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AN OPTIMUM DISCRETE SPACE SEQUENTIAL SEARCH PROCEDURE WHICH CONSIDERS FALSE ALARM AND FALSE DISMISSAL INSTRUMENT ERRORS

by

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An Optimum Discrete Space Sequential Search Procedure Which Considers False Alarm and False Dismissal Instrument Errors

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An optimum sequential search policy is developed for locating an object in a discrete search space assuming a search instrument having both false alarm and false dismissal errors.

The search policy obtained specifies that the experimenter always takes his next observation at the potential location which has the largest a posteriori probability of being the true location as computed from the prior probability distribution and previous experimental evidence via Bayes Rule.

The results were derived under the following assumptions. A single object is assumed to be located in 1 of N locations and to remain stationary during the search process. The experimenter is presumed to have an initial discrete a priori probability function describing the location of the object. Also available is a single search instrument which the experimenter can employ to examine each of the potential locations. Only one location can be observed at any one time, however. The instrument is assumed to perform a simple, fixed sample size, hypothesis test

having known conditional error probabilities of both Type I and Type II. The preliminary decision made by the instrument is thus a noisy decision which is only statistically related to the presence or absence of the object at the location in question.

The resulting search policy is based on an optimality criterion which states that the experimenter desires to minimize the number of unfavorable observational responses he receives while simultaneously maximizing the probability that the first favorable observation occur at the true object location. Conditions are derived whereby the experimenter can select a set of instrument parameters which will assure him of achieving any desired degree of certainty in his final decision.

Several properties are derived for the search policy specified including the average duration of the search and the probability of successful search termination.

A communication system search example is also considered.

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CHAPTER I

STATEMENT OF THE PROBLEM

1.1 Introduction

A sequential search process is a search process in which an observer is called upon to select a sequence of options, or equivalently to make a series of preliminary decisions in the process of using a measuring instrument to locate something. For ease of discussion the article being sought will be called an object and the person, machine or combination of the two making the decisions will be called the experimenter.

The total region which may potentially be occupied by the object will be called the search space. It will be assumed that only a portion of the space may be searched at any one time. The instrument will be employed by the experimenter to make observations, or equivalently take samples, from the environment occupied by the object. The instrument is assumed to have an output which is statistically related to the presence or absence of the object at the location observed.

The objects being sought can be real objects such as balls in urns or failed mechanical or electrical parts in a complex physical system, or they can be properties of a more general nature. Examples of this second type are the coordinate locations associated with a radar target such as an incoming missile or the operating frequency and time origin of a complex coded military communication signal. This last example will be considered in some detail since it is of considerable practical interest.

In each of these cases the objective of the experimenter is to formulate a search policy which will locate the object in an efficient manner. Determining the optimum search policy for some meaningful criterion of optimality becomes a valuable end in itself. This dissertation addresses itself to this problem. The search policy must specify the search sequence followed by the experimenter as a function of the previous observation history. The sets of information which constitute complete descriptions of what are meant by the terms search policy and observation history are strongly dependent on the type of search being conducted. The complexity of any particular search policy depends on the type of search being performed. Only the specific search policy of interest in this dissertation will be discussed in any detail in what follows. However, most search policies must provide the answers to the following basic questions:

i) how does the experimenter select the location(s) which will next be observed by the instrument?

ii) what portion of the previous search history must be retained and what form should the stored information have to be most relevant to subsequent decisions?

iii) how does the experimenter decide when to stop sampling and terminate the search procedure?

iv) what terminal decision concerning the object location does the experimenter make?

There are many additional choices which could potentially be included in this list. Additional questions pertaining to how best to

select the instrument characteristics for each observation, what constitutes a location, etc. could also be included. Rather than include such general considerations, the specific search problem treated in this dissertation will be introduced.

A conceptual framework which will aid in localizing the specific problem within the broader framework of search problems is shown in Figure 1-1. The figure is adapted from the article by Beiman and Eisner (1).* The search process which will be considered in this work is indicated by the heavy black line in Figure 1-1 and the problem description will essentially trace this path from top to bottom.

1.2 The Search Problem Considered

The search space will be assumed discrete with N possible object locations, each labeled with an appropriate integer, $\ell = 1,2,\ldots,N$. The object can be located in one and only one of the locations. Also considered is the case where no object is present. This may be considered to be an (N+1)-location search space where the $(N+1)^{st}$ location cannot be observed. An initial probability distribution is assumed which describes the experimenter's a priori knowledge of the object location. The prior distribution will be denoted by the row vector $\underline{p}_0 = (p_0(1), p_0(2), \ldots, p_0(N))$.

The prior distribution will be assumed to be a true conditional probability distribution known to the experimenter. No consideration is given to the case where $\underline{p}_0$ is unknown or inaccurately known. It is conditional in the sense that it describes the prior distribution given that an object is present. For many cases of practical interest the discrete

*References are consecutively numbered and found in the List of References.

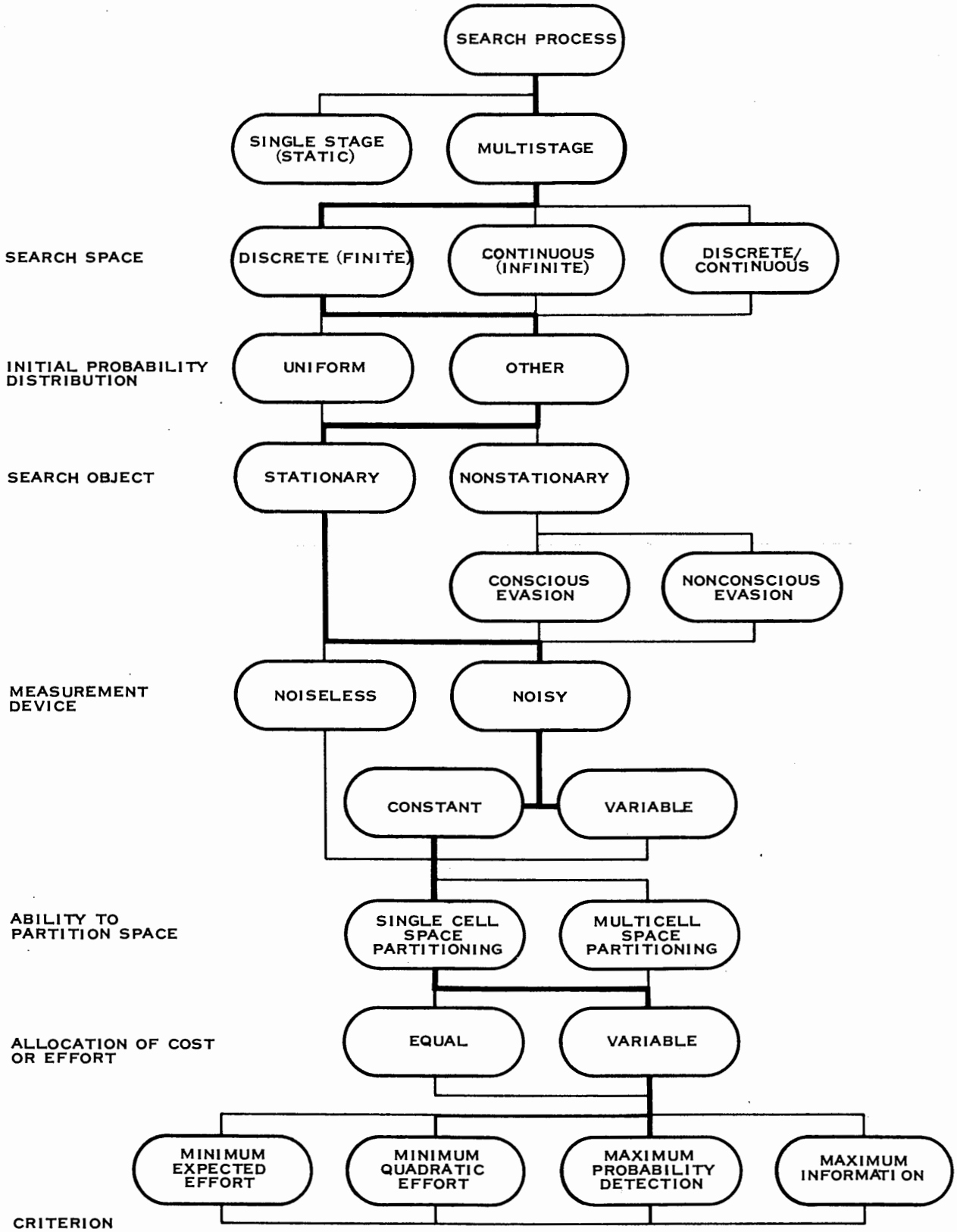

Figure 1-1. Search Algorithm Framework

character of the search space is a true description of the situation, in others it is an approximation made possible by the finite resolution of the search instrument.

The search object is assumed to be stationary during the duration of the search. This implies that the object remains at one location during the search and eliminates consideration of evasion of the searcher by the object. In many real situations a stationary target assumption is valid, particularly if the search can be consummated in a time short compared to the time required for the object to move to a new location.

The measurements which are made during the search sequence are assumed to be noisy. Furthermore, the noise statistics are assumed constant with respect to both the location being observed and the temporal position of the latest observation in the search sequence. It is assumed that only one location can be observed for any one experiment and that the same instrument is used for each. The experimenter is assumed to have complete freedom in the choice of experiment location for each experiment. He selects each experiment sequentially assuming the results of all previous experiments are known. Determining a search algorithm which optimally incorporates the past search history into the choice of the next experiment location is the primary result which will be derived. The search instrument is assumed to perform a simple hypothesis test at each location and to employ a fixed sample size. The allocation of search effort is thus variable in the sense that the locations may be observed in any order and a different number of times during the search process. However, the allocation is also quantized since each experiment allocates a fixed unit of search effort. The experimenter's choice deals with the order in which these search quanta are allocated to the N potential locations.

Each experiment has two potential outcomes denoted x_i. The first is considered favorable to the presence of the object at the observed location, is denoted "yes" or "x_i=1", the alternative is denoted "no" or "x_i=0". Because of the simple form of the experiments the experimenter obtains either a "yes" or "no" response from each experiment. He is also assumed to know the statistical validity of the information received; the behavior of the observation instrument and the conditions of the experiment are assumed to be well known so that statistical errors of type I and type II are known. Their respective conditional probabilities of occurrence are denoted β and α respectively and are constant with both the location observed and temporal position in the observation sequence as noted above. The two types of errors are illustrated schematically in Figure 1-2.

Each experiment yields two types of information, the outcome of the experiment, x_i, and the location, $e_i=\ell$; $\ell=1,2,\ldots,N$, where the observation was taken. This information, plus knowledge of the α and β error probabilities comprise the result of an experiment.

Selection of the instrument parameters α,β may or may not be at the disposal of the experimenter. In either case they are assumed known and constant in the following paragraphs.

The total accumulated experiment history $\underline{h}_n$ at any point in the search, n=1,2,..., is represented by knowledge of which locations have been observed and the order of the observations, $\underline{e}_n$, and the responses $\underline{x}_n$ received at each, thus $\underline{h}_n=(\underline{e}_n,\underline{x}_n)$.

The search history, which represents all information collected during the prior portions of the search, must be maintained and updated

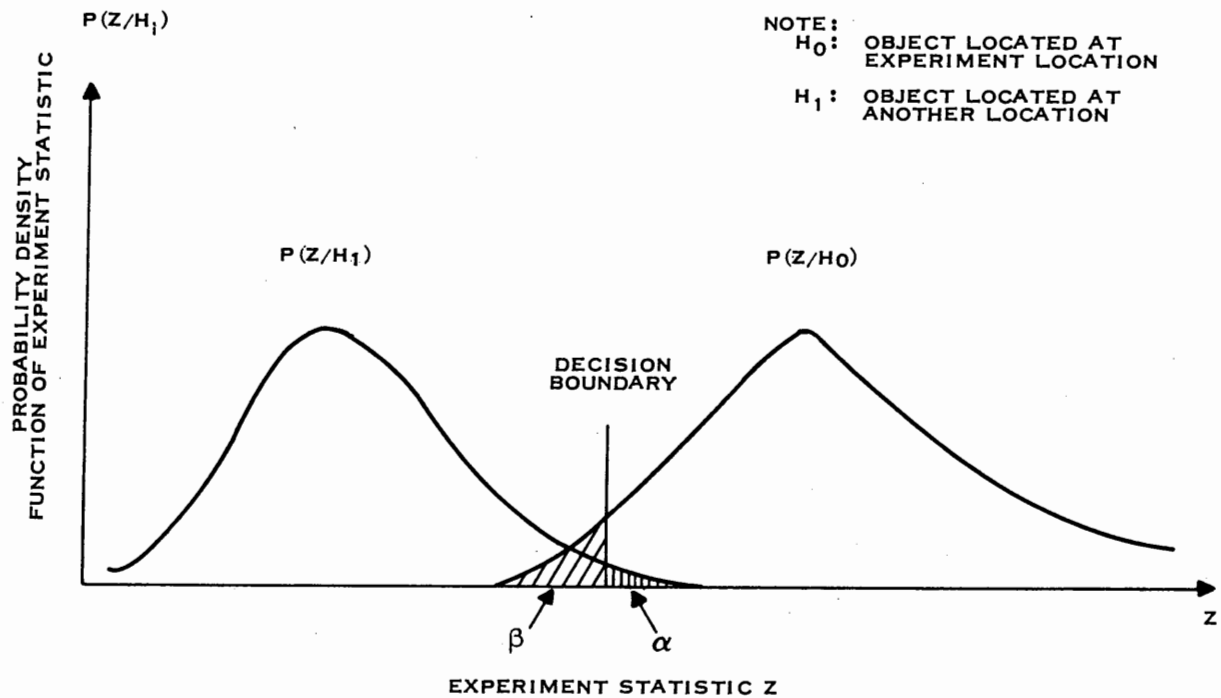

Figure 1-2. Schematic Illustration of Experimental Error Probabilities (β and α) of Types I and II Assuming Fixed Instrument and Experimental Conditions

during the search process. The most information which is available and hence all which could possibly be employed for the $(n+1)^{st}$ experiment decision is the total prior history, $\underline{h}_n=(\underline{e}_n,\underline{x}_n)$, and the a priori probability vector, $\underline{p}_0$. Since $\underline{h}_n$ grows linearly with the length, n, of the search history, storing $\underline{h}_n$ is clearly not desirable. Considering that the posterior probability, given $\underline{h}_n$, represents all that is known about the object location after n observations, the posterior probability vector $\underline{p}_n(\theta|\underline{h}_n)$, representing modifications of the prior distribution made in response to the observations obtained is certainly a sufficient set of information and clearly superior from the storage viewpoint. Storage requirements are then determined by the size of the search space N and the accuracy of the probability values which are stored. The next chapter discusses a method of sequentially computing the N dimensional vector representing the experimenter's current estimate of the true probability distribution, $\underline{p}_n$; these results are based on $\underline{h}_n$ and $\underline{p}_0$ as one would expect. It is furthermore shown that the probabilities $\underline{p}_n$ can be equivalently calculated by storing the N number pairs $(s_n(\ell),r_n(\ell))$, $\ell=1,2,\ldots,N$ representing the number of yes and no responses obtained at each location in the first n observations. The N number pairs $(s_n(\ell),r_n(\ell))$ are thus sufficient statistics for the search problem by the Bayesian definition of sufficiency given in Raiffa & Schlaifer (2).

Since the posterior distribution can be updated as the search progresses one would suspect that the experimenter would always select the next experiment location to correspond to the current most probable point, providing search costs and instrument parameters were neither location nor time dependent.

This expected result is shown to be true in Chapter III for the criterion discussed below. The next chapter discusses methods of computing the posterior probabilities.

The criterion selected in this dissertation for optimizing the search policy is that of maximizing the probability of correctly detecting the true object location with an associated side condition which enables the experimenter to employ a simple stopping rule and yet be predictably certain of a correct decision. The search is stopped whenever the first yes observational response is received. To assure the experimenter that the above stopping rule will correctly locate the true object location with satisfactory veracity requires that a side condition be placed on the instrument parameters (α,β). It may be observed that the results derived in Chapter III are valid irrespective of whether the side condition is imposed or not. However, the choice of optimality criterion is neither satisfying nor convincing unless one is using a reliable instrument.

One notes the analogy between the search process described above and the usual sequential hypothesis test, particularly in the special case of two locations. Determining a search policy on the basis of the previous search history is similar to determining a sequential hypothesis test stopping/terminal decision rule on the basis of a statistic incorporating previous sample values. A search policy introduces the additional requirement for specifying the experiment location whenever an additional sample is to be taken. One must decide both when to continue sampling and where the sample can most advantageously be selected.

1.3 Related Search Problems Considered in the Literature

Search problems have previously been considered by a great number of authors. A univeral supposition in each of these works has been the assumption of a zero conditional probability of false alarm, $\alpha=0$. The instrument is assumed to never give a "yes" response at an erroneous location. Under this assumption it is not possible to erroneously locate the object unless the search resources (time, energy, money, etc.) are somehow limited so that the search cannot always be completed. The desire to maximize some measure of search success by the appropriate allocation of resources has motivated research of this family of search problems.

An early study of the optimal distribution of search effort was made during World War II by the Antisubmarine Warfare Operations Research Group (ASWORG) of the Navy Department. The available portions of this work are documented in three articles by Koopman (3,4,5). The fundamental problem considered by ASWORG was how best to simultaneously allocate a given amount of search effort in the potential search region so as to maximize the overall probability of locating the object. Koopman assumed that the conditional probability of detecting the object at a location, given that it was present, was a negative exponential function of the search effort density expended at the location in question. De Guenin (6) extended the result to a general nondecreasing detection probability function (detection probability never decreases with increasing search effort). Neither of these results provided information about how best to allocate the search effort sequentially; it was assumed that the searcher had an instrument which could simultaneously distribute varying amounts

of a fixed total search effort throughout the search region. Lyubatov (7) considered the problem of how best to sequentially distribute search effort, assuming a discrete search space, a nondecreasing detection probability function, and a search instrument which could only observe one location at a time but for any arbitrary duration greater than zero (not quantized). He obtained the optimum limiting strategy which maximized the probability of locating the object and exhibited a constructive method of generating an approximation to it.

Chu (8) developed optimal minimum cost search strategies assuming a discrete search space. He considered the additional complication introduced by uncertain prior location probabilities and an uncertain conditional detection probability, β, for the search instrument. Optimal adaptive search decision making policies were developed which had minimum expected loss for the case of known search parameters, and minimum expected mean loss when the parameters were not known. A Bayesian learning approach was used to update the parameters through learning observations (results of past searches). Ross (9) considered a related minimum cost problem with known parameters. In each of the above studies, $\alpha=0$, was assumed.

Many other special search problems have also been considered, some assumed a random time of object appearance, others considered an intelligent and evasive object. A bibliography and brief abstract of these results can be found in the survey paper by Dobbie (10), or the earlier bibliography by Enslow (11).

The major difference between the problem considered in this dissertation and the majority of previous search problems is represented by the non-zero probability of false alarm, α. Assuming α non-zero implies that

the experimenter always remains uncertain about the true object location. He must now not only consider how to allocate his resources during the search process but also must decide when to terminate the search since a yes observational response is no longer a guarantee that the object has been found.

CHAPTER II

POSTERIOR PROBABILITY EXPRESSIONS

2.1 Introduction

This chapter is concerned with formulating the posterior probability expressions representing the experimenter's current best information about the true location of the object being sought given the previous experiment history. By employing Bayes theorem to incorporate observed data into the prior probability model, historical evidence concerning search problems of the type considered may be modified in response to actual data to better represent the particular search procedure being executed.

Two different types of experiment information are available as previously noted; experiment locations and observed outcomes. The experiment locations are sequentially selected by the experimenter based on the previous search history. Observed data consists of Bernoulli events coming from one of two different distributions. By assumption, a single sample distribution applies to all erroneous locations; this distribution is also assumed different from the distribution associated with the true object location.

In practice, the posterior probabilities would be computed sequentially, each based on that portion of the experiment history previously observed. At any arbitrary time during the search, the experimenter has selected the experiment location for each of the previous observations and hence influenced the corresponding samples observed. The posterior

distribution thus depends on both the outcomes observed and the experimenter's choice of experiment locations. Based on this previous history the experimenter must decide where to take the next sample.

Because of the interaction between experiment locations and potential outcomes one can see that the number of potential experiment histories and hence the number of posterior probability vectors grows rapidly with the number of locations and samples, n. This result is also discussed in this chapter.

2.2 Single Experiment Posterior Probability Results

It is necessary for subsequent use, to determine the form of the posterior probabilities given an experiment history consisting of the set of experiment locations e_1, e_2, e_3, ... and the corresponding experimental observations x_1, x_2, ... Each experiment location e_i can be sequentially chosen by the experimenter to correspond to any one of the N potential object locations, $e_i=\ell$ for some $\ell=1,2,\ldots,N$. The observation x_i corresponding to each experiment location e_i is constrained by the observation instrument to have values $x_i = 0,1$. The n step experiment history $\underline{h}_n=(\underline{e}_n,\underline{x}_n) = (e_1,e_2,\ldots e_n,x_1,x_2,\ldots,x_n)$, the instrument error parameters (α,β) and the prior probabilities $\underline{p}_0=(p_0(1), p_0(2), \ldots p_0(N))$ represent all the information that is required to compute the posterior probability vector $\underline{p}_n$. The elements of $\underline{p}_n$ will be denoted

$$p_n(\theta=\ell|\underline{e}_n,\underline{x}_n) \qquad \ell=1,2,\ldots,N \tag{2-1}$$

where θ denotes the true but unknown object location and the two n dimensional row vectors $\underline{e}_n$ and $\underline{x}_n$ represent the prior search history. Consider the case for a single experiment, e_1 with outcome x_1. Then by Bayes Rule the posterior probability that the object is located at location ℓ given the seach history $\underline{h}_1$, $p_1(\theta=\ell|e_1,x_1)$, may be expressed in terms of the prior probabilities and the conditional detection probabilities associated with the instrument. Consider the case for $e_1=k$, and $x_1=0$, implying a "no" response was received while observing location k. For this case,

$$p_1(\theta=\ell|e_1=k,x_1=0) = \frac{p(x_1=0|e_1=k,\theta=\ell)p_0(\theta=\ell)}{p(x_1=0|e_1=k,\theta=k)p_0(\theta=k)+p(x_1=0|e_1=k,\theta\neq k)p_0(\theta\neq k)} ;$$

$$\ell=1,2,\ldots,N \qquad (2\text{-}2)$$

There are two specific expressions to consider, the posterior probabilities associated with locations other than the one observed, $\ell\neq k$, and the probability associated with the experiment location, k. Hence,

$$p_1(\theta=\ell|e_1=k,x_1=0) = \begin{cases} \left(\frac{1-\alpha}{1-\alpha-\beta}\right) \cdot \dfrac{p_0(\ell)}{\frac{1-\alpha}{1-\alpha-\beta} - p_0(k)} , & \ell\neq k \\[2ex] \left(\frac{1-\alpha}{1-\alpha-\beta}\right) \cdot \dfrac{\left(\frac{\beta}{1-\alpha}\right) p_0(\ell)}{\frac{1-\alpha}{1-\alpha-\beta} - p_0(\ell)} , & \ell=k \end{cases} \qquad (2\text{-}3)$$

Similarly, for a "yes", $x_1=1$, response at location k, the posterior probabilities may be expressed

$$p_1(\theta=\ell|e_1=k,x_1=1) = \begin{cases} \dfrac{\alpha}{1-\alpha-\beta} \cdot \dfrac{p_0(\ell)}{\dfrac{\alpha}{1-\alpha-\beta} + p_0(k)} & , \ \ell\neq k \\ \\ \dfrac{\alpha}{1-\alpha-\beta} \cdot \dfrac{(\frac{1-\beta}{\alpha})\ p_0(\ell)}{\dfrac{\alpha}{1-\alpha-\beta} + p_0(\ell)} & , \ \ell=k \end{cases} \tag{2-4}$$

These results hold equally well for assessing the effect of any single observation on a prior probability vector $\underline{p}_{n-1}$. The above expressions can be applied iteratively for a sequence of observations of arbitrary length; knowing $\underline{p}_{n-1}$, one can compute $\underline{p}_n$ for each experiment history $\underline{h}_n$.

Note in the above results that the experiment location $e_1=k$ is the only location whose relative probability value is changed by the observation. All other locations have posterior probabilities which are simply scaled versions of the corresponding prior probabilities. For example, given a negative response, $x_1=0$, while observing location k, the prior probability $p_0(k)$ will be reduced by the factor $\beta/(1-\alpha)$ relative to all $p_0(\ell),\ell\neq k$. Given the prior probabilities $p_0(k)$, and $p_0(\ell)$, the respective posterior probabilities given a no response at $e_1=k$, namely $p_1(k|e_1=k,x_1=0)$, and $p_1(\ell|e_1=k,x_1=0)$ will have changed relative value by $\beta/(1-\alpha)$,

$$\frac{p_1(k|\cdot)}{p_1(\ell|\cdot)} = \left(\frac{\beta}{1-\alpha}\right) \cdot \frac{p_0(k)}{p_0(\ell)} \tag{2-5}$$

Note that $\beta/(1-\alpha) < 1$ for $\alpha,\beta<0.5$ which is the only case of real interest. In contrast, any two locations different from the one which was observed

do not change relative value (although the individual values obviously change).

$$\frac{p_1(j|e_1=k,x_1=0)}{p_1(i|e_1=k,x_1=0)} = \frac{p_0(j)}{p_0(i)} \tag{2-6}$$

A similar result holds true for yes observational responses but the scale factor is different, e.g. $(1-\beta)/\alpha>1$,

$$\frac{p_1(k|e_1=k,x_1=1)}{p_1(\ell|e_1=k,x_1=1)} = \left(\frac{1-\alpha}{\beta}\right) \frac{p_0(k)}{p_0(\ell)} \tag{2-7}$$

The above results assume that the instrument error parameters α,β are identical for all locations and for every sample, n=1,2,... . For location dependent error parameters α,β would be functions of e_i; $\alpha(e_i),\beta(e_i)$. The simple relative probability changes discussed above still hold and the ratio of any two posterior probabilities have the same form with $\alpha(k)$, $\beta(k)$ substituted appropriately. Subsequent calculations become considerably more complicated, however. For this reason (α,β) are assumed to be independent of location. One notes that this assumption is satisfied for many real problems and could be approximated in others by varying the instrument to compensate for location dependent changes in experimental conditions.

The single observation posterior probabilities for the experiment location k are illustrated in Figure 2-1 for the special case $\alpha=0.1$, $\beta=0.05$. The strong dependence of p_n on p_{n-1} is evident from the figure. The knee or unity slope conditions of the two curves both occur on the

line $p_n=1-p_{n-1}$. The unity slope (us) conditions occur for prior probabilities p_{n-1} given by

$$\text{us}\,|\,x=1 \text{ occurs @ } p_{n-1} = \frac{\left(\frac{1-\beta}{\alpha}\right)^{\frac{1}{2}} - 1}{\left(\frac{1-\beta}{\alpha}\right) - 1}$$

and (2-8)

$$\text{us}\,|\,x=0 \text{ occurs @ } p_{n-1} = \frac{1 - \left(\frac{\beta}{1-\alpha}\right)^{\frac{1}{2}}}{1 - \left(\frac{\beta}{1-\alpha}\right)}$$

for yes and no observational responses respectively. Improving the search instrument by reducing α and/or β can be seen to accentuate the sharpness of the corner in both cases. In the limit as α and β go to zero the two curves of Figure 2-1 become L shaped and appropriately oriented.

$$\lim_{\frac{1-\beta}{\alpha}\to\infty}\left[\frac{\left(\frac{1-\beta}{\alpha}\right)^{\frac{1}{2}} - 1}{\left(\frac{1-\beta}{\alpha}\right) - 1}\right] = 0$$

(2-9)

$$\lim_{\frac{\beta}{1-\alpha}\to 0}\left[\frac{1 - \left(\frac{\beta}{1-\alpha}\right)^{\frac{1}{2}}}{1 - \left(\frac{\beta}{1-\alpha}\right)}\right] = 1$$

As might be expected α and β are not always at the disposal of the experimenter nor independent of one another. The search parameters α and β are frequently constrained so that they cannot be independently adjusted; for such cases decreasing α will cause β to increase and vice versa. In such cases the experimenter may be able to exercise his limited control to improve some overall measure of search success such as the total average time required. A particular example of this type is considered in Chapter IV. As will be noted from the example, minimizing the average time consumed by the search can be a tiresome numerical problem and as such doesn't necessarily yield a unique minimum.

A useful observation may be made at this point about the effect of a yes/no (or no/yes) experiment combination on the posterior probability associated with the observed location. Sequentially applying the appropriate posterior probability expressions given above one can easily arrive at the following equation for the posterior probability of location ℓ given a prior probability $p_0(\ell)$ and two co-located observations with

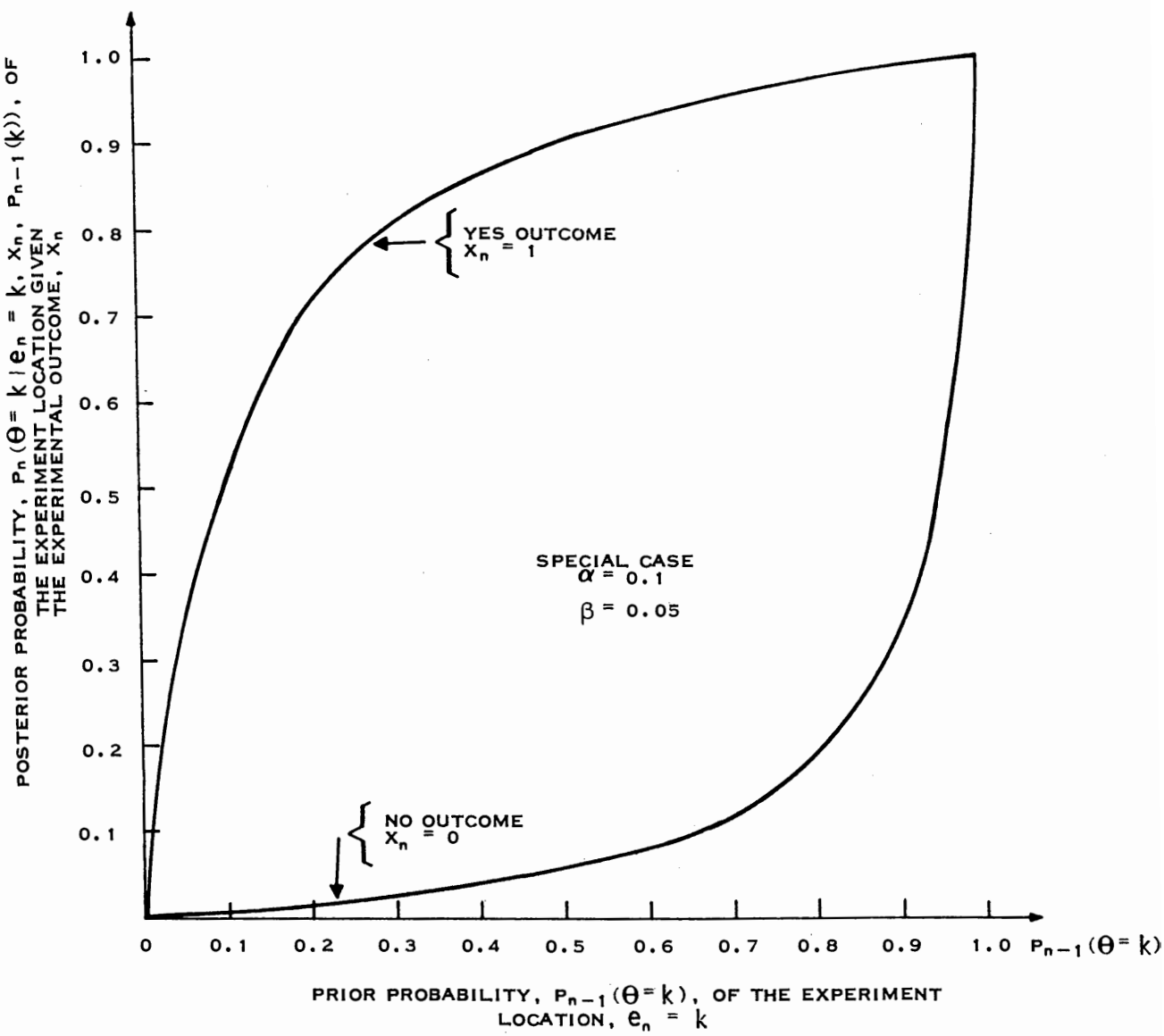

Figure 2-1. Single Experiment Posterior Probabilities vs. Prior Probabilities

outcomes $x_1=1$ and $x_2=0$ (or $x_1=0$, $x_2=1$).

$$p_2(\ell|\underline{e}_2=\underline{\ell},x_1=1,x_2=0) = \frac{(\frac{\beta}{1-\alpha})(\frac{1-\beta}{\alpha})\ p_0(\ell)}{1 + \frac{1-\alpha-\beta}{1-\alpha}\ (\frac{\beta-\alpha}{\alpha})\ p_0(\ell)} \tag{2-10}$$

The ratio of this probability to its prior value $p_0(\ell)$ is of interest because it illustrates the relative influence of sequential yes and no responses for a given set of instrument parameters, α,β.

Let the ratio under discussion be denoted R, where

$$R = \frac{p_2(\ell|\underline{e}_2=\underline{\ell},x_1=1,x_2=0)}{p_0(\ell)} = \frac{1}{p_0(\ell) + \frac{\alpha(1-\alpha)}{\beta(1-\beta)}[1 - p_0(\ell)]} \tag{2-11}$$

One sees that R depends on the prior probability $p_0(\ell)$ as well as α and β unless $\alpha=\beta$. For the special case $\alpha=\beta$, R=1 results, implying that no cumulative probability change resulted from the observation pair. For a more general choice of $\alpha,\beta < 0.5$ one notes that $\alpha \gtrless \beta$ correspondingly implies $\alpha(1-\alpha) \gtrless \beta(1-\beta)$. Observing that $0<p_0(\ell) \leq 1$ the following conclusions may be drawn about R independent of the value of $p_0(\ell)$;
$\alpha<\beta \rightleftharpoons R>1$, $\alpha>\beta \rightleftharpoons R\leq 1$.

Thus, if the experimenter has a choice of instrument parameters he may heuristically select $\alpha<\beta$ to lessen the effect of temporary setbacks ($x_i=0$) which may occur while observing the true location. Reducing α also simultaneously reduces the number of false alarms and is beneficial in this way also. No convincing argument may be given for either alternative, however, unless α and β are both related to some common measure of search effort which is to be minimized or maximized. This

choice is not discussed at any length in this chapter. A particular example is given in Chapter IV. One notes for future reference that selecting $\alpha<\beta$ and assuming that the experimenter will always choose the a posteriori most probable location for observation implies that multiple "no" responses are required to cancel the effect of a single "yes." The choice $\beta<\alpha$ implies the opposite result.

2.3 Multiple Experiment Posterior Probabilities

The iterative nature of the posterior probability relationships can be used to advantage to sequentially calculate the vector of posterior probabilities as one proceeds through a given search sequence. The probability of the observed location e_n is modified in response to the experimental outcome x_n by employing the appropriate expression given above, all other locations are simply multiplicatively scaled in value so that they all sum to unity. Knowing $\underline{p}_{n-1}$, and the nth experiment location and outcome (e_n, x_n), and α and β, one can determine $\underline{p}_n$. Each computation of the next probability vector will require computations including a single addition and division, about N multiplications and N storage accesses. For real time problems these operations must be completed during an observation time.

If the experimenter was to exhaustively compute all possible probability vectors resulting in a problem having N potential object locations $\ell=1,2,\ldots,N$ and considering n=J previous experimental observations, $e_1, e_2, \ldots, e_J$, the total number, T(N,J), of different posterior probability vectors $\underline{p}_J(\theta|\underline{e}_J, \underline{x}_J)$ which could be computed is given by the following expression

$$T(N,J) = \binom{2N + J - 1}{J} \tag{2-12}$$

This expression is illustrated in Figure 2-2 as a function of the number of experimental observations, J. Several different values of N are illustrated. The top of the figure represents a rough bound on the rapid access storage size available with large present day computers such as the IBM 360-65. One notes that the size of the probability vector array resulting for the case N=5, J=10 is typical of the problem parameters for which the probability array can be exhaustively computed.

The number T(N,J) also represents the largest array required if one is to use the backward computation method for finding an optimal <u>truncated</u> strategy with the truncation point (J+1) as discussed in references (12,13). Such a method is feasible for numerical problems of modest size such as the case N=5, J=10 illustrated in Figure 2-2.

One can use the iterative nature of the posterior probability array which evolves during a search process to determine analytical probability results assuming n experiments (n=1,2,...) have been completed. Three results of this type which will later prove of interest are given below. The first shows the form of the posterior probabilities given that a sequence of n consecutive "no" responses, $\underline{x}_n = \underline{\phi}_n$, have been obtained at the n experiment locations $\underline{e}_n$. The second result reports the form of the posterior probabilities given n consecutive "yes" responses, $\underline{x}_n = \underline{1}_n$. The third result assumes a more general and unspecified sequence of experimental outcomes, $\underline{x}_n$.

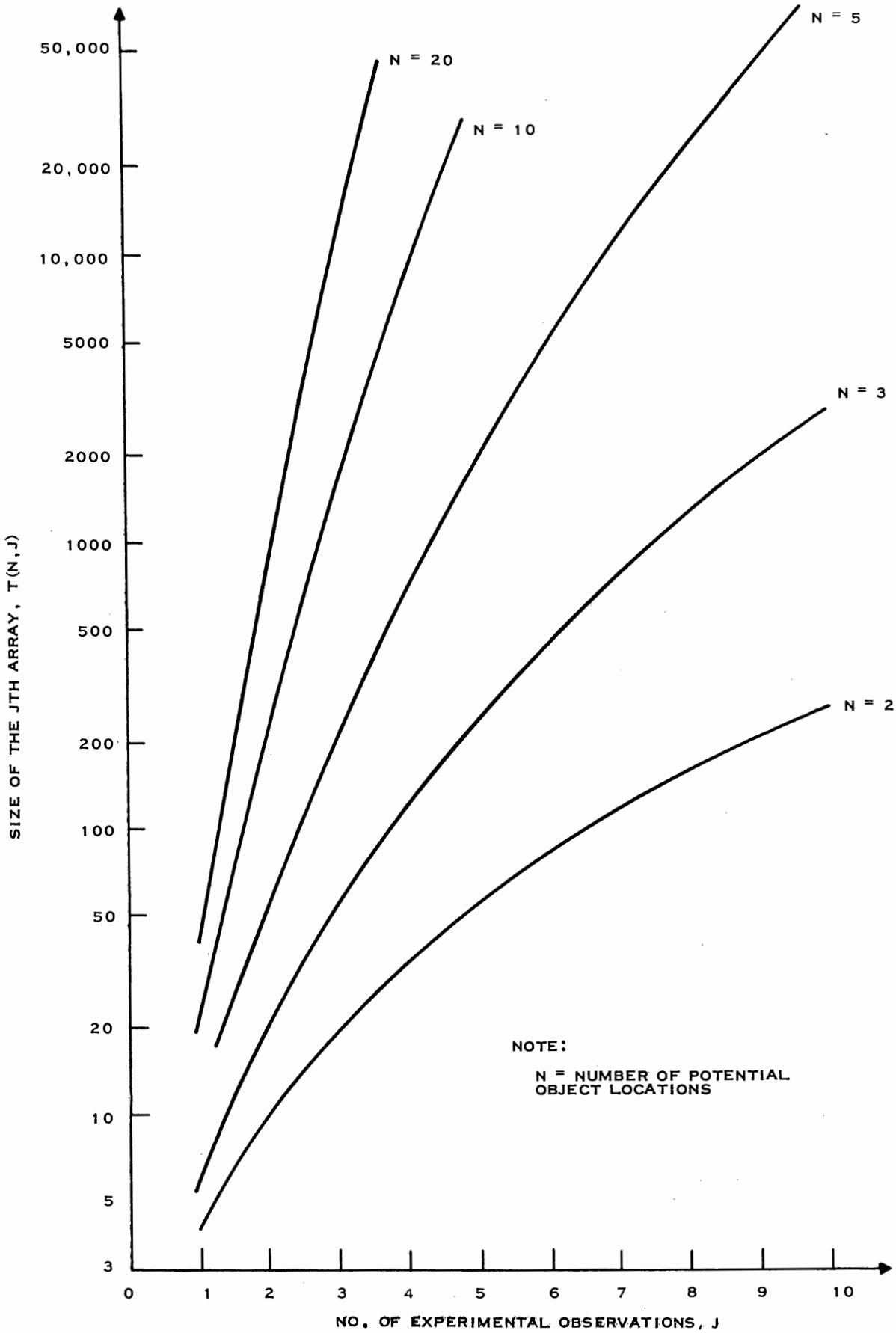

Figure 2-2. Total Size of the Jth Posterior Probability Array vs Total Number of Experiments, J

Assume first that n consecutive "no" responses, $\underline{x}_n=\underline{\phi}_n$, have been observed. For this case the posterior probabilities associated with locations $\ell=1,2,\ldots,N$ may be shown to be of the general form

$$p_n(\theta=\ell|\underline{e}_n,\underline{x}_n=\underline{\phi}_n)=\left(\frac{1-\alpha}{1-\alpha-\beta}\right)\frac{\left(\frac{\beta}{1-\alpha}\right)^{m_n(\ell)+1}p_0(\ell)}{\left[\frac{1-\alpha}{1-\alpha-\beta}-\sum_{i=1}^{n}\left(\frac{\beta}{1-\alpha}\right)^{m_i(e_i)}p_0(e_i)\right]}\ ;\ \ell=1,\ldots,N \tag{2-13}$$

The coefficients $(\beta/(1-\alpha))^{m_i(e_i)}$ in the denominator are related to the number of times a particular location has been observed during the first i observations. The experiment locations e_i; $i=1,2,\ldots,n$ are subscripted to indicate temporal position in the sequence (not observation location) and each e_i will be equal to some location $\ell=1,2,\ldots,N$. The counting indices $m_i(\ell)$, $-1 \leq m_i(\ell) \leq i-1$, for $\ell=1,2,\ldots,N$, and $i=1,2,\ldots n$ are employed to keep track of the number of times each location ℓ has been observed up to and including the current observation, e_i. After i total observations, any particular location ℓ can have been observed $(m_i(\ell)+1)$ times where $m_i(\ell)=-1,0,1,\ldots,(i-1)$. Thus, $m_i(e_i)$ is one less than the accumulated total of the number of times location $e_i=\ell$ has been observed during the first i observations. Any location ℓ which hasn't previously been observed must not be included in the denominator summation (which represents the previous search history) so that $m_i(\ell)=-1$ never appears in the denominator by definition. Note that $\sum_{\ell=1}^{N}(m_i(\ell)+1)=i$; which expresses the obvious fact that each of the i observations was taken at some location ℓ.

The coefficient $(\beta/(1-\alpha))^{m_n(\ell)+1}$ in the numerator has a counting index which is one greater than the largest coefficient $m_i(e_i)$ corresponding to location ℓ which appears in the denominator series if

location ℓ has previously been observed. If location ℓ has not been previously observed, then $m_n(\ell)\bar{=}-1$. The numerator exponent $(m_n(\ell)+1)$ thus represents the total number of times that location ℓ has been observed in the first n experiments.

One may note that the n terms in the denominator sum may be reordered in a way which reflects point location correspondence instead of their temporal position in the observation sequence. Assume locations $\ell=1,2,\ldots,N$ have each been observed $r_n(1), r_n(2),\ldots,r_n(N)$ times respectively in the first n observations where $(\sum_{i=1}^{N} r_i=n)$. Then

$$\sum_{i=1}^{n} \left(\frac{\beta}{1-\alpha}\right)^{m_i(e_i)} p_0(e_i) = \sum_{i=1}^{N} \sum_{j=0}^{r_n(i)-1} \left(\frac{\beta}{1-\alpha}\right)^{j} p_0(i)$$

$$= \left(\frac{1-\alpha}{1-\alpha-\beta}\right) \sum_{i=1}^{N} \left[1 - \left(\frac{\beta}{1-\alpha}\right)^{r_n(i)}\right] p_0(i) \qquad (2\text{-}14)$$

Incorporating this result in the above expression, one finds

$$p_n(\theta=\ell|\underline{e}_n,\underline{x}_n=\underline{\phi}_n) = \frac{\left(\frac{\beta}{1-\alpha}\right)^{r_n(\ell)} p_0(\ell)}{\sum_{i=1}^{N} \left(\frac{\beta}{1-\alpha}\right)^{r_n(i)} p_0(i)} \quad ; \quad \ell=1,\ldots,N \qquad (2\text{-}15)$$

which is an alternative way of expressing $p_n(\theta=\ell|\cdot\cdot)$. Note that $r_n(\ell)=m_n(\ell)+1$, $\ell=1,2,\ldots,N$.

A similar result obtains for the posterior probability given n consecutive "yes" observational responses, n=1,2,... . The general form in this case is given below,

$$p_n(\theta=\ell|\underline{e}_n,\underline{x}_n=\underline{1}_n) = \left(\frac{\alpha}{1-\alpha-\beta}\right) \frac{\left(\frac{1-\beta}{\alpha}\right)^{y_n(\ell)+1} p_0(\ell)}{\left[\frac{\alpha}{1-\alpha-\beta} + \sum_{i=1}^{N} \left(\frac{1-\beta}{\alpha}\right)^{y_i(e_i)} p_0(e_i)\right]} \; ; \; \ell=1,2,\ldots,N \tag{2-16}$$

The interpretation of $y_i(e_i)$ is similar to the previous interpretation of the $m_i(e_i)$; the index $y_i(e_i)$, $y_i(e_i)=-1,0,\ldots i-1$, is employed to keep track of the number of yes responses obtained at location $e_i=\ell$ in the first i observations. The value $y_i(\ell)=-1$ does not appear in the denominator by definition. The notation once again emphasizes the sequential nature of the responses. The terms may likewise be reordered to emphasize the location dependence of the observations. Assuming locations $\ell=1,2,\ldots,N$ are observed $s_n(1)$, $s_n(2)$, ..., $s_n(N)$ times respectively in the first n observations, the posterior probabilities may be expressed

$$p_n(\theta=\ell|\underline{e}_n,\underline{x}_n=\underline{1}_n) = \frac{\left(\frac{1-\beta}{\alpha}\right)^{s_n(\ell)} p_0(\ell)}{\sum_{i=1}^{N} \left(\frac{1-\beta}{\alpha}\right)^{s_n(i)} p_0(i)} \; ; \; \ell=1,2,\ldots,N \tag{2-17}$$

The above results have assumed consecutive sequences of "no" or "yes" responses. The general experiment situation could have responses of both types occurring in any sequence of n experiments taken at experiment locations e_1, e_2, ..., e_n. The resulting posterior probabilities, given the general experiment history $\underline{h}_n=(\underline{e}_n,\underline{x}_n)$, can be written in the following two ways: the first form is descriptive of the temporal dependence of the observations, and the second form descriptive of the location dependence,

$$p_n(\theta=\ell|\underline{e}_n,\underline{x}_n)=\frac{\left(\frac{\beta}{1-\alpha}\right)^{m_n(\ell)+1}\left(\frac{1-\beta}{\alpha}\right)^{y_n(\ell)+1}p_0(\ell)}{1-\frac{1-\alpha-\beta}{1-\alpha}\sum_{i=1}^{n}\left(\frac{\alpha-1}{\alpha}\right)^{x_i}\left(\frac{\beta}{1-\alpha}\right)^{m_i(e_i)}\left(\frac{1-\beta}{\alpha}\right)^{y_i(e_i)}p_0(e_i)}$$

$$\ell=1,2,\ldots,N \qquad (2\text{-}18)$$

$$p_n(\theta=\ell|\underline{e}_n,\underline{x}_n)=\frac{\left(\frac{\beta}{1-\alpha}\right)^{r_n(\ell)}\left(\frac{1-\beta}{\alpha}\right)^{s_n(\ell)}p_0(\ell)}{\sum_{i=1}^{N}\left(\frac{\beta}{1-\alpha}\right)^{r_n(i)}\left(\frac{1-\beta}{\alpha}\right)^{s_n(i)}p_0(i)}\ ; \quad \ell=1,2,\ldots,N \qquad (2\text{-}19)$$

The exponents $m_i(e_i)$, $y_i(e_i)$, $r_n(\ell)$, $s_n(\ell)$ have the same interpretation as for the previous results. The sum over ℓ of the terms $[s_n(\ell)+r_n(\ell)]$ must add to n since every observation was taken at some location $e_i=\ell$, $\ell=1,2,\ldots,N$ and resulted in either $x_i=0$ or $x_i=1$.

$$\sum_{\ell=1}^{N}\left[s_n(\ell)+r_n(\ell)\right]=n \qquad (2\text{-}20)$$

One sees that knowledge of α,β, $s_n(\ell)$, $r_n(\ell)$, and $p_0(\ell)$ for $\ell=1,2,\ldots,N$ permits the calculation of the posterior probability vector $\underline{p}_n(\theta=\ell|\cdot\cdot)$ given n ovservations. Calculation of all previous sets of posterior probabilities for observations e_i, $i=1,2,\ldots,n$ requires knowledge of the total previous experiment history $(\underline{e}_n,\underline{x}_n)$, including temporal order however. The total number of possible histories $\underline{h}_n$ has previously been noted to increase rapidly with the number of potential object locations N and the total number of observations, n=j.

2.4 Proof of the Posterior Probability Expression for a General Experiment History

The various general forms for the posterior probability vectors have previously been stated and discussed but not proven. A proof of the most general of these results will now be given. The case considered assumes a general experiment history $\underline{h}_n$.

THEOREM: The ℓth term (ℓ=1,2,...,N) of the N element posterior probability vector, $\underline{p}_n(\theta|\underline{h}_n)$, given the general experiment history $\underline{h}_n=(\underline{e}_n,\underline{x}_n)$ and the discrete prior distribution $\underline{p}_0$, is expressed by equation (2-18) which was previously given.

PROOF: The proof will be by mathematical induction. The case for a single experiment (n=1) and either observation $x_1=0$ or $x_1=1$ have both been derived in detail at the beginning of this chapter. Both results may be observed to conform to the general result given in equation (2-18). Hence the contention is true for n=1.

Assume the result has been derived and shown to be of the desired form for an arbitrary n-observation experiment history, $\underline{h}_n=(\underline{e}_n,\underline{x}_n)$. Consider the possible elements, $\underline{p}_{n+1}(\theta=k|\cdot)$, of the $(n+1)^{st}$ probability vector after the $(n+1)^{st}$ observation is taken at an arbitrary experiment location, $e_{n+1}=\ell$; ℓ=1,2,...,N. Two observational outcomes are possible. Consider first the "no" result $x_{n+1}=0$. As has previously been noted (see equation (2-1))

$$p_{n+1}(\theta=k|\underline{e}_n,e_{n+1}=\ell,\underline{x}_n,x_{n+1}=0) = \frac{\left(\frac{\beta}{1-\alpha}\right)^{\delta(k-\ell)} p_n(k|\underline{e}_n,\underline{x}_n)}{1 - \frac{1-\alpha-\beta}{1-\alpha}\, p_n(\ell|\underline{e}_n,\underline{x}_n)} \; ; \; k=1,2,\ldots,N \tag{2-21}$$

But $p_n(k|\cdot)$ and $p_n(\ell|\cdot)$ have been assumed to be of the stated form. Substituting into the above expression and clearing fractions results in the following expression.

$$p_{n+1}(\theta=k|\underline{e}_n,e_{n+1}=\ell,\underline{x}_n,x_{n+1}=0) =$$

$$\frac{\left(\frac{\beta}{1-\alpha}\right)^{\delta(k-\ell)}\left(\frac{\beta}{1-\alpha}\right)^{m_n(k)+1}\left(\frac{1-\beta}{\alpha}\right)^{y_n(k)+1}p_0(k)}{1-\frac{1-\alpha-\beta}{1-\alpha}\sum_{i=1}^{n}\left(\frac{\alpha-1}{\alpha}\right)^{x_i}\left(\frac{\beta}{1-\alpha}\right)^{m_i(e_i)}\left(\frac{1-\beta}{\alpha}\right)^{y_i(e_i)}p_0(e_i)-\frac{1-\alpha-\beta}{1-\alpha}\left(\frac{\beta}{1-\alpha}\right)^{m_n(\ell)+1}\left(\frac{1-\beta}{\alpha}\right)^{y_n(\ell)+1}p_0(\ell)} \tag{2-22}$$

Note that the denominator could be combined into a single sum of the desired form since $e_{n+1}=\ell$ and $x_{n+1}=0$ imply that $m_{n+1}(\ell)=m_n(\ell)+1$ and $[(\alpha-1)/\alpha]^{x_{n+1}}=1$. Furthermore, for the particular term $\theta=\ell$ the Kronecker delta has value unity so that the numerator becomes

$$\left(\frac{\beta}{1-\alpha}\right)^{m_n(\ell)+2}\left(\frac{1-\beta}{\alpha}\right)^{y_n(\ell)+1}p_0(\ell) \tag{2-23}$$

Since the $(n+1)^{st}$ observation at location $e_{n+1}=\ell$ resulted in a no response, $x_{n+1}=0$, we observe that $m_{n+1}(\ell)+1=m_n(\ell)+2$ and $y_{n+1}(\ell)+1=y_n(\ell)+1$. Thus, for $\theta=\ell$, the numerator also has the desired form. For $\theta=k\neq\ell$, $\delta(k-\ell)=0$ so that the numerator of each of these (N-1) terms is unaffected by the additional observation at location ℓ, i.e. $m_{n+1}(k)+1=m_n(k)+1$. Incorporating these results into the above equation, one finds that $p_{n+1}(\theta=k|\cdot\cdot)$ has the stated form.

Consider now the case for the alternative, "yes," observational response, $x_{n+1}=1$. In this case, as has previously been shown (equation (2-4)),

$$p_{n+1}(\theta=k|\underline{e}_n,e_{n+1}=\ell,\underline{x}_n,x_{n+1}=1) = \frac{\left(\frac{1-\beta}{\alpha}\right)^{\delta(k-\ell)} p_n(k\ \underline{e}_n,\underline{x}_n)}{1+\frac{1-\alpha-\beta}{\alpha}\ p_n(\ell|\underline{e}_n,\underline{x}_n)}\ ;\ k=1,2,\ldots,N. \tag{2-24}$$

Again substituting $p_n(k|\cdot)$ and $p_n(\ell|\cdot)$ as above, one finds

$$p_{n+1}(\theta=k|\underline{e}_n,e_{n+1}=\ell,\underline{x}_n,x_{n+1}=1) =$$

$$\frac{\left(\frac{1-\beta}{\alpha}\right)^{\delta(k-\ell)}\left(\frac{\beta}{1-\alpha}\right)^{m_n(k)+1}\left(\frac{1-\beta}{\alpha}\right)^{y_n(k)+1} p_0(\ell)}{1-\frac{1-\alpha-\beta}{1-\alpha}\sum_{i=1}^{n}\left(\frac{\alpha-1}{\alpha}\right)^{x_i}\left(\frac{\beta}{1-\alpha}\right)^{m_i(e_i)}\left(\frac{1-\beta}{\alpha}\right)^{y_i(e_i)} p_0(e_i)+\frac{1-\alpha-\beta}{\alpha}\left(\frac{\beta}{1-\alpha}\right)^{m_n(\ell)+1}\left(\frac{1-\beta}{\alpha}\right)^{y_n(\ell)+1} p_0(\ell)} \tag{2-25}$$

Noting that $\left[-\left(\frac{1-\alpha-\beta}{1-\alpha}\right)\left(\frac{\alpha-1}{\alpha}\right)^1\right] = \frac{1-\alpha-\beta}{\alpha}$ and observing several other results paralleling those discussed in the previous case, one observes that the response $x_{n+1}=1$ also results in posterior probabilities $p_{n+1}(\theta=k|\cdot)$ of the stated form.

Thus, assuming the general form for $p_n(\theta=k|\cdot\cdot)$ one can show it true for $p_{n+1}(\theta=j|\cdot\cdot)$. Since the result is also true for n=1, then by mathematical induction the result is true for all n=1,2,..., QED.

The special cases where a particular experiment outcome history is assumed, $x_n=\underline{\phi}_n$ for example, can be gotten as special cases of the above general expression. Alternatively, the above proof could be applied in a very similar manner.

This completes the discussion of the general and specific forms taken by the posterior probabilities given a specific search history. These results will be used in subsequent chapters.

CHAPTER III

DERIVATION OF THE OPTIMUM SEARCH POLICY

3.1 Introduction and Statement of the Optimality Criterion

This section describes the major results which have been derived in this dissertation. An optimum search policy is derived for locating the true object under the optimality criterion and problem assumptions previously discussed. The criterion employed is that of minimizing the probability of obtaining n consecutive "no" responses for all n=1,2,... while simultaneously maximizing the probability that the first "yes" response will occur at the true object location. The stopping rule employed is as follows; the object is declared to be found when the first favorable response is received. Instrument parameter conditions which assure the experimenter that the object can be correctly located with any desired probability of success are also determined.

Expressions are derived for determining the average search duration. Several example cases are evaluated for a uniform prior distribution. A computational method is discussed for approximating the average search duration to any desired accuracy for an arbitrary prior distribution.

The optimum search procedure which results from an optimization employing the above criterion is as follows:

> Always observe the experiment location corresponding to the current most probable location* as determined by the posterior probability expressions of Chapter II and stop searching whenever a favorable response is received.

*Ties are broken by employing a separate random device as discussed in Section 3.2.

This result is very similar to results obtained in several previous studies discussed in Chapter I where the special assumption of a zero false alarm probability, $\alpha=0$, was made. This chapter thus generalizes these results to $\alpha\neq 0$ for the particular problem considered.

The basis for selecting the search optimization criterion given above is based partly on the author's heuristic feeling that such a search criterion is reasonable for many searches and partly because such a choice significantly simplifies the form and reduces the number of past histories which must be considered. The criterion may be argued as follows: If the instrument has good statistical validity in the sense that α and β are small* then a good search procedure should quickly discard potential locations which initially appear highly probable but are not the true location so as to arrive at the true but unknown object location in a minimum number of trials. Since the probability of falsely accepting an erroneous location is α, one obviously requires α to be small if errors are to be minimized. Once the search procedure arrives (specifies that an observation be taken) at the true location, a favorable response will be observed with probability $(1-\beta)$. The probability of observing a "yes" response at any of the other $(N-1)$ locations is α so that whenever the ratio $(1-\beta)/(N-1)\alpha$ is large compared to unity the above criterion is a heuristically reasonable choice.

One should note that the results which will be obtained are exact for any $\alpha,\beta<\frac{1}{2}$ so long as one accepts the above criterion. The choice of criterion is not convincing, however, unless $(1-\beta)/(N-1)\alpha$ is large.

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*The qualifiers "good" and "small" will be discussed in subsequent paragraphs after the principal results are derived.

If $(1-\beta)/(N-1)\alpha$ is not large the criterion will still have the advertised properties but the probability of stopping the search at the true object location may not be satisfactorily large. An inequality is given relating the instrument parameters (α,β) to the probability of correctly terminating the search. The exact meaning of the qualifier large will be examined after the derivation.

Selection of the instrument parameters (α,β) may not be totally at the disposal of the experimenter so that the ratio given above need not be large in all cases. However, assuming a fixed instrument the above criterion is still useful, providing that the experimenter employs the above instrument to localize probable object locations and then switches to a second verification instrument (or second parameter set (α,β) verification mode) to assure oneself that the declared location is truely valid at some acceptable probability level. The above dual mode search technique is typical of the assumptions employed for results previously derived for the communications search problem which will subsequently be considered. See for example the article by Bohacek (14).

The problem considered assumes that there are N potential object locations. Only one location may be observed for any one experiment. The experimenter is assumed to have complete freedom in the choice of which location is to be observed for each of the experiments. Each experiment performed is assumed to be a simple hypothesis test having identical, known characteristics which are independent of both the location observed and the temporal position of the observation in the sequence of observations.

Each experiment yields two types of information, the outcome of the experiment and the location where the observation was taken. This

information, plus knowledge of the α and β error probabilities comprise the result of an experiment.

The total accumulated experiment history, $\underline{h}_n$, at any point, n, in the search is represented by knowledge of which locations have been observed, the order of the observations, and the responses received at each. As has been shown in Chapter II, the above information may be employed in combination with the a priori probabilities to compute the a posteriori conditional probabilities associated with each of the N locations after an arbitrary number of search steps. The a posteriori probabilities associated with locations ℓ, ℓ=1,2,...,N after n observations are written

$$p_n(\ell|\underline{e}_n,\underline{x}_n)$$

where the prior experiment history is denoted $\underline{h}_n=(\underline{e}_n,\underline{x}_n)$ and

$$\underline{e}_n = (e_1, e_2, \ldots, e_n) \ , \ \underline{x}_n = (x_1, x_2, \ldots, x_n).$$

Each experiment location corresponds to one of the potential object locations,

$$e_i = \ell \quad \text{for some } \ell = 1,2,\ldots,N.$$

The analytical advantage of the assumed criterion is great since the observation history can now be specified to be of the form $\underline{x}_n=(0,0,0,\ldots 0,1)$ at the point when the search is terminated and $\underline{x}_j=(0,0,0,\ldots 0)$ at any previous time, $j<n$. Determining the optimum search procedure consists of determining the optimum experiment sequence $\underline{e}_n=(e_1,\ldots,e_n)$ for all n.

A prior distribution $p_0(\ell)$, ℓ=1,2,...,N is assumed known and of a general discrete form. It will also be assumed that the prior distribution

has been initially ordered in terms of nonincreasing probability values, $p_0(1) \geq p_0(2) \geq \ldots \geq p_0(N)$.

3.2 Proof of the Optimum Search Policy

The optimum search policy for the criterion outlined above consists of always selecting the next experiment location such that it corresponds to the potential object location which is currently most probable as based on the prior distribution and previously obtained data.* The posterior probability associated with each location may be computed sequentially from the posterior probability expressions of Chapter II. The search is terminated whenever a "yes" observation is obtained. The search policy so defined is heuristically very satisfying since it corresponds both to intuition and previous results derived for related problems for which, $\alpha=0$.

This section presents a proof of the above result. The proof is not very compact since a good deal of explanatory material is interspersed in the presentation.

THEOREM: Assuming a given prior distribution, $\underline{p}_0$, and known instrument parameters (α,β) the experiment sequence $\underline{e}_n$, $n=1,2,\ldots$, which minimizes the probability of obtaining n consecutive unfavorable responses and simultaneously maximizes the probability that the first favorable response is obtained at the true object location is the set e_i, $i=1,2,\ldots,n$ such that each of the experiment locations e_i which are sequentially selected correspond to the a posteriori most probable location given the previous search history $\underline{h}_{i-1}=(\underline{e}_{i-1},\underline{x}_{i-1})$, that is

$$e_i = \{\ell \mid p(\ell \mid \underline{e}_{i-1},\underline{x}_{i-1}) = \max_{j=1,\ldots N} p(j \mid \underline{e}_{i-1},\underline{x}_{i-1})\}; \; i=1,\ldots,n \quad (3\text{-}1)$$

---

*Ties are discussed below.

In case of ties, one of the locations is randomly selected by employing any stochastic device which assigns equal probability to each of the tied locations.

The proof will consist of two parts. The first proof determines the experiment sequence which minimizes the probability of obtaining n consecutive "no" responses. The second portion of the proof consists of showing that this result also simultaneously maximizes the probability that the first yes response will be obtained at the correct location.

LEMMA: The choice of experiment locations $\underline{e}_n$, n=1,2,..., which minimizes the probability of obtaining n consecutive "no" responses is that given in the above theorem for all n.

PROOF: Denote the conditional probability of obtaining n consecutive no responses given the experiment history, $\underline{e}_n$ as $p_n(\underline{x}_n=\underline{\phi}_n|\underline{e}_n)$.

This probability may be written in an expanded form as the product of n previous conditional probabilities (see for example Parzen(15)),

$$p_n(\underline{x}_n=\underline{\phi}_n|\underline{e}_n)=p(x_n=0|e_n,\underline{e}_{n-1},\underline{x}_{n-1}=\underline{\phi}_{n-1})p(x_{n-1}=0|e_{n-1},\underline{e}_{n-2},\underline{x}_{n-2}=\underline{\phi}_{n-2})\ \cdots$$

$$p(x_2=0|e_2,e_1,x_1=0)p(x_1=0|e_1). \tag{3-2}$$

The true object location will be denoted θ where $\theta=\ell$ for some $\ell=1,2,\ldots,N$. The proof will proceed by mathematical induction. Consider n=1. Then

$$p_1(x_1=0|e_1=\ell)=[p(x_1=0|e_1=\ell,\theta=\ell)p_0(\theta=\ell|e_1=\ell)+p(x_1=0|e_1=\ell,\theta\neq\ell)p_0(\theta\neq\ell|e_1=\ell)] \tag{3-3}$$

One notes that $p(x_1=0|e_1=\ell,\theta=\ell)=\beta$ and that $p(x_1=0|e_1=\ell,\theta\neq\ell)=(1-\alpha)$, where these conditional probabilities depend only on the instrument parameters and experimental conditions. Similarly, since the true object location does not depend on the choice of experiment location $p_0(\theta=\ell|e_1=\ell)=p_0(\ell)$ and $p_0(\theta\neq\ell|e_1=\ell)=1-p_0(\ell)$. Hence

$$p_1(x_1=0|e_1=\ell) = \left[\beta p_0(\ell) + (1-\alpha)(1-p_0(\ell))\right]$$

$$= (1-\alpha-\beta)\left[\frac{1-\alpha}{1-\alpha-\beta} - p_0(\ell)\right] \qquad (3\text{-}4)$$

The experiment location, $e_1=\ell$, which minimizes this expression may be noted, by inspection, to be the location corresponding to $p_0(1)$ the a'priori most probable location.

Consider the case for n=2. While this case is unnecessary to the proof it does yield additional insight into the form of the general result. It is included for this reason. From above

$$p_2(\underline{x}_2=\underline{\phi}|\underline{e}_2) = p(x_2=0|e_1,e_2,x_1=0)\, p_1(x_1=0|e_1). \qquad (3\text{-}5)$$

(Note that the general expression for the rightmost term has been determined above.) Consider the term $p(x_2=0|e_1,e_2,x_1=0)$ which may alternatively be expressed

$$p(x_2=0|e_1=\ell,e_2=k,x_1=0)=p(x_2=0|e_1=\ell,e_2=k,x_1=0,\theta=k)p_1(\theta=k|e_1=\ell,x_1=0) +$$

$$p(x_2=0|e_1=\ell,e_2=k,x_1=0,\theta\neq k)p_1(\theta\neq k|e_1=\ell,x_1=0) \qquad (3\text{-}6)$$

Once again this may be written in the alternative form

$$p(x_2=0|e_2=k,e_1=\ell,x_1=0) = (1-\alpha-\beta)\left[\frac{1-\alpha}{1-\alpha-\beta} - p_1(\theta=k|e_1=\ell,x_1=0)\right] \qquad (3\text{-}7)$$

where $p_1(\theta=k|e_1=\ell,x_1=0)$ represents the posterior probability associated with location k, k=1,2,...,N. given that a no response, $x_1=0$, was observed at experiment location $e_1=\ell$. The posterior probability expressions for a single observation have been developed in Chapter II and shown to be

$$p_1(\theta=k|e_1=\ell,x_1=0) = \frac{(1-\alpha)}{(1-\alpha-\beta)} \frac{\left(\frac{\beta}{1-\alpha}\right)^{\delta(k-\ell)} p_0(k)}{\left[\frac{1-\alpha}{1-\alpha-\beta} - p_0(\ell)\right]} \tag{3-8}$$

where $\delta(k-\ell)$ denotes the Kronecker delta function. Substituting (3-8) in the above expression (3-7) results in the following equation,

$$p(x_2=0|e_2=k,e_1=\ell,x_1=0) = (1-\alpha)\left(\frac{\frac{1-\alpha}{1-\alpha-\beta} - p_0(\ell) - \frac{\beta}{1-\alpha}^{\delta(k-\ell)} p_0(k)}{\left[\frac{1-\alpha}{1-\alpha-\beta} - p_0(\ell)\right]}\right) \tag{3-9}$$

Note that the denominator of this result is within a constant of being equal to the term $p_1(x_1=0|e_1=\ell)$. Thus, from equation (3-5) the probability of obtaining two consecutive no responses may be written

$$p_2(\underline{x}_2=\underline{\phi}_2|e_1=\ell,e_2=k) = (1-\alpha-\beta)(1-\alpha)\left(\frac{1-\alpha}{1-\alpha-\beta} - \left[p_0(\ell) + \frac{\beta}{1-\alpha}^{\delta(k-\ell)} p_0(k)\right]\right) \tag{3-10}$$

Note the summation form representing the experiment dependence. Minimizing the result is accomplished by selecting experiment locations ℓ and k which will maximize the pair of terms in the inner bracket.

$$\min_{\substack{e_1=\ell \\ e_2=k \\ \ell,k=1,2,\ldots,N}} \left[p_2(\underline{x}_2=\underline{\phi}_2|..)\right] \qquad \max_{\substack{e_1=\ell \\ e_2=k \\ \ell,k=1,2,\ldots,N}} \left[p_0(\ell) + \left(\frac{\beta}{1-\alpha}\right)^{\delta(k-\ell)} p_0(k)\right] \tag{3-11}$$

One notes that both terms in the sum refer to prior probability values and that $p_0(1) \geq p_0(i)\ \forall i \neq 1$ so that $\ell=1$ is one of the two selections. One would then select k=2 if $p_0(2) > \left(\frac{\beta}{1-\alpha}\right)p_0(1)$. Otherwise k=1 would be

observed a second time since $(\frac{\beta}{1-\alpha})p_0(1) \geq p_0(2)$. Any alternative choices can be observed to be no better at best since $p_0(3) \leq p_0(2)$.

Since $\left[(\frac{\beta}{1-\alpha})^{\delta(k-\ell)} p_0(k)\right]$ is within a scale factor of equalling the corresponding posterior probabilities $p_1(k|e_1=\ell,x_1=0)$ for $k=1,2,\ldots N$ given the first experiment one notes that selecting observation locations which each correspond to the current most probable point will simultaneously minimize the probabilities of obtaining both one and two consecutive no responses.

Note too that the resulting probability of obtaining two consecutive no responses, $p_2(\underline{x}_2=\underline{\phi}_2|..)$, is independent of the order in which the two experiments locations ℓ and k were selected. Simultaneous minimization of both $p_1(\underline{x}_1=\underline{\phi}_1|..)$ and $p_2(\underline{x}_2=\underline{\phi}_2|..)$ is not independent of order, however, except for the special case $\ell=k$.

The proof proceeds now by mathematical induction. Assume that the probability of obtaining (n-1) consecutive no responses has been determined to be of the following form and further shown to be minimized by always observing the experiment location corresponding to the current most probable location.

$$p_{n-1}(\underline{x}_{n-1}=\underline{\phi}_{n-1}|\underline{e}_{n-1})=(1-\alpha-\beta)\cdot(1-\alpha)^{n-2}\left[\frac{1-\alpha}{1-\alpha-\beta}-\sum_{i=1}^{n-1}K_i p_0(e_i)\right] \qquad (3\text{-}12)$$

In this expression the constants K_i are all of the form $K_i=(\beta/(1-\alpha))^{m_i(e_i)}$, $m_i(e_i)=0,1,2,\ldots,i-1$. The sum contains (n-1) terms representing the experiment location history through $\underline{e}_{n-1}$. Each term employs a prior probability $p_0(e_i)$ corresponding to the particular location which was observed in the ith experiment $e_i=\ell$, for some $\ell=1,2,\ldots,N$. Since e_i is a temporal indicator, e_i and $e_j, j\neq 1$, may or

may not refer to the same location ℓ. The constants K_i account for multiple observations of the same location ℓ and were discussed in Chapter II. The first observation of each location $\ell=1,2,\ldots,N$ will employ $m(\ell)=0$, the second $m(\ell)=1$, etc.

Before proceeding with the proof, several observations will be made about the nature of $p_{n-1}(\underline{x}_{n-1}=\underline{\phi}_{n-1}|\underline{e}_{n-1})$. The terms in the partial sum, $S_{n-1}=\sum_{i=1}^{n-1} K_i p_0(e_i)$, may be observed to be simply related to the corresponding posterior distributions for experiments $e_1,e_2,\ldots,e_{n-1}$. Recall from Chapter II that the posterior probability associated with each of the N locations after j unsuccessful experiments $\underline{x}_j=\underline{\phi}_j$, $j=1,2,..$ may be expressed

$$p_j(\theta=\ell|\underline{e}_j,\underline{x}_j=\underline{\phi}_j) = \left(\frac{1-\alpha}{1-\alpha-\beta}\right) \frac{\left(\frac{\beta}{1-\alpha}\right)^{m_j(\ell)+1} p_0(\ell)}{\left[\frac{1-\alpha}{1-\alpha-\beta} - \sum_{i=1}^{j} K_i p_0(e_i)\right]} \quad \ell=1,2,\ldots,N \tag{3-13}$$

where $(m_j(\ell)+1)$ is equal to the total number of previous observations of location ℓ in the first j trials and is independent of when they occurred. Observe that the largest exponent in S_{n-1} which refers to location ℓ is one less than $(m_j(\ell)+1)$. Note that the posterior probabilities are all equally influenced by the previous experiment history via the denominator. Hence the relative values of the probabilities at any stage j after j unfavorable responses are independent of the scale factor and depend on only the associated prior probabilities and the number of observations at each location,

$$\frac{p(\theta=\ell|\underline{e}_i,\underline{x}_i=\underline{\phi}_i)}{p(\theta=k|\underline{e}_i,\underline{x}_i=\underline{\phi}_i)} = \left(\frac{\beta}{1-\alpha}\right)^{[m_i(\ell)-m_i(k)]} \frac{p_0(\ell)}{p_0(k)} . \tag{3-14}$$

The sequence of observations which maximize the partial sum S_j and hence minimize the probability of obtaining j consecutive no responses (and also coincidentally maximize the denominator of $p_j(\theta=\ell|..)$ above) may thus be easily predicted. Consider the N infinite geometric sequences s_i shown below, each of which may be generated from the corresponding prior probability $p_0(1) \geq p_0(2) \geq \ldots \geq p_0(N)$.

$$s_1 = \{p_0(1), (\tfrac{\beta}{1-\alpha})p_0(1), (\tfrac{\beta}{1-\alpha})^2 p_0(1), \ldots, (\tfrac{\beta}{1-\alpha})^k p_0(1), \ldots\}$$

$$s_2 = \{p_0(2), \ldots \qquad , (\tfrac{\beta}{1-\alpha})^k p_0(2), \ldots\} \quad (3\text{-}15)$$

$$\vdots$$

$$s_N = \{p_0(N), \ldots \qquad , (\tfrac{\beta}{1-\alpha})^k p_0(N), \ldots\}$$

The partial sum, $S_j = \sum_{i=1}^{j} K_i p_0(e_i)$ may be noted to be the sum of any j terms selected without replacement from the above N sequences under the condition that each new selection is the leftmost term remaining in the row (experiment location) selected. Maximizing the partial sum S_j is thus equivalent to selecting the j largest terms in the above array.

A single nonincreasing infinite sequence can be constructed from the above N sequences by first selecting (without replacement) the largest value from the above array ($p_0(1)$ by the previous order assumptions), then the next largest, etc. Ties are broken by randomly selecting from the subset of points having equal value. Any random device which assigns equal probability to each of the tied points may be used. Once selected, the term is removed from consideration.

Having constructed such an ordered sequence one notes that a related sequence of experiment locations may be derived from it. As an example, consider the experiment sequence $\underline{e}$ implied by the sequence $\underline{P}$ shown below,

$$\underline{P} = \{p_0(1),\ p_0(2),\ p_0(3),\ (\tfrac{\beta}{1-\alpha})p_0(1),\ p_0(4),\ \ldots\ \}$$
$$\underline{e} = \{e_1=1,\ e_2=2,\ e_3=3,\ e_4=1,\ e_5=4,\ \ldots\ \} \tag{3-16}$$

The order in which experiments should be performed to minimize the sequence $p_j(\underline{x}_j=\underline{\phi}_j|\underline{e}_j)$ is thus specified by the associated sequence $\underline{P}$, and is entirely predictable for the criterion under discussion.

Continuing now with the proof of the Lemma, recall that both the form of $p_{n-1}(\underline{x}_n=\underline{\phi}_{n-1}|\underline{e}_{n-1})$ and the terms in the partial sum S_{n-1} have been assumed to be specified as elaborated above. Furthermore it is assumed that these choices minimize $p_{n-1}(\underline{x}_{n-1}=\underline{\phi}_{n-1}|\underline{e}_{n-1})$.

The remainder of the proof requires a demonstration that the probability of obtaining n consecutive unfavorable responses has this same form and further that the sum S_n includes all terms in S_{n-1} plus an additional term which is proportional to the maximum a posteriori probability $p_n(\theta=\ell|\underline{e}_n,\underline{x}_n=\underline{\phi}_n)$

We note that the probability of obtaining n consecutive no responses, $p_n(\underline{x}_n=\underline{\phi}_n|\underline{e}_n)$, may be written in the equivalent form,

$$p_n(\underline{x}_n=\underline{\phi}_n|\underline{e}_n) = p(x_n=0|\underline{e}_{n-1},e_n,\underline{x}_{n-1}=\underline{\phi}_{n-1})p_{n-1}(\underline{x}_{n-1}=\underline{\phi}_{n-1}|\underline{e}_{n-1}) \tag{3-17}$$

The first term in the above product may be written

$$p(x_n=0|\underline{e}_{n-1},e_n=\ell,\underline{x}_{n-1}=\underline{\phi}_{n-1})=p(x_n=0|\underline{e}_{n-1},e_n=\ell,\theta=\ell,\underline{x}_{n-1}=\underline{\phi}_{n-1})\cdot$$

$$p_{n-1}(\theta=\ell|\underline{e}_{n-1},\underline{x}_{n-1}=\underline{\phi}_{n-1}) + p(x_n=0|\underline{e}_{n-1},e_n=\ell,\theta\neq\ell,\underline{x}_{n-1}=\underline{\phi}_{n-1})\cdot$$

$$p_{n-1}(\theta\neq\ell|\underline{e}_{n-1},\underline{x}_{n-1}=\underline{\phi}_{n-1}) \tag{3-18}$$

But this may be observed to be of the same general form as the result of equation (3-7), namely

$$p(x_n=0|\underline{e}_{n-1},e_n=\ell,\underline{x}_{n-1}=\underline{\phi}_{n-1})=(1-\alpha-\beta)\left[\frac{1-\alpha}{1-\alpha-\beta}-p_{n-1}(\theta=\ell|\underline{e}_{n-1},\underline{x}_{n-1}=\phi_{n-1})\right] \tag{3-19}$$

Then, from the equation (2-13) of Chapter II the a posteriori probability associated with points ℓ, $\ell=1,2,\ldots,N$ given the experiment history $\underline{h}_{n-1}=(\underline{e}_{n-1},\underline{\phi}_{n-1})$ may be written in the following form,

$$p_{n-1}(\theta=\ell|\underline{e}_{n-1},\underline{x}_{n-1}=\underline{\phi}_{n-1}) = \frac{1-\alpha}{1-\alpha-\beta}\cdot\frac{\left(\frac{\beta}{1-\alpha}\right)^{m_{n-1}(\ell)+1}p_0(\ell)}{\left[\frac{1-\alpha}{1-\alpha-\beta}-\sum_{i=1}^{n-1}K_i p_0(e_i)\right]};\ \ell=1,2,\ldots,N \tag{3-20}$$

Recall that the numerator term $(\beta/1-\alpha)^{m_{n-1}(\ell)+1}p_0(\ell)$ refers to the location ℓ where the nth experiment e_n will be performed. The exponent $m_{n-1}(\ell)$, $0 \leq m_{n-1}(\ell) \leq n-1$, has a value one less than the total number of observations taken at location ℓ in the first n-1 experiments, $\underline{e}_{n-1}=(e_1,e_2,\ldots,e_{n-1})$. If location ℓ has not previously been observed, then $m_{n-1}(\ell)=-1$ as previously noted, implying $(\beta/1-\alpha)^{m_{n-1}(\ell)+1}=1$.

Combining the last two major equations given above one arrives at the following expression which denotes the probability of obtaining a no response on observation n given the specified history,

$$p(x_n=0|\underline{e}_{n-1},e_n=\ell,\underline{x}_{n-1}=\underline{\phi}_{n-1})=(1-\alpha)\frac{\left[\frac{1-\alpha}{1-\alpha-\beta}-\sum_{i=1}^{n-1}K_ip_0(e_i)-(\frac{\beta}{1-\alpha})^{m_{n-1}(\ell)+1}p_0(\ell)\right]}{\left[\frac{1-\alpha}{1-\alpha-\beta}-\sum_{i=1}^{n-1}K_ip_0(e_i)\right]} \tag{3-21}$$

Finally, this result multiplied by $p_{n-1}(\underline{x}_{n-1}=\underline{\phi}_{n-1}|\underline{e}_{n-1})$ as indicated in (3-17) yields the desired expression,

$$p_n(\underline{x}_n=\underline{\phi}_n|\underline{e}_n)=(1-\alpha-\beta)(1-\alpha)^{n-1}\left[\frac{1-\alpha}{1-\alpha-\beta}-\sum_{i=1}^{n-1}K_ip_0(e_i)-(\frac{\beta}{1-\alpha})^{m_{n-1}(\ell)+1}p_0(\ell)\right] \tag{3-22}$$

By interpreting $(\beta/1-\alpha)^{m_{n-1}(\ell)+1}p_0(\ell)$ as the general term $K_np_0(e_n)$ since $m_{n-1}(\ell)+1=m_n(\ell)$, one notes that the probability of obtaining n consecutive no responses has the desired form as indicated in equation (3-12),

$$p_n(\underline{x}_n=\underline{\phi}_n|\underline{e}_n)=(1-\alpha-\beta)(1-\alpha)^{n-1}\left[\frac{1-\alpha}{1-\alpha-\beta}-\sum_{i=1}^{n}K_ip_0(e_i)\right]. \tag{3-23}$$

Furthermore, one sees that $S_n=S_{n-1}+(\beta/1-\alpha)^{m_{n-1}(\ell)+1}p_0(\ell)$. S_n can be maximized by selecting an experiment location $e_n=\ell$ which will maximize the final term. The best one can do to maximize S_n is to select the next term appearing in the ordered sequence $\underline{P}$ since all larger terms have already been selected. Other rearrangements or alternative choices for the elements of S_n may do equally well but none can result in a larger value for S_n.

Thus, by finite induction the above expression represents the probability of obtaining n consecutive no responses for all n=1,2,... .

Furthermore, the specified choice of experiment locations minimizes the above probability for every n. This completes the proof of the Lemma.

Since the search process ends with the first yes response for the assumed criterion, one is obviously interested in following an experiment sequence which will maximize the probability that the first yes will occur at the true object location. The following paragraphs are devoted to demonstrating that the experiment sequence derived above as an optimum for minimizing the number of sequential no responses has simultaneously served this desiderata for all n.

LEMMA: The probability that the first yes observational response will occur at the true object location is maximized by selecting the experiment sequence $\underline{e}_n$, n=1,2,... which always observes the location which is a posteriori most probable prior to the observation in question.*

PROOF: The probability that the first yes response occurs at experiment n and that e_n is the true object location will be denoted,

$$p_n(\underline{x}_{n-1}=\underline{\phi}_{n-1}, x_n=1, \theta=\ell \,|\, \underline{e}_{n-1}, e_n=\ell). \tag{3-24}$$

This result may be written in an alternative form in terms of previously derived results,

$$p_n(\underline{x}_{n-1}=\underline{\phi}_{n-1}, x_n=1, \theta=\ell \,|\, \underline{e}_n=(\underline{e}_{n-1}, \ell)) = p(x_n=1 \,|\, \theta=\ell, \underline{e}_{n-1}, e_n=\ell, \underline{x}_{n-1}=\underline{\phi}_{n-1}) \cdot$$
$$p_{n-1}(\theta=\ell \,|\, \underline{e}_{n-1}, e_n=\ell, \underline{x}_{n-1}=\underline{\phi}_{n-1}) p_{n-1}(\underline{x}_{n-1}=\underline{\phi}_{n-1} \,|\, \underline{e}_{n-1}, e_n=\ell) \tag{3-25}$$

Eliminating independent conditioning statements this may be written in terms of known expressions,

*See previous comments about methods for breaking ties.

$$p_n(\underline{x}_{n-1}=\underline{\phi}_{n-1},x_n=1,\theta=\ell|\underline{e}_n=(\underline{e}_{n-1},\ell))=(1-\beta)p_{n-1}(\theta=\ell|\underline{e}_{n-1},\underline{x}_{n-1}=\underline{\phi}_{n-1})\cdot$$

$$p_{n-1}(\underline{x}_{n-1}=\underline{\phi}_{n-1}|\underline{e}_{n-1}) \qquad (3\text{-}26)$$

$$=(1-\beta)\left(\frac{1-\alpha}{1-\alpha-\beta}\right)\frac{\left(\frac{\beta}{1-\alpha}\right)^{m_{n-1}(\ell)+1}p_0(\ell)}{\left[\frac{1-\alpha}{1-\alpha-\beta}-\sum_{i=1}^{n-1}K_ip_0(e_i)\right]}(1-\alpha-\beta)(1-\alpha)^{n-2}\left[\frac{1-\alpha}{1-\alpha-\beta}-\sum_{i=1}^{n-1}K_ip_0(e_i)\right]$$

Simplifying, the desired result is given by

$$p_n(\underline{x}_{n-1}=\underline{\phi}_{n-1},x_n=1,\theta=\ell|\underline{e}_n=(\underline{e}_{n-1},\ell))=(1-\beta)(1-\alpha)^{n-1}\left(\frac{\beta}{1-\alpha}\right)^{m_{n-1}(\ell)+1}p_0(\ell)$$

$$n=1,2,\ldots$$

or

$$=(1-\beta)(1-\alpha)^{n-1}\left(\frac{\beta}{1-\alpha}\right)^{m_n(\ell)}p_0(\ell) \qquad n=1,2,\ldots \qquad (3\text{-}27)$$

This expression represents the probability that the first yes will be obtained on the nth experiment, $e_n=\ell$, and that it will occur at the correct location. This result assumes a value of $m_{n-1}(\ell)+1$ equal to the total number of times location ℓ has been observed during the first (n-1) experiments. Since $e_n=\ell$, $(m_{n-1}(\ell)+1)=m_n(\ell)$. The above result is conditionally maximized, given $\underline{e}_{n-1}$ and $\underline{x}_{n-1}=\underline{\phi}_{n-1}$, by observing the location $e_n=\ell$ corresponding to the largest posterior probability.

The probability that the first yes response will occur at the true location, independent of when it occurs, may be obtained by summing the above equation over all n. Let P_∞ denote the probability that the first yes occurs at the correct location, then

$$P_\infty=(1-\beta)\sum_{i=1}^{\infty}(1-\alpha)^{i-1}\left(\frac{\beta}{1-\alpha}\right)^{m_i(e_i)}p_0(e_i)=(1-\beta)\sum_{i=1}^{\infty}(1-\alpha)^{i-1}K_ip_0(e_i) \qquad (3\text{-}28)$$

It is apparent that the above sum is maximized by always selecting the experiment location corresponding to the largest remaining relative probability, $K_i p_0(e_i)$. Except for the addition of the geometric weighting function $(1-\alpha)^{i-1}$ the terms in this sum are identical to those in S_∞.

The effect of α and β on P_∞ may be noted in the above expression. The multiplicative factor $(1-\beta)$ influences the total sum no matter what the value of α or the form of the prior distribution. Similarly, α influences the result but more strongly for a uniform prior distribution than other alternatives due to the geometric weighting $(1-\alpha)^{i-1}$. Note that the sum may be easily approximated by a finite sum for any numerical problem since all terms are known. Furthermore the error can be bounded above since all terms $K_j p_0(e_j) \leq K_n p_0(e_n)$ for $j > n$ if the optimum search policy is observed. Thus

$$P_\infty = P_n + (1-\beta) \sum_{i=n+1}^{\infty} (1-\alpha)^{i-1} K_i p_0(e_i)$$

$$P_n < P_\infty < P_n + (1-\beta)\frac{(1-\alpha)^n}{\alpha} K_n p_0(e_n) \tag{3-29}$$

$$P_n < P_\infty < P_n + \varepsilon_n$$

where ε_n represents the error bound and can be made as small as is desired by increasing n.

The partial sum for i=1 to i=n, say P_n, represents the probability that the first yes occurs in the first n observations and that it occurs at the correct location. P_n thus represents the probability that the search will terminate correctly at the true object location in n or fewer observations. A plot of P_n vs. n would thus represent a cumulative probability of correct search termination.

3.3 Properties of the Optimum Search Policy, Computational Approximations, and Instrument Parameter Conditions

As an example of the computations required to approximate P_∞ consider the problem illustrated in Table 3-1. The example assumes α=0.05, β=0.1, N=10 and a nonuniform prior distribution as shown. For the problem parameters given the sum of the first twenty terms is a good approximation to the probability of correctly terminating the search, P_∞; namely

$$P_\infty < P_{20} + (1-\beta)\,\frac{(1-\alpha)^{20}}{\alpha}\,K_{20}\,p_0(e_{20})$$

$$P_\infty < .83646 + .01072$$

or

$$.83646 < P_\infty < .84718$$

Thus, the sum of the first twenty terms gives an estimate of successful search termination which is in error by slightly more than one percent.

A closed form expression for P_∞ may be determined in the special case of a uniform prior distribution if the optimum search policy is followed. In this special case search orders which observe each location i times before any other location is observed (i+1) times, i=1,2,..., are equivalent.

Assume a search routine where each location is observed in numerical order 1,2,...,N until all have been examined and then the routine begins anew in the same order as initially. The search continues until the first yes occurs.

For the above search procedure and a uniform prior distribution a closed form expression may be derived for the probability of correctly terminating the search process. The result, denoted P_∞^u, is derived in Appendix A and may be shown to be,

Table 3-1. Example Calculation

Location	1	2	3	4	5	6	7	8	9	10	$\alpha = 0.05$
Prior Probability	.3	.15	.1	.1	.1	.05	.05	.05	.05	.05	$\beta = 0.1$

Sequences

$s_1 = \{3.0\text{x}10^{-1}, 3.15789\text{x}10^{-2}, 3.32410\text{x}10^{-3}, 3{,}49905\text{x}10^{-4}, 3{,}68832\text{x}10^{-5}, \ldots\}$

$s_2 = \{1.5\text{x}10^{-1}, 1.57895\text{x}10^{-2}, 1.66205\text{x}10^{-3}, 1.74953\text{x}10^{-4}, 1.84161\text{x}10^{-5}, \ldots\}$

$s_3 = \{1.0\text{x}10^{-1}, 1.05263\text{x}10^{-2}, 1.10803\text{x}10^{-3}, 1.16635\text{x}10^{-4}, 1.22774\text{x}10^{-5}, \ldots\}$

$s_4 = \{\text{same as } s_3\}$

$s_5 = \{\text{same as } s_3\}$

$s_6 = \{5.0\text{x}10^{-2}, 5.26316\text{x}10^{-3}, 5.54017\text{x}10^{-4}, 5.83175\text{x}10^{-5}, 6.13869\text{x}10^{-5}, \ldots\}$

$\vdots$

$s_{10} = \{\text{all identical to } s_6\}$

$\underline{P} = \{3.0\text{x}10^{-1}, 1.5\text{x}10^{-1}, 1.0\text{x}10^{-1}, 1.0\text{x}10^{-1}, 1.0\text{x}10^{-1}, 5.0\text{x}10^{-2}, 5.0\text{x}10^{-2}, 5.0\text{x}10^{-2},$
$5.0\text{x}10^{-2}, 5.0\text{x}10^{-2}, 3.15789\text{x}10^{-2}, 1.57895\text{x}10^{-2}, 1.0526\text{x}10^{-2}, 1.0526\text{x}10^{-2},$
$1.0546\text{x}10^{-2}, 5.26316\text{x}10^{-3}, \ldots\}$

$\underline{e} = \{1, 2, \ldots 10, 1, 2, \ldots 10, 1, 2, \ldots 10 \ldots\}$

$$P_{\infty}^{u} = \frac{1-\beta}{N\alpha}\left[\frac{1-(1-\alpha)^{N}}{1-\beta(1-\alpha)^{N-1}}\right] \tag{3-30}$$

This result can usually be made as close to unity as one desires if α is selected to be small enough. For $N\alpha << 1$ and β not unity

$$P_{\infty}^{u} \cong 1 - \tfrac{1}{2} N\alpha\left(\frac{1+\beta}{1-\beta} - \frac{1}{N}\right) = 1 - \tfrac{1}{2}\frac{(N-1)\alpha}{(1-\beta)}\left[1 + \beta\left(\frac{N+1}{N-1}\right)\right] \tag{3-31}$$

One observes from this approximate result that a condition may be developed specifying what is meant by a "large" value of the quantity $(N-1)\alpha/(1-\beta)$. If the experimenter can specify a satisfactory value of P_{∞}^{u}, the above expression may be employed to find instrument parameters (α,β) which will be acceptable. Any nonuniform prior distribution will result in a larger probability of successful search termination. This result is discussed below. Equation (3-30) could also be used for this purpose but with greater difficulty. Note that the controlling instrument parameter is α, the false alarm probability since the last term is relatively insensitive to β over a wide range of β for small values of $N\alpha$. The product $(N-1)\alpha$ may be interpreted as the average number of false alarms which occur in a complete search of N cells.

Assuming the same parameters as for the previous example, N=10, α=0.05, and β=0.1 but a uniform prior distribution the probability of successful search termination is less than it was for the nonuniform case, namely P_{∞}^{u}=0.77086 as compared with P_{20}=0.83646.

Figure 3-1 illustrates P_{∞}^{u} vs. α for the case N=100 and assuming both $\alpha=2\beta$ and $\alpha=0.5\beta$. The relative insensitivity of P_{∞}^{u} to variations in β is also indicated in Figure 3-1, at least over a range of four in β.

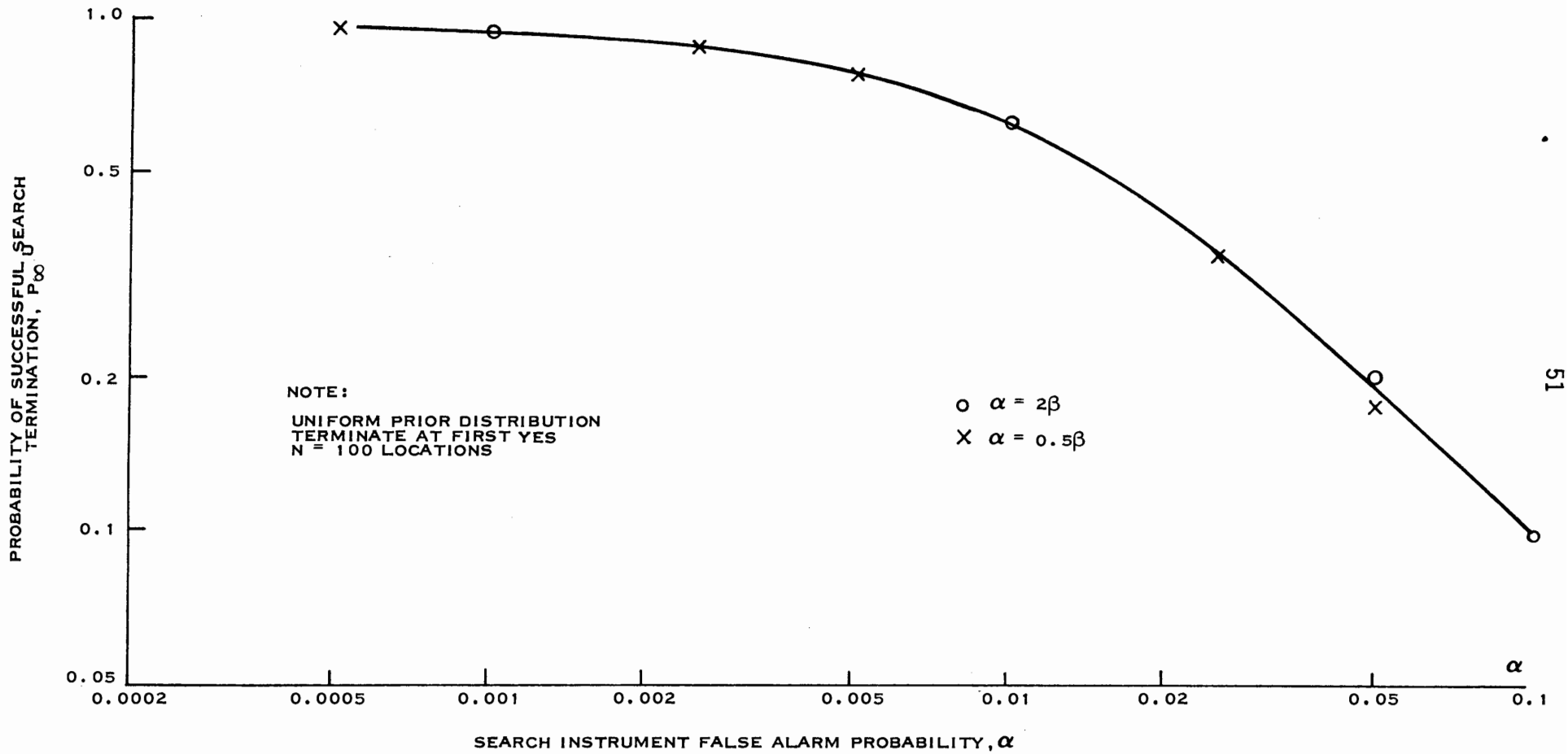


Figure 3-1. Probability of Successfully Locating the Correct Location vs False Alarm Probability

Any known nonuniform prior distribution will result in a probability of successful search termination which is always greater than that determined for the uniform distribution provided the optimum search procedure is followed and the search parameters α,β, and N are identical. This result is a natural consequence of the geometric weighting of terms in P_∞. The first few terms $K_i p_0(e_i)$ i=1,2,3 are most influential if the optimum search procedure is followed. Increasing the value of the first few terms at the expense of later ones increases the value of P_∞.

One may also be interested in assuring that the posterior probability associated with the location which has been observed always exceeds a specified probability standard P_s whenever the first yes occurs, where $0 < P_s < 1$. This may be assured by appropriate selection of α,β for any P_s and N (providing of course that α,β are at the experimenter's disposal). Given any search history of consecutive no responses, $\underline{h}_{n-1}=(\underline{e}_{n-1},\underline{\phi}_{n-1})$, the worst case a posteriori distribution for exceeding P_s would be uniform; in this case $p_{n-1}(1)=p_{n-1}(2)=\ldots=p_{n-1}(N)=\frac{1}{N}$.

A yes observation at experiment location $e_n=\ell$ would enhance the observed location in the following manner as previously determined (see equation (2-4)),

$$p_n(\ell) = \frac{\alpha}{1-\alpha-\beta} \; \frac{(\frac{1-\beta}{\alpha})\,\frac{1}{N}}{(\frac{\alpha}{1-\alpha-\beta} + \frac{1}{N})} \tag{3-33}$$

The condition $p_n(\ell) \geq P_s$ implies the following inequality relating α,β,N, and P_s.

$$\frac{(N-1)\alpha}{(1-\beta)} \leq \frac{(1-P_s)}{P_s} \tag{3-34}$$

This inequality implies a condition very similar to that previously noted for assuring successful search termination. One again notes that the critical parameter for satisfying this inequality is the probability of false alarm, α.

If the experimenter requires that a single yes be sufficient to cause a worst case initial probability of 1/N to exceed P_s, then for given N, suitable instrument parameter combinations (α,β) may be chosen to satisfy the above inequality. For example, with N=100 and P_s=.99 the ratio $(\alpha/1-\beta)$ must be less than 1.02×10^{-4}. With the special condition $2\beta=\alpha$ this implies that $\beta \leq 5\times10^{-5}$ and $\alpha \leq 10^{-4}$. For these same choices of α,β and N the probability of correctly terminating the search fora uniform prior is P_∞^U=.995 although this result is not plotted in Figure 3-1 because of scale limitations.

3.4 Probability of Termination in n Experiments

Another property of interest for the optimum search algorithm described above is the probability that the first yes is obtained at stage n after (n-1) consecutive no responses. This result will indicate the probability of terminating the search at stage n irrespective of whether the nth experiment is performed at the correct location. This may be alternatively described as the probability of terminating the search at stage n irrespective of whether the termination is correct or incorrect.

Denote the probability of obtaining the first yes observational response on experiment n and at location $e_n=\ell$ in the following manner where $\underline{e}_n=(\underline{e}_{n-1},\ell)$.

$$p_n(\underline{x}_{n-1}=\underline{\phi}_{n-1},x_n=1|\underline{e}_n)=p(x_n=1|\underline{e}_n,\underline{x}_{n-1}=\underline{\phi}_{n-1})p_{n-1}(\underline{x}_{n-1}=\underline{\phi}_{n-1}|\underline{e}_{n-1})$$

$$=\Big[p(x_n=1|\underline{e}_{n-1},e_n=\ell,\theta=\ell,\underline{x}_{n-1}=\underline{\phi}_{n-1})p_{n-1}(\theta=\ell|\underline{e}_{n-1},\underline{x}_{n-1}=\underline{\phi}_{n-1})$$

$$+\ p(x_n=1|\underline{e}_{n-1},e_n=\ell,\theta\neq\ell,\underline{x}_{n-1}=\underline{\phi}_{n-1})p_{n-1}(\theta\neq\ell|\underline{e}_{n-1},\underline{x}_{n-1}=\underline{\phi}_{n-1})\Big]\cdot$$

$$p_{n-1}(\underline{x}_{n-1}=\underline{\phi}_{n-1}|\underline{e}_{n-1}) \qquad (3\text{-}35)$$

Substituting in terms of previously derived expressions, this becomes

$$= \begin{cases} [\alpha + (1-\alpha-\beta)p_0(\ell)] & , n=1 \\ [\alpha + (1-\alpha-\beta)p_{n-1}(\theta=\ell|\underline{e}_{n-1},\underline{x}_{n-1}=\underline{\phi}_{n-1})](1-\alpha-\beta)(1-\alpha)^{n-2}\cdot \\ \qquad \cdot\left[\frac{1-\alpha}{1-\alpha-\beta}-\sum_{i=1}^{n-1}K_ip_0(e_i)\right] & , n=2,3,\ldots \end{cases} \qquad (3\text{-}36)$$

Employing equation (2-13) and clearing fractions this expression may be written in the form,

$$= \alpha(1-\alpha)^{n-2}(1-\alpha-\beta)\left[\frac{1-\alpha}{1-\alpha-\beta}-\sum_{i=1}^{n-1}K_ip_0(e_i)\right]+(1-\alpha)^{n-1}(1-\alpha-\beta)\left(\frac{\beta}{1-\alpha}\right)^{m_{n-1}(\ell)+1}p_0(\ell)$$

$$n=1,2,3,\ldots \qquad (3\text{-}37)$$

where $\left[\beta/(1-\alpha)\right]^{m_{n-1}(\ell)+1}=K_np_0(e_n)$ by the previous definition. The above expression, and the equivalent results shown below, represent the probability of terminating the search at experiment step n, with no conditioning as to the accuracy of the termination decision.

$$p_n(\underline{x}_{n-1}=\underline{\phi}_{n-1},x_n=1|(\underline{e}_{n-1},\ell))=\alpha(1-\alpha)^{n-2}(1-\alpha-\beta)\left[\frac{1-\alpha}{1-\alpha-\beta}-\sum_{i=1}^{n-1}K_ip_o(e_i)\right]+$$

$$\left(\frac{1-\alpha-\beta}{1-\beta}\right)p(\underline{x}_{n-1}=\underline{\phi}_{n-1},x_n=1,\theta=\ell|\underline{e}_n=(e_{n-1},\ell)) \qquad n=1,2,\ldots \quad (3\text{-}38)$$

$$p_n(\underline{x}_{n-1}=\underline{\phi}_{n-1}, x_n=1, |\underline{e}_n) = \frac{1-\alpha-\beta}{1-\beta}\left[\alpha(1-\beta)(1-\alpha)^{n-2}\left(\frac{1-\alpha}{1-\alpha-\beta} - \sum_{i=1}^{n-1} K_i p_0(e_i)\right) + \right.$$

$$\left. p(\underline{x}_{n-1}=\underline{\phi}_{n-1}, x_n=1, \theta=\ell | (\underline{e}_{n-1}, \ell))\right] \quad ; \; n=1,2,\ldots \qquad (3\text{-}39)$$

The last equation shows the relationship between correct search termination at stage n and incorrect termination at this same stage since the term $p(\underline{x}_{n-1}=\underline{\phi}_{n-1}, x_n=1, \theta=\ell | \underline{e}_n=(\underline{e}_{n-1}, \ell))$ represents the probability of correct termination. The result also coincidentally illustrates the fact that the optimum search policy minimizes the probability of erroneous termination for every n. This may be noted from the inclusion of the partial sum S_{n-1} in the first term inside the brackets. Since the optimum search policy maximizes S_{n-1}, it will also minimize this first term for every n.

The probability that the search will terminate on or before stage n is the partial sum of the first n such terms, say P_n',

$$P_n' = \left(\frac{1-\alpha-\beta}{1-\beta}\right) \sum_{j=1}^{n} \left(\alpha(1-\beta)(1-\alpha)^{j-1}\left[\frac{1-\alpha}{1-\alpha-\beta} - \sum_{i=1}^{j-1} K_i p_0(e_i)\right] + \right.$$

$$\left. p(\underline{x}_{j-1}=\underline{\phi}_{j-1}, x_j=1, \theta=\ell | (\underline{e}_{j-1}, \ell))\right) \qquad (3\text{-}40)$$

The form of P_n' may be simplified by interchanging the order of the double summation term, performing the inner sum using the book by Jolley (16), and combining several other terms in an appropriate manner. The result is derived in Appendix A and shown below.

$$P_n' = 1-(1-\alpha)^n + (1-\alpha-\beta)(1-\alpha)^{n-1} \sum_{i=1}^{n} K_i p_0(e_i) \qquad (3\text{-}41)$$

The above expression can be easily evaluated for any number of search steps, n.

Employing the example previously illustrated in Table 3-1, the probability of terminating the search in n=20 or fewer experiments may be easily calculated. The result, $P_{20}'=0.99603$, is somewhat greater than the equivalent probability of correct search termination, $P_{20}=0.83646$, as it must be.

The probability that the search process will ultimately terminate is unity as one would expect. This may be most conveniently noted by observing that the probability of obtaining n consecutive no responses (equation (3-23)) is the product of a monotonically decreasing function which is going to zero multiplied by $(1-\alpha)^n$ which also goes to zero with increasing n (for nonzero α). Hence the probability of obtaining n consecutive no responses approaches zero as n increases without limit.

For the special case of a uniform prior distribution the expression for P_n' may be simplified. Writing the sample size as n=rN+s, where r=0,1,2,... and $0 \leq s \leq N-1$, the probability of search termination on or before experiment n, denoted $P_n^{U'}$, may be expressed in the following forms (see Appendix A for the proof),

$$P_n^{U'} = 1 - (1-\alpha)^n \left(\frac{\beta}{1-\alpha}\right)^r \left[1 - \frac{s}{N} \frac{(1-\alpha-\beta)}{(1-\alpha)}\right] \tag{3-42}$$

or equivalently

$$P_n^{U'} = 1 - (1-\alpha)^{n-r} \beta^r \left[1 - \frac{s}{N} \frac{(1-\alpha-\beta)}{(1-\alpha)}\right]$$

Because of the order of the search procedure and for large n, r is approximately equal to the number of observations of the true location and n-r the number of observations of other locations.

3.5 Average Search Duration for the Optimum Search Policy

Having derived the probability of terminating the search procedure in exactly n experiments one can use this result to find the average length of the search, $\underline{L}$. The average search duration is obtained by performing the following summation,

$$\underline{L} = \sum_{j=1}^{\infty} j \; p(x_j=1, \underline{x}_{j-1}=\underline{\phi}_{j-1} | \underline{e}_j) \tag{3-43}$$

Substituting from equation (3-36), interchanging the order of the double sum term, and performing various algebraic simplifications, one arrives at the following expression for the average search duration. The proof is also given in Appendix A.

$$\underline{L} = \frac{1}{\alpha}\left[1-(1-\alpha-\beta)\cdot\sum_{i=1}^{\infty}(1-\alpha)^{i-1} \; K_i p_0(e_i)\right] \tag{3-44}$$

In the special case of a uniform prior distribution the following equation results.

$$\underline{L}^U = \frac{1}{\alpha}\left[1-(1-\alpha-\beta)\;\frac{1}{N\alpha}\left[\frac{1-(1-\alpha)^N}{1-\beta(1-\alpha)^{N-1}}\right]\right] \tag{3-45}$$

For the example of Table 3-1 which has previously been considered the average duration of the search is $\underline{L}$=4.161 experiments. This result assumes a summation truncated after 20 terms and hence will be slightly larger than the true value.

If one employed a perfect instrument and observed the locations in order of decreasing prior probability values the average search duration would be 3.8 experiments. In the second case the search duration is related to the average object location within the specified search order.

A perfect instrument correctly locates the object every time the opportunity arises but several experiments are required, on the average, to arrive at the true object location. The numerical difference between the average search durations for the cases may be considered a "search excess" caused by the imperfect instrument. The probability of successfully consummating the search is also different for the two cases as has previously been noted.

3.6 A Hypothesis Test for the Case of No Object

In some instances the experimenter may be uncertain as to whether there is an object located within the N location search space when the search is undertaken. This case was briefly discussed in Chapter I. When no object is present, the nature of the search policy discussed above assures the experimenter of erroneously locating an object if he does not truncate the search procedure after a finite number of steps. The above results may be employed to determine an experiment truncation point, n, at which the experimenter is willing to claim that no object is present and yet be reasonably certain of finding it if it is there.

If no object is present the search policy employed has no effect on the probability of erroneously locating an object, denoted P_n^E, since all locations appear identical. The probability, P_n^E, may be simply expressed in terms of the instrument false alarm probability, α, and the number of experiments, n, viz.,

$$P_n^E = 1 - (1-\alpha)^n \tag{3-46}$$

The experimenter would obviously like to make this probability small if there is a reasonable chance of no object being present. This implies choosing n to be small.

However, from equation (3-28) representing the probability of successful search termination, P_n, one sees that P_n increases with n. The experimenter must thus weigh the two alternatives and select a value of n accordingly. For the example of Table 3-1 with an instrument parameter α=0.05 and selecting ten experiments, n=10, the two probabilities of interest have values P_{10}^E=0.40126 and P_{10}=0.69843. The importance of employing an instrument which has a small false alarm probability, α, is apparent from the example. If the experimenter was forced to employ the stated value of α the number of experiments performed may well be reduced below 10 to further reduce the probability of falsely claiming an object to be present. The probability of successful search termination assuming the presence of an object will unfortunately also be reduced.

For the case in which the experimenter has some option about the instrument parameter set (α,β) which is employed, the value of α may be selected to achieve a satisfactory probability of erroneously locating an object in n experiments, P_n^E. Figure 3-2 illustrates the improvement in P_n^E caused by reducing α for several values of n. For the range of values $P_n^E < 0.1$, $P_n^E \cong n\alpha$ is an excellent approximation as may be noted. Small values of P_n^E thus imply small values of $n\alpha$. By using α=0.005 instead of α=0.05 in the above example the probability of erroneously locating an object in n=10 experiments is reduced to a more respectable value, P_{10}^E=0.05. The same change is also beneficial in increasing P_n so long as the probability of false dismissal remains the same, β=0.1. For the example distribution of Table 3-1, P_{10}=0.8506 for α=0.005.

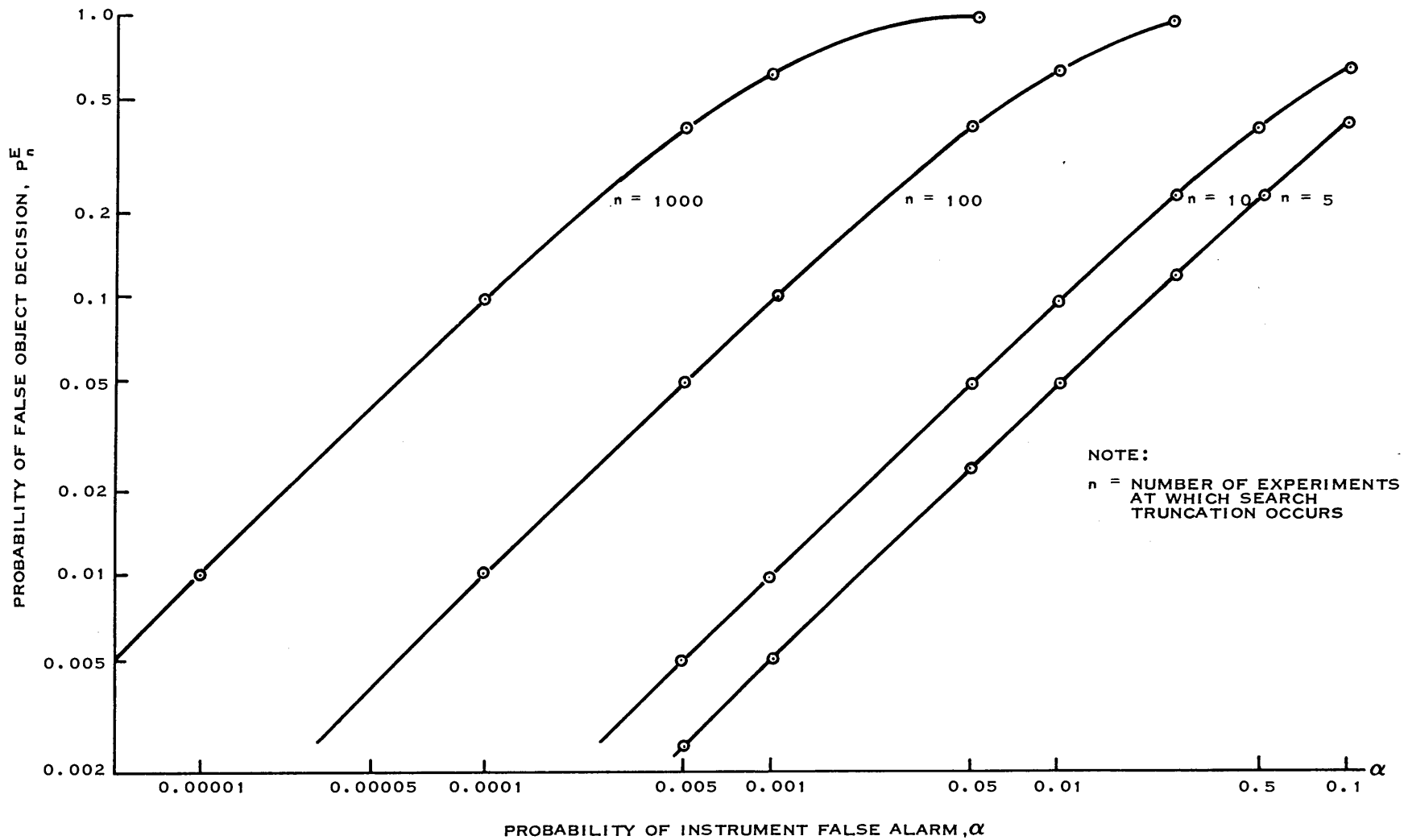


Figure 3-2. Probability of Erroneously Locating an Object in n Experiments vs. Instrument False Alarm Probability

It is not usually possible to treat the instrument parameters α and β as independent quantities as has been done above. In many instances α and β are directly related so that decreasing α simultaneously increases β. In such cases α and β can only be decreased by increasing the number of observation samples (sample size) employed by the instrument for each preliminary hypothesis test. A particular example of such practical considerations is illustrated in the next chapter.

CHAPTER IV

A COMMUNICATIONS SEARCH EXAMPLE

The results given in Chapter III can be applied to search problems of many different types as was indicated in Chapter I. As an illustration of such an application the author has selected a communication system search problem. The preceding search results will be applied to this example. The example chosen is of interest in several areas of digital data communications, particularly for interference resistant military communication systems and multiple user satellite communications systems employing code division multiple access techniques to separate the user signals from one another.

For systems of these types the transmitter employs a specially constructed binary code sequence to scramble and significantly increase the redundancy of the radiated signal. The signal alterations are performed in ways which are known to the intended receiver and hence can be neutralized by him in a useful way.

A major difficulty encountered in using such signal structures is that of initial synchronization. The receiver must locate the time origin of the received scrambled sequence before its effects can be neutralized. (The frequency and phase of the waveform may also be unknown in some cases, further complicating the problem.) The receiver's privileged information concerning the redundant signal structure is of value only after he has located the correct time origin since this information is necessary for proper system operation.

Random disturbances of the separate timing clocks employed by the transmitter and receiver, plus signal propagation time uncertainties caused by the intervening propagation medium prevents the receiver from knowing the time origin and forces him to search through a bounded uncertainty region until he locates the appropriate signal epoch. This problem is frequently modeled in terms of the general search problem previously discussed. A description of the relationships between the technical aspects of this special search problem and the search parameters α, β, and N as discussed in this thesis are thoroughly described in the previously referenced dissertation by Bohacek (14) and a related paper by Selin and Tuteur (17). The problem model suggested in each of these studies fits naturally into the framework of this dissertation and will be employed.

In this context, the receiver performs an experiment by selecting one of the N potential signal parameter sets and employing the signal corresponding to it in an instrument (detector) which performs a simple, fixed sample, hypothesis test on the received signal. It will be assumed that the receiver employs ideal noncoherent detection of a known signal combined with additive white normal noise to perform the preliminary decision making function (experiment). For this case the design of the instrument and the expressions giving the resulting conditional error probabilities α and β are well known, see for example Van Trees (18). The technical parameter of interest for this problem is the energy to noise density ratio, denoted $d^2=(E_r/N_o/2)$. The numerator of this ratio represents the total received signal energy E_r upon which the preliminary decision is based. The total energy E_r is simply related to the average received signal power, S, and the duration of the

experiment, T; namely E_r=ST. Average power is ordinarily a constant for a particular search and depends on the equipments involved and the problem geometry. The experiment duration T may be considered to be at the disposal of the experimenter and is related to a distribution parameter of the test statistic employed. The fixed preliminary test assumptions of Chapter I thus implies that T be constant for all experiments; its value may, however, be selected by the experimenter before the search begins. The noise power density, $N_0/2$, may be shown to be equal to the variance of the additive, zero mean, normal noise assumed as a condition of the problem. The noise spectrum is assumed flat and located in the detector bandwidth of W Hz with W=1/T for all T so that WT is a constant, WT=1.

The above assumptions and conditions imply that the ratio d, called the performance index, is a constant of the instrument once T is selected. Furthermore, each value of d constrains the parameters (α,β) to have a fixed and known, although complicated, relationship to one another. The functional relationship between α and β for contours of constant performance index, d, is illustrated in Figure 4-1. This figure is adapted from Figure 4-58 in Van Trees and is usually called the receiver operating characteristic.

This illustration merely indicates the fact that decreasing β increases α and vice versa for a fixed test distribution. Changing the experiment duration (prior to the search) corresponds to selecting a different contour for the experiments.

The hypothesis test which results for the above problem has a test statistic which has a central χ^2 distribution with 2 degrees of freedom for locations not containing the object and a noncentral χ'^2 distribution

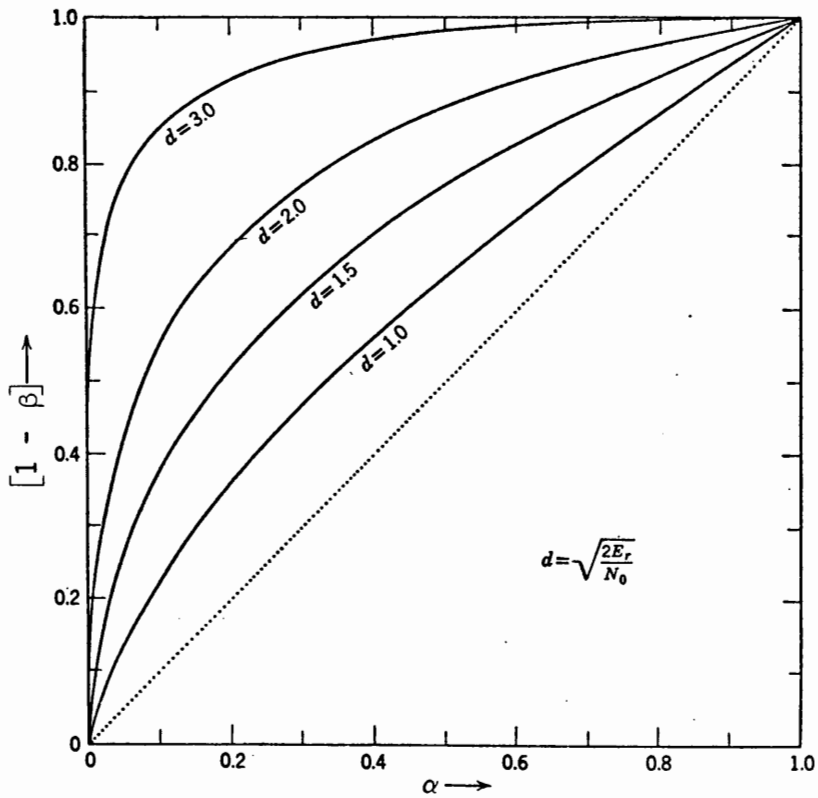


Figure 4-1. Receiver Operating Characteristic, (1-β) vs α for Constant Performance Index, d

with 2 degrees of freedom for the true object location. The noncentrality parameter λ may be related to the previous technical parameters, namely $\lambda=2E_r/N_o=d^2$. The conditional error probabilities α and β are thus integrals over appropriate regions of the real line as determined by the experimenter's choice of decision boundary (or threshold). The two probabilities α and β are related by a numerically integrable and tabulated function, Marcum's Q function (19),

$$1 - \beta = Q(d, \sqrt{-2\ln_e \alpha})$$

An alternative source is the χ'^2 table by Haynam et. al. (20) although this source is limited to $0.1 \geq \alpha \geq 0.001$.

Having selected the duration of each experiment, T, the experimenter can vary the instrument error parameters along the appropriate contour of constant d. This is accomplished by altering the instrument decision boundary or threshold (selecting the size of the test).

Alternatively, increasing the experiment duration T for a fixed decision boundary reduces β thereby reducing the average number of experiments required but at a cost of increasing the duration of each experiment.

The above experimental situation will be employed, together with a uniform prior distribution, to illustrate the fact that the preceding theoretical results have practical application and furthermore that the experimenter can sometimes exercise the limited control that he has over the instrument parameters α, β and T to improve the search operation. For example, it may be beneficial for the experimenter to increase the time expended for each experiment T, thereby reducing β, in order to

reduce the average number of experiments required, $\underline{L}^U$. It is possible that the product of these two, say $TL = T\underline{L}^U$, may be reduced by doing so. This implies that the total time required to locate the object, the product TL, is of importance to the experimenter. This implication is true for the case at hand and reducing this time is usually of critical importance to the receiving terminal.

Because of the complexity of the integral involved in determining α and β the minimization cannot conveniently be accomplished analytically and a brief numerical example is given. The minimum TL product which is determined is thus approximate (and perhaps only a local minimum). The primary purpose of the example is to illustrate that beneficial tradeoffs can frequently be made by the experimenter in a real situation.

Fixing the performance index d at several constant values, specifically d=4,5,6, and 7 for the computations which were made, and then appropriately varying α and β along a contour of constant d permits one to examine the resulting variation in the average number of experiments, $\underline{L}^U$, and the probability of successful search termination P_∞^U. The numerical results were determined by employing the expressions given in Chapter III, equations (3-45) and (3-30), respectively. Both $\underline{L}^U$ and P_∞^U increase monotonically with decreasing α (increasing β) along a contour of constant d in the region of interest. These results are shown in Figures 4-2 and 4-3, respectively. Although it is not shown in these figures, recall that β increases with decreasing α, for constant d, ala Figure 4-1. The monotonic nature of the average search duration is not unexpected since decreasing α decreases the probability of erroneously

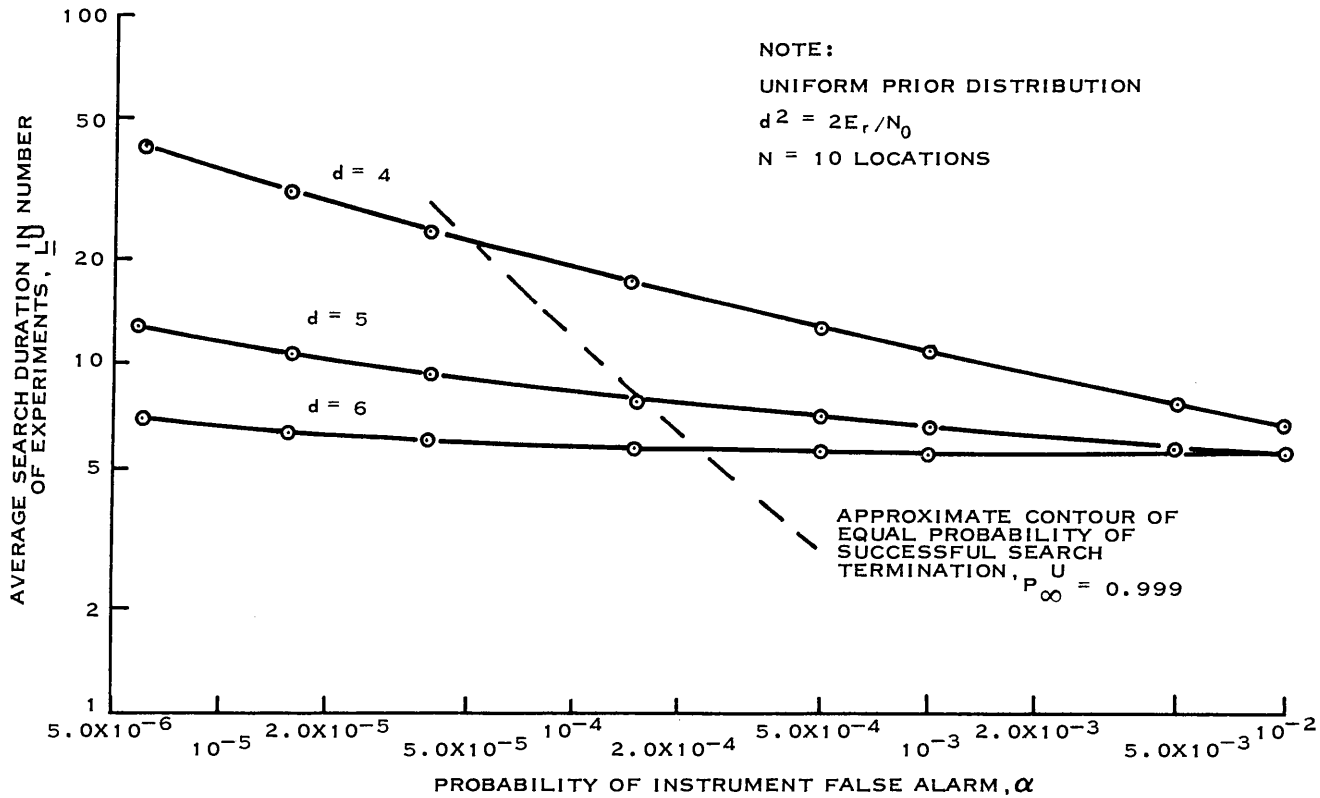


Figure 4-2. Average Number of Experiments vs False Alarm Probability for Constant Performance Index, d

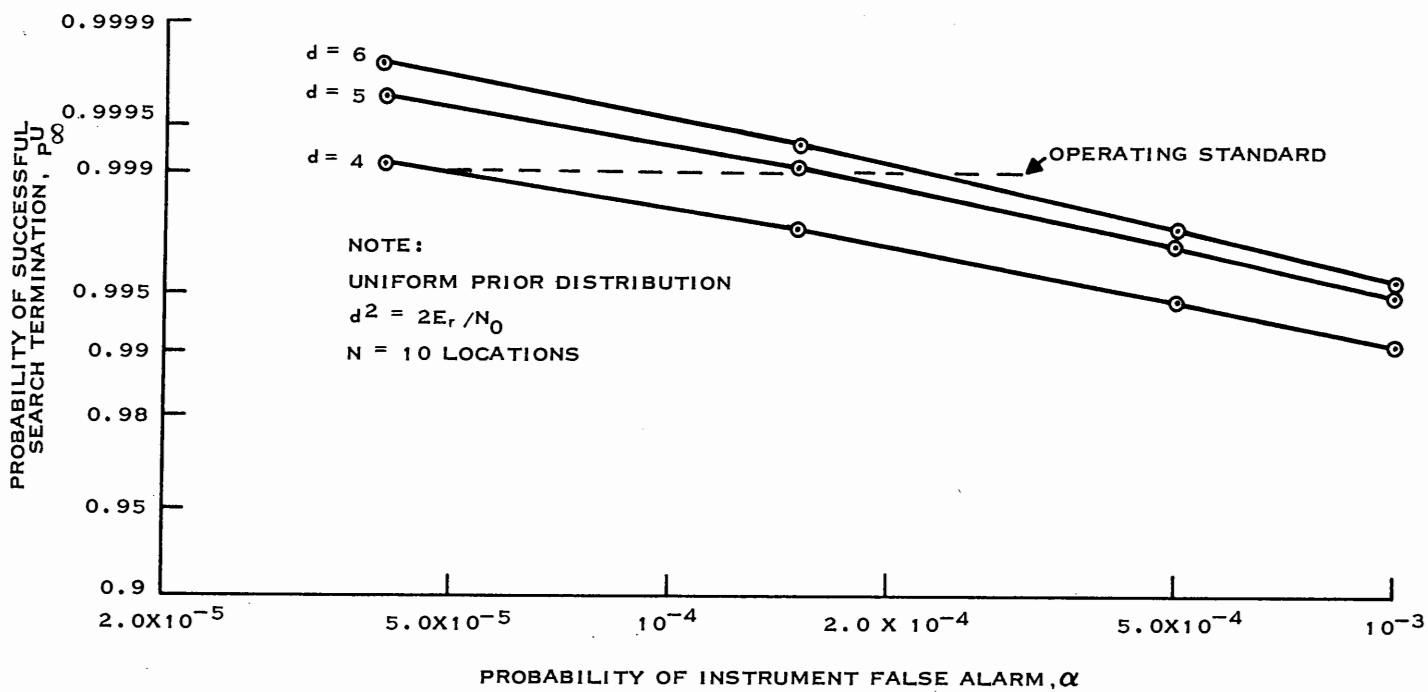


Figure 4-3. Probability of Successful Search Termination vs Instrument False Alarm Error for Constant Performance Index, d

locating the object while simultaneously increasing β, thus increasing the probability of passing the true location several times before it's finally recognized.

Setting an operating standard by selecting a value for the probability of correct search termination, $P_{\infty}^{U}=0.999$, one can immediately select the appropriate conditional values of α given d from Figure 4-3. Having done so for each value of d the result can be transferred to Figure 4-2 and a contour of equal $P_{\infty}^{U}=0.999$ drawn. One notes from Figure 4-2 that the average number of experiments required to obtain the same measure of search success, $P^{U}=0.999$, varies inversely with the performance index d. The duration of each experiment varies directly with d^2 for constant S and N_0, $d^2 \sim T$, as previously discussed. One might well expect that the total length of the search, TL, (the duration of each experiment times the average number of experiments), could be minimized by appropriately selecting T. This is indeed the case, at least locally, as is illustrated in Figure 4-4. The result shown is only relative since S and N_0 are unspecified. The experiment duration was assumed unity for d=4, i.e. $T_4=1$ for convenience. The relative values of the other experiment durations are then $T_5=25/16$, $T_6=36/16$ and $T_7=49/16$, respectively.

The result of Figure 4-4 indicates that there is at least a local minimum for the total relative search duration, TL, in the neighborhood of d=5. The corresponding instrument parameters are $\alpha=1.7\times10^{-4}$, $\beta=0.11$.

The experimenter may thus select a value of T according to Figure 4-4 for a known ratio S/N_0. Further numerical explorations could pinpoint the result still further but with little real benefit since TL varies slowly with d.

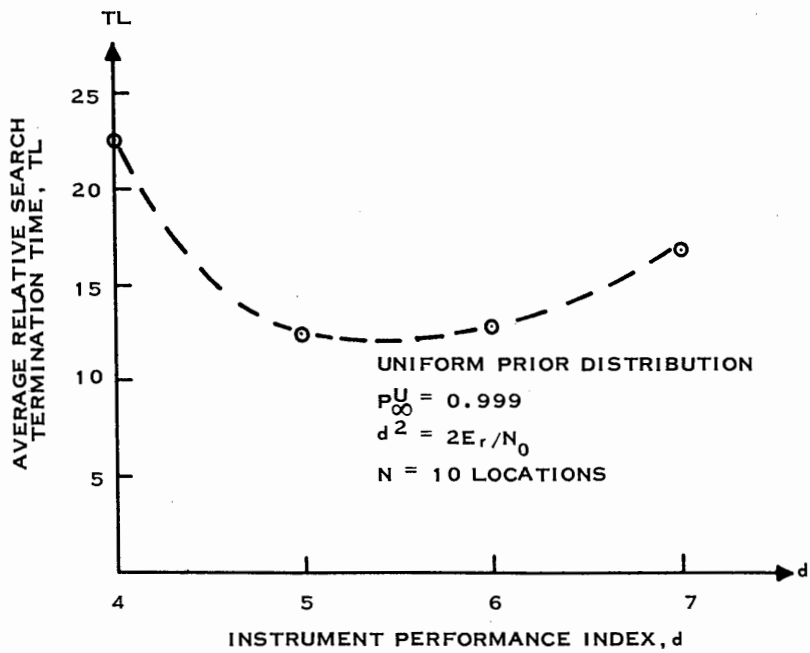


Figure 4-4. Average Search Time vs Instrument Performance Index

The preceding example has indicated how the theory derived in preceding chapters can be applied to a real problem and furthermore that for each such problem it may be possible to intelligently select the parameters of the search instrument to minimize the total time required to find the object. Each such problem must, however, be examined on its own merits and no general results of this sort are possible.

CHAPTER V

SUMMARY

The preceding chapters have developed an optimum sequential search policy for locating a single stationary object situated in a discrete search space having N potential locations. A single search instrument was employed to perform the search. The instrument was assumed to perform a simple, fixed sample, hypothesis test for each experiment; both false alarm and false dismissal errors were considered and the conditional probability of each was assumed known; α and β respectively. A completely specified discrete probability distribution was assumed to describe the experimenter's a priori knowledge of the true object location.

The search policy obtained specifies that the experimenter always takes his next observation at the location which has the largest a posteriori probability of being the true location as computed from the prior probability distribution and previous experimental evidence via Bayes Rule. General expressions for the posterior probabilities were derived for any given search history, instrument parameter set, and prior probability distribution.

The stopping rule employed stated that the experimenter was to stop whenever a yes instrument response was obtained. The policy derived was shown to minimize the number of consecutive no responses received and also to maximize the probability that the first yes response was received at the correct location.

Conditions were derived enabling the experimenter to specify a set of instrument parameters which would at least assure him of attaining a desired probability of successful search termination. Several properties were derived for the optimum search policy including the average duration of the search, the probability of correct search termination, and the probability of terminating the search in a fixed number of experiments.

Properties of major interest for the optimum search polity are summarized below. The cumulative probability of successful search termination in n or fewer experiments was developed in Section 3.2 equation (3-28) and may be written

$$P_n = (1-\beta) \sum_{i=1}^{N} (1-\alpha)^{i-1} K_i p_0(e_i)$$

where the terms in the sum are defined in the referenced section and the products $K_i p_0(e_i)$ are representative of the search history. The probability of successful termination is obtained by letting n grow without bound and can be closely approximated with a finite number of terms for any specific numerical problem. For the special case of a uniform prior distribution the corresponding $(n \to \infty)$ result is given by

$$P^U = \frac{1-\beta}{N\alpha} \left[\frac{1 - (1-\alpha)^N}{1 - \beta(1-\alpha)^{N-1}} \right]$$

The uniform distribution represents a lower bound on P_∞ in the sense that $P_\infty \geq P_\infty^U$ for all <u>known</u> prior distributions if the optimum search policy is followed. This result and the approximation given in equation (3-31) can be used to establish a satisfactory instrument

parameter set (α,β) to satisfy a desired termination probability standard, $P_\infty^U \geq P_s$. The closed expression P_∞^U would be computationally more convenient for numerically estimating whether the given (or proposed) set of instrument parameters would perform to the experimenter's satisfaction.

Another result of interest is the probability of terminating the search (correctly or incorrectly) in n or fewer experiments. This result, given in equation (3-41) is reproduced below,

$$P_n' = 1 - (1-\alpha)^n + (1-\alpha-\beta)(1-\alpha)^{n-1} \sum_{i=1}^{n} K_i p_0(e_i)$$

If the prior distribution is uniform, then for n=rN+s, the above result becomes

$$P_n' = 1 - (1-\alpha)^n \left(\frac{\beta}{1-\alpha}\right)^r \left[1 - \frac{s}{N}\left(\frac{1-\alpha-\beta}{1-\alpha}\right)\right]$$

The average duration of the search, $\underline{L}$, was given in Section 3.5 and is shown in the following equation,

$$\underline{L} = \frac{1}{\alpha}\left[1 - (1-\alpha-\beta) \cdot \sum_{i=1}^{\infty} (1-\alpha)^{i-1} K_i p_0(e_i)\right]$$

This general result takes on the following form for a uniform prior distribution

$$\underline{L}^U = \frac{1}{\alpha}\left[1 - (1-\alpha-\beta)\frac{1}{N\alpha}\left[\frac{1-(1-\alpha)^N}{1-\beta(1-\alpha)^{N-1}}\right]\right]$$

A hypothesis test was discussed in Section 3.6 which provided a logical basis for enabling the experimenter to select an experiment truncation point if he is uncertain as to the presence of an object.

For the no-object case the untruncated optimum search policy is sure to locate an object erroneously since the search continues until the instrument responds favorably. This result thus provides a method for extending the search policy to this case also.

Chapter IV discussed the numerical minimization of the total search time for a realistic communication system search problem. This result illustrated a dependence of the instrument parameters (α,β) which is typical for many problems and showed that the experimenter could use his limited design freedom to improve the search operation with this constraint.

There are many avenues which could be followed for extending the results obtained in this dissertation. The assumption of location and time independent instrument parameters could be altered. Several alternative stopping rules could be considered which would be potentially better choices for the case of a poor and unalterable instrument.

One could also consider the multiple object problem or alternatively the case of a continuous search space with a finite resolution instrument. Answers to each of these would be of practical interest in problems which are not well suited to the assumptions and conditions required for the results derived herein.

APPENDIX A

MATHEMATICAL DERIVATIONS OF THE PROBABILITY OF SEARCH TERMINATION AND THE AVERAGE SEARCH DURATION

Secion 3.3, equation (3-30), presented a result representing the probability of successful search termination for a uniform prior distribution, P_∞^U. The derivation of this result is given below. From equation (3-28) the general form, P_∞, may be written

$$P_\infty = (1-\beta) \sum_{i=1}^{\infty} (1-\alpha)^{i-1} K_i p_0(e_i) \tag{A-1}$$

For the special prior distribution $p_0(i) = 1/N$, $\forall$ i and the search procedure discussed in Section 3.3 the above sum may be equivalently written as follows,

$$\sum_{i=1}^{\infty} (1-\alpha)^{i-1} \left(\frac{\beta}{1-\alpha}\right)^{m_i(e_i)} p_0(e_i) = \frac{1}{N} \sum_{j=1}^{\infty} \left[\beta(1-\alpha)^{N-1}\right]^{j-1} \cdot \sum_{i=1}^{N} (1-\alpha)^{i-1}$$

$$= \frac{1}{N\alpha} \cdot \frac{1 - (1-\alpha)^N}{1 - \beta(1-\alpha)^{N-1}} \tag{A-2}$$

Equation (3-30) given in the text follows immediately

$$P_\infty^U = \frac{1-\beta}{N\alpha} \left[\frac{1 - (1-\alpha)^N}{1 - \beta(1-\alpha)^{N-1}}\right] \tag{A-3}$$

Section 3.4 presented an expression in equation (3-41) representing the probability of search termination on or before the nth experiment, denoted P_n'. This result is derived below.

From equation (3-40) which was developed in the text the desired result may be initially written

$$P'_n = \left(\frac{1-\alpha-\beta}{1-\beta}\right) \sum_{j=1}^{n} \left(\alpha(1-\beta)(1-\alpha)^{j-1} \left[\frac{1-\alpha}{1-\alpha-\beta} - \sum_{i=1}^{j-1} K_i p_0(e_i) \right] + \right.$$

$$\left. p(\underline{x}_{j-1}=\underline{\phi}_{j-1}, x_j=1, \theta=\ell \,|\, (\underline{e}_{j-1}, \ell)) \right) \quad n=1,2,\ldots \tag{A-4}$$

Substituting from equation (3-27) and changing the form somewhat, this becomes

$$P'_n = \alpha \sum_{j=1}^{n} (1-\alpha)^{j-1} - (1-\alpha-\beta)\alpha \sum_{j=1}^{n} (1-\alpha)^{j-2} \sum_{i=1}^{j-1} K_i p_0(e_i) +$$

$$+ (1-\alpha-\beta) \sum_{j=1}^{n} (1-\alpha)^{j-1} K_j p_0(e_j) \tag{A-5}$$

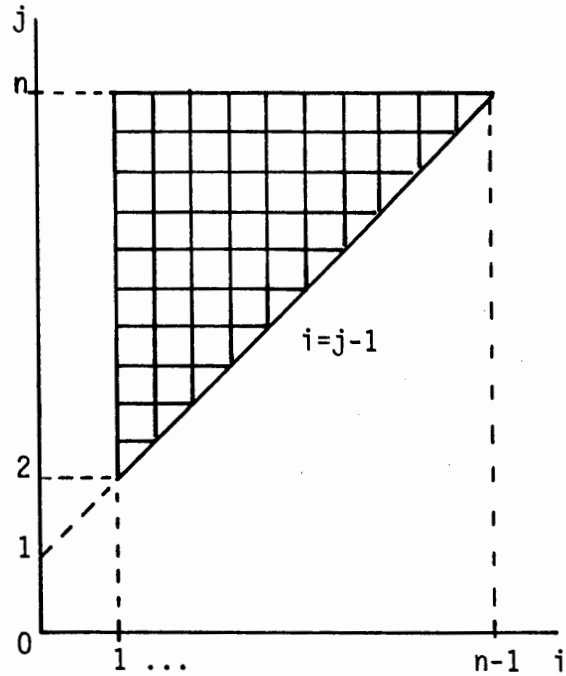
The first term can be immediately summed. By interchanging the order of summation for the second term one of the two summations can be performed for this term also. Note that the second term has value zero for j=1.

$$P'_n = \alpha \frac{1-(1-\alpha)^n}{1-(1-\alpha)} - (1-\alpha-\beta)\alpha \sum_{i=1}^{n-1} K_i p_0(e_i) \sum_{j=i+1}^{n} (1-\alpha)^{j-2} +$$

$$+ (1-\alpha-\beta) \sum_{j=1}^{n} (1-\alpha)^{j-1} K_j p_0(e_j) \tag{A-6}$$

The sketch in Figure (A-1)a illustrates the region of summation and the interchange in the order of summation. Performing the inner sum in the second term results in the expression

a)



b)

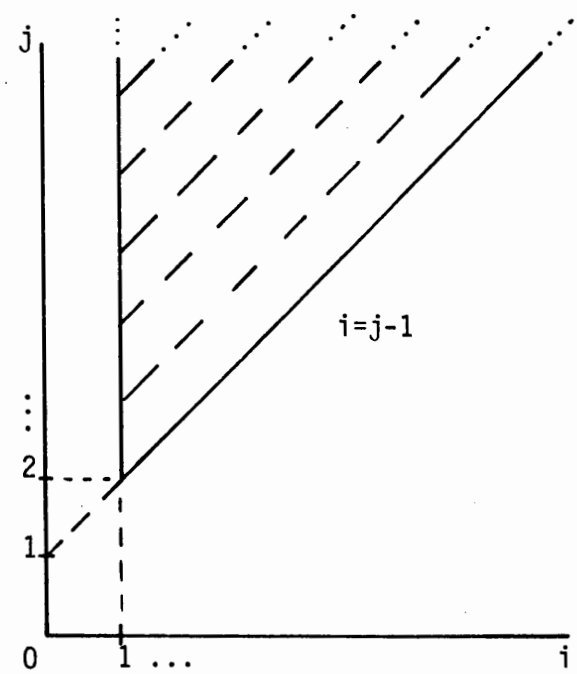


Figure A-1. Regions of Summation, Finite a) and Infinite b)

$$P'_n = 1-(1-\alpha)^n - \alpha(1-\alpha-\beta)\sum_{i=1}^{n} K_i p_0(e_i)(1-\alpha)^{i-1}\left[\frac{1-(1-\alpha)^{n-1}}{1-(1-\alpha)}\right]+$$

$$+ (1-\alpha-\beta)\sum_{j=1}^{n}(1-\alpha)^{j-1} K_j p_0(e_i) \qquad \text{(A-7)}$$

$$= 1-(1-\alpha)^n + (1-\alpha-\beta)(1-\alpha)^{n-1}\sum_{i=1}^{n} K_i p_0(e_i)$$

This last result is the form shown in equation (3-41) and thus substantiates the result shown. Q.E.D.

Consider now the special case of a uniform prior distribution and assume the search procedure employed is that discussed in Section 3.3. For this case, writing $n=rN+s$, where $r=0,1,\ldots$; $0\leq s\leq(N-1)$, this may be written

$$\sum_{i=1}^{n}\left(\frac{\beta}{1-\alpha}\right)^{m_i(e_i)} p_0(e_i) = \sum_{j=1}^{r}\left(\frac{\beta}{1-\alpha}\right)^{j-1}\sum_{i=1}^{N} p_0(i)+\left(\frac{\beta}{1-\alpha}\right)^{r}\sum_{i=1}^{s} p_0(i)$$

$$= \sum_{j=1}^{r}\left(\frac{\beta}{1-\alpha}\right)^{j-1} + \left(\frac{\beta}{1-\alpha}\right)^{r}\sum_{i=1}^{s} p_0(i)$$

$$= \frac{1-\left(\frac{\beta}{1-\alpha}\right)^{r}}{1-\left(\frac{\beta}{1-\alpha}\right)} + \left(\frac{\beta}{1-\alpha}\right)^{r}\sum_{i=1}^{s} p_0(i)$$

$$= \frac{1-\alpha}{1-\alpha-\beta}\left[1-\left(\frac{\beta}{1-\alpha}\right)^{r}\right]+\left(\frac{\beta}{1-\alpha}\right)^{r}\frac{s}{N} \qquad \text{(A-8)}$$

since $p_0(i) = 1/N$, $\forall$ i.

Incorporating this special case into the above general result, one obtains the probability of terminating the search in n or fewer experiments for a uniform prior distribution

$$P_n^{U'} = 1-(1-\alpha)^n+(1-\alpha-\beta)(1-\alpha)^{n-1}\left[\frac{1-\alpha}{1-\alpha-\beta}\left[1-\left(\frac{\beta}{1-\alpha}\right)^r\right]+\left(\frac{\beta}{1-\alpha}\right)^r \frac{s}{N}\right]$$

$$= 1-(1-\alpha)^n+(1-\alpha)^n\left[1-\left(\frac{\beta}{1-\alpha}\right)^r\right]+(1-\alpha-\beta)(1-\alpha)^{n-1}\left(\frac{\beta}{1-\alpha}\right)^r \frac{s}{N}$$

$$= 1 - (1-\alpha)^n\left(\frac{\beta}{1-\alpha}\right)^r\left[1 - \frac{s}{N}\frac{(1-\alpha-\beta)}{(1-\alpha)}\right] \qquad \text{(A-9)}$$

This last result is the form shown in equation (3-42) and completes the derivation. Q.E.D.

Another result which was stated without proof in the text was the average search duration, see Section 3.5. From equation (3-43), the defining equation, the expression we wish to determine is denoted $\underline{L}$, where

$$\underline{L} = \sum_{j=1}^{\infty} jp\ (x_j=1, \underline{x}_{j-1}=\underline{\phi}_{j-1} | \underline{e}_j) \qquad \text{(A-10)}$$

Then from equation (3-36)

$$\underline{L} = 1[\alpha+(1-\alpha-\beta)p_0(e_1)]+\alpha\sum_{j=2}^{\infty} j(1-\alpha)^{j-1}-\alpha(1-\alpha-\beta)\sum_{j=2}^{\infty} j(1-\alpha)^{j-2}\sum_{i=1}^{j-1} K_i p_0(e_i) +$$

$$+ (1-\alpha-\beta)\sum_{j=2}^{\infty} j(1-\alpha)^{j-1}\ K_j p_0(e_j) \qquad \text{(A-11)}$$

The second term may again be summed by employing the book by Jolley (p.2) and reduces to the following form.

$$II = \alpha \sum_{j=1}^{\infty} j(1-\alpha)^{j-1} = \frac{1}{\alpha} - \alpha$$

The third, double sum, term can similarly be partially reduced by interchanging the order of summation. See Figure (A-1)b.

$$III = -\alpha(1-\alpha-\beta) \sum_{j=2}^{\infty} j(1-\alpha)^{j-2} \sum_{i=1}^{j-1} K_i p_0(e_i)$$

$$= -\alpha(1-\alpha-\beta) \sum_{i=1}^{\infty} K_i p_0(e_i) \sum_{j=i+1}^{\infty} j(1-\alpha)^{j-2}$$

$$III = -(1-\alpha-\beta)\left[\sum_{j=1}^{\infty} i(1-\alpha)^{i-1} K_i p_0(e_i) + \sum_{i=1}^{\infty} (1-\alpha)^{i-1} K_i p_0(e_i) + \right.$$

$$\left. + \left(\frac{1-\alpha}{\alpha}\right) \sum_{i=1}^{\infty} (1-\alpha)^{i-1} K_i p_0(e_i)\right]$$

Combining these results one obtains

$$\underline{L} = \alpha + (1-\alpha-\beta)p_0(e_1) + \frac{1}{\alpha} - \alpha - (1-\alpha-\beta) \sum_{i=1}^{\infty} i(1-\alpha)^{i-1} K_i p_0(e_i) +$$

$$-(1-\alpha-\beta) \sum_{i=1}^{\infty} (1-\alpha)^{i-1} K_i p_0(e_i) - (1-\alpha-\beta)\left(\frac{1-\alpha}{\beta}\right) \sum_{i=1}^{\infty} (1-\alpha)^{i-1} K_i p_0(e_i) +$$

$$+(1-\alpha-\beta) \sum_{j=2}^{\infty} j(1-\alpha)^{j-1} K_j p_0(e_j) \qquad \text{(A-12)}$$

This expression may be simplified to the following form which is identical to equation (3-44).

$$\underline{L} = \frac{1}{\alpha}\left[1-(1-\alpha-\beta) \cdot \sum_{i=1}^{\infty}(1-\alpha)^{i-1} K_i p_0(e_i)\right] \qquad \text{(A-13)}$$

The above general equation may once again be simplified by assuming a uniform prior distribution and the previously specified search policy. For a uniform distribution the above sum may be evaluated by again letting n=rN+s,

$$\sum_{i=1}^{\infty}(1-\alpha)^{i-1}K_i p_0(e_i) = \sum_{i=1}^{N}(1-\alpha)^{i-1}p_0(i)\sum_{j=0}^{\infty}\left[(1-\alpha)^N\left(\frac{\beta}{1-\alpha}\right)\right]^j$$

$$= \sum_{i=1}^{N}(1-\alpha)^{i-1}p_0(i)\left[\frac{1}{1-(1-\alpha)^N\left(\frac{\beta}{1-\alpha}\right)}\right]$$

$$= \frac{1}{N\alpha}\left[\frac{1-(1-\alpha)^N}{1-\beta(1-\alpha)^{N-1}}\right]$$

since $p_0(i) = 1/N$, $\forall$ i.

Substituting into equation (A-13) above yields $\underline{L}^U$, the average search duration for a uniform prior distribution

$$\underline{L}^U = \frac{1}{\alpha}\left[1-(1-\alpha-\beta)\,\frac{1}{N\alpha}\left[\frac{1-(1-\alpha)^N}{1-\beta(1-\alpha)^{N-1}}\right]\right] \qquad \text{(A-14)}$$

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13. ABSTRACT

An optimum sequential search policy is developed for locating an object in a discrete search space assuming a search instrument having both false alarm and false dismissal errors.

The search policy obtained specifies that the experimenter always takes his next observation at the potential location which has the largest a posteriori probability of being the true location as computed from the prior probability distribution and previous experimental evidence via Bayes Rule.

The results were derived under the following assumptions. A single object is assumed to be located in 1 of N locations and to remain stationary during the search process. The experimenter is presumed to have an initial discrete a priori probability function describing the location of the object. Also available is a single search instrument which the experimenter can employ to examine each of the potential locations. Only one location can be observed at any one time, however. The instrument is assumed to perform a simple, fixed sample size, hypothesis test having known conditional error probabilities of both Type I and Type II. The preliminary decision made by the instrument is thus a noisy decision which is only statistically related to the presence or absence of the object at the location in question.

A communication system search example is also considered.

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