DIVISION OF MATHEMATICAL SCIENCES
Southern Methodist University
Dallas, Texas 75275

THEMIS SIGNAL ANALYSIS STATISTICS RESEARCH PROGRAM

EXACT EXISTENCE OF EVERY POSSIBLE DISTRIBUTION FOR ANY SAMPLE ORDER STATISTIC

by

John E. Walsh

Technical Report No. 6
Department of Statistics THEMIS Contract

Department of Statistics
Southern Methodist University
Dallas, Texas 75222

THEMIS SIGNAL ANALYSIS STATISTICS RESEARCH PROGRAM

EXACT EXISTENCE OF EVERY POSSIBLE DISTRIBUTION FOR ANY SAMPLE ORDER STATISTIC

by

John E. Walsh

Technical Report No. 6
Department of Statistics THEMIS Contract

September 9, 1968

Research sponsored by the Office of Naval Research
Contract N00014-68-A-0515
Project NR 042-260

DEPARTMENT OF STATISTICS
Southern Methodist University

EXACT EXISTENCE OF EVERY POSSIBLE DISTRIBUTION FOR ANY SAMPLE ORDER STATISTIC

John E. Walsh
Southern Methodist University*

ABSTRACT

Given are any stated value for sample size and an arbitrary but specified univariate distribution. Let an arbitrary but specified order statistic for a univariate random sample of this size be considered. It is shown that a statistical population always exists, for yielding the sample, such that the distribution of the order statistic is exactly the specified distribution. Some asymptotic implications of these results are outlined.

INTRODUCTION AND RESULTS

The problem of the possible distributions for the largest and the smallest order statistics of sets of independent observations has been considered by Juncosa (1949) for nonsample cases. He showed that the order statistic distribution can come as close as desired to any distribution as the number of observations increases sufficiently.

This paper, with stronger results, considers the more restricted situation of a random sample, any sample size, and all order statistics. It is shown that an order statistic distribution can exactly equal any distribution (not necessarily continuous). More specifically, consider an arbitrary but specified univariate distribution, any given sample size, and any stated order statistic. A distribution is identified such that the order statistic of a sample of the given size from the corresponding population has a distribution exactly equal to the specified distribution.

---

*Research partially supported by NASA Grant NGR 44-007-028.

Some implications of these results are for asymptotic situations where the sample size, specified, is arbitrarily large. The population sampled is absolutely continuous (density function exists) for the cases considered. First, the cumulative distribution function (cdf) of an extreme can have any possible form that is absolutely continuous. Also, the cdf of an extreme can have an unlimited number of forms when the density function of the population sampled is required to be nonzero and analytic (all derivatives exist, etc.) over its range of nonzero values. Thus, the usual three limiting forms, often called asymptotes (for example, see Gumbel, 1958), are only a few of an infinite number of possibilities.

Second, for asymptotic situations, consider a sample percentage point and suppose that the density function of the population sampled is nonzero throughout its range of nonzero values. The cdf of this percentage point can have an infinite number of absolutely continuous forms (including the normality form that often occurs). In fact, it can have any such form that is nonzero over its range of nonzero values. Moreover, the percentage point cdf can have an infinite number of forms when the density function is also required to be analytic over its range of nonzero values.

Although these implications are for specified sample size, they apply to any sample size (no matter how large). Thus, taking n extremely large does not necessarily reduce the possible forms of distributions that can occur for an extreme or a sample percentage point.

VERIFICATION

Use n for the sample size and let the t-th order statistic be considered (t = 1 for the smallest sample value, etc.). Also, let F(x) be the cdf that is to occur exactly for the t-th order statistic. The function $G(x)$ is defined by

$$\sum_{i=t}^{n} \binom{n}{i} G(x)^{i} [1-G(x)]^{n-i} \equiv F(x). \tag{1}$$

The function $G(x)$ satisfies the properties of a cdf (monotonically increasing, $G(-\infty) = 0$, $G(\infty) = 1$, etc.).

Consider the cdf for the t-th order statistic of a random sample of size n from a population with cdf $G(x)$. This distribution is given by the lefthand expression of equation (1) and thus exactly equals $F(x)$.

REFERENCES

Gumbel, Emil J., Statistics of Extremes, Columbia University Press, 1958.

Juncosa, M. L., "On the distribution of the minimum of a sequence of mutually independent random variables," Duke Math. Jour., Vol.16, 1949, pp. 609-618.

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

NATING ACTIVITY *(Corporate author)* SOUTHERN METHODIST UNIVERSITY	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP UNCLASSIFIED

ORT TITLE
Exact Existence of Every Possible Distribution for Any Sample Order Statistic

CRIPTIVE NOTES *(Type of report and inclusive dates)*
Technical Report

HOR(S) *(First name, middle initial, last name)*

John E. Walsh

ORT DATE September 9, 1968	7a. TOTAL NO. OF PAGES 3	7b. NO. OF REFS 2
NTRACT OR GRANT NO. N00014-68-A-0515 OJECT NO. NR 042-260	9a. ORIGINATOR'S REPORT NUMBER(S) 6. 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*	

STRIBUTION STATEMENT

No limitations

UPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Office of Naval Research

BSTRACT

Given are any stated value for sample size and an arbitrary but specified univariate distribution. Let an arbitrary but specified order statistic for a univariate random sample of this size be considered. It is shown that a statistical population always exists, for yielding the sample, such that the distribution of the order statistic is exactly the specified distribution. Some asymptotic implications of these results are outlined.

D FORM 1 NOV 65 1473

UNCLASSIFIED
Security Classification