THEMIS SIGNAL ANALYSIS STATISTICS RESEARCH PROGRAM

A NOTE ON MATRIX RENEWAL FUNCTION

by

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Technical Report No. 55
Department of Statistics THEMIS Contract

January 23, 1970

Research sponsored by the Office of Naval Research
Contract N00014-68-A-0515
Project NR 042-260

DEPARTMENT OF STATISTICS
Southern Methodist University

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ABSTRACT

The Laplace-Steiltjes Transform of the matrix renewal function $M(t)$ of a Markov Renewal process expanded in powers of the argument s, in this paper, by using a generalized inverse of the matrix $I-P_o$, where P_o is the transition probability matrix of the imbedded markov chain. This helps in obtaining the values of moments of any order of the number of renewals and also of the moments of the first passage times, for large values of t, the time. All the results of renewal theory are hidden under the Laplacian curtain and this expansion helps to lift this curtain at least for large values of t and is thus useful in applications of Markov Renewal processes to inventory control of repairable items and to counter theory.

A note on Matrix Renewal Function

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1. INTRODUCTION: Let $P_o = [p_{ij}]$ $(i,j = 1,2,\ldots,m)$ be the transition probability matrix of the imbedded Markov Chain of a Markov Renewal Process (M.R.P.), involving a finite number m of states. Let F_{ij} be the distribution function (d.f.) of the time spent in state i by the process before going to the next transition state j $(i,j = 1,2,\ldots m)$. Let J_t denote the state of the process at time t and $N_j(t)$ denote the number of visits to state j in the interval (o,t). The matrix $M(t) = [M_{ij}(t)]$, where

$$M_{ij}(t) = E\{N_j(t) \mid J_o = i\} \tag{1.1}$$

is called the matrix renewal function of the M.R.P. Its Laplace-Stieltjes transform (L.-S.T.) is given by (Pyke 1961)

$$m(s) = \int_o^\infty e^{-st} d_t M(t) = (I-q(s))^{-1} - I \tag{1.2}$$

where

$$Q(t) = [Q_{ij}(t)],\ Q_{ij}(t) = p_{ij}F_{ij}(t),\ q(s) = \int_o^\infty e^{-st} d_t Q(t). \tag{1.3}$$

The asymptotic behaviour of the renewal function in an ordinary renewal process (m=1) is well-known (see for example Cox 1962 or Smith 1959) and

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*This research was sponsored by the office of Naval Research, Contract No. N00014-68-A-0515 project No. NR 042-260.

the simplest way of deriving it is to expand its Laplace transform in powers of the argument s, in the neighborhood of s=0 and use Tauberian arguments. Since all higher order moments of the number of renewals in an ordinary renewal process are expressible in terms of the renewal function, their asymptotic behaviour can also be investigated from this, provided a sufficient number of terms in the expansion of the Laplace transform are retained.

These results and methods can be extended to an M.R.P. The present authors demonstrated this by expanding the L.‾S.T. $(I-q(s))^{-1}-I$ given by (1.2) in powers of s. The expansion was obtained from

$$I-q(s) = I-P_o+sP_1-\frac{s^2}{2!}P_2+\frac{s^3}{3!}P_3 \text{ ——.....} \quad (1.4)$$

where

$$P_k = \int_o^\infty x^k dQ(x), \quad (k=0,1,2,...,). \quad (1.5)$$

subject, of course, to the existence of these moments. The main difficulty in this is obviously the singularity of the matrix $I-P_o$. The required expansion of m(s) was obtained in terms of certain matrices H_r (r=0,1,2,...) obtainable from the adjoint of $I-Po+sP_1$. Recently Hunter (1968) and Keilson (1969) have independently solved this problem by different methods. Hunter uses a generalized inverse (Rao 1966) of $I-P_o$ along with the relationship between the moments of Q_{ij} and G_{ij}, the d.f. of the first passage time of the M.R.P. from state i to state j (Pyke 1961). Keilson uses the spectral decomposition of the matrix q(s) and then finds the limiting values of the first two derivatives of $(I-q(s))^{-1}-I$ with respect to s, at s=0, which are nothing but the coefficients of powers of s in its expansion. The spectral decomposition by itself is elegant but obtaining the limits from it becomes more and more involved, if further terms in the expansion of m(s) are needed (which are in fact needed for investigation of higher moments of $N_j(t)$ or of moments of G_{ij}) Hunter's generalized inverse method seems to be the best but unfortunately he uses it indirectly, on the moments of G_{ij} first and then using the derived results to expand m(s). His method has become, therefore, more laborious and unnecessarily complicated. The present note is intended to use the generalized inverse method directly on $(I-q(s))^{-1}$. This method yields the matrix coefficients in its expansion very easily, recursively.

2. Generalized Inverse of $I-P_o$

Let $\underline{U}' = [U_1, U_2, \ldots, U_m]$ be the vector of the stationary state probabilities ($\Sigma U_i=1$) of the imbedded Markov Chain and let $\underline{e}'=[1,\ldots,1]$, a vector of m components. Then it is well-known that

$$P_o\underline{e} = \underline{e}\ , \qquad \underline{U}'\ P_o = \underline{U}' \tag{2.1}$$

Let $$L = \underline{e}\ \underline{U}' \tag{2.2}$$

The matrix $Z = (I-P_o + L)^{-1}$ is known as the fundamental matrix of the Markov Chain (Kemmeny and Snell 1960) and the following results are easily derivable. (Hunter 1968) ....(2.3)

(i) Z is a generalized inverse of $I - P_o$ i.e.

$$(I-P_o)Z(I-P_o) = (I-P_o)$$

(ii) $P_oZ = ZP_o$

(iii) $Z\underline{e} = \underline{e}$ (iv) $\underline{U}'Z = \underline{U}'$

(v) $ZL = LZ = L$

(vi) $Z(I-P_o) = I - L$

We also define

$$k_r = \underline{U}'\ P_r\underline{e}\ (r = 1,2,\ldots)\quad , \tag{2.4}$$

so that

$$LP_r L = k_r L \quad (r = 1,2,\ldots) \tag{2.5}$$

We shall also need the following result (Rao 1966)

The general solution of the equation $AX = B$ is

$$X = A^- B + (I-A^- A)W \tag{2.6}$$

where A^- is any generalized inverse of A and W is any arbitrary matrix. This result holds if and only if the consistency condition

$$AA^- B = B \tag{2.7}$$

is satisfied.

3. Expansion of $(I - q(s))^{-1}$.

It is well-known from the theory of ordinary renewal processes (see for example Keilson 1969) that, if the Markov chain is ergodic and if the P_k's exist, $(I-q(s))^{-1}$ can be expanded in the form

$$\frac{1}{s} A_{-1} + A_o + s A_1 + s^2 A^2 + \ldots\ldots \quad . \tag{3.1}$$

We shall now show how the A's can be determined. From (1.4),

$$(I-P_o + sP_1 - \frac{s^2}{2!} P_2 + \ldots)\ (\frac{1}{s} A_{-1} + A_o + sA_1 + \ldots) \equiv I \tag{3.2}$$

Equating coefficients of s^r on both sides,

$$(I-P_o)A_r - \sum_{\alpha=1}^{r+1} \frac{(-1)^\alpha}{\alpha!} P_\alpha A_{r-\alpha} = \delta_{ro} I, \quad (r = -1,0,1,2,\ldots) \tag{3.3}$$

where δ_{ro} is the kronecker delta. Premultiply (3.3) by L and observing that $L(I-P_o) = 0$, we obtain

$$\sum_{\alpha=1}^{r+1} \frac{(-1)^\alpha}{\alpha!} L P_\alpha A_{r-\alpha} + \delta_{ro} L = 0 \quad (r = -1,0,1,\ldots) \tag{3.4}$$

Writing (3.3) as an equation in A_r, in terms of $A_{r-1}, A_{r-2},\ldots A_{-1}$, one can easily verify from (3.4) and (2.3) (vi) that the consistency condition (2.7) is satisfied and hence the general solution of (3.3) is by (2.6),

$$A_r = Z\{\sum_{\alpha=1}^{r+1} \frac{(-1)^\alpha}{\alpha!} P_\alpha A_{r-\alpha} + \delta_{ro} I\} + LW_r, \ \ldots \tag{3.5}$$

where W_r is some arbitrary matrix. To eliminate W_r from this, we substitute this value of A_r in the equation obtained from (3.4) by changing r to r+1 and we get, after a little algebra and the use of (2.5),

$$LW_r = -\frac{1}{k_1} \sum_{\alpha=1}^{r+1} \frac{(-1)^\alpha}{\alpha!} L\, P_1 Z P_\alpha A_{r-\alpha} - \frac{1}{k_1} \delta_{ro} LP_1 Z$$

$$+ \frac{1}{k_1} \sum_{\alpha=1}^{r+1} \frac{(-1)^{\alpha+1}}{(\alpha+1)!} LP_{\alpha+1} A_{r-\alpha} + \frac{1}{k_1} \delta_{r+1,0} L \tag{3.6}$$

Substituting this back in A_r, we get finally,

$$A_r = \sum_{\alpha=1}^{r+1} \frac{(-1)^\alpha}{\alpha!} \{ZP_\alpha - \frac{1}{k_1} LP_1 ZP_\alpha - \frac{1}{k_1(\alpha+1)} LP_{\alpha+1}\} A_{r-\alpha}$$

$$+ \delta_{ro}(I - \frac{1}{k_1} LP_1) Z + \frac{1}{k_1} \delta_{r+1,o} L \qquad (r = -1,0,1,\ldots) \tag{3.7}$$

In particular

$$A_{-1} = \frac{1}{k_1} L. \tag{3.8}$$

$$A_o = (I - \frac{1}{k_1} LP_1)\ Z\ (I - \frac{1}{k_1} P_1 L) + \frac{k_2}{2k_1^2} L, \tag{3.9}$$

$$A_1 = \{-ZP_1 + \frac{1}{k_1} LP_1 ZP_1 + \frac{1}{2k_1} LP_2\} A_o$$

$$+ \frac{1}{2k_1} \{ZP_2 - \frac{1}{k_1} LP_1 ZP_2 - \frac{1}{3k_1} LP_3\} L. \tag{3.10}$$

4. Remarks: This expansion of $(I-q(s))^{-1}$ yields immediately, the following asymptotic expression for the renewal function M(t), by inverting the L.S.T.

for large t:

$$M(t) = tA_{-1} + A_o + 0(1) \tag{4.1}$$

The Laplace-Stieltjes transform of the second factorial moment $E\{N_j(t)\overline{N_j(t)-1}\}$ is given by (Kshirsagar and Gupta 1967)

$$2\{(I-q(s)^{-1}-I\}_d\{(I-q(s)^{-1}-I\} \tag{4.2}$$

where $_dB$ stands for the diagonal matrix obtained from a matrix B, by removing all the off-diagonal elements. By using (3.7), (3.1), in (4.2), it can be readily shown that, for large t, the second factorial moment, conditional on $J_o = i$ is of the form

$$t^2 G_1 + t G_2 + G_3 + 0(1) \tag{4.3}$$

The matrices G_1, G_2, G_3 can be easily obtained in terms of A_{-1}, A_0, A_1,... and then in terms of P_k's and Z, L from (3.7). Extension to higher order moments is also possible in exactly a similarly manner, as their L.-S.T.'s all turn up in terms of $(I-q(s))^{-1}$ and q(s).

Hunter (1968) obtained a relationship between the moments of G_{ij} and of Q_{ij}. However, it is easier to work in terms of their moment generating functions, which are nothing but their L.-S.T.'s and the relationship between them is (Pyke 1961)

$$q(s) = \{(I-q(s))^{-1}-I\}\{_d(I-q(s))^{-1}\}^{-1}$$

Again one can use (3.1) and (3.7) to expand q(s) and hence obtain moments of G_{ij}. The expansion of $(I-q(s))^{-1}$ has therefore several uses, especially to lift the Laplacian curtain on Markovian renewal theory, at least for large t.

A remark on the expansion obtained by the present authors earlier (1967) and this expansion here is probabily not out of place here.

Hunter (1968) has derived the relationship between the matrices H_r mentioned in the introduction and Z and P_k's. Thereby he shows that the two expansions are identical. In particular, the matrix H_o in the earlier papers is nothing but L used in this paper, except for a constant. Also the constant $1/\alpha$ used by the authors in the earlier papers and defined as

$$|p_1| \cdot \text{product of latent roots (non-zero) of } p_1^{-1} (I-P_o)$$

can be easily seen to be

$$\lim_{s \to 0} \frac{1}{s} \; |I - P_o + sP_1|$$

and reduces to (by the application of L' Hospital's rule)

$$\lim_{s \to 0} \frac{d}{ds} |I - P_o + sP_1| = \underline{U}'P_1\underline{e} = k_1 ,$$

which is used in this paper.

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UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (*Corporate author*) SOUTHERN METHODIST UNIVERSITY	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP UNCLASSIFIED

3. REPORT TITLE

A note on Matrix Renewal Function

4. DESCRIPTIVE NOTES (*Type of report and inclusive dates*)

Technical Report

5. AUTHOR(S) (*First name, middle initial, last name*)

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6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
January 23, 1970	9	8

8a. CONTRACT OR GRANT NO. N00014-68-A-0515	9a. ORIGINATOR'S REPORT NUMBER(S) 55
b. PROJECT NO. NR 042-260	
c.	9b. OTHER REPORT NO(S) (*Any other numbers that may be assigned this report*)
d.	

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11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Office of Naval Research

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DD FORM 1473 1 NOV 65 (PAGE 1)

S/N 0101-807-6811

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Security Classification

A-31408