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MINIMAL SUFFICIENT STATISTICS FOR THE GROUP DIVISIBLE PARTIALLY BALANCED INCOMPLETE BLOCK DESIGN (GD-PBIBD) WITH INTERACTION UNDER AN EISENHART MODEL II

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Minimal Sufficient Statistics for the Group Divisible Partially Balanced Incomplete Block Design (GD-PBIBD) with Interaction Under an Eisenhart Model II

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1. Introduction and Summary: This paper exhibits a set of minimal sufficient statistics together with the distribution of each statistic in the set for the GD-PBIBD with p replications per cell under the assumption that there is an Eisenhart Model II [4] with interaction. The Rao-Blackwell theorem [1,11] states that, if a minimum variance unbiased estimator exists, it must be an explicit function of the minimal sufficient statistics. It will be shown that the family of joint distributions of the minimal sufficient statistics is not complete, and so the question of existence of uniform minimum variance unbiased estimators is yet to be solved. This article should be regarded as a generalization of [6], [12] and some results given in [5] and [13].

2. Method: It is clear from [5], [6], [12], [13] that the derivation of a minimal sufficient set of statistics essentially hinges on the construction of an orthogonal matrix. Regarding the problem under consideration, the construction of an orthogonal matrix was based on the results of [6] and [7]. When the quadratic form of the joint probability density function of the normal variables in the vector of observations, say Y, is operated on by the orthogonal matrix, the quadratic form is reduced to a form which gives a set of sufficient statistics [9, 10]. The minimality of the set is then established by

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the scheme given by Lehmann and Scheffé [8].

3. Definitions and Assumptions: The definition of GD-PBIBD and various relationships which hold for such a design can be found in [2]. Since the purpose of this paper is to consider block treatment interaction, we shall assume that we have p observations per cell. Specifically we assume the model

(I) $$y_{ijk} = \mu + \beta_i + \tau_j + (\beta\tau)_{ij} + e_{ijk}$$

where $i = 1, 2, \ldots. b;\ j = 1,2,\ldots. t;\ k = n_{ij}$

$$n_{ij} = \begin{cases} 0 \text{ if treatment } j \text{ does not appear in block } i, \\ 1,2,\ldots. p \text{ if treatment } j \text{ appears in block } i. \end{cases}$$

The observations y_{ijo} do not exist.

In matrix notation, the bkp observations described by (I) can be written as:

(II) $Y = \mu\, j_1^{bkp} + X_1\beta + X_2\tau + X_3(\beta\tau) + e$ where the dimension of the matrices in (II) are:

Y(bkp x 1), X_1(bkp x b), β(b x 1), X_2(bkp x t), τ(t x 1), X_3(bkp x bk), $(\beta\tau)$ (bk x 1), e(bkp x 1), μ(1 x1), and j_1^u or j_u^1 is a u x 1 or 1 x u vector with all elements equal to one. This notation will be used throughout this paper and J_v^u will denote an u x v matrix with all elements equal to one. Under an Eisenhart Model II, we have the following distributional properties for the vectors $e, \beta, \tau, (\beta\tau)$.

(a) e is distributed (~) as the multivariate normal (MVN) mean $0(\emptyset)$ and covariance matrix $\sigma^2 I_{bkp}$, that is, $e \sim \text{MVN}(\emptyset, \sigma^2 I_{bkp})$;

(b) $\beta \sim \text{MVN}(\emptyset, \sigma_1^2 I_b)$;

(c) $\tau \sim \text{MVN}(\emptyset, \sigma_2^2 I_t)$;

(d) $(\beta\tau) \sim MVN\ (\emptyset, \sigma_3^2 I_{bk})$

(e) Cov $(\beta, \tau) = \emptyset$, Cov$[\beta, (\beta\tau)] = \emptyset$, Cov $[\tau, (\beta\tau)] = \emptyset$, Cov $(e, \beta) = \emptyset$,
Cov $(e, \beta) = \emptyset$, Cov$[e, (\beta\tau)] = \emptyset$

(f) μ is a scalar constant.

The following relationships hold for the matrix model and the prime(') will denote the transpose of a matrix throughout this paper:

(1) $X_1'X_1 = pkI_b$, (2) $X_2'X_2 = rpI_t$, (3) $X_3'X_3 = pI_{bk}$,

(4) $J_{bkp}^{bkp}X_1 = pkJ_b^{bkp}$, (5) $J_b^{bkp}X_1' = J_{bkp}^{bkp}$, (6) $J_{bkp}^{bkp}X_2 = rpJ_t^{bkp}$,

(7) $J_t^{bkp}X_2' = J_{bkp}^{bkp}$, (8) $J_{bkp}^{bkp}X_3 = bJ_{bk}^{bkp}$, (9) $J_{bk}^{bkp}X_3' = J_{bkp}^{bkp}$,

(10) If $X_2'X_1 = N$, $NN' = p^2[rB_0+\lambda_1B_1+\lambda_2B_2]$ where B's are defined in [3] and $B_0 = I_t$, $B_0 + B_1 + B_2 = J_t^t$, (11) If $X_3'X_1 = M'$, $MM' = p^2kI_b$

(12) If $X_3'X_2 = L'$, $LL' = p^2rI_t$,

(13) If $(X_2' - p^{-1}k^{-1}X_1N') = A$, $A'X_2 = (prI_t - p^{-1}k^{-1}NN')$

$$= [prI_t - pk^{-1}(rB_0 + \lambda_1B_1 + \lambda_2B_2)]$$

$$= \frac{p}{k}[r(k - 1)B_0 - \lambda_1B_1 - \lambda_2B_2]$$

(14) $A'X_1 = \emptyset$, (15) $ML' = pN$; (16) $J_t^tN = pkJ_b^t$, (17) $L'J_t^tN = p^2kJ_b^{tk}$

(18) $J_t^tL = pJ_{bk}^t$, (19) $L'J_t^tNM = p^3kJ_{bk}^{bk}$, (20) $L'J_t^tL = p^2J_{bk}^{bk}$,

(21) $N'J_t^t N = p^2 k J_b^b$ (22) $M'N'J_t^t = p^2 k J_t^{bk}$

(23) $X_1X_1'X_3X_3' = X_3X_3'X_1X' = pX_1X_1'$, (24) $X_2X_2'X_3X_3' = X_3X_3'X_2X_2' = pX_2X_2'$,

(25) If $F' = X_3' - p^{-1}k^{-1}M'X_1' - k[(rk-r+\lambda_1)p]^{-1}(L'-p^{-1}k^{-1}M'N')$

$$\cdot [I_t - (\lambda_1-\lambda_2)(\lambda_2 t)^{-1}(B_0+B_1)]A'$$

then $F'j_1^{bkp} = \emptyset$, $F'X_1 = \emptyset$, $F'X_2 = \emptyset$ and $p^{-1}F'F$ is an idempotent matrix of rank $bk-b-t+1$.

It is clear from the assumptions made for the matrix model (II) that Y is distributed as the multivariate normal with mean $\bar{\mu}$ and covariance matrix Σ where

$$\bar{\mu} = E(Y) = \mu j_1^{bkp} \quad \text{and}$$

$$\Sigma = E(Y - \bar{\mu}) \ (Y - \bar{\mu})' = (X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I_{bkp})$$

The joint density of the elements of Y is given by

(III) $g(Y,\gamma) = (2\pi)^{-bkp/2}|\Sigma|^{-1/2} \ \exp[-2^{-1}(Y - \bar{\mu})' \ \Sigma^{-1}(Y - \bar{\mu})]$

where γ is a vector of parameters $\mu, \sigma, \sigma_1, \sigma_2$, and σ_3.

4. Development of a Minimal Sufficient Set of Statistics:

Let $X = (j_1^{bkp}, X_1, X_2, X_3 \ )$. Since XX' is symmetric, there exists an orthogonal matrix Q such that $Q'XX'Q = D$ where D is a diagonal matrix. The rank of XX' for the GD-PBIBD is bk. Hence, we can have

$$Q'XX'Q = \begin{bmatrix} C' \\ P_5' \end{bmatrix} [J_{bkp}^{bkp} + X_1X_1' + X_2X_2' + X_3X_3'](C,P_5) = \begin{bmatrix} D^* & \emptyset \\ \emptyset & \emptyset \end{bmatrix}$$

where we have partitioned Q into (C,P_5) and D^* is bk x bk. From this it follows that $P_5'(J_{bkp}^{bkp} + X_1X_1' + X_2X_2' + X_3X_3')P_5 = \emptyset$. Since the matrices J_{bkp}^{bkp}, X_1X_1', X_2X_2', X_3X_3' are positive semi-definite, it follows that P_5 is a set of bk(p-1) orthogonal vectors such that

$$P_5'j_1^{bkp} = \emptyset, \quad P_5'X_1 = \emptyset, \quad P_5'X_2 = \emptyset \text{ and } P_5'X_3 = \emptyset.$$

The characteristic roots of NN' [3] are p^2rk, $p^2(rk - \lambda_2 t)$, $p^2(r - \lambda_1)$ with multiplicities 1, m-1, and m(n-1) respectively. Let $p^2(rk - \lambda_2 t) = \theta_1$ and $p^2(r - \lambda_1) = \theta_2$. The GD-PBIBD is classified as Singular (S), Semi-Regular (SR), or Regular (R), according as $\theta_2 = 0$. $\theta_1 = 0$, or neither of them is equal to zero. For brevity, a minimal set of sufficient statistics for the R-GD-PBIBD will be derived and from this result it will be easy to exhibit the minimal sets of sufficient statistics for S and SR-GD-PBIBD. Since NN' is symmetric there exists an orthogonal matrix Q_3 such that

$$Q_3'NN'Q_3 = \begin{bmatrix} p^2rk & \emptyset & \emptyset \\ \emptyset & \theta_1 I_{m-1} & \emptyset \\ \emptyset & \emptyset & \theta_2 I_{m(n-1)} \end{bmatrix}.$$

Partition Q_3 into (P_{30}, P_{31}, P_{32}) such that

$$\begin{bmatrix} P_{30}' \\ P_{31}' \\ P_{32}' \end{bmatrix} (NN')(P_{30}, P_{31}, P_{32}) = \begin{bmatrix} p^2rk & \emptyset & \emptyset \\ \emptyset & \theta_1 I_{m-1} & \emptyset \\ \emptyset & \emptyset & \theta_2 I_{m(n-1)} \end{bmatrix}.$$

Corresponding to the unique root p^2rk, there corresponds the unique characteristic vector P_{30}. Since $(1/\sqrt{t})j_t^1 NN'(1/\sqrt{t})j_1^t = p^2rk$,

$P_{30} = (1/\sqrt{t})j_1^t$, then $j_t^1 P_{31} = \emptyset$ and $j_t^1 P_{32} = \emptyset$.

The non-characteristic roots of NN' and N'N are equal and are of the same multiplicities. Hence, let Q_2 be an orthogonal matrix such that

$$Q_2' N' N Q_2 = \begin{bmatrix} p^2 rk & \emptyset & \emptyset \\ \emptyset & \emptyset_{c_0+c_1'} & \emptyset \\ \emptyset & \emptyset & D_3^* \end{bmatrix},$$

where:

i) D_3^* is a diagonal matrix of the non-zero characteristic roots of NN' excluding p^2rk

ii) c_0 = multiplicity of zero characteristic root of NN'

iii) $c_1' = b-t$

Partition Q_2 into (P_{20}, P_{21}, Q_{22}) such that

$$\begin{bmatrix} P_{20}' \\ P_{21}' \\ Q_{22}' \end{bmatrix} N'N (P_{20}, P_{21}, Q_{22}) = \begin{bmatrix} p^2 rk & \emptyset & \emptyset \\ \emptyset & \emptyset_{c_0+c_1'} & \emptyset \\ \emptyset & \emptyset & D_3^* \end{bmatrix},$$

where the dimensions of P_{20}, P_{21} and Q_{22} are $b\times 1$, $b\times(c_0+c_1')$ and $b \times \sum_{i=1}^{2} c_i$ respectively, where c_i denotes the multiplicity of the i^{th} non-zero characteristic root of NN' other than p^2rk. If $Q_{22} = (P_{22}, P_{23})$, the following relationships can be established among the partitions of Q_3 and Q_2

i) $P_{22}' = \theta_1^{-1/2} P_{31}' N$

ii) $P_{23}' = \theta_2^{-1/2} P_{32}' N.$

Since $A'A = prI_t - p^{-1}k^{-1}NN'$, the orthogonal matrix which diagonalizes NN', will also diagonalize $A'A$, that is,

$$Q_3' A' A Q_3 = \begin{bmatrix} 0 & \emptyset & \emptyset \\ \emptyset & [pr-(pk)^{-1}\theta_1] I_{m-1} & \emptyset \\ \emptyset & \emptyset & [pr-(pk)^{-1}\theta_2] I_{m(n-1)} \end{bmatrix}.$$

Consider now $F' = X_3' - p^{-1}k^{-1}M'X_1' - k(rk-r+\lambda_1)^{-1}(L' - p^{-1}k^{-1}M'N')$
$\cdot [I-(\lambda_1-\lambda_2)(\lambda_2 t)^{-1}(B_0+B_1)]A'$.
Since $p^{-1}F'F = p^{-1}F'X_3$ is an idempotent matrix of rank bk-b-t+1, let P_4' be bk-b-t+1 orthogonal vectors from a bk x bk orthogonal matrix which diagonalizes $p^{-1}F'F$. This can be done since we can always choose P_4 corresponding to the non-zero characteristic roots of the idempotent matrix.

From the above results one can verify that the matrix P as defined below is an orthogonal matrix.

$$P' = \begin{bmatrix} (bkp)^{-1/2} j_{bkp}^1 \\ (pk)^{-1/2} P_{21}' X_1' \\ (pk\theta_1)^{-1/2} P_{31}' N X_1' \\ (pk\theta_2)^{-1/2} P_{32}' N X_1' \\ [pr-(pk)^{-1}\theta_1]^{-1/2} P_{31}' A' \\ [pr-(pk)^{-1}\theta_2]^{-1/2} P_{32}' A' \\ p^{-1} P_4' F \\ P_5' \end{bmatrix} .$$

It should be noted here that the idempotency property of $p^{-1}F'F$ makes P an orthogonal matrix. All attempts to construct a similar matrix in order to exhibit a set of sufficient statistics for general PBIBD with two associate classes have been unsuccessful so far, although the results can be conjectured.

Consider now $g(Y,\gamma) = (2\pi)^{-\frac{bkp}{2}} |\Sigma|^{-\frac{1}{2}} \exp[-2^{-1}(Y-\bar{\mu})'PP'\Sigma^{-1}PP'(Y-\bar{\mu})]$.

With P' as defined above we can show that:

$$P'(Y-\bar{\mu}) = \begin{bmatrix} (bkp)^{\frac{1}{2}}(y_{\cdots} - \mu) \\ (pk)^{-\frac{1}{2}}P_{21}'X_1'Y \\ (pk\theta_1)^{-\frac{1}{2}}P_{31}'NX_1'Y \\ (pk\theta_2)^{-\frac{1}{2}}P_{32}'NX_1'Y \\ [pr-(pk)^{-1}\theta_1]^{-\frac{1}{2}}P_{31}'A'Y \\ [pr-(pk)^{-1}\theta_2]^{-\frac{1}{2}}P_{32}'A'Y \\ p^{-\frac{1}{2}}P_4'F'Y \\ P_5'Y \end{bmatrix},$$

where $y_{\cdots} = (bkp)^{-1}J_{bkp}^{1}Y$.

Find $P'\Sigma P$ and invert it to derive $P'\Sigma^{-1}P$ as follows:

$$\begin{bmatrix} W_{11} & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\ \emptyset & W_{22} & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & W_{33} & \emptyset & W_{35} & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & W_{44} & \emptyset & W_{46} & \emptyset & \emptyset \\ \emptyset & \emptyset & W_{53} & \emptyset & W_{55} & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & W_{64} & \emptyset & W_{66} & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & W_{77} & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & W_{88} \end{bmatrix},$$

where :

$W_{11} = (\sigma^2+pk\sigma_1{}^2+pr\sigma_2{}^2+p\sigma_3{}^2)^{-1}$,

$W_{22} = (\sigma^2+pk\sigma_1{}^2+p\sigma_3{}^2)^{-1}I_{c_o+c_1}$, ,

$$W_{33} = [(\sigma^2+\{pr-(pk)^{-1}\theta_1\}\sigma_2^2+p\sigma_3^2)]d_1^{-1}I_{m-1} \quad ,$$

$$W_{35} = W_{53} = -[k^{-2}\theta_1(rk-p^{-2}\theta_1)]^{1/2}d_1^{-1}\sigma_2^2I_{m-1} \quad ,$$

$$W_{44} = [\sigma^2+\{pr-(pk)^{-1}\theta_2\}\sigma_2^2+p\sigma_3^2]d_2^{-1}I_{m(n-1)} \quad ,$$

$$W_{46} = W_{64} = -[k^{-2}\theta_2(rk-p^{-2}\theta_2)]^{1/2}d_2^{-1}\sigma_2^2I_{m(n-1)} \quad ,$$

$$W_{55} = [\sigma^2+pk\sigma_1^2+(pk)^{-1}\theta_1\sigma_2^2+p\sigma_3^2]d_1^{-1}I_{m-1} \quad ,$$

$$W_{66} = [\sigma^2+pk\sigma_1^2+(pk)^{-1}\theta_2\sigma_2^2+p\sigma_3^2]d_2^{-1}I_{m(n-1)} \quad ,$$

$$W_{77} = [\sigma^2+p\sigma_3^2]^{-1}I_{bk-b-t+1} \quad ,$$

$$W_{88} = \sigma^2I_{bk(p-1)} \quad ,$$

where:

$$d_1 = [\sigma^2+p\sigma_3^2][\sigma^2+pk\sigma_1^2+pr\sigma_2^2+p\sigma_3^2] + (p^2rk-\theta_1)\sigma_1^2\sigma_2^2$$

and

$$d_2 = [\sigma^2+p\sigma_3^2][\sigma^2+pk\sigma_1^2+pr\sigma_2^2+p\sigma_3^2] + (p^2rk-\theta_2)\sigma_1^2\sigma_2^2 \quad .$$

Letting $q = (Y-\bar{\mu})'PP'\Sigma^{-1}PP'(Y-\bar{\mu})$, we have

$$q = (bkp)(\sigma^2+pk\sigma_1^2+pr\sigma_2^2+p\sigma_3^2)^{-1}(y... - \mu)^2$$

$$+ [pk(\sigma^2+pk\sigma_1^2+p\sigma_3^2)^{-1}Y'X_1P_{21}P_{21}'X_1'Y$$

$$+ (pkd_1)^{-1}[\sigma^2+\{pr-(pk)^{-1}\theta_1\}\sigma_2^2+p\sigma_3^2]Y'X_1P_{22}P_{22}'X_1'Y$$

$$+ (pkd_2)^{-1}[\sigma^2+\{pr-(pk)^{-1}\theta_2\}\sigma_2^2+p\sigma_3^2]Y'X_1P_{23}P_{23}'X_1'Y$$

$$+ [p(\sigma^2+ p\sigma_3^2)]^{-1}Y'FP_4P_4'F'Y + \sigma^{-2}Y'P_5P_5'Y$$

$$+ [pr - (pk)^{-1}\theta_1d_1]^{-1}[\sigma^2+pk\sigma_1^2+ (pk)^{-1}\theta_1\sigma_2^2+p\sigma_3^2]\ Y'A\ P_{31}P_{31}'A'Y$$

$$+ [pr - (pk)^{-1}\theta_2d_2]^{-1}[\sigma^2+pk\sigma_1^2+(pk)^{-1}\theta_2\sigma_2^2+p\sigma_3^2]\ Y'\ A\ P_{32}P_{32}'A'Y$$

$$-2\,(d_1kp^2)^{-1}\theta_1^{\frac{1}{2}}\sigma_2^{\,2}Y'X_1P_{22}P_{31}'A'Y\ -2(d_2kp^2)^{-1}\theta_2^{\frac{1}{2}}\sigma_2^{\,2}Y'X_1P_{23}P_{32}'A'Y\quad .$$

Define the ten statistics as follows:

$$
\text{(IV)}\quad
\begin{aligned}
s_1 &= y_{\cdots}\ , \\
s_2 &= (pk)^{-1}Y'X_1P_{21}P_{21}'X_1'Y \text{ if } b > t - c_0, \text{ not defined for } b = t-c_0\ , \\
s_3 &= (pk)^{-1}Y'X_1P_{22}P_{22}'X_1'Y\ , \\
s_4 &= (pk)^{-1}Y'X_1P_{23}P_{23}'X_1'Y\ , \\
s_5 &= p^{-1}Y'FP_4P_4'F'Y\ , \\
s_6 &= Y'P_5P_5'Y\ , \\
s_7 &= [pr - (pk)^{-1}\theta_1]\ Y'AP_{31}P_{31}'A'Y\ , \\
s_8 &= [pr - (pk)^{-1}\theta_2]\ Y'AP_{32}P_{32}'A'Y\ , \\
s_9 &= (p^2k)^{-1}\theta_1^{\frac{1}{2}}Y'X_1P_{22}P_{31}'A'Y\ , \\
s_{10} &= (p^2k)^{-1}\theta_2^{\frac{1}{2}}Y'X_1P_{23}P_{32}'A'Y.
\end{aligned}
$$

From the Neyman-Pearson [9, 10] factorization condition for sufficiency, it follows that these ten statistics are sufficient for the parameters μ. σ^2, $\sigma_1^{\,2}$, $\sigma_2^{\,2}$, $\sigma_3^{\,2}$.

At this stage it should be noted that in order to exhibit a set of sufficient statistics for the S-GD-PBIBD or SR-GD-PBIBD, $\theta_2 = 0$ or $\theta_1 = 0$ should be substituted in (IV). Hence, there will be eight statistics (after deleting s_4, s_{10} or s_3, s_9 accordingly, as θ_2 or θ_1 is zero) sufficient for μ, σ^2, $\sigma_1^{\,2}$, $\sigma_2^{\,2}$, $\sigma_3^{\,2}$.

$g(Y,\gamma)$ may be written in the form

$$g(Y,\gamma) = P(\gamma)\ \exp\ [-2^{-1}(bkpv_I s_1^{\,2} - 2bkpv_{11}s_1 + \sum_{i=2}^{10} v_i s_i)]$$

where

$$v_1 = (\sigma^2 + pk\sigma_1^{\,2} + pr\sigma_2^{\,2} + p\sigma_3^{\,2})^{-1}\ ,$$

$$v_2 = (\sigma^2 + pk\sigma_1^{\,2} + p\sigma_3^{\,2})^{-1}\ ,$$

$$v_3 = [\sigma^2 + \{pr - (pk)^{-1}\theta_1\}\sigma_2^2 + p\sigma_3^2]d_1^{-1},$$

$$v_4 = [\sigma^2 + \{pr - (pk)^{-1}\theta_2\}\sigma_2^2 + p\sigma_3^2]d_2^{-1},$$

$$v_5 = (\sigma^2 + p\sigma_3^2)^{-1},$$

$$\text{(V)} \quad v_6 = \sigma^{-2},$$

$$v_7 = [\sigma^2 + pk\sigma_1^2 + (pk)^{-1}\theta_1\sigma_2^2 + p\sigma_3^2]d_1^{-1},$$

$$v_8 = [\sigma^2 + pk\sigma_1^2 + (pk)^{-1}\theta_2\sigma_2^2 + p\sigma_3^2]d_2^{-1},$$

$$v_9 = -2\sigma_2^2 d_1^{-1},$$

$$v_{10} = -2\sigma_2^2 d_2^{-2},$$

$$v_{11} = v_1\mu .$$

Lehmann and Scheffé [8] have set forth a procedure by which a set of sufficient statistics may be shown to be minimal. It follows then that a sufficient condition that $s_1, s_2, \ldots, s_{10}$ form a minimal set is that there exist no constants $b_1, b_2, \ldots, b_{11}, c$ (not all zero) such that

$$\text{(VI)} \quad \sum_{i=1}^{11} b_i v_i(\gamma) = c.$$

(VI) is not true for any $b_1, b_2, \ldots, b_{11}$ and c except when all vanish.

In (V) it is clear that μ appears only in v_{11}. Since $v_1, v_2, \ldots, v_{10}$ are homogeneous functions of $\sigma, \sigma_1, \sigma_2, \sigma_3$ of degree -2, the constant c can only be zero.

In order to prove that v_i's are linearly independent, effect the following transformation;

$$x = \sigma^2, \quad y = \sigma^2 + pk\sigma_1^2 + p\sigma_3^2, \quad z = \sigma^2 + pk\sigma_1^2 + pr\sigma_2^2 + p\sigma_3^2, \quad w = \sigma^2 + p\sigma_3^2.$$

The functions of (V) become:

$$v_1 = xyz^2w^3 [1 + \alpha_1 u] [1 + \alpha_2 u] D^{-1} ,$$

$$v_2 = xz^3w^3 [1 + \alpha_1 u] [1 + \alpha_2 u] D^{-1} ,$$

$$v_3 = xyz^2w^2 [w + \alpha_1 (z-y)] [1 + \alpha_2 u] D^{-1} ,$$

$$v_4 = xyz^2w^2 [w + \alpha_2 (z-y)] [1 + \alpha_1 u] D^{-1} ,$$

$$v_5 = xyz^3w^2 [1 + \alpha_1 u] [1 + \alpha_2 u] D^{-1} ,$$

$$v_6 = yz^3w^3 [1 + \alpha_1 u] [1 + \alpha_2 u] D^{-1} ,$$

$$v_7 = xyz^2w^2 [y + \beta_1 (z-y)] [1 + \alpha_2 u] D^{-1} ,$$

$$v_8 = xyz^2w^2 [y + \beta_1 (z-y)] [1 + \alpha_1 u] D^{-1} ,$$

$$v_9 = -2xyz^2w^2\delta_1 (z-y) [1 + \alpha_2 u] D^{-1} ,$$

$$v_{10} = -2xyz^2w^2\delta_1 (z-y) [1 + \alpha_1 u] D^{-1} .$$

where : $u = (z-y)(y-w)(zw)^{-1}$, $\alpha_1 = (p^2rk)^{-1}(p^2rk - \theta_1)$, $\alpha_2 = (p^2rk)^{-1}(p^2rk - \theta_2)$,

$$\beta_1 = (p^2rk)^{-1}, \quad \delta_1 = pr^{-1}, \quad D = xyz^3w^3[1 + \alpha_1 u][1 + \alpha_2 u].$$

Since $xyz^2w^3u^2$, $xz^3w^3u^2$, $xyz^3w^2u^2$, and $yz^3w^3u^2$ occur only in v_1, v_2, v_5 and v_6 respectively, they are themselves mutually independent and are linearly independent of v_3, v_4, v_8, v_9, and v_{10}. In order to show that v_3, v_4, v_7, v_8, v_9, and v_{10} are mutually linearly independent, form a 6 x 6 determinant corresponding to the coefficients of w, uw, z, y, uz, uy after cancelling the common factor xyz^2w^2 in the above six functions. It can be shown that the determinant does not vanish, and hence this implies v_3, v_4, v_7, v_8, v_9, and v_{10} are linearly independent. This conclusion then implies that the set of sufficient statistics defined in (IV) is minimal.

For the benefit of the readers, it should be pointed out here that the above method of establishing a minimality of the set of the sufficient statistics for the R-GD-PBIBD can be simplified in the case of S-GD-PBIBD and SR-GD-PBIBD. The method consists of keeping appropriate v_i's from (V) corresponding to S-GD-PBIBD or SR-GD-PBIBD; effecting the same transformation, and following the technique given in [7].

In Table I, various properities of the statistics in a minimal set are given.

It should be noted in Table I that:

(a) s_2 is not defined if $b = t-c_0$, and the relationship for s_2 holds if $b > t-c_0$.

(b) $\chi^2(s)$ denotes the central chi-square with s degrees of freedom.

(c) The g_i's are the non-zero characteristic roots of $2^{-1}(G_1 + G'_1)\not{Z}$ where $G_1 = k^{-1}X_1N'P_{31}A'$ and h_i's are the non-zero characteristic roots of $2^{-1}(H_1 + H'_1)\not{Z}$ where $H_1 = k^{-1}X_1N'P_{32}P'_{32}A'$.

The results of this paper will be summarized in the following theorem and corollaries:

<u>Theorem</u>: If an Eisenhart Model II is assumed in a GD-PBIBD with interaction and p observations per cell, then (i) for S-GD-PBIBD or SR-GD-PBIBD there are eight statistics in a minimal set of sufficient statistics if $b > t-c_0$, and there are seven statistics in a minimal set of sufficient statistics if $b = t-c_0$ (ii) for R-GD-PBIBD there

Table I

Properties of Sufficient

Statistics in a Minimal Set

Statistics	Expectation	Distribution	Independent of
s_1	μ	$N[\mu;(bkp)^{-1}(\sigma^2+pk\sigma_1^2+pr\sigma_2^2+p\sigma_3^2)]$	$s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}$
s_2	$(c_0+c'_1)[\sigma^2+pk\sigma_1^2+p\sigma_3^2]$	$(\sigma^2+pk\sigma_1^2+p\sigma_3^2)\chi^2_{(c_0+c'_1)}$	$s_1, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}$
s_3	$(m-1)[\sigma^2+pk\sigma_1^2+(pk)^{-1}\theta_1\sigma_2^2+p\sigma_3^2]$	$[\sigma^2+pk\sigma_1^2+(pk)^{-1}\theta_1\sigma_2^2+p\sigma_3^2]\chi^2_{(m-1)}$	$s_1, s_2, s_4, s_5, s_6, s_8, s_{10}$
s_4	$m(n-1)[\sigma^2+pk\sigma_1^2+(pk)^{-1}\theta_2\sigma_2^2+p\sigma_3^2]$	$[\sigma^2+pk\sigma_1^2+(pk)^{-1}\theta_1\sigma_2^2+p\sigma_3^2]\chi^2_{[m(n-1)]}$	$s_1, s_2, s_3, s_5, s_6, s_7, s_9$
s_5	$(bk-b-t+1)\ (\sigma^2+p\sigma_3^2)$	$[\sigma^2+p\sigma_3^2]\ \chi^2_{(bk-b-t+1)}$	$s_1, s_2, s_3, s_4, s_6, s_7, s_8, s_9, s_{10}$
s_6	$bk(p-1)\sigma^2$	$\sigma^2\chi^2_{[bk(p-1)]}$	$s_1, s_2, s_3, s_4, s_5, s_7, s_8, s_9, s_{10}$
s_7	$(m-1)[\sigma^2+(pk)^{-1}\theta_1\sigma_2^2+p\sigma_3^2]$	$[\sigma^2+(pk)^{-1}\theta_1\sigma_2^2+p\sigma_3^2]\ \chi^2_{(m-1)}$	$s_1, s_2, s_4, s_5, s_6, s_8, s_{10}$
s_8	$m(n-1)[\sigma^2+(pk)^{-1}\theta_1\sigma_2^2+p\sigma_3^2]$	$[\sigma^2+(pk)^{-1}\theta_2\sigma_2^2+p\sigma_3^2]\ \chi^2_{[m(n-1)]}$	$s_1, s_2, s_3, s_5, s_6, s_7, s_9$
s_9	$(m-1)(p^2rk-\theta_1)(pk)^{-2}\theta_1\sigma_2^2$	$\Sigma g_i\ \chi^2(1)$	$s_1, s_2, s_4, s_5, s_6, s_8, s_{10}$
s_{10}	$m(n-1)(p^2rk - \theta_2)(pk)^{-2}\theta_2\sigma_2^2$	$\Sigma h_i\ \chi^2(1)$	$s_1, s_2, s_3, s_5, s_6, s_7, s_9$

are ten statistics in a minimal set of sufficient statistics if $b > t-c_0$ and there are nine statistics in a minimal set of sufficient statistics if $b = t-c_0$.

Corollary 1: The explicit form of the statistics in a minimal set is given in (IV).

Corollary 2: The expectation of each of the statistics as defined in (V) is given in column two of the table.

Corollary 3: The distribution of each of the statistics of the minimal set as defined in (IV) is given in column three of the table.

Corollary 4: The statistics defined in (IV) are pairwise independent except for the pairs (s_3, s_7), (s_3, s_9), (s_4, s_8), (s_4, s_{10}), (s_7, s_9), (s_8, s_{10}), for (R-GD-PBIBD); (s_3, s_7), (s_3, s_9), (s_7, s_9), for S-GD-PBIBD; (s_4, s_8), (s_4, s_{10}), (s_8, s_{10}) for SR-GD-PBIBD.

COMMENTS:

(i) It is clear from the table that there are six unbiased estimators of σ_2^2 and hence there are non-trivial unbiased estimators of zero. This implies that the minimal sufficient set of statistics proposed in this paper is not complete.

(ii) If it exists, a minimum variance unbiased estimator must be an explicit function of the (minimal sufficient) statistics in the table.

(iii) In [6] and [12], the components (except one) proposed in the minimal sets of sufficient statistics were associated with the quantities normally calculated from the A.O.V. table. All attempts to establish similar relationships in this case have not been successful. This implies, in order to make the maximum use of the results given in this article, that a new computing technique must be derived or the existing analysis of variance method must be written in terms of the components of the minimal sufficient statistics.

(iv) Further work using the results of this paper is necessary to investigate the problem of combination of the several unbiased estimators of the variance components and the related question, the existence or non-existence of uniform minimum variance unbiased estimators of the variance components.

At present, various problems similar to those given in [5]and [13] are being investigated.

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13. ABSTRACT

An Eisenhart Model II with interaction for a group divisible partially balanced incomplete block design with p replocates per cell is considered. Specifically, the model $y_{ijk} = \mu + \beta_i + \tau_j + (\beta\tau)_{ij} + e_{ijk}$ is assumed, where $i = 1, 2, \cdots, b$; $j = 1, 2, \cdots, t$; $k = n_{ij}$ and $n_{ij} = 0$ if treatment j does not appear in block i, and $n_{ij} = 1, 2, \cdots, p$ if treatment j appears in block i.

If β_i, τ_j, $(\beta\tau)_{ij}$, and e_{ijk} are normally and independently distributed, then a set of minimal sufficient statistics together with the distribution of each statistic in the set is derived. It is shown that the family of joint distributions of the minimal sufficient statistics is not complete and so the question of existence of uniformly minimum variance unbiased estimates is yet to be solved.

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