THEMIS SIGNAL ANALYSIS STATISTICS RESEARCH PROGRAM

GENERALLY APPLICABLE TWO-PERSON PERCENTILE GAME THEORY

by

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Technical Report No. 59
Department of Statistics THEMIS Contract

March 31, 1970

Research sponsored by the Office of Naval Research
Contract N00014-68-A-0515
Project NR 042-260

DEPARTMENT OF STATISTICS
Southern Methodist University

GENERALLY APPLICABLE TWO-PERSON PERCENTILE GAME THEORY

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ABSTRACT

Considered is discrete two-person game theory where the players choose their strategies separately and independently. Payoffs can be of a very general nature and are not necessarily numbers. However, the totality of outcomes (pairs of payoffs), corresponding to the possible combinations of strategies, can be ranked separately by each player according to their desirability to that player. For specified α_i, a largest level of desirability (corresponds to one or more outcomes O_i) occurs for the i-th player such that he can assure, with probability at least α_i, that an outcome with at least this desirability is obtained; this can be done simultaneously for i=1,2. Game theory of a median nature occurs when $\alpha_1=\alpha_2=1/2$. A method is given for determining O_i and an optimum (mixed) strategy for each player. Practical aspects of applying this percentile game theory are examined. Results for $\alpha_1=\alpha_2=1/2$ are compared with those previously developed for discrete median game theory.

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*Research partially supported by Mobil Research and Development Corporation. Also associated with ONR contract N00014-68-A-0515 and NASA Grant NGR 44-007-028.

INTRODUCTION AND DISCUSSION

The case of two players with finite numbers of strategies is considered. Each player selects his strategy separately and independently of the strategy chosen by the other player.

A pair of payoffs, one to each player, occurs for every combination of strategy choice by the players. These pairs are the possible outcomes for the game. The payoffs can be of an extremely general nature. Some of them may not even be numerical (for example, they may denote categories). However, the outcomes are considered to be such that they can be ranked, according to relative desirability level, separately by each player.

The ranking of outcomes can be tedious but should usually be achievable on a paired comparison basis. That is, for each two outcomes, a player expresses his preference (with equal desirability a possibility). A ranking is obtained when no circularity of definite preference occurs. Often, reasonable rules can be imposed that will eliminate curcularity of definite preference. For example, a suitable function of the two payoffs might be used for ranking the outcomes. This approach avoids the practical difficulty of requiring a player to make a huge number of paired comparisons.

It is to be emphasized that a ranking of outcomes not only considers the payoff to the player doing the ranking but also the corresponding payoff to the other player. Thus, a ranking provides the relative desirability of what can occur for the game, including results for the other player.

The basis for percentile game theory is that each player should want the occurrence of an outcome with a high level of desirability (to him). However, a player does not fully control the outcome choice and needs a criterion (to guide him in strategy choice) that reflects his desires and also is usable. The class of criteria considered in this paper is always usable and, for each player, should often include a criterion that reflects the player's desires.

For player i ($i=1,2$), let the outcomes be ordered according to increasing desirability to him. Also, player i specifies a probability α_i which represents the assurance with which he wishes to obtain an outcome with reasonably high desirability. A largest level of desirability occurs among the outcomes such that player i can assure, with probability at least α_i, that an outcome with at least this desirability is obtained. This can be done simultaneously for both players. The symbol O_i designates the outcome, or outcomes, with this largest level of desirability for player i.

A method (oriented toward minimum effort) is given for identifying O_i and determining an optimum mixed strategy for each player. This method of solution tends to maximize α_i for a given level of desirability for O_i.

Only a finite number of values are attainable for α_i. A value is attainable for α_i when use of a corresponding optimum strategy (by player i) cannot assure some outcome at least as desirable as O_i with probability in excess of α_i. For a given player (and method of solution), attainable values are determined by the ordering for the outcomes and the locations of the outcomes in the payoff matrix for the player. It would seem advantageous to use only α_i that are attainable. For example, an attainable α_i whose value is nearest the stated α_i should be a satisfactory choice in some cases.

Some results are developed for helping reduce the effort needed to identify O_i and determine an optimum mixed strategy for player i. That is, consider all outcomes that are at least as desirable as a specified outcome. Let the locations of these outcomes be marked in the payoff matrix for player i. Depending on the locations of the marks, a bound is developed for the probability with which player i can assure the occurrence of an outcome in the set that is marked.

It is to be noted that assuring at least the level for O_i with probability at least α_i is the best that can be "forced" by player i when α_i is given. However, the mixed strategy used by the other player can be such that the probability of at least O_i substantially exceeds α_i. This could happen even when the other player uses a mixed strategy that is optimum for him (in the sense of this paper). In fact, evaluation of the true probabilities of at least O_i, when both players use optimum strategies, provides information that can be useful. Suppose, for example, that player 1 has only one optimum strategy but player 2 has several strategies that are optimum. Also, player 2 knows the value used for α_1. Then, player 2 might choose among his optimum strategies by considering what happens when player 1 uses his optimum strategy.

Some results have already been developed for the case of discrete median game theory. In fact, the idea of ranking the outcomes, which led to the material of this paper, was initially used in ref. 1 for median game theory. In this paper, median game theory occurs as the special case where $\alpha_1=\alpha_2=1/2$. A comparison is made between this special case and previous material for median game theory.

The next section contains a statement of the results for this paper. The following sections provides a comparison of the case $\alpha_1=\alpha_2=1/2$ with previous material for median game theory. The final section contains some verification for the stated results.

STATEMENT OF RESULTS

The same results apply to each player and are stated for player i. Material is first given for the general case where α_i can have any value in the interval $0<\alpha_i\leq 1$. Then, modifications for the case where α_i is an attainable value are considered. The results are stated in terms of a marking of outcome locations in the payoff matrix for player i. The $r(\geq 2)$ rows of this payoff matrix correspond to the strategies for player i while the $c(\geq 2)$ columns are the strategies for the other player.

The case of $\alpha_i\leq 1/2$ is considered first. As the initial step, mark the position(s) in the payoff matrix for player i of the outcome(s) with the highest level of desirability to player i. Next, also mark the position(s) of the outcome(s) with the next to highest desirability. Continue this marking, according to decreasing level of desirability, until the first time that marks in all the columns can be obtained from a set of rows whose number does not exceed $1/\alpha_i$. Then, if r-s is the smallest number of rows in such a set, a marked outcome can be assured with probability at least $1/(r-s)\geq\alpha_i$, perhaps greater than $1/(r-s)$. Now, remove the mark(s) for the least desirable outcome(s) of those that received marks. Then, by the following procedure, determine whether some one of the remaining outcomes can be assured with probability at least α_i. The procedure is to replace every marked position in the matrix by unity and all others by zero. The resulting matrix of ones and zeroes is considered to be for a zero-sum game with an expected-value basis and is solved for the value of the game to player i. Some one of the outcomes corresponding to the marked positions can be obtained with probability at least α_i if and only if this game value is at least α_i.

Suppose that the game value is less than α_i. Then the maximum level of desirability that can be assured with probability at least α_i is the level corresponding to the outcome(s) with marking(s) removed at this step. Otherwise (game value at least α_i), remove the mark(s) for the least desirable outcome(s) of those still having marks. Then, as just described, determine whether some one of the remaining marked outcomes can be assured with probability at least α_i. If not, the maximum level of desirability that can be assured with probability at least α_i is the level corresponding to the outcome(s) with marking(s) removed at this step. If a probability of at least α_i can be assured, continue in the same way until the first time some one of the remaining marked outcomes cannot be assured with probability at least α_i. Then, the maximum desirability level that can be assured with probability at least α_i is the level for the outcome(s) with marking(s) removed at this step.

Now consider the case of $\alpha_i > 1/2$. Mark the matrix positions of outcomes (as for $\alpha_i \leq 1/2$), according to decreasing desirability level, until the last time that unmarked positions in all rows can be obtained from a set of columns whose number does not exceed $1/(1-\alpha_i)$. If c-s' is the smallest number of columns in such a set, player i can assure a marked outcome with probability at most $1 - 1/(c-s')$, and perhaps less than this value, where it is to be noticed that $\alpha_i \geq 1 - 1/(c-s')$. When $\alpha_i = 1 - 1/(c-s')$, replace all marked positions by unity and all unmarked positions by zero. Then, treating the resulting payoff matrix as for a zero-sum game with an expected-value basis, solve for the game value to player i. If this game value is α_i, a desirability level at least equal to the last (and lowest) level marked can be assured with probability α_i.

When this game value is less than α_i, or when $\alpha_i > 1 - 1/(c-s')$, the procedure is to also mark the position(s) of the outcome(s) with the highest desirability level among those whose positions are still unmarked. Replace all marked positions by unity and all unmarked positions by zero in the resulting marking of the matrix. This matrix of ones and zeroes is considered to be for a zero-sum game with an expected-value basis and is solved for the value to player i. If the game value is at least α_i, a desirability level at least equal to that for the outcome(s) marked at the last step can be assured with probability at least α_i. If the game value is less than α_i, continue in the same way until the first time some one of the marked outcomes can be assured with probability at least α_i. Then a desirability level at least equal to that for the marking at the last step can be assured with probability at least α_i. Incidentally, if $\alpha_i > 1 - 1/c$, the marking needs to be continued until the first time that a pure strategy of all marked outcomes occurs for player i.

The method of solution determines the outcome(s) O_i. Now consider determination of an optimum strategy for player i. Use the matrix marking that (ultimately) resulted in the smallest set of marked outcomes (by the procedure used) such that an outcome of this set can be assured with probability at least α_i. Replace the marked positions by unity and the others by zero. Treat the resulting matrix as that for a zero-sum game with an expected-value basis. An optimum strategy for player i in this zero-sum game is α_i-optimum for him. The value of the game for player i is an attainable α_i that is at least equal to the stated α_i being used.

Next, consider situations where a desired value is stated for α_i but the requirement that α_i must be attainable is imposed. Then, the attainable value used is ordinarily: The nearest value at most equal to the stated α_i, the nearest value at least equal to it, or the nearest attainable value to the stated α_i. The nearest attainable value at least equal to the stated α_i is directly determined by the procedure given for the case of general α_i. When the stated α_i is not attainable, the nearest attainable value at most equal to it is determined by first removing the mark(s) for the outcome(s) with lowest desirability level (in the final marking for general solution using the stated α_i). Then, marked positions are replaced by unity, unmarked positions by zero, and the resulting matrix treated as for a zero-sum game with an expected-value basis. The value of this game for player i determines the attainable α_i that is at most equal to the stated α_i. The procedure for determination of the attainable α_i to be used also provides O_i and a corresponding optimum strategy for player i.

The set of all available α_i such that $0 < \alpha_i \leq 1$ can be determined in a straightforward but tedious fashion. As the new marks for decreasing levels of desirability are made, they are replaced by unity and the unmarked positions are replaced by zero, in the matrix for player i. The resulting matrix is considered to be for a zero-sum game with an expected-value basis. Solution of this game for the value to player i provides an attainable value for α_i. This is done for all levels of desirability in the ordering of the outcomes by player i. Of course, more than one level of desirability can provide the same value for α_i. Also, α_i is zero when the markings do not occur in all columns, and α_i is unity for a marking, and all further markings, when there is at least one row that is fully marked.

The method used requires that all outcomes with equal desirability to player i be simultaneously marked in his payoff matrix. This tends to reduce the amount of computation and also to maximize the probability of obtaining at least a stated level of desirability. However, other ways could be used in which not all the outcomes of equal desirability are marked at the same time. In fact, the preferred sequence approach of ref. 2 could be used to mark each outcome separately. These special approaches might possibly be useful in some cases but are not considered here.

COMPARISON WITH MEDIAN MATERIAL

Median game theory occurs as the special case of percentile game theory where $\alpha_1 = \alpha_2 = 1/2$. Several results have already been developed for median game theory (refs. 1, 2, 3, 4, 5). The first results are given in ref. 2. The application advantages of these first results are expounded in ref. 3. Results with increased applicability are given in ref. 4, and results with general applicability are given in ref. 5. For all of these cases, the players are assumed to select their strategies separately and independently.

The possibility of cooperation is considered in ref. 1 and some rules are developed for deciding when cooperation is preferable to optimum use of median game theory. In development of these rules, the idea of a general ranking of outcomes arose (a restricted form of ranking is used in ref. 5). This idea is the basis for the material on generally applicable two-person percentile game theory given in this paper.

The results for $\alpha_1 = \alpha_2 = 1/2$ are the most useful that have been obtained for median game theory with no cooperation (and include the results of ref. 5 as a special case). They are generally applicable subject only to the ability of the players to rank the outcomes according to increasing desirability level. Any kind of outcomes that can be ranked by the players are eligible for use. The ranking can be according to any type of preference. For example, an increase in the payoff to the other player might represent an increase in the desirability of an outcome.

Only the case of separate and independent choice of strategies is considered here. However, an approach similar to that of ref. 1 should be usable in deciding on situations where cooperation is preferable to two-person percentile game theory, and is a subject for further research.

VERIFICATIONS

The statements about the probability properties when marks in all the columns can be obtained from a set of r-s rows follow from

THEOREM I. When the marked positions of outcomes in the matrix for player i are such that marks in all columns are obtained from r-s rows, this player can assure occurrence of a marked outcome with probability at least 1/(r-s).

PROOF. When r-s = 1, so that a row is fully marked, the probability is unity that some one of its outcomes can be assured by the player.

Suppose that r-s ≥ 2. Let $p_1,\ldots,p_r$ and $q_1,\ldots,q_c$ be the mixed strategies used, with a unit probability being possible. The probability of obtaining a marked outcome is

$$\sum_{i=1}^{r} p_i Q_i,$$

where Q_i is the sum of the q's for the columns that have marked outcomes in the i-th row. The largest value of this probability that the player can assure, through choice of $p_1,\ldots,p_r$, is

$$G = \min_{q_1,\ldots,q_c} (\max_i . Q_i).$$

Let $i(1),\ldots,i(r-s)$ be r-s rows that together contain marked values in all columns. For any minimizing choice of the values for $q_1,\ldots,q_c$, all of $Q_{i(1)},\ldots,Q_{i(r-s)}$ are at most G. Thus,

$$(r-s)G \geq Q_{i(1)} + \ldots + Q_{i(r-s)} \geq 1$$

and a probability of at least 1/(r-s) can be assured by the player.

COROLLARY When the unmarked positions of outcomes in the matrix for player i are such that unmarked positions in all rows are obtained from c-s' columns, the other player can assure an unmarked outcome with probability at least 1/(c-s').

When the circumstances for the Corollary hold, player i can assure a marked value with probability at most $1 - 1/(c-s')$. If $c-s' \le 1/(1-\alpha_i)$, then $\alpha_i \ge 1 - 1/(c-s')$, so that player i can assure a marked outcome with probability at most α_i (perhaps less). Also a probability as high as α_i can possibly occur only when $\alpha_i = 1 - 1/(c-s')$.

The remaining results can be verified by suitable use of

THEOREM II. A sharp lower bound on the probability that player i can assure some outcome of a specified set that is marked in his payoff matrix, and one or more corresponding optimum strategies for him in accomplishing this, can be determined from solution of the value to player i of a zero-sum game with an expected-value basis. The payoff matrix for player i in this game has the value unity at all marked positions and zero at all other positions.

PROOF. Let each player use an arbitrary but specified mixed strategy (with a unit probability possible). The expression for the expected payoff to player i with these strategies is also the expression for the probability of the occurrence of some one of the outcomes that are marked in the original payoff matrix for player i.

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UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* SOUTHERN METHODIST UNIVERSITY	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP UNCLASSIFIED	
3. REPORT TITLE Generally applicable two-person percentile game theory.		
4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)* Technical report		
5. AUTHOR(S) *(First name, middle initial, last name)* John E. Walsh		
6. REPORT DATE March 31, 1970	7a. TOTAL NO. OF PAGES 13	7b. NO. OF REFS 5
8a. CONTRACT OR GRANT NO. N00014-68-A-0515 b. PROJECT NO. NR 042-260 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) 59 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*	
10. DISTRIBUTION STATEMENT This document has been approved for public release and sale; its distribution is unlimited. Reproduction in whole or in part is permitted for any purpose of the United States Government.		
11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Office of Naval Research	

13. ABSTRACT

Considered is discrete two-person game theory where the players choose their strategies separately and independently. Payoffs can be of a very general nature and are not necessarily numbers. However, the totality of outcomes (pairs of payoffs), corresponding to the possible combinations of strategies, can be ranked separately by each player according to their desirability to that player. For specified α_i, a largest level of desirability (corresponds to one or more outcomes O_i) occurs for the i-th player such that he can assure, with probability at least α_i, that an outcome with at least this desirability is obtained; this can be done simultaneously for $i=1,2$. Game theory of a median nature occurs when $\alpha_1=\alpha_2=1/2$. A method is given for determining O_i and an optimum (mixed) strategy for each player. Practical aspects of applying this percentile game theory are examined. Results for $\alpha_1=\alpha_2=1/2$ are compared with those previously developed for discrete median game theory.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6811

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A-31408