THEMIS SIGNAL ANALYSIS STATISTICS RESEARCH PROGRAM

OPTIMUM PROPERTY OF THE PARTIAL SEQUENTIAL PROBABILITY RATIO TEST

by

Campbell B. Read

Technical Report No. 53
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DEPARTMENT OF STATISTICS
Southern Methodist University

Optimum Property of the Partial Sequential Probability Ratio Test

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Southern Methodist University and Southwestern Medical School of the University of Texas.

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1. Introduction. Let θ be the parameter of interest in the distribution of an i.i.d. sequence $\{X_n\}$ of r.v.'s having common p.d.f. $p_\theta(\cdot)$. The Partial Sequential Probability Ratio Test (PSPRT) procedure is defined on the class $\mathcal{C}_n$ of test procedures requiring at least n observations to be made. Beyond stage n, observations are continued one at a time according as

$$B < p_{1n'}/p_{on'} < A \qquad (n' \geq n)$$

or not, where $0 < B < 1 < A$ is assumed, and we are testing the simple hypothesis H_o: $\theta = \theta_o$ against the simple alternative H_1: $\theta = \theta_1$. $p_{in'}$ is the joint p.d.f. of the first n' observations $\underline{x}_{n'} = (x_1, \ldots, x_{n'})$ when $\theta = \theta_i$ $(i = 0, 1)$. When we stop observing, H_o is accepted if $p_{1n'}/p_{on'} \leq B$, and H_1 is accepted if $p_{1n'}/p_{on'} \geq A$.

Essentially the PSPRT is a two-stage procedure, in which the second stage operates exactly like a Wald SPRT [4]. The latter has an optimum property, established by Wald and Wolfowitz ([5], [6]), namely, that among all tests for which

$$P(\text{reject } H_o|\theta_o) \leq \alpha, \; P(\text{reject } H_1|\theta_1) \leq \beta,$$

and for which $E(N|\theta_i) < \infty$ $(i = 0,1)$, the SPRT with error probabilities α and β minimizes $E(N|\theta_o)$ and $E(N|\theta_1)$.

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and for which $E(N|\theta_i) < \infty$ $(i = 0,1)$, the SPRT with error probabilities α and β minimizes $E(N|\theta_o)$ and $E(N|\theta_1)$.

If θ has a prior distribution $\lambda = (w, 1-w)$ on $\Omega = (\theta_o, \theta_1)$, and if the losses for wrong rejection of H_i are L_i $(i = 0,1)$, with cost per observation c, then the Bayes risk of any test procedure δ is

$$r(w,\delta) = w\left[\alpha L_o + cE(N|\theta_o)\right] + (1-w)\left[\beta L_1 + cE(N|\theta_1)\right] \tag{1}$$

where α and β depend on δ. Wald and Wolfowitz proved the auxiliary result that the Bayes risk (1) is minimized by a SPRT.

In this paper we show that the same results hold for a PSPRT procedure if attention is restricted to tests in the class $\mathcal{C}_n$. The proofs directly extend those of Lehmann and LeCam [1].

In a separate paper [2], Wald's approximations to the OC function and ASN of a SPRT are extended to a PSPRT, and several numerical results are given for the problem of testing a normal mean with known variance. There is a clear indication that the PSPRT may result in a lower ASN than that resulting from a SPRT with the same error probabilities, for some values of θ satisfying $\theta_o < \theta < \theta_1$. This last feature may make the PSPRT an attractive procedure to follow, if the experimenter wishes to allow for the possibility that θ may take some value other than θ_o or θ_1.

2. Main results.

<u>Theorem 1</u>. There exist numbers w' and w", determined by L_o, L_1, and c, but independent of n and w such that the Bayes risk (1) in the class of procedures $\mathcal{C}_n$ terminating with probability one, is minimized by a PSPRT with boundaries

$$B = \frac{w}{1-w} \cdot \frac{1-w''}{w''}, \quad A = \frac{w}{1-w} \cdot \frac{1-w'}{w'} \quad .$$

<u>Corollary 1</u>. If A and B are the boundaries of the SPRT which minimizes

the Bayes risk (1) in the class $\mathcal{C}_o$ of decision procedures, then they are also the boundaries of the PSPRT which minimizes (1) in the class $\mathcal{C}_n$, for any given n.

Theorem 2. Given any two numbers w_o' and w_o'' such that $0 < w_o' < w_o'' < 1$, there exist numbers ℓ $(0 < \ell < 1)$ and $c > 0$ such that the Bayes risk (1) in the class $\mathcal{C}_n$ with $L_o = 1 - \ell$, $L_1 = \ell$ and a prior probability w for θ_o satisfying $w_o' < w < w_o''$ is minimized by a PSPRT with boundaries

$$B = \frac{w}{1-w} \cdot \frac{1-w_o''}{w_o''} \quad , \qquad A = \frac{w}{1-w} \cdot \frac{1-w_o'}{w_o'} \quad .$$

Theorem 3. Among all procedures in the class $\mathcal{C}_n$ for which

$P(\text{reject } H_o|\theta_o) \leq \alpha$,

$P(\text{reject } H_1|\theta_1) \leq \beta$,

and for which $E(N|\theta_o)$ and $E(N|\theta_1)$ are finite, the PSPRT with error probabilities α and β minimizes both $E(N|\theta_o)$ and $E(N|\theta_1)$.

The proofs which follow in Section 3 are closely related to Lehmann's Lemmas 5 and 6, and his Theorem 8 [1], and it is of interest first to examine an intuitive argument. If the PSPRT is denoted by T_n, with boundaries A and B, and the conditional procedure, given $\underline{x}_n$ has been observed at stage n, by $T(\underline{x}_n)$, then $T(\underline{x}_n)$ is a SPRT (possibly with no further observations) having the continuation region

$$B_n' = B \cdot \frac{p_{on}}{p_{1n}} < \frac{p_1(x_{n+1},\ldots, x_n')}{p_o(x_{n+1},\ldots, x_n')} < A \cdot \frac{p_{on}}{p_{1n}} = A_n'$$

If $\alpha(\underline{x}_n)$, $\beta(\underline{x}_n)$ and $E(N|\underline{x}_n,\theta_i)$ are the error probabilities of this conditional procedure, and

$W_n = wp_{on}/(wp_{on}+(1-w)p_{1n})$

is the posterior probability of θ_o at stage n, then it follows from Lehmann's Lemma 5 that $T(\underline{x}_n)$ minimizes the conditional (or posterior) Bayes risk

$$r(w_n, \delta(\underline{x}_n)) = w_n\left[\alpha(\underline{x}_n)L_o + cE(N|\underline{x}_n, \theta_o)\right] + (1-w_n)\left[\beta(\underline{x}_n)L_1 + cE(N|\underline{x}_n, \theta_1)\right]$$

if $B_n' = \frac{w_n}{1-w_n} \cdot \frac{1-w''}{w''}$, $A_n' = \frac{w_n}{1-w_n} \cdot \frac{1-w'}{w'}$ for certain numbers $0 < w' < w'' < 1$.

But it is easy to show that the continuation region

$$\frac{w_n}{1-w_n} \cdot \frac{1-w''}{w''} < \frac{p_1(x_{n+1},\ldots,x_{n'})}{p_o(x_{n+1},\ldots,x_{n'})} < \frac{w_n}{1-w_n} \cdot \frac{1-w'}{w'}$$

is equivalent to

$$B = \frac{w}{1-w} \cdot \frac{1-w''}{w''} < \frac{p_{1n'}}{p_{on'}} < \frac{w}{1-w} \cdot \frac{1-w'}{w'} = A \quad , n' \geq n. \tag{2}$$

Hence, whatever $\underline{x}_n$ may be observed, the SPRT $T(\underline{x}_n)$ which minimizes the conditional Bayes risk, is derived from the same PSPRT T_n. Hence, intuitively, it would appear that T_n minimizes the (unconditional) Bayes risk (1). While the above equivalence relations demonstrate the likelihood principle, the uniqueness of T_n arises also from the fact that w' and w" are independent of w_n, and hence independent of $\underline{x}_n$. Finally, Lehmann's proof shows that the SPRT minimizes the conditional Bayes risk at every stage of the experiment. In the class $\mathcal{C}_n$, then, it seems natural that the PSPRT should do the same, from stage n onwards.

3. Proofs of theorems. In what follows, some familiarity with Lehmann's treatment ([1], 3.10 and 3.12) is assumed.

Proof of Theorem 1. Suppose that n observations have been taken, with outcome $\underline{x}_n$. We first wish to decide whether or not to stop. The loss nc already incurred does not affect the problem, since no further action can retrieve it. An unlimited sequence $X_{n+1}, \ldots$ is available.

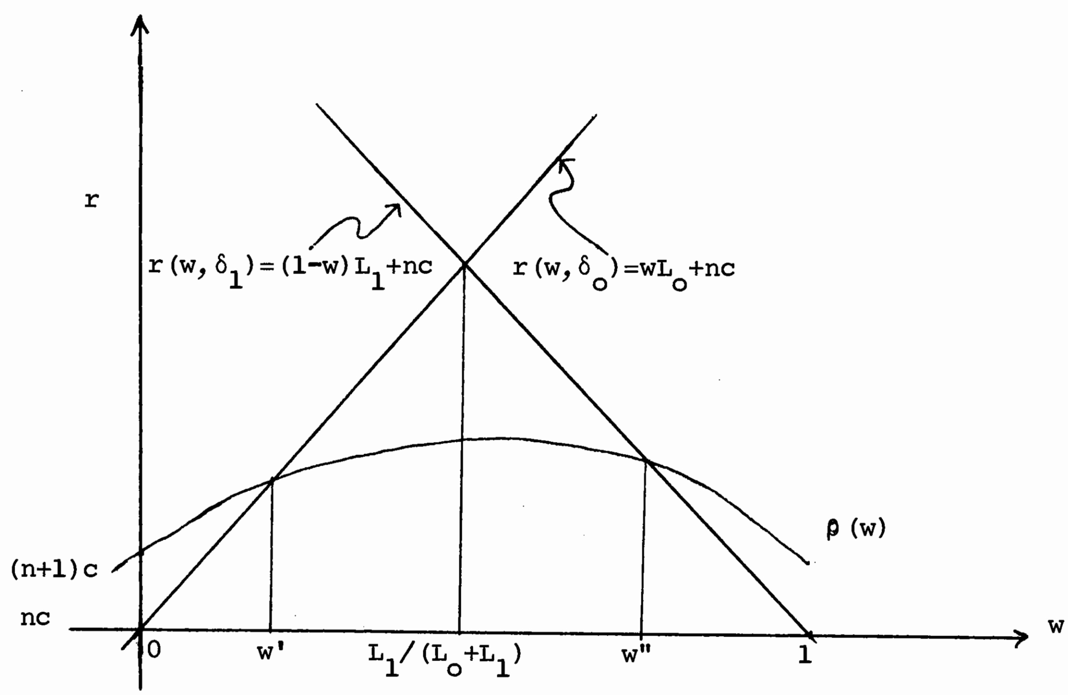
Let δ_o be the procedure which rejects H_o without taking any further observations beyond stage n, and δ_1 the procedure that similarly rejects H_1.

Thus $r(w,\delta_o) = wL_o + cn$

and $r(w,\delta_1) = (1-w)L_1 + cn$.

Let $\rho(w) = \inf_{\delta \in \mathcal{C}_{n+1}} r(w,\delta)$.

Then ρ is concave and continuous in (0,1) (see [1], Lemma 5).



If $\rho(L_1/(L_o+L_1)) < L_oL_1/(L_o+L_1) + nc$, define w' and w" by $r(w', \delta_o) = \rho(w')$, and $r(w", \delta_1) = \rho(w")$.

Otherwise let

$$w' = w" = L_1/(L_o + L_1) .$$

Figure 1 shows that we restrict attention to cases in which $0 < w' < w" < 1$. δ_o minimizes (1) if and only if $w \le w'$, and δ_1 minimizes (1) if and only if $w \ge w"$. So the optimum step at stage n is uniquely established as: if $Pr(H_o$ true) $< w'$ or $> w"$, take no further observations and reject or accept H_o respectively. If $w' < Pr(H_o$ true) $< w"$, observe X_{n+1}.

The proof is completed by induction as in Lehmann's Lemma 5. A similar situation arises at stage n' if $\underline{x}_{n'}$ has been observed $(n' \ge n)$, and the optimum rule is the same, except that

$$Pr(H_o \text{ true}) = w_{n'} = wp_{on'}/\left[wp_{on'}+(1-w)p_{1n'}\right].$$

The rule for continuing to observe, i.e. $w' < Pr(H_o$ true) $< w"$, is equivalent to (2), which defines the required PSPRT.

This establishes Theorem 1.

Corollary 1 follows from the analysis in the preceding proof. w' and w" are uniquely determined, and the argument for deriving them is not affected by n. Lehmann's Lemma 5 leads to the same continuation region (2) in the class C_o (where for the sake of completeness, we may take $p_{1n'}/p_{on'} = 1$ when $n' = 0$).

Proof of Theorem 2. Consider first the class C_o of decision procedures. Then Theorem 2 is Lehmann's Lemma 6.

But the corollary to Theorem 1 implies that a PSPRT with the same boundaries A and B minimizes the Bayes risk (1) in the class C_n, for any given finite positive integer n, and for losses $L_o = \ell$, $L_1 = 1-\ell$, and cost c.

Theorem 2 follows.

Proof of Theorem 3. Lehmann's proof applies with very little change.

For the sake of clarity, it is repeated here.

Consider any PSPRT δ with $0 < B < 1 < A$, and any constant w such that $0 < w < 1$. If

$$w' = w/\left[A(1-w)+w\right], \qquad w'' = w/\left[B(1-w)+w\right],$$

then $0 < w' < w < w'' < 1$ and (2) is satisfied. By Theorem 2, there exist ℓ and c such that $0 < \ell < 1$ and $c > 0$; and δ is Bayes in the class C_n for the situation in which w is the prior probability of H_o, $L_o = 1-\ell$, $L_1=\ell$, and cost c. Let this PSPRT have error probabilities α and β, and ASN $E(N|\theta)$; and consider any other procedure δ^* with error probabilities $\alpha^* \le \alpha$, $\beta^* \le \beta$, and ASN $E^*(N|\theta_i) < \infty$, $i = 0,1$. Since δ minimizes the Bayes risk,

$$w[(1-\ell)\alpha+cE(N|\theta_o)] + (1-w)[\ell\beta + cE(N|\theta_1)]$$
$$\le w[(1-\ell)\alpha^*+cE^*(N|\theta_o)] + (1-w)[\ell\beta^*+cE^*(N|\theta_1)]$$

and so

$$wE(N|\theta_o) + (1-w)E(N|\theta_1) \le wE^*(N|\theta_o) + (1-w)E^*(N|\theta_1) .$$

Since this holds uniformly for all w in $(0,1)$,

$$E(N|\theta_o) \le E^*(N|\theta_o)$$

and $E(N|\theta_1) \le E^*(N|\theta_1)$.

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13. ABSTRACT

$\{X_n\}$ is a sequence of i.i.d. r.v.'s with common p.d.f. $p_\theta(\cdot)$, and tests of $H_0:\theta = \theta_0$ against $H_1:\theta = \theta_1$ are discussed. A Partial Sequential Probability Ratio Test operates exactly like a Wald SPRT, except that n observations are made before the stopping rule is applied. It is shown that in the class C_n of test procedures requiring at least n observations, (1) the Bayes risk with fixed losses for wrong decision and fixed cost is minimized by a PSPRT, (2) the boundaries of this PSPRT are those of the Wald SPRT which minimizes the Bayes risk in the class C_0, and (3) for all tests having upper bounds α and β on the probabilities of wrong dicision, and having finite ASN's, the PSPRT with error probabilities α and β minimizes the ASN at $\theta = \theta_0$ and $\theta = \theta_1$. These are the optimum properties which Wald's SPRT has in the class C_0 .

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