	.437	.473	.467
1.69	.473	.475	.535	.540	.583	.576
1.96	.539	.540	.619	.624	.681	.675
2.25	.598	.598	.693	.696	.763	.759
3.24	.736	.732	.848	.842	.916	.915
4.0	.800	.794	.907	.910	.962	.961
5.76	.882	.876	.967	.962	.993	.992
9.0	.943	.939	.992	.992	.999	.999
16.0	.979	.978	.999	.999	1.000	1.000

c_1	n=5 m=21 F test	n=5 m=21 v test	n=10 m=21 F test	n=10 m=21 v test	n=21 m=21 F test	n=21 m=21 v test
1.0	.250	.250	.250	.250	.250	.250
1.21	.337	.338	.369	.370	.399	.390
1.44	.422	.423	.487	.488	.551	.555
1.69	.501	.501	.595	.594	.685	.689
1.96	.571	.570	.687	.684	.790	.790
2.25	.633	.630	.762	.757	.866	.863
3.24	.770	.764	.900	.894	.971	.968
4.0	.830	.824	.945	.940	.990	.989
5.76	.903	.898	.983	.981	.999	.999
9.0	.954	.952	.996	.996	1.000	1.000
16.0	.983	.983	1.000	1.000	1.000	1.000

Kendall and Stuart (1967) when they discuss distribution-free statistics:

> "If we examine power against the alternatives considered in normal distribution theory, we obtain a measure of how much we can lose by using a distribution-free test if the assumptions of normal theory really are valid. If this loss is small we are encouraged to sacrifice the little extra efficiency of the standard normal theory methods for the extended range of validity attached to the use of the distribution-free test."

And Box and Andersen (1955) indicate that when considering any alternative to normal theory test criteria the power loss under normality should be considered. Using the comparison of the power functions as a measure of goodness the small loss of power, which results from using the v-test when the underlying populations are normal, leads to the conclusion that there is little reason to prefer the F-test over the v-test when the normal theory assumptions are valid. The incompleteness of the information on the power of the modified F-test allows only a conjecture that the v-test is slightly more powerful. Therefore the behavior of these tests under non-normality will determine which is to be preferred.

CHAPTER IV

THE EFFECTS OF NON-NORMALITY UPON V

When the two populations, Π_1 and Π_2, are not normally distributed the significance levels associated with the critical points for the v-test are different from the assigned values. If this difference in significance levels is not too large then the v-test will be said to be robust, otherwise it is non-robust. Just how large "large" is will depend upon the level and the situation, generally "large" will be taken to be greater than .01 or .02. It was precisely the non-robustness of the F-test that led to the consideration of V as a test statistic, therefore the v-test should be examined for robustness and compared with the F-test.

To study the robustness of the v-test directly under various types of non-normality some properties of the V-statistic or the related Z-statistic need to be known. Since Z is a linear combination of order statistics it would seem to be rather straight forward to obtain some properties of Z at least for some particular non-normal distributions. But if the v-test is to be considered in several non-normal situations, the properties of the Z-statistic will need to be known for a class of non-normal distributions. Subrahmaniam (1969) considers order statistics from a class of non-normal distributions which have density functions given by Edgeworth series. This paper would seem to provide the background needed to obtain the moments of Z, but unfortunately it does not. If the moments of Z are to be obtained when Π_1 and Π_2 are distributed according to some distributions in Subrahmaniam's class of non-normal distributions then the expectations of products of order statistics from distributions in this class are needed, and these are not given in nor are they easily obtainable from Subrahmaniam's paper.

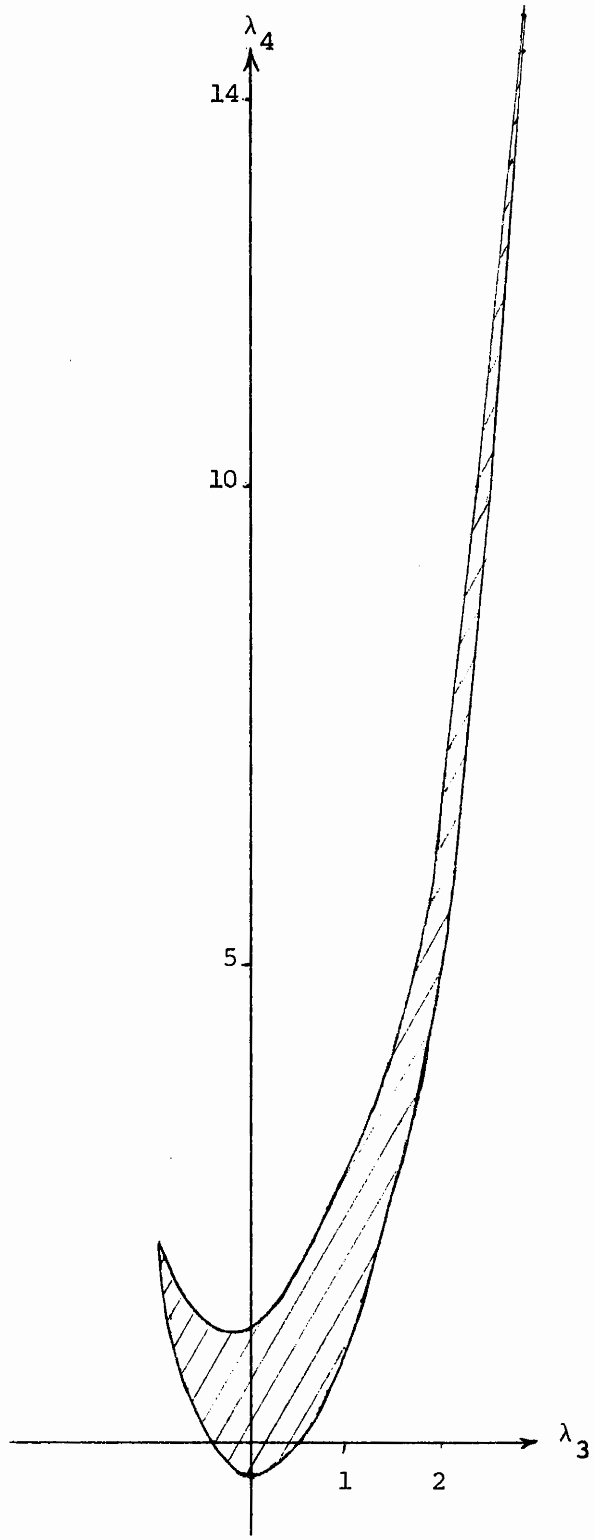
Another class of non-normal distributions which suggests itself for a study of the robustness of the v-test is the family of Burr distributions. In Figure 1, the range of skewness and kurtosis for which Burr distributions exist is shown. The parameter for skewness, λ_3, and the kurtosis parameter, λ_4, are as defined earlier, and the normal distribution values of λ_3 and λ_4 occur at the origin. Therefore the Burr distribution covers a region surrounding the normal distribution point.

The moments of order statistics from any Burr distribution may be obtained by numerical integration as can the expected values of cross products of order statistics, so in theory, close approximations to the moments of Z are possible. However both the number of numerical integrations involved and the fact that the resulting moments would then only help in an approximation of the distribution of Z combine to discourage this approach. Finally the class of Burr distributions is more restrictive than the general class of non-normals considered by Subrahmaniam (1969), and the results obtained by using the Burr distributions would be dependent to some extent upon the actual properties of the Burr distribution and upon the properties of the approximating distribution.

Therefore, since exact results are not possible by either of the two approaches above and since the approximate results are rather difficult to obtain, sampling seems to be an acceptable approach for a preliminary study of the robustness of the v-test.

Because of the large range of shapes possible with the family of Burr distributions, this family will contain most of the distributions which

FIGURE 1.



Range of Skewness, λ_3, and Kurtosis, λ_4, for Burr Distributions will be sampled. In order to obtain symmetric distributions with large values of λ_4 some mixtures of Burr distributions were used. These mixtures are given in Appendix A.

Another reason for using Burr distributions is apparent when the generation of a sample is considered. The c.d.f. of a Burr distribution is a closed form and can be solved explicitly in terms of the argument x to give

$$x = \left[\left(1 - F(x)\right)^{-1/k} - 1 \right]^{1/c} . \tag{29}$$

Thus, given a value of a random variable which is uniformly distributed on the interval [0, 1], it is a simple matter to obtain a corresponding value for a random variable distributed according to a given Burr distribution.

In the sampling experiments the V and F statistics were calculated according to the following procedure. A pseudo-random number generator (see Appendix B) was used to generate a sample of size n + m from a uniform distribution on [0, 1]. The first n values were used to generate a sample of size n from Π_1, and the remaining m values were used to generate a sample of size m from Π_2. Transformations of the form of (29) were used to obtain samples from Burr distributions and then samples were standardized to obtain the samples from Π_1 and Π_2. The standardization was performed in order to consider effects of non-normality when the variances of Π_1 and Π_2 were equal. Let y_i and z_i denote the i^{th} samples from Π_1 and Π_2 respectively. Upon obtaining the y_i's and the z_i's the statistics

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j| \quad \text{and} \quad \sum_{i=1}^{m} \sum_{j=1}^{m} |z_i - z_j| \tag{30}$$

were computed, and the V and F statistics obtained by the formulae

$$v = \left[m(m-1) \sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j|\right] \Big/ \left[n(n-1) \sum_{i=1}^{m} \sum_{j=1}^{m} |z_i - z_j|\right] \tag{31}$$

$$f = \left[m(m-1) \sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j|^2\right] \Big/ \left[n(n-1) \sum_{i=1}^{m} \sum_{j=1}^{m} |z_i - z_j|^2\right] . \tag{32}$$

For a given set of sample sizes and a given pair of distributions for Π_1 and Π_2 the V and F statistics were calculated 2500 different times and the number of times each of these statistics exceeded its respective critical point was observed. These numbers were then used to obtain observed significance levels. For each pair of distributions for Π_1 and Π_2 observed significance levels were obtained for nominal α-levels of .25, .10 and .05 for each of the nine combinations of n and m values given in Table 1.

Seven non-normal distributions were used and for each of these the following deviations from normality for Π_1 and Π_2 were considered. First population one was normally distributed while population two was non-normal. Next population one was non-normally distributed while Π_2 was normally distributed, and finally both populations were distributed according to the same non-normal distribution. Among the seven non-normal distributions one was skewed only ($\lambda_3 = 0.47$), three were symmetric and leptokurtic ($\lambda_4 = 1.02, 2.01, 3.01$) and three were both skewed and leptokurtic (the ordered pairs for λ_3 and λ_4 were (.78, .73), (.95, 1.19) and (1.18, 2.02)). Therefore there were 21 different non-normal situations for which observed significance levels were obtained.

Since the V and F statistics are both symmetric in the sample values and in their treatment of the two samples the lower tail of the distribution of both V and F was also used to obtain observed significance levels. For if Π_1 and Π_2 are both normal and are sampled n and m times respectively and if w_α is the nominal α-level critical point for the test (either V or F) when the numerator is based on a sample of size m and the denominator is based on a sample of size n, then the statistic will fail to exceed $1/w_\alpha$ exactly α percent of the time.

Therefore if Π_1 is a normal population and Π_2 is a given non-normal population from which samples of sizes a and b have been taken respectively, then the observed significance levels for this case are correlated with the observed significance levels for the case where Π_1 is the given non-normal population and Π_2 is the normal population, sampled b and a times respectively. Similarly when both Π_1 and Π_2 are distributed according to the same non-normal distribution with sample sizes a and b there is a correlation between the observed significance levels here and the observed significance levels for the case with sample sizes b and a. If b = a when both Π_1 and Π_2 are non-normal then a second set of observed significance levels is obtained which is correlated with the first set. In the absence of this correlation this last case would effectively give the same results as repeating the calculation of the V and F statistics 5000 times. In all of these cases the amount of the correlation between sets of observed significance levels is undetermined, but consideration of cases where both Π_1 and Π_2 are non-normal with n = m indicates that this correlation is small.

The observed significance levels for these upper tail tests are tabled in Appendix C and summarized in Table 3 below. In order to reduce the volume of data to be considered, the results of the sampling experiments were treated as observations in sampling from a multinomial and accordingly chi-square statistics were obtained as a measure of lack of fit. These chi-square statistics were obtained as a measure of the lack of fit for the upper tail of the distributions and therefore are a measure of the robustness of the tests as upper tail tests.

The chi-square statistics were calculated twice for each set of data, the first chi-square statistic having 3 degrees of freedom and being a measure of combined robustness at the .05, .10 and .25 levels and the second chi-square statistic having two degrees of freedom and being a measure of combined robustness at the .05 and .10 levels only. The three d.f. and two d.f. chi-square statistics were computed for the observed significance levels for both the v-test and the F-test. Thus for a given set of sample sizes and a given type of non-normality there were four chi-square statistics, namely the two and three d.f. chi-squares for V, $\chi_2^2(V)$, $\chi_3^2(V)$, and the two and three d.f. chi-squares for F, $\chi_2^2(F)$, $\chi_3^2(F)$. In order to measure the relationship between corresponding chi-square statistics the differences, $\chi_3^2(F) - \chi_3^2(V)$ and $\chi_2^2(F) - \chi_2^2(V)$, and the ratios, $\chi_3^2(V)/\chi_3^2(F)$ and $\chi_2^2(V)/\chi_2^2(F)$, were obtained. The average of these differences and the geometric mean of these ratios over the nine different pairs of sample sizes for each of the twenty-one different non-normal situations is given in Table 3.

The first four columns of the upper part of Table 3 give the values of the skewness and kurtosis parameters, λ_3 and λ_4, for Π_1 and Π_2 respectively, the last four columns give the average of the differences and the geometric mean of the ratios of the corresponding chi-square statistics for three degrees of freedom and two degrees of freedom respectively. The twenty-one non-normal situations are organized into seven groups of three rows. Each group of three rows follows the same pattern, in the first row Π_1 is normally distributed while Π_2 is distributed according to a given one of the non-normal distributions, in the second row Π_1 and Π_2 switch distributions and in the third row both Π_1 and Π_2 are distributed according to the given non-normal distribution. The last three rows of Table 3 give averages of the means of differences and the geometric mean of the ratio means for the situations identified on the left hand side.

In Table 3 one asterisk denotes a non-normal situation where a test of robustness for the F-test using the chi-square statistic would fail to reject robustness at the .05 level. Two asterisks denote a situation where the same criterion would indicate robustness for the v-test and three asterisks denote a situation where both tests can be considered robust by the above criterion. Since for each row there were nine sets of sample sizes for which both the three d.f. and the two d.f. chi-square statistics were computed the asterisks represent cases where the chi-square test for robustness failed to reject robustness for at least five of the nine sets of sample sizes.

The .05 level was chosen as the level to use for testing robustness on the basis of the power of the chi-square test. If the actual significance levels of the v-test or the F-test are, for example, .06 and .11, then the power of the two d.f. chi-square test based on a sample of size 2500 is approximately 0.8. This value is obtained from the limiting power function for the chi-square test given by Kendall and Stuart (1967) in paragraph 30.27 of their work, The Advanced Theory of Statistics.

The situations where either the F-test or the v-test or both can be considered robust will be listed next, and following this listing these situations will be discussed. It should be remembered throughout the following that the robustness described is for upper tailed tests only.

TABLE 3

SUMMARY OF SAMPLING EXPERIMENTS FOR ROBUSTNESS

Shape Parameters				3 d.f. Chi-squares			2 d.f. Chi-squares		
num., Π_1		denom., Π_2		means of differences		geometric means of ratios	means of differences		geometric means of ratios
λ_3	λ_4	λ_3	λ_4						
0	0	0.47	0.04	-1.159	***	1.322	-0.424	***	1.296
0.47	0.04	0	0	0.840	***	1.248	0.046	***	0.926
0.47	0.04	0.47	0.04	0.416	***	0.855	1.617	***	0.604
0	0	0.01	1.02	-3.622		1.321	-1.913		1.301
0.01	1.02	0	0	3.802	***	0.649	3.843	***	0.293
0.01	1.02	0.01	1.02	13.678		0.635	13.826		0.582
0	0	0	2.01	-30.770		1.277	-18.360		1.183
0	2.01	0	0	-15.460		1.764	-8.370	*	2.458
0	2.01	0	2.01	5.380		0.879	5.960		0.865
0	0	0	3.01	-97.510		1.204	-76.530		1.171
0	3.01	0	0	-24.320		2.248	-9.270		2.800
0	3.01	0	3.01	15.550		0.911	16.480		0.905
0	0	0.78	0.73	0.487		1.074	3.046	**	0.656
0.78	0.73	0	0	1.274		1.509	0.983	*	1.554
0.78	0.73	0.78	0.73	11.262	**	0.420	10.712		0.374
0	0	0.95	1.19	-2.584		1.146	0.882		0.951
0.95	1.19	0	0	0.172	*	1.473	2.524	***	0.941
0.95	1.19	0.95	1.19	20.971		0.406	20.841		0.386
0	0	1.18	2.02	1.390		0.985	7.740		0.902
1.18	2.02	0	0	9.530		0.564	14.920	**	0.174
1.18	2.02	1.18	2.02	49.560		0.468	46.780		0.467
Summary of rows 4-21									
Numerator Normal				-19.110		1.184	-12.222		1.040
Denominator Normal				-3.555		1.221	0.668		0.899
Both Non-normal				16.688		0.618	16.602		0.566

* The F-test is robust in this situation for the levels indicated.
** The v-test is robust in this situation for the levels indicated.
*** Both tests may be considered robust in this situation.

1) The first three rows of Table 3 represent situations where either one or both of the distributions involved are skewed with only a very small amount of leptokurtosis. For these three situations both the three d.f. chi-square tests and the two d.f. chi-square tests indicate robustness for both the F-test and the v-test at least eight times out of every nine, and this is in keeping with the previous work on the robustness of the F-test (Pearson (1931), Geary (1947)) which showed that non-robustness is a problem only when non-normal kurtosis is present.

2) The next situation where some robustness is indicated is when Π_1 has shape parameters $\lambda_3 = .01$, $\lambda_4 = 1.02$ and Π_2 is normally distributed. Both the F-test and the v-test are noted as robust at all three α-levels and at the lower two α-levels alone. Both the three d.f. and the two d.f. chi-square tests show the F-test as robust in six of the nine cases. The v-test is indicated as robust at all three α-levels seven times and at the lower two α-levels by eight of the nine separate tests. It should be noted that in this situation every time the F-test could be considered as robust the v-test could also be considered robust.

3) The next place in Table 3 where robustness is noted is when the denominator is normally distributed and the numerator population is symmetric and leptokurtic with $\lambda_4 = 2.01$. The F-test is marked as being robust for α-levels of .05 and .10 only and this is based upon seven of the nine two d.f. chi-square tests failing to reject robustness at the .05 level. This case is singular in that it is the only time the F-test was strongly indicated as robust and the v-test was not indicated as robust by several of the chi-square tests. This is due to the more conservative nature of the v-test as may be seen from the data tabled

in Appendix C.

4) When one or both populations are moderately skewed and leptokurtic, $\lambda_3 = .78$, $\lambda_4 = .73$, either the F-test or the v-test is marked as robust. Namely, when the numerator is normal and when both numerator and denominator are non-normal the v-test is marked as robust on the basis of six out of nine two d.f. chi-square tests in the first case and five out of nine three d.f. tests in the second case. When the denominator is normal and the numerator has the moderately skewed and leptokurtic population then the F-test is marked as robust at the lower two α-levels on the basis of five out of nine two d.f. chi-square tests.

5) The F-test is again marked as robust when the numerator is slightly more skewed and leptokurtic than above ($\lambda_3 = .95$, $\lambda_4 = 1.19$) and the denominator is again normal. Here the three d.f. and the two d.f. chi-square tests failed to reject robustness for the F-test seven out of nine times and the two d.f. chi-square test indicated robustness for the v-test six of the nine times.

6) Finally when the numerator is even more skewed and leptokurtic ($\lambda_3 = 1.18$, $\lambda_4 = 2.02$) and the denominator is normal the chi-square tests indicated that the v-test was robust for the lower two α-levels five times out of nine.

The most striking thing about the list above is the number of times the F-test is indicated to be robust by a majority of the chi-square tests involved. Upon consideration of the situations in which this happens it appears that in each situation where the F-test might be considered as robust the denominator is based upon a normal population and the numerator population is the one which departs from normality. Gayen (1950) indicated that in some situations the F-test on variances may be robust when he said, "a distinction should always be made between samples from the same population and those from two different populations," and "It is therefore not unlikely that the presence of deviations from normality in both the measures ... may sometimes, far from disturbing the normal-theory law of the variance ratio, contribute towards its stability." This has not been pointed out more clearly because the literature has tended to deal with situations where, when there are two populations, both populations have the same shape. When only one of two populations is non-normal it would be expected that non-normality for the denominator would have more serious effects upon the F-test than would non-normality for the numerator, and this is precisely what the sampling experiments showed.

The second thing to notice from the listing above is that the v-test tended to be robust when the F-test was robust. The v-test was robust on its own only three times, namely when Π_1 and Π_2 were moderately skewed and leptokurtic, when the numerator was normal and the denominator was again only moderately skewed and leptokurtic and finally when the denominator was normal and the numerator was more extremely skewed and leptokurtic. In short it seems that the situations and regions for which the v-test is robust are slightly more general and more broad than those for which the F-test is robust, but that the v-test also has problems with non-robustness. The next question is how badly non-robust is the v-test compared to the F-test?

The ratios and differences of the chi-square statistics were used because they contain the same information as the pair of chi-square statistics and this information is in a form that is easier to use for comparison and summary purposes. Table 3 lists the means of the differences and the geometric means of the ratios for each of the non-normal situations. These two types of means summarize the relationship between the lack of fit statistics for the v-test and the F-test for each situation.

The robustness of the F-test has already been noted when the denominator is normal and the numerator only moderately non-normal. The more conservative nature of the v-test, which is apparent from the unabridged data, is brought out in Table 3 in these situations where the F-test is robust. In these situations the mean of the differences is generally negative or close to zero and the geometric mean of the ratios is either greater than one or close to one. In general, then, when the denominator is normal the v-test is no better than the F-test.

When the numerator is normal and the denominator is non-normal the cumulative measures given in Table 3 indicate that the v-test is worse than the F-test when the non-normality is due to leptokurtosis only and that the v-test is a little better than the F-test in its degree of non-robustness when the non-normality is due to both skewness and leptokurtosis.

When both numerator and denominator have the same non-normal distribution the v-test appears to be better than the F-test. In particular the chi-square statistics for the v-test are consistently smaller than those for the F-test and the overall geometric means for the two d.f. and three d.f. chi-squares are close to one-half. The conclusion that the v-test is not as non-robust as the F-test in these situations is warranted by the unabridged data itself, in fact the unabridged data indicates that the v-test tends to be only slightly non-robust when both populations have the same non-normal distribution and in particular when the numerator sample size is not less than the denominator sample size.

In order to compare the v-test with the modified F-test some further sampling experiments were run. Before describing the results of these experiments, the nature of the modified F-test will be reviewed. Box and

Andersen (1955) developed a combined measure of kurtosis for the two populations involved and then proposed using this measure (by means of a certain formula) to adjust the degrees of freedom to be used in a usual F-test. In particular the modified F-test consists of the following; the usual F-statistic is computed from the data, the measure of kurtosis is used to adjust the degrees of freedom, and the computed F-statistic is compared with the critical points for an F distribution having the adjusted degrees of freedom.

In the sampling experiments which compared the v-test and the modified F-test the two populations were both sampled either 5 or 21 times. For each of these two pairs of sample sizes two non-normal situations were considered. First the numerator distribution was symmetric with a kurtosis of 2.02 while the denominator was normally distributed, secondly both the numerator and the denominator came from a symmetric distribution with a kurtosis of 2.02. In each of these four sampling situations the V- and modified F-statistic were calculated 1000 times each. The results of these experiments are expressed as observed cumulative probabilities and are given in Table 4.

The first six columns of Table 4 give the identifying parameters for the adjoining block of observed cumulative probabilities. The normal cumulative probabilities are the theoretical probabilities for the v-test under normality for both populations.

On the basis of the data in Table 4 it appears that the v-test is more robust than the modified F-test when the sample size is small but that the reverse is true for large sample sizes. Furthermore, even at the larger of the sample sizes, the upper tail probabilities for the v-test are closer to the normal theory values than are the corresponding values for the modified F-test when the denominator is normal and the numerator is non-normal.

TABLE 4

CUMULATIVE PROBABILITIES FOR THE V-STATISTIC AND THE MODIFIED F-STATISTIC

Numerator λ_3 λ_4 sample size	Denominator λ_3 λ_4 sample size	Test	Normal Cumulative Probabilities .05	.10	.25	.75	.90	.95
			Observed Cumulative Probabilities					
0 2.02 5	0 0 5	v	.075	.151	.317	.760	.913	.963
		mod F	.087	.159	.353	.718	.909	.971
0 2.02 5	0 2.02 5	v	.060	.134	.282	.696	.862	.929
		mod F	.061	.123	.315	.653	.867	.928
0 2.02 21	0 0 21	v	.107	.170	.358	.801	.922	.967
		mod F	.088	.146	.319	.783	.933	.974
0 2.02 21	0 2.02 21	v	.081	.138	.298	.697	.857	.923
		mod F	.052	.105	.252	.711	.879	.926

Because of the preliminary nature of the comparison study the results above should be considered as indicative in nature only. To define areas where each of the two alternatives to the F-test is best would require a more extensive comparison between the two tests.

On the basis of all of the sampling experiments performed it would seem to be advisable to use an alternative test whenever both populations would tend to have the same shape, the v-test for small sample sizes and the modified F-test for large sample sizes, and to use the regular F-test otherwise.

CHAPTER V

SUMMARY

Because in most situations the population distributions are very rarely known the use of criteria based upon specific distributions involves certain risks. In order to avoid some of these risks criteria are sought which will not mislead the statistician, and hopefully will even allow him to obtain the desired information when the underlying assumptions are not satisfied.

In choosing a test of hypotheses the statistician risks making one of two errors. A powerful test based upon some specific family of distributions may be chosen at the risk of encountering a distribution which is not from that specific family, or a more general and less powerful test may be chosen which is good for a larger class of distributions at the risk of encountering a member of that specific family of distributions for which the better test could have been used.

In the first case the level of the test will usually be wrong, in the second the power will be less than it could have seen.

In order to circumvent the first of these errors the idea of robustness was developed. In particular, tests for which the probability of Type I error remains relatively constant regardless of the original distribution are considered robust. Some of the standard tests, such as the analysis of variance test with fixed effects have proved to be robust, and therefore extremely useful. However in many cases the standard tests are not robust and so non-parametric tests have been suggested as robust alternatives. This however ignores the second of the two errors mentioned above, namely loss of power when the more powerful test could have been used. Therefore robustness alone is not an adequate criterion for judging an alternative procedure.

The way to take the second kind of error into account is to consider the power of the alternative test when the standard test would be valid. If an alternative procedure is powerful under the standard test assumptions and is robust for departures from those assumptions, then it should be used instead of the standard test.

In Chapter III the power function of the v-test was approximated under normality assumptions and this approximation was found to be very close to the power function for the F-test. Therefore the problems represented by the second kind of error are small when the v-test is used as an alternative to the F-test.

Departures from normality can seriously affect the F-test on variances, but unfortunately they also affect the v-test. In particular when the denominator of the F-test comes from a population with non-normal kurtosis and the numerator is normal, or when both the denominator and numerator are from the same type of distribution with non-normal kurtosis the actual type I error probability for the F-test changes greatly. It appears that the v-test is similarly affected when the denominator is leptokurtic and the numerator is normal, but when both numerator and denominator are from the same type of distribution with non-normal kurtosis the v-test is not as seriously affected as the F-test. Namely, the probability of a type I error for the v-test does not differ from the normal theory value as much as does the type I error probability for the F-test.

Whenever the denominator is normally distributed and the numerator is only moderately leptokurtic both the F-test and the v-test appear to be robust. Therefore the v-test extends the region of robustness of the F-test a certain amount and is less severely non-robust than the F-test for an even larger region.

The modified F-test suggested by Box and Andersen (1955) was shown to be better than an F-test for sample sizes of 20 for both numerator and denominator. This was verified by the work done in comparing the modified F-test with the v-test. However when the numerator and denominator do not have the same shape it seems that the modified F-test is not as good as when the numerator and denominator are both from the same non-normal population. Furthermore when the sample size drops to around 5 for both numerator and denominator the v-test is clearly better than the modified F-test for the cases considered.

In Appendix C the unabridged data from the sampling experiments is given. Included along with the observed significance levels for the v-test and the F-test is the combined observed significance levels for the two tests considered as one test. These numbers were obtained by counting the number of times that both the F-statistic and the V-statistic were significant at the same level for the same sample. This count was divided by 2500 to obtain the numbers given in the tables in the appendix.

These numbers were originally obtained as an indicator of the similarity between the v-test and the F-test. If these numbers were very small, then the F-test and the v-test would have been giving different conclusions for the same set of data, whereas if these numbers were generally near or equal to their maximum possible values then a direct relationship between the two tests would have been indicated. As it is these numbers indicate that the v-test and the F-test are indeed different, in accordance with the way they are obtained, but that the two tests do test basically the same thing. In short, for a given sample, when one of these tests shows significance or non-significance then the other will very likely show significance or non-significance also.

Because these combined observed significance levels are sampling observations upon the joint significance levels for both the v- and F-tests they are smaller than either of the corresponding significance levels. Since the non-robustness of both the F-test and the v-test in the cases studied is in the direction of type I error probilities being larger than stated, the combined significance levels will be closer to the individual nominal significance level than will either of the actual individual levels. This suggests that the combined test might be a robust alternative to the F-test. The results of the sampling experiments do not conclusively indicate the outcome of this question. In particular, in every case but one where the data indicated that the combined test could be considered robust by the same criterion used for the individual F-tests and v-tests either one or both of the individual tests could also be considered robust. Because of this further consideration of the combined test will be postponed until a later time.

Furthermore, as an indirect outcome of this work the usefulness and versatility of the Burr distribution as an approximation to various distributions was established. Because only two parameters are required to determine the Burr distribution and because the Burr can be used to obtain fits to four moments it seems that this family of distributions should find a great deal of use when approximations are required.

In conclusion, the v-test, when considered as an alternative to the F-test, seems to be an appropriate upper tail test in small sample situations where both the numerator and denominator are believed to follow distributions with the same non-normal kurtosis. If the v-test is used in this situation the departure from robustness will be less than would occur with the F-test, and if prechance the distributions of numerator and denominator are normal the power lost by using the v-test is inconsequential.

While risking a slightly greater loss in power under normality, the modified F-test might be the best alternative to the usual F-test when both populations have the same non-normal kurtosis and the sample sizes for both populations are at least as great as 20. However any definite conclusions on this matter will have to await further work on the modified F-test.

When considering upper tail tests in situations where only one of the two populations is likely to have non-normal kurtosis it is preferable to have the numerator non-normal instead of the denominator. If the denominator is normally distributed and the numerator is only moderately non-normal in kurtosis then either the F- or the v-test would be appropriate, and possibly even robust as an upper tail test. But if the denominator has non-normal kurtosis and the numerator is normal then both the F-test and the v-test are equally bad. The behavior of the modified F-test, when only one population is non-normal, is not known.

Finally whenever the only non-normality possible is skewness it appears that both the F-test and the v-test are robust and equally appropriate.

APPENDIX A

A SYMMETRIC DOUBLE BURR DISTRIBUTION

In order to examine the effects of non-normality upon the v-test and the F-test a symmetric distribution with variable kurtosis was desired. Symmetric Burr distributions with kurtosis less than $\lambda_4 = 1.25$ are obtainable but higher values of λ_4 were desired. Because skewed Burr distributions with large values of λ_4 exist (see Figure 1) and because the cumulative distribution function, c.d.f., of a Burr distribution is easy to work with it was natural to consider mixing two Burr distributions to obtain a distribution of the desired shape.

Let the c.d.f. and the probability density function, p.d.f., of a regular Burr distribution be denoted by F(x) and f(x) respectively, then

$$F(x) = 1 - (1 + x^c)^{-k} \qquad x > 0$$
$$= 0 \qquad x \leq 0 \qquad (33)$$

and

$$f(x) = ckx^{c-1}(1 + x^c)^{-k-1} \qquad x > 0 \qquad (34)$$
$$= 0 \qquad x \leq 0$$

and the first derivative of the p.d.f. is

$$f'(x) = ckx^{c-2}(1 + x^c)^{-k-2}[(c-1)(1+x^c) - c(k+1)x^c] \qquad x > 0$$
$$= 0 \qquad x \leq 0 \qquad (35)$$

Thus by equating this last expression to zero and solving for x, the mode is found to be

$$\left(\frac{c-1}{ck+1}\right)^{1/c} . \tag{36}$$

The transformation

$$u = x - [(c-1)/(ck+1)]^{1/c} \tag{37}$$

will translate the regular Burr distribution so that the mode of the new distribution lies at the origin. This transformed p.d.f. and c.d.f. are respectively

$$\begin{aligned} g(u) &= ck(u+a)^{c-1}[1 + (u+a)^c]^{-k-1} & u > -a \\ &= 0 & u \leq -a \end{aligned} \tag{38}$$

and

$$\begin{aligned} G(u) &= 1 - [1 + (u+a)^c]^{-k} & u > -a \\ &= 0 & u \leq -a \end{aligned} \tag{39}$$

where

$$a = [(c-1)/(ck+1)]^{1/c} .$$

Consider the distribution with a c.d.f. given by

$$H(u) = 1/2[G(u) + 1 - G(-u)] . \tag{40}$$

The corresponding p.d.f. is

$$h(u) = 1/2\{g(u) + g(-u)\} . \tag{41}$$

And if $\mathbf{U}$ is distributed according to H(u) then

$$E(U^r) = \int_{-\infty}^{\infty} u^r h(u)\,du = \frac{1}{2}\int_{-\infty}^{\infty} u^r g(u)\,du + \frac{1}{2}\int_{-\infty}^{\infty} u^r g(-u)\,du . \tag{42}$$

When r is even, letting z = -u gives

$$E(U^r) = \frac{1}{2}\int_{-\infty}^{\infty} u^r g(u)\,du + \frac{1}{2}\int_{-\infty}^{\infty} z^r g(z)\,dz$$

$$= \int_{-\infty}^{\infty} u^r g(u)\,du \qquad (43)$$

and when r is odd letting z = -u gives

$$E(U^r) = \frac{1}{2}\int_{-\infty}^{\infty} u^r g(u)\,du - \frac{1}{2}\int_{-\infty}^{\infty} z^r g(z)\,dz = 0 \ . \qquad (44)$$

Therefore H(u) is symmetric and by virtue of the fact that the mode of g(u) is at the origin h(u) is unimodal. Furthermore by changing the parameters of the original Burr distribution the even moments about the origin, namely the kurtosis, can be changed within a wide range of values. Figure 2 shows the relationship between the kurtosis of the original Burr distribution, F(x), plotted as the ordinate, and the kurtosis of the symmetric double Burr distribution, H(u), derived from F(x).

Two distributions of the form of H(u), with λ_4 = 2.02 and 3.02, were used in the sampling experiments. Since the c.d.f. for distributions of the form of H(u) cannot be solved explicitly in terms of the argument, u, the following approach had to be used in order to transform a random sample from a uniform distribution into a random sample from H(u).

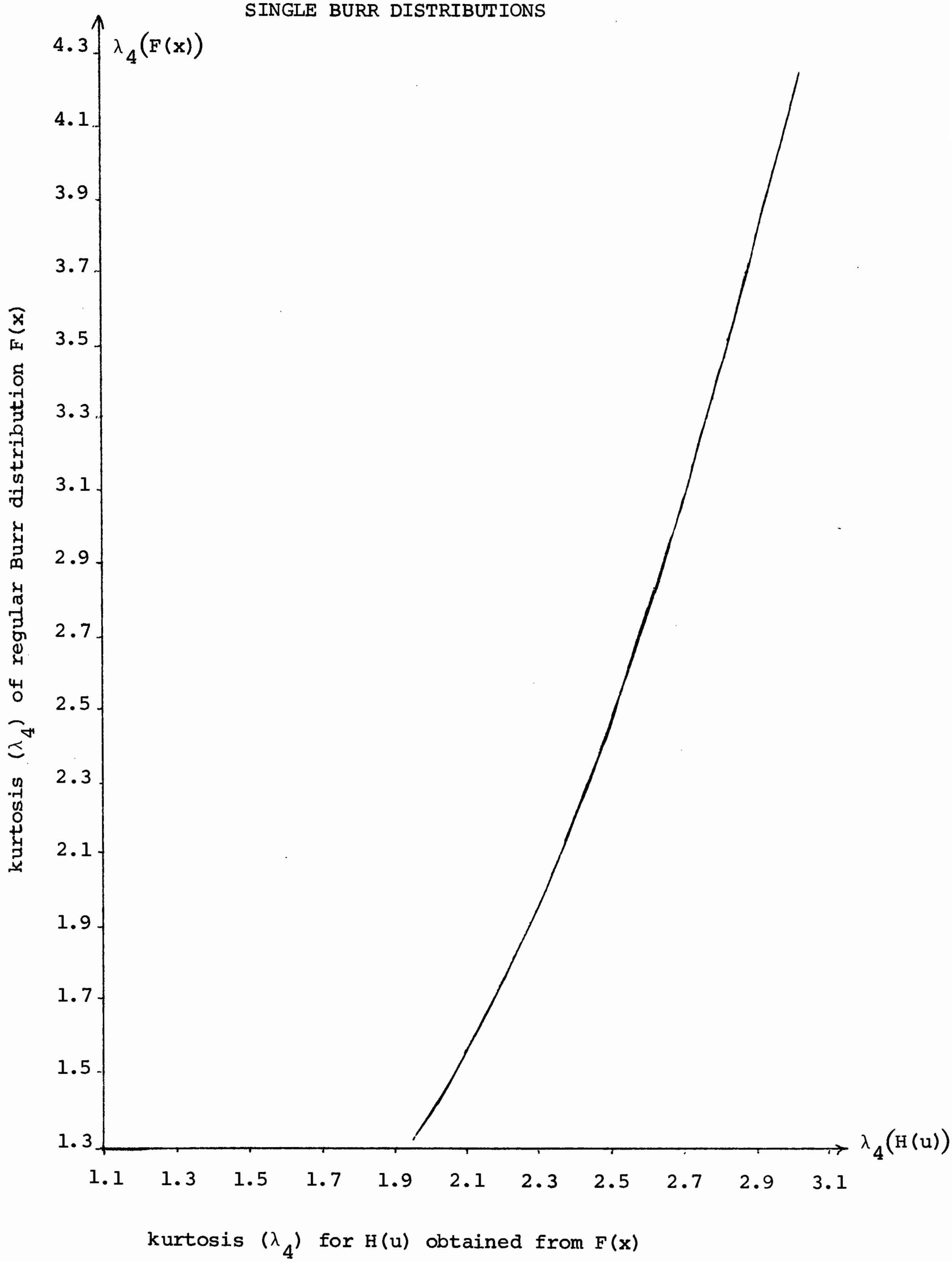
The c.d.f. for H(u) is

$$\begin{aligned} H(u) &= \frac{1}{2}\{1 - [1 + (u+a)^c]^{-k} + [1 + (a-u)^c]^{-k}\} && \text{if } -a < u < a \\ &= 0 && \text{if } u \leq -a \\ &= 1 && \text{if } u \geq -a \end{aligned} \qquad (45)$$

where a is the same as above. Breaking the non-trivial part of H(u) down

FIGURE 2.

RELATION OF KURTOSIS OF DOUBLE AND SINGLE BURR DISTRIBUTIONS



into the two portions G(u) and 1-G(-u), solving each one explicitly for u and averaging the result an approximation, say $\hat{u}$, to the solution of H(u) in terms of u is obtained. Next a smoothing function is chosen for the particular H(u) of interest which will improve $\hat{u}$ as much as desired. This is done by choosing H(u) to take on a set of values (such as the values .01 to 1.0 in steps of .01) finding the approximate solution, $\hat{u}$, at each of these values, and then using $\hat{u}$ in H(u) to obtain a value to compare with the chosen value of H(u). By the proper choice of a piece-wise linear function with which to operate upon the original values of H(u) the approximate solution, $\hat{u}$, can be made as accurate as is desired.

In the work done the smoothing functions were chosen so that the final value of H(u) was within .005 of the original value of H(u) at each of the 100 values used. This seemed to be justifiable in the light of Box and Andersen's (1955) statement to the effect that

> "Since the assumptions on which 'exact' distributions are determined are seldom justified in practice, and since the mind cannot appreciate small differences in probability, reasonable approximations to probability distributions are all that are really required."

This approach to the use of H(u) seemed to provide reasonable approximations to the exact probability distribution for sampling purposes, and so it was used for the two symmetric leptokurtic distributions mentioned above.

APPENDIX B

A PSEUDO-RANDOM NUMBER GENERATOR

The pseudo-random number generator used comes from one developed for the UNIVAC 1108 by Marsaglia and Bray (1968). The form actually used is a modified form of the generator given by Marsaglia and Bray, the modification being given by Grosenbaugh (1969).

It is called a pseudo-random number generator in that it can regenerate exactly the same set of numbers given the same starting value, otherwise the values generated by this generator are uniformly distributed between zero and one. In order to assure fresh samples the last value of one run was used as a starting value of the next run throughout the sampling work.

The 1108 dependent version of this random number generator is given below, Marsaglia and Bray (1968) give versions for other types of computers.

```
      FUNCTION URAND(KEY)
      DIMENSION NN(128)
      IF(KEY) 3, 3, 1
 1    L=234175
      M=616223
      ML=65539
      MM=33554433
      MK=362436069
      K=KEY
      KEY=0
      DO 2 I=1, 128
      K=K*MK
 2    NN(I)=K
```

```
 3    L=L*ML+(ISIGN(1, L*ML)-ISIGN(1, L))/2
      M=M*MM+(ISIGN(1, M*MM)-ISIGN(1, M))/2
      J=1+IABS(L)/268435456
      URAND=.5+FLOAT(NN(J)+L+M)*.145519152E-10
      K=K*MK+(ISIGN(1, K*MK)-ISIGN(1, K))/2
      NN(JJ)=K
      RETURN
      END
```

As can be seen from this listing this generator is a combination of three separate generators which are combined to increase the periodic length of the generator to the order of 10^8 or more. The starting value is KEY which should be an odd positive integer of six places or more.

APPENDIX C

TABLES OF OBSERVED SIGNIFICANCE LEVELS

The following tables give the observed significance levels as obtained from the sampling experiments. Each group of three rows contains the observed significance levels coming from one group of 2500 V and F values along with some additional information and the identification information for that group of test statistics. As was mentioned previously both the upper and lower tails of the empirical distributions of F and V were used to obtain observed significance levels. Therefore for each group of 2500 values of V and F there are six observed significance levels for each of the two tests.

The first three significance levels for V and F come from the lower tail and were obtained by using the inverse of the upper tail critical points for situations where the numerator and denominator are reversed from the way they are given in the identification information accompanying the significance levels. Thus these first three observed significance levels are upper tail levels for a situation where the distributions and sample sizes are reversed. The last three significance levels for V and F are the upper tail levels for the situation identified at the left of that group of three rows. Because they come from the same group of observations there is some correlation between the two sets of observed significance levels, just how much has not been determined.

The identification information for each group of observed significance levels is contained in the first two columns of each three row groups.

The information given is respectively the numerator sample size and the denominator sample size on the first row, the numerator population shape parameters on the second row and the denominator population shape parameters on the third row.

After the word both in each group of three rows, six numbers which resemble observed significance levels are given. These numbers represent the number of times both the V and the F statistics exceeded their respective critical points for the same set of sample values. The number of times both statistics exceeded their critical points was divided by 2500 to obtain these six numbers which are directly comparable with the two observed significance levels directly above them. As these numbers indicate the v- and the F-tests treat a given sample in very much the same way, and when one test is significant the other is very likely to be significant also.

			.05	.10	.25	.25	.10	.05
n=5	m=5	V	.0508	.0880	.2404	.2560	.0972	.0532
$\lambda_3=0$	$\lambda_4=0$	F	.0532	.0952	.2384	.2508	.1040	.0532
$\lambda_3=.467$	$\lambda_4=.045$	Both	.0492	.0844	.2240	.2396	.0936	.0476
n=5	m=10	V	.0432	.0936	.2612	.2344	.0976	.0432
$\lambda_3=0$	$\lambda_4=0$	F	.0452	.0984	.2604	.2428	.0976	.0496
$\lambda_3=.467$	$\lambda_4=.045$	Both	.0408	.0892	.2460	.2228	.0896	.0416
n=10	m-10	V	.0428	.0884	.2420	.2444	.0936	.0428
$\lambda_3=0$	$\lambda_4=0$	F	.0456	.0952	.2436	.2456	.0976	.0444
$\lambda_3=.467$	$\lambda_4=.045$	Both	.0396	.0824	.2260	.2312	.0896	.0384
n=5	m=21	V	.0392	.0952	.2440	.2488	.1020	.0488
$\lambda_3=0$	$\lambda_4=0$	F	.0416	.0992	.2392	.2548	.0992	.0460
$\lambda_3=.467$	$\lambda_4=.045$	Both	.0372	.0920	.2300	.2392	.0912	.0416
n=10	m=21	V	.0548	.1052	.2632	.2484	.0960	.0468
$\lambda_3=0$	$\lambda_4=0$	F	.0572	.1092	.2660	.2484	.0944	.0468
$\lambda_3=.467$	$\lambda_4=.045$	Both	.0496	.0980	.2496	.2336	.0856	.0420
n=21	m=21	V	.0432	.0944	.2372	.2604	.0976	.0444
$\lambda_3=0$	$\lambda_4=0$	F	.0484	.0948	.2376	.2572	.1036	.0496
$\lambda_3=.467$	$\lambda_4=.045$	Both	.0388	.0856	.2164	.2420	.0900	.0416
n=5	m=5	V	.0452	.0988	.2528	.2332	.0892	.0468
$\lambda_3=.467$	$\lambda_4=.045$	F	.0484	.1072	.2476	.2388	.0908	.0500
$\lambda_3=0$	$\lambda_4=0$	Both	.0412	.0968	.2404	.2240	.0852	.0464
n=5	m=10	V	.0424	.0904	.2372	.2528	.1004	.0440
$\lambda_3=.467$	$\lambda_4=.045$	F	.0464	.0888	.2432	.2540	.0996	.0464
$\lambda_3=0$	$\lambda_4=0$	Both	.0396	.0848	.2252	.2412	.0904	.0400
n=10	m-10	V	.0436	.0876	.2388	.2612	.0952	.0464
$\lambda_3=.467$	$\lambda_4=.045$	F	.0456	.0908	.2412	.2604	.1080	.0516
$\lambda_3=0$	$\lambda_4=0$	Both	.0392	.0808	.2264	.2480	.0896	.0432
n=5	m=21	V	.0496	.1016	.2492	.2420	.0880	.0448
$\lambda_3=.467$	$\lambda_4=.045$	F	.0520	.1052	.2504	.2420	.0868	.0440
$\lambda_3=0$	$\lambda_4=0$	Both	.0460	.0968	.2352	.2268	.0796	.0400
n=10	m=21	V	.0476	.0916	.2440	.2592	.0988	.0544
$\lambda_3=.467$	$\lambda_4=.045$	F	.0500	.0924	.2436	.2572	.1004	.0556
$\lambda_3=0$	$\lambda_4=0$	Both	.0428	.0856	.2300	.2420	.0896	.0488
n=21	m=21	V	.0340	.0904	.2536	.2356	.0948	.0520
$\lambda_3=.467$	$\lambda_4=.045$	F	.0408	.0928	.2528	.2404	.0972	.0536
$\lambda_3=0$	$\lambda_4=0$	Both	.0316	.0820	.2340	.2200	.0872	.0468

			.05	.10	.25	.25	.10	.05
n=5	m=5	V	.0504	.0948	.2324	.2524	.0968	.0480
λ_3=.467	λ_4=.045	F	.0536	.1008	.2316	.2556	.0988	.0504
λ_3=.467	λ_4=.045	Both	.0476	.0920	.2196	.2440	.0912	.0448
n=5	m=10	V	.0528	.1028	.2564	.2356	.0960	.0512
λ_3=.467	λ_4=.045	F	.0576	.1048	.2624	.2360	.0968	.0580
λ_3=.467	λ_4=.045	Both	.0488	.0976	.2456	.2164	.0896	.0492
n=10	m=10	V	.0516	.1092	.2616	.2612	.0928	.0440
λ_3=.467	λ_4=.045	F	.0548	.1120	.2620	.2560	.1016	.0504
λ_3=.467	λ_4=.045	Both	.0480	.1012	.2492	.2424	.0872	.0416
n=5	m=21	V	.0520	.0984	.2408	.2244	.0952	.0436
λ_3=,467	λ_4=.045	F	.0564	.1020	.2432	.2316	.0912	.0452
λ_3=.467	λ_4=.045	Both	.0504	.0944	.2268	.2140	.0836	.0380
n=10	m=21	V	.2456	.0972	.2440	.2424	.1008	.0508
λ_3=.467	λ_4=.045	F	.0500	.1052	.2476	.2416	.0976	.0564
λ_3=.467	λ_4=.045	Both	.0436	.0904	.2292	.2260	.0908	.0476
n=21	m=21	V	.0456	.0956	.2492	.2452	.0908	.0408
λ_3=.467	λ_4=.045	F	.0544	.1020	.2544	.2460	.0956	.0444
λ_3=.467	λ_4=.045	Both	.0440	.0892	.2340	.2280	.0820	.0360
n=5	m=5	V	.0556	.1024	.2420	.2764	.1120	.0628
λ_3=0	λ_4=0	F	.0588	.1088	.2520	.2760	.1180	.0660
λ_3=.011	λ_4=1.02	Both	.0532	.0976	.2332	.2624	.1084	.0604
n=5	m=10	V	.0492	.0912	.2464	.2856	.1344	.0700
λ_3=0	λ_4=0	F	.0512	.0984	.2480	.2800	.1280	.0712
λ_3=.011	λ_4=1.02	Both	.0456	.0880	.2368	.2660	.1176	.0652
n=10	m=10	V	.0476	.0988	.2508	.2856	.1380	.0804
λ_3=0	λ_4=0	F	.0556	.1092	.2572	.2820	.1336	.0804
λ_3=.011	λ_4=1.02	Both	.0428	.0932	.2372	.2684	.1284	.0760
n=5	m=21	V	.0484	.0984	.2392	.2680	.1084	.0604
λ_3=0	λ_4=0	F	.0508	.1056	.2448	.2628	.1048	.0512
λ_3=.011	λ_4=1.02	Both	.0464	.0944	.2288	.2504	.0960	.0472
n=10	m=21	V	.0484	.0920	.2324	.2748	.1344	.0748
λ_3=0	λ_4=0	F	.0564	.1052	.2584	.2636	.1272	.0680
λ_3=.011	λ_4=1.02	Both	.0476	.0888	.2248	.2524	.1204	.0636
n=21	m=21	V	.0524	.1000	.2500	.3068	.1352	.0700
λ_3=0	λ_4=0	F	.0628	.1124	.2648	.2960	.1332	.0728
λ_3=.011	λ_4=1.02	Both	.0492	.0920	.2372	.2824	.1200	.0628

			.05	.10	.25	.25	.10	.05
n=5	m=5	V	.0628	.1228	.2832	.2324	.0944	.0484
λ_3=.011	λ_4=1.02	F	.0640	.1280	.2812	.2336	.1016	.0524
λ_3=0	λ_4=0	Both	.0584	.1188	.2708	.2224	.0912	.0460
n=5	m=10	V	.0576	.1200	.2816	.2492	.1124	.0628
λ_3=.011	λ_4=1.02	F	.0604	.1232	.2812	.2500	.1096	.0664
λ_3=0	λ_4=0	Both	.0548	.1136	.2700	.2372	.1036	.0592
n=10	m=10	V	.0592	.1212	.2784	.2548	.1012	.0504
λ_3=.011	λ_4=1.02	F	.0648	.1212	.2728	.2624	.1128	.0556
λ_3=0	λ_4=0	Both	.0556	.1124	.2600	.2416	.0960	.0476
n=5	m=21	V	.0580	.1052	.2692	.2400	.1036	.0532
λ_3=.011	λ_4=1.02	F	.0560	.1084	.2660	.2476	.1040	.0548
λ_3=0	λ_4=0	Both	.0544	.0996	.2532	.2300	.0940	.0472
n=10	m=21	V	.0692	.1312	.2964	.2392	.1080	.0564
λ_3=.011	λ_4=1.02	F	.0732	.1336	.2984	.2500	.1152	.0648
λ_3=0	λ_4=0	Both	.0640	.1248	.2808	.2276	.1024	.0536
n=21	m=21	V	.0744	.1400	.2960	.2456	.1024	.0504
λ_3=.011	λ_4=1.02	F	.0776	.1360	.2864	.2608	.1172	.0672
λ_3=0	λ_4=0	Both	.0680	.1248	.2728	.2328	.0964	.0480
n=5	m=5	V	.0652	.1132	.2676	.2732	.1200	.0604
λ_3=.011	λ_4=1.02	F	.0664	.1180	.2760	.2760	.1260	.0680
λ_3=.011	λ_4=1.02	Both	.0604	.1084	.2588	.2640	.1124	.0588
n=5	m=10	V	.0556	.1100	.2716	.2576	.1316	.0792
λ_3=.011	λ_4=1.02	F	.0604	.1192	.2820	.2600	.1288	.0836
λ_3=.011	λ_4=1.02	Both	.0524	.1052	.2632	.2424	.1232	.0752
n=10	m=10	V	.0724	.1232	.2808	.2668	.1276	.0692
λ_3=.011	λ_4=1.02	F	.0764	.1340	.2824	.2724	.1364	.0752
λ_3=.011	λ_4=1.02	Both	.0664	.1172	.2648	.2508	.1200	.0640
n=5	m=21	V	.0592	.1080	.2720	.2676	.1256	.0744
λ_3=.011	λ_4=1.02	F	.0600	.1156	.2776	.2620	.1224	.0700
λ_3=.011	λ_4=1.02	Both	.0560	.1048	.2588	.2476	.1128	.0668
n=10	m=21	V	.0652	.1220	.2724	.2908	.1424	.0864
λ_3=.011	λ_4=1.02	F	.0736	.1336	.2844	.2952	.1464	.0920
λ_3=.011	λ_4=1.02	Both	.0620	.1164	.2620	.2696	.1332	.0788
n=21	m=21	V	.0676	.1228	.2704	.2808	.1296	.0708
λ_3=.011	λ_4=1.02	F	.0764	.1360	.2776	.2868	.1432	.0840
λ_3=.011	λ_4=1.02	Both	.0640	.1144	.2500	.2584	.1200	.0660

			.05	.10	.25	.25	.10	.05
n=5	m=5	V	.0364	.0868	.2280	.3060	.1452	.0780
$\lambda_3=0$	$\lambda_4=0$	F	.0424	.0944	.2392	.2972	.1492	.0836
$\lambda_3=0$	$\lambda_4=2.01$	Both	.0352	.0816	.2204	.2868	.1392	.0756
n=5	m=10	V	.0412	.0824	.2208	.3080	.1484	.0772
$\lambda_3=0$	$\lambda_4=0$	F	.0444	.0916	.2304	.2984	.1420	.0792
$\lambda_3=0$	$\lambda_4=2.01$	Both	.0380	.0796	.2108	.2860	.1340	.0736
n=10	m=10	V	.0396	.0868	.2116	.3308	.1572	.0968
$\lambda_3=0$	$\lambda_4=0$	F	.0468	.0932	.2228	.3088	.1548	.0944
$\lambda_3=0$	$\lambda_4=2.01$	Both	.0376	.0800	.2020	.2084	.1460	.0872
n=5	m=21	V	.0384	.0764	.1944	.3192	.1524	.0836
$\lambda_3=0$	$\lambda_4=0$	F	.0436	.0856	.2088	.2972	.1352	.0736
$\lambda_3=0$	$\lambda_4=2.01$	Both	.0364	.0724	.1872	.2896	.1300	.0696
n=10	m=21	V	.0336	.0752	.1968	.3384	.1676	.0960
$\lambda_3=0$	$\lambda_4=0$	F	.0420	.0908	.2108	.3100	.1484	.0788
$\lambda_3=0$	$\lambda_4=2.01$	Both	.0324	.0732	.1872	.3004	.1424	.0760
n=21	m=21	V	.0300	.0656	.1848	.3688	.1940	.1152
$\lambda_3=0$	$\lambda_4=0$	F	.0384	.0800	.2104	.3312	.1788	.1080
$\lambda_3=0$	$\lambda_4=2.01$	Both	.0284	.0628	.1768	.3240	.1700	.0984
n=5	m=5	V	.0888	.1476	.3088	.2236	.0968	.0516
$\lambda_3=0$	$\lambda_4=2.01$	F	.0896	.1556	.3052	.2316	.1016	.0536
$\lambda_3=0$	$\lambda_4=0$	Both	.0844	.1432	.2924	.2156	.0924	.0484
n=5	m=10	V	.0784	.1516	.3244	.2348	.1044	.0464
$\lambda_3=0$	$\lambda_4=2.01$	F	.0808	.1548	.3200	.2408	.1104	.0480
$\lambda_3=0$	$\lambda_4=0$	Both	.0724	.1452	.3096	.2236	.0992	.0424
n=10	m=10	V	.0892	.1596	.3332	.1984	.0764	.0356
$\lambda_3=0$	$\lambda_4=2.01$	F	.0940	.1548	.3212	.2164	.0876	.0412
$\lambda_3=0$	$\lambda_4=0$	Both	.0840	.1432	.3068	.1908	.0736	.0336
n=5	m=21	V	.0892	.1548	.3496	.2104	.0864	.0492
$\lambda_3=0$	$\lambda_4=2.01$	F	.0932	.1616	.3496	.2188	.0916	.0472
$\lambda_3=0$	$\lambda_4=0$	Both	.0868	.1496	.3340	.2016	.0808	.0436
n=10	m=21	V	.1116	.1772	.3516	.2056	.0804	.0392
$\lambda_3=0$	$\lambda_4=2.01$	F	.1120	.1748	.3348	.2200	.0880	.0396
$\lambda_3=0$	$\lambda_4=0$	Both	.1024	.1664	.3240	.1948	.0724	.0348
n=21	m=21	V	.1028	.1824	.3704	.1704	.0640	.0328
$\lambda_3=0$	$\lambda_4=2.01$	F	.0920	.1640	.3308	.2016	.0820	.0368
$\lambda_3=0$	$\lambda_4=0$	Both	.0852	.1560	.3212	.1656	.0616	.0300

			.05	.10	.25	.25	.10	.05
n=5	m=5	V	.0740	.1376	.2936	.2684	.1244	.0684
$\lambda_3=0$	$\lambda_4=2.01$	F	.0796	.1440	.2920	.2736	.1272	.0716
$\lambda_3=0$	$\lambda_4=2.01$	Both	.0700	.1348	.2788	.2552	.1192	.0656
n=5	m=10	V	.0688	.1308	.2792	.2776	.1404	.0780
$\lambda_3=0$	$\lambda_4=2.01$	F	.0768	.1424	.2896	.2732	.1340	.0788
$\lambda_3=0$	$\lambda_4=2.01$	Both	.0656	.1256	.2712	.2564	.1276	.0724
n=10	m=10	V	.0860	.1400	.2888	.2884	.1392	.0816
$\lambda_3=0$	$\lambda_4=2.01$	F	.0888	.1496	.2864	.2872	.1452	.0852
$\lambda_3=0$	$\lambda_4=2.01$	Both	.0796	.1332	.2688	.2680	.1280	.0740
n=5	m=21	V	.0728	.1372	.2964	.2724	.1400	.0856
$\lambda_3=0$	$\lambda_4=2.01$	F	.0808	.1472	.3036	.2672	.1276	.0784
$\lambda_3=0$	$\lambda_4=2.01$	Both	.0716	.1320	.2848	.2524	.1208	.0724
n=10	m=21	V	.0736	.1372	.2948	.2728	.1384	.0888
$\lambda_3=0$	$\lambda_4=2.01$	F	.0848	.1480	.2992	.2664	.1308	.0800
$\lambda_3=0$	$\lambda_4=2.01$	Both	.0704	.1280	.2776	.2456	.1232	0724
n=21	m=21	V	.0796	.1400	.3016	.2748	.1404	.0856
$\lambda_3=0$	$\lambda_4=2.01$	F	.0824	.1464	.3068	.2744	.1400	.0880
$\lambda_3=0$	$\lambda_4=2.01$	Both	.0704	.1284	.2796	.2520	.1260	.0752
n=5	m=5	V	.1080	.1736	.3572	.2092	.0732	.0372
$\lambda_3=0$	$\lambda_4=3.01$	F	.1080	.1796	.3516	.2212	.0816	.0400
$\lambda_3=0$	$\lambda_4=0$	Both	.1028	.1664	.3392	.2008	.0680	.0356
n=5	m=10	V	.1120	.1884	.3700	.2120	.0988	.0456
$\lambda_3=0$	$\lambda_4=3.01$	F	.1180	.1904	.3636	.2240	.1000	.0516
$\lambda_3=0$	$\lambda_4=0$	Both	.1080	.1812	.3524	.2056	.0916	.0432
n=10	m=10	V	.1384	.2144	.3888	.1856	.0704	.0332
$\lambda_3=0$	$\lambda_4=3.01$	F	.1296	.2076	.3620	.2116	.0852	.0380
$\lambda_3=0$	$\lambda_4=0$	Both	.1228	.1968	.3548	.1820	.0656	.0316
n=5	m=21	V	.1140	.1880	.3692	.2064	.0952	.0552
$\lambda_3=0$	$\lambda_4=3.01$	F	.1144	.1896	.3660	.2204	.1016	.0576
$\lambda_3=0$	$\lambda_4=0$	Both	.1076	.1804	.3552	.1976	.0904	.0500
n=10	m=21	V	.1440	.2192	.3984	.1972	.0864	.0488
$\lambda_3=0$	$\lambda_4=3.01$	F	.1344	.2144	.3600	.2192	.0968	.0560
$\lambda_3=0$	$\lambda_4=0$	Both	.1292	.2000	.3516	.1888	.0792	.0452
n=21	m=21	V	.1572	.2456	.4224	.1804	.0688	.0352
$\lambda_3=0$	$\lambda_4=3.01$	F	.1280	.1960	.3728	.2164	.0908	.0492
$\lambda_3=0$	$\lambda_4=0$	Both	.1240	.1928	.3668	.1752	.0668	.0328

			.05	.10	.25	.25	.10	.05
n=5	m=5	V	.0956	.1548	.3012	.2892	.1436	.0884
λ_3=0	λ_4=3.01	F	.1016	.1608	.3012	.2956	.1460	.0960
λ_3=0	λ_4=3.01	Both	.0932	.1500	.2848	.2756	.1372	.0868
n=5	m=10	V	.0908	.1468	.3068	.3152	.1668	.1036
λ_3=0	λ_4=3.01	F	.1024	.1616	.3160	.3072	.1604	.032
λ_3=0	λ_4=3.01	Both	.0872	.1440	.2924	.2956	.1496	.0924
n=10	m=10	V	.1100	.1656	.3064	.2988	.1568	.0984
λ_3=0	λ_4=3.01	F	.1108	.1712	.3044	.3052	.1580	.1000
λ_3=0	λ_4=3.01	Both	.1000	.1548	.2824	.2788	.1428	.0888
n=5	m=21	V	.1236	.1808	.3172	.3024	.1716	.1128
λ_3=0	λ_4=3.01	F	.1324	.1972	.3316	.2960	.1548	.0980
λ_3=0	λ_4=3.01	Both	.1204	.1772	.3112	.2788	.1464	.0928
n=10	m=21	V	.1008	.1652	.3152	.2884	.1704	.1088
λ_3=0	λ_4=3.01	F	.1120	.1824	.3232	.2880	.1596	.1028
λ_3=0	λ_4=3.01	Both	.0960	.1572	.2976	.2664	.1484	.0944
n=21	m=21	V	.0996	.1684	.3148	.3004	.1620	.0996
λ_3=0	λ_4=3.01	F	.1016	.1608	.3104	.2920	.1600	.1056
λ_3=0	λ_4=3.01	Both	.0872	.1424	.2876	.2732	.1440	.0880
n=5	m=5	V	.0488	.0908	.2380	.2744	.1108	.0552
λ_3=0	λ_4=0	F	.0516	.0968	.2416	.2740	.1132	.0616
λ_3=.782	λ_4=.729	Both	.0472	.0880	.2248	.2616	.1036	.0540
n=5	m=10	V	.0452	.0900	.2392	.2596	.1076	.0524
λ_3=0	λ_4=0	F	.0492	.0984	.2448	.2596	.1076	.0540
λ_3=.782	λ_4=.729	Both	.0428	.0872	.2276	.2448	.0980	.0500
n=10	m=10	V	.0496	.1044	.2420	.2740	.1080	.0564
λ_3=0	λ_4=0	F	.0552	.1124	.2556	.2752	.1132	.0600
λ_3=.782	λ_4=.729	Both	.0440	.0976	.2296	.2588	.1024	.0536
n=5	m=21	V	.0476	.1020	.2302	.2824	.1236	.0672
λ_3=0	λ_4=0	F	.0508	.1108	.2416	.2700	.1216	.0628
λ_3=.782	λ_4=.729	Both	.0432	.0988	.2268	.2596	.1132	.0568
n=10	m=21	V	.0428	.0924	.2456	.2740	.1164	.0604
λ_3=0	λ_4=0	F	.0504	.1056	.2540	.2644	.1140	.0556
λ_3=.782	λ_4=.729	Both	.0396	.0864	.2312	.2480	.1056	.0520
n=21	m=21	V	.0448	.0944	.2408	.2672	.1068	.0596
λ_3=0	λ_4=0	F	.0576	.1096	.2636	.2580	.1076	.0564
λ_3=782	λ_4=.729	Both	.0428	.0892	.2292	.2400	.0976	.0500

			.05	.10	.25	.25	.10	.05
n=5	m=5	V	.0584	.1040	.2696	.2392	.1012	.0536
$\lambda_3=.782$	$\lambda_4=.729$	F	.0600	.1088	.2636	.2444	.1152	.0576
$\lambda_3=0$	$\lambda_4=0$	Both	.0544	.0992	.2512	.2284	.0980	.0504
n=5	m=10	V	.0552	.1108	.2756	.2356	.0996	.0588
$\lambda_3=.782$	$\lambda_4=.729$	F	.0564	.1192	.2724	.2416	.1020	.0632
$\lambda_3=0$	$\lambda_4=0$	Both	.0512	.1072	.2608	.2220	.0904	.0560
n=10	m=10	V	.0560	.1164	.2820	.2360	.0840	.0428
$\lambda_3=.782$	$\lambda_4=.729$	F	.0572	.1196	.2844	.2420	.0980	.0500
$\lambda_3=0$	$\lambda_4=0$	Both	.0504	.1080	.2676	.2212	.0784	.0416
n=5	m=21	V	.0548	.1084	.2752	.2440	.1116	.0628
$\lambda_3=.782$	$\lambda_4=.729$	F	.0564	.1184	.2756	.2488	.1144	.0648
$\lambda_3=0$	$\lambda_4=0$	Both	.0508	.1028	.2656	.2364	.1028	.0564
n=10	m=21	V	.0640	.1196	.2792	.2332	.0968	.0560
$\lambda_3=.782$	$\lambda_4=.729$	F	.0692	.1296	.2800	.2448	.1092	.0564
$\lambda_3=0$	$\lambda_4=0$	Both	.0596	.1144	.2616	.2176	.0900	.0488
n=21	m=21	V	.0624	.1300	.2972	.2216	.0924	.0536
$\lambda_3=.782$	$\lambda_4=.729$	F	.0684	.1304	.2860	.2412	.1144	.0644
$\lambda_3=0$	$\lambda_4=0$	Both	.0572	.1192	.2752	.2092	.0884	.0488
n=5	m=5	V	.0564	.1056	.2540	.2608	.1148	.0576
$\lambda_3=.782$	$\lambda_4=.729$	F	.0628	.1120	.2556	.2676	.1184	.0640
$\lambda_3=.782$	$\lambda_4=.729$	Both	.0532	.1032	.2428	.2512	.1084	.0576
n=5	m=10	V	.0544	.1196	.2644	.2420	.1052	.0524
$\lambda_3=.782$	$\lambda_4=.729$	F	.0612	.1260	.2784	.2404	.1056	.0584
$\lambda_3=.782$	$\lambda_4=.729$	Both	.0512	.1136	.2544	.2228	.0968	.0492
n=10	m=10	V	.0472	.1060	.2616	.2728	.1156	.0596
$\lambda_3=.782$	$\lambda_4=.729$	F	.0568	.1156	.2708	.2768	.1292	.0704
$\lambda_3=.782$	$\lambda_4=.729$	Both	.0440	.0996	.2484	.2572	.1108	.2576
n=5	m=21	V	.0560	.1104	.2640	.2744	.1292	.0680
$\lambda_3=.782$	$\lambda_4=.729$	F	.0608	.1204	.2780	.2756	.1268	.0700
$\lambda_3=.782$	$\lambda_4=.729$	Both	.0540	.1064	.2556	.2564	.1168	.0616
n=10	m=21	V	.0548	.1168	.2620	.2644	.1120	.0596
$\lambda_3=.782$	$\lambda_4=.729$	F	.0664	.1300	.2852	.2684	.1212	.0616
$\lambda_3=.782$	$\lambda_4=.729$	Both	.0524	.1104	.2528	.2428	.1032	.0540
n=21	m=21	V	.0600	.1208	.2756	.2636	.1136	.0612
$\lambda_3=.782$	$\lambda_4=.729$	F	.0724	.1376	.2792	.2668	.1296	.0764
$\lambda_3=.782$	$\lambda_4=.729$	Both	.0576	.1140	.2524	.2394	.1076	.0584

			.05	.10	.25	.25	.10	.05
n=5	m=5	V	.0444	.0856	.2316	.2824	.1152	.0620
$\lambda_3=0$	$\lambda_4=0$	F	.0492	.0892	.2368	.2788	.1232	.0660
$\lambda_3=.948$	$\lambda_4=1.19$	Both	.0424	.0800	.2224	.2696	.1112	.0584
n=5	m=10	V	.0416	.0860	.2352	.2808	.1324	.0616
$\lambda_3=0$	$\lambda_4=0$	F	.0496	.0948	.2452	.2796	.1284	.0680
$\lambda_3=.948$	$\lambda_4=1.19$	Both	.0404	.0816	.2232	.2628	.1188	.0584
n=10	m=10	V	.0432	.0836	.2080	.3104	.1304	.0648
$\lambda_3=0$	$\lambda_4=0$	F	.0540	.0944	.2204	.3016	.1308	.0720
$\lambda_3=.948$	$\lambda_4=1.19$	Both	.0420	.0788	.1948	.2896	.1204	.0616
n=5	m=21	V	.0468	.1008	.2412	.2776	.1260	.0704
$\lambda_3=0$	$\lambda_4=0$	F	.0544	.1104	.2492	.2704	.1192	.0632
$\lambda_3=.948$	$\lambda_4=1.19$	Both	.0456	.0984	.2316	.2560	.1100	.0600
n=10	m=21	V	.0476	.0896	.2256	.2900	.1292	.0656
$\lambda_3=0$	$\lambda_4=0$	F	.0588	.1052	.2436	.2776	.1204	.0620
$\lambda_3=.948$	$\lambda_4=1.19$	Both	.0460	.0860	.2120	.2608	.1124	.0580
n=21	m=21	V	.0484	.0996	.2304	.2864	.1232	.0612
$\lambda_3=0$	$\lambda_4=0$	F	.0604	.1204	.2552	.2800	.1236	.0652
$\lambda_3=.948$	$\lambda_4=1,19$	Both	.0452	.0952	.2200	.2644	.1084	.0540
n=5	m=5	V	.0600	.1084	.2796	.2552	.0956	.0500
$\lambda_3=.948$	$\lambda_4=1.19$	F	.0628	.1164	.2796	.2600	.1056	.0548
$\lambda_3=0$	$\lambda_4=0$	Both	.0568	.1052	.2648	.2444	.0908	.0484
n=5	m=10	V	.0668	.1208	.2924	.2296	.0960	.0480
$\lambda_3=.948$	$\lambda_4=1.19$	F	.0728	.1276	.2908	.2368	.0976	.0548
$\lambda_3=0$	$\lambda_4=0$	Both	.0640	.1156	.2772	.2232	.0892	.0444
n=10	m=10	V	.0628	.1240	.3060	.2212	.0888	.0412
$\lambda_3=.948$	$\lambda_4=1.19$	F	.0668	.1264	.2996	.2364	.0992	.0540
$\lambda_3=0$	$\lambda_4=0$	Both	.0604	.1156	.2824	.2124	.0836	.0392
n=5	m=21	V	.0628	.1228	.2928	.2376	.1180	.0680
$\lambda_3=.948$	$\lambda_4=1.19$	F	.0660	.1248	.2956	.2456	.1212	.0744
$\lambda_3=0$	$\lambda_4=0$	Both	.0596	.1172	.2852	.2280	.1104	.0624
n=10	m=21	V	.0632	.1260	.2940	.2520	.1164	.0672
$\lambda_3=.948$	$\lambda_4=1.19$	F	.0720	.1256	.2904	.2648	.1336	.0756
$\lambda_3=0$	$\lambda_4=0$	Both	.0612	.1172	.2756	.2384	.1096	.0628
n=21	m=21	V	.0752	.1352	.3152	.2128	.0760	.0356
$\lambda_3=.948$	$\lambda_4=1.19$	F	.0768	.1360	.2984	.2392	.0940	.0504
$\lambda_3=0$	$\lambda_4=0$	Both	.0676	.1252	.2852	.2008	.0724	.0340

			.05	.10	.25	.25	.10	.05
n=5	m=5	V	.0604	.1092	.2644	.2696	.1160	.0620
$\lambda_3=.948$	$\lambda_4=1.19$	F	.0664	.1148	.2716	.2720	.1208	.0668
$\lambda_3=.948$	$\lambda_4=1.19$	Both	.0592	.1036	.2512	.2556	.1104	.0576
n=5	m=10	V	.0652	.1220	.2760	.2544	.1120	.0584
$\lambda_3=.948$	$\lambda_4=1.19$	F	.0732	.1320	.2852	.2564	.1084	.0636
$\lambda_3=.948$	$\lambda_4=1.19$	Both	.0640	.1180	.2644	.2408	.1008	.0544
n=10	m=10	V	.0732	.1264	.2804	.2684	.1152	.0608
$\lambda_3=.948$	$\lambda_4=1.19$	F	.0868	.1396	.2852	.2704	.1332	.0740
$\lambda_3=.948$	$\lambda_4=1.19$	Both	.0692	.1196	.2648	.2544	.1100	.0588
n=5	m=21	V	.0556	.1136	.2708	.2540	.1244	.0708
$\lambda_3=.948$	$\lambda_4=1.19$	F	.0656	.1256	.2904	.2584	.1184	.0716
$\lambda_3=.948$	$\lambda_4=1.19$	Both	.0540	.1080	.2632	.2400	.1100	.0640
n=10	m=21	V	.0640	.1212	.2744	.2664	.1272	.0676
$\lambda_3=.948$	$\lambda_4=1.19$	F	.0768	.1440	.2940	.2736	.1352	.0728
$\lambda_3=.948$	$\lambda_4=1.19$	Both	.0600	.1152	.2644	.2456	.1168	.0596
n=21	m=21	V	.0612	.1188	.2660	.2700	.1228	.2656
$\lambda_3=.948$	$\lambda_4=1.19$	F	.0792	.1336	.2812	.2872	.1396	.0864
$\lambda_3=.948$	$\lambda_4=1.19$	Both	.0584	.1120	.2424	.2512	.1160	.0604
n=5	m=5	V	.0804	.1400	.3040	.2308	.0968	.0512
$\lambda_3=1.18$	$\lambda_4=2.02$	F	.0832	.1420	.3036	.2376	.1064	.0564
$\lambda_3=0$	$\lambda_4=0$	Both	.0772	.1320	.2908	.2212	.0936	.0496
n=5	m=10	V	.0700	.1336	.3124	.2364	.1112	.0568
$\lambda_3=1.18$	$\lambda_4=2.02$	F	.0716	.1372	.3108	.2428	.1156	.0624
$\lambda_3=0$	$\lambda_4=0$	Both	.0668	.1252	.3004	.2224	.1032	.0532
n=10	m=10	V	.0800	.1436	.3092	.2308	.0976	.0528
$\lambda_3=1.18$	$\lambda_4=2.02$	F	.0832	.1480	.3044	.2508	.1200	.0684
$\lambda_3=0$	$\lambda_4=0$	Both	.0720	.1356	.2892	.2204	.0932	.0508
n=5	m=21	V	.0716	.1480	.3324	.2212	.1096	.0628
$\lambda_3=1.18$	$\lambda_4=2.02$	F	.0756	.1512	.3304	.2332	.1088	.0680
$\lambda_3=0$	$\lambda_4=0$	Both	.0704	.1416	.3164	.2124	.0988	.0584
n=10	m=21	V	.0944	.1632	.3420	.2324	.1024	.0540
$\lambda_3=1.18$	$\lambda_4=2.02$	F	.1000	.1680	.3348	.2536	.1248	.0716
$\lambda_3=0$	$\lambda_4=0$	Both	.0892	.1528	.3212	.2236	.0960	.0508
n=21	m=21	V	.0868	.1504	.3144	.2152	.0904	.0472
$\lambda_3=1.18$	$\lambda_4=2.02$	F	.0852	.1392	.2820	.2600	.1232	.0752
$\lambda_3=0$	$\lambda_4=0$	Both	.0772	.1304	.2708	.2088	.0876	.0448

			.05	.10	.25	.25	.10	.05
n=5	m=5	V	.0676	.1248	.2808	.2760	.1244	.0676
$\lambda_3=1.18$	$\lambda_4=2.02$	F	.0752	.1340	.2884	.2804	.1348	.0780
$\lambda_3=1.18$	$\lambda_4=2.02$	Both	.0656	.1188	.2696	.2652	.1204	.0648
n=5	m=10	V	.0704	.1260	.2828	.2824	.1448	.0808
$\lambda_3=1.18$	$\lambda_4=2.02$	F	.0792	.1360	.3012	.2844	.1448	.0888
$\lambda_3=1.18$	$\lambda_4=2.02$	Both	.0688	.1188	.2740	.2656	.1308	.0768
n=10	m=10	V	.0788	.1252	.2776	.2916	.1328	.0760
$\lambda_3=1.18$	$\lambda_4=2.02$	F	.0900	.1428	.2892	.3056	.1540	.0928
$\lambda_3=1.18$	$\lambda_4=2.02$	Both	.0760	.1176	.2596	.2776	.1260	.0724
n=5	m=21	V	.0716	.1480	.3324	.2212	.1096	.0628
$\lambda_3=1.18$	$\lambda_4=2.02$	F	.0756	.1512	.3304	.2332	.1088	.0680
$\lambda_3=1.18$	$\lambda_4=2.02$	Both	.0704	.1416	.3164	.2124	.0988	.0584
n=10	m=21	V	.0772	.1344	.2976	.2708	.1348	.0876
$\lambda_3=1.18$	$\lambda_4=2.02$	F	.0968	.1644	.3180	.2704	.1496	.0972
$\lambda_3=1.18$	$\lambda_4=2.02$	Both	.0736	.1284	.2808	.2500	.1256	.0816
n=21	m=21	V	.0820	.1408	.2944	.2784	.1384	.0792
$\lambda_3=1.18$	$\lambda_4=2.02$	F	.1000	.1644	.3112	.2964	.1572	.1036
$\lambda_3=1.18$	$\lambda_4=2.02$	Both	.0768	.1308	.2760	.2576	.1276	.0740

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13. ABSTRACT

The usual F-test on variances has been shown to be non-robust when the underlying populations are not normally distributed. An alternative test for equality of variances is developed and compared with the F-test by means of sampling experiments. This alternative test is shown to be less seriously affected by non-normal populations than is the F-test and a brief comparison of this alternative test and a modified F-test developed earlier is given.

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