Concomitant of Multivariate Order Statistics With Application to Judgment Post-Stratification

Xinlei WANG, Lynne STOKES, Johan LIM and Min CHEN*

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Abstract

We generalize the definition of a concomitant of an order statistic in the multivariate case, develop general expressions for its density, and establish related properties. The concomitant of a normal random vector is studied in detail, and methods for calculating its moments are discussed. Furthermore, we apply the theory to develop new estimators of the mean from a judgement post-stratified sample, where post-strata are formed by rank classes of auxiliary variables. Our estimators are shown to be more efficient than existing ones and robust against violations of the normality assumption. They are also well suited to applications requiring cost efficiency.

Keywords: Best linear unbiased estimator; Gaussian quadrature; Least squares estimator; Ranked set sampling; Selection differential.

*Xinlei Wang is Assistant Professor and Lynne Stokes is Professor, Department of Statistical Science, Southern Methodist University, 3225 Daniel Avenue, P O Box 750332, Dallas, Texas 75275-0332, swang@mail.smu.edu and slstokes@mail.smu.edu. Johan Lim is Assistant Professor, Department of Applied Statistics, Yonsei University, Seoul, Korea; on leave from Department of Statistics, Texas A&M University, johanlim@stat.tamu.edu. Min Chen is Ph.D candidate, McCombs School of Business, University of Texas at Austin, minchen@mail.utexas.edu. The authors thank Professor Shin-Jae Lee at Seoul National University for providing us data of human teeth size.

1 Introduction

Let $(X_h, Y_h)_{h=1}^H$ be H independent random vectors from a common bivariate distribution. Denote by $X_{(r:H)}$ the rth ordered X-variate, $1 \leq r \leq H$. The concomitant of the rth order statistic of X is defined to be the Y-variate paired with $X_{(r:H)}$ and is denoted $Y_{[r:H]}$. Properties of concomitants have been studied by many authors (e.g., Bhattacharya 1974; Sen 1976, 1981; David et al. 1977; Yang 1977; Goel and Hall 1994, Nagaraja and David 1994); see David and Nagaraja (2003, Section 6.8) for an overview. Applications of concomitants include their use in estimating correlation (Barnett et al. 1976), in ranking and selection (Yeo and David 1984; David 1993), and ranked set sampling (Stokes 1977).

In this article, we extend the definition of concomitants to the multivariate case, develop general expressions for their distributions, and establish related properties. That is, we study the distribution of a Y-variate associated with ordered components of an absolutely continuous $\mathbf{X}$-vector. For example, suppose $\mathbf{X}_h$ contains the scores of the hth employee on two pre-employment screening measures and Y_h his or her score on a later job-performance measure, for a sample of H employees. Our theory would allow evaluation of the distribution of the job-performance measure for an employee ranked, say, best on both screening tests. It would also allow comparison of that distribution to the concomitant job-performance measure for an unscreened employee or to one scoring best on a single screening measure, in order to evaluate our selection procedure.

Our theory was motivated by an application of concomitants to judgement post-stratification (JP-S) (MacEachern et al. 2004), a method closely related to ranked set sampling (RSS). Both are useful when the variable of interest, Y, is expensive to measure, but can be ranked, at least approximately, much more cheaply. The ranking is referred to as judgement ranking. Both RSS and JP-S allow better estimation of the mean of Y, where the reduction in variance is provided by stratification. A ranked set sample can be thought of as a stratified sample, where judgement ranks define the strata. A judgment post-stratified sample can be thought of as a simple random sample (SRS), where judgement ranks define the post-strata.

This makes JP-S more practical than RSS for some applications, where the researcher may be amenable to beginning with an SRS with the option of using auxiliary data later, but reluctant to beginning with a non-standard design, such as a RSS (MacEachern et al. 2004).

A common method of judgement ranking in RSS is via an accessible auxiliary variable X, making Y a concomitant. We introduce a similar idea for JP-S in Section 5. As in conventional post-stratification, one can use multiple auxiliary variables for forming post-strata. When the ranks of these auxiliary variables jointly define post-strata, we need the theory and properties of the concomitant of multivariate order statistics in order to develop and compute JP-S estimators of the mean and investigate their properties.

The article will proceed as follows. In Section 2, we introduce concomitants of bivariate $\mathbf{X}$-vectors and present analytical results. In Section 3, we apply these results to the normal case, and show how to compute means and variances of the concomitant. In Section 4, we extend our methods with straightforward modifications to the higher-dimensional case. Section 5 first reviews methods of mean estimation that have been suggested for JP-S samples using ranking information from more than one auxiliary variable. Then we propose new estimators with attractive properties, which are available when certain distributional assumptions about the data can be made. Results of simulation and empirical studies comparing the estimators are reported. We conclude with a brief discussion in Section 6.

2 Concomitant of Bivariate Order Statistics

2.1 The General Theory

Let $(X_{h1}, X_{h2}, Y_h)_{h=1}^{H}$ be an iid random sample from a trivariate distribution, where the random variables X_1 and X_2 are absolutely continuous. Denote the order of X_{h1} among $X_{11}, \ldots, X_{H1}$ by $R_{h:H}$, and the order of X_{h2} among $X_{12}, \ldots, X_{H2}$ by $S_{h:H}$. We consider the random variable Y_h given the ranks $R_{h:H} = r$ and $S_{h:H} = s$, called the *concomitant of the rth order statistic of X_1 and the sth order statistic of X_2* and denoted by $Y_{h[r,s:H]}$.

For simplicity, we ignore the subscripts H and h and denote the concomitant as $Y_{[r,s]}$, its pdf as $f_{[r,s]}(y)$, the rank random variables as R and S, and the bivariate rank distribution $Pr[R_{h:H} = r, S_{h:H} = s]$ as π_{rs}, whenever no ambiguity exists.

Theorem 1. *Suppose (X_1, X_2, Y) follows a trivariate distribution with a joint pdf $f(x_1, x_2, y)$. Let $m(X_1, X_2)$ and $v(X_1, X_2)$ denote the conditional mean and variance of Y, $E[Y|X_1, X_2]$ and $Var[Y|X_1, X_2]$, respectively. Then, the distribution of the concomitant $Y_{[r,s]}$ among the H iid random vectors, is given by*

$$f_{[r,s]}(y) = \frac{\sum_{k=\mathcal{L}}^{\mathcal{U}} C_k \int\int_{\mathcal{X}} \theta_1^k \theta_2^{r-1-k} \theta_3^{s-1-k} \theta_4^{H-r-s+1+k} f(x_1, x_2, y) dx_1 dx_2}{\sum_{k=\mathcal{L}}^{\mathcal{U}} C_k \int\int_{\mathcal{X}} \theta_1^k \theta_2^{r-1-k} \theta_3^{s-1-k} \theta_4^{H-r-s+1+k} f(x_1, x_2) dx_1 dx_2} \tag{1}$$

where $\mathcal{U} = \min(r-1, s-1)$ and $\mathcal{L} = \max(0, r+s-H-1)$, $\mathcal{X}$ is the support of the distribution of the $\mathbf{X}$-vector,

$$C_k = \frac{(H-1)!}{k!(r-1-k)!(s-1-k)!(H-r-s+1+k)!}$$

$$\theta_1(x_1, x_2) = Pr(X_1 < x_1, X_2 < x_2) \quad ; \quad \theta_2(x_1, x_2) = Pr(X_1 < x_1, X_2 > x_2);$$

$$\theta_3(x_1, x_2) = Pr(X_1 > x_1, X_2 < x_2) \quad ; \quad \theta_4(x_1, x_2) = Pr(X_1 > x_1, X_2 > x_2).$$

The mean and variance of $Y_{[r,s]}$ can be expressed by

$$\mu_{[r,s]} = E\left[m(X_{1(r,s)}, X_{2(r,s)})\right] \tag{2}$$

$$\sigma^2_{[r,s]} = E\left[v(X_{1(r,s)}, X_{2(r,s)})\right] + Var\left[m(X_{1(r,s)}, X_{2(r,s)})\right] \tag{3}$$

where $(X_{1(r,s)}, X_{2(r,s)})$ are bivariate order statistics of (X_1, X_2).

Proof. First, we can write

$$f_{[r,s]}(y) = \frac{\int\int_{\mathcal{X}} f(y|x_1, x_2, r, s) p(r, s|x_1, x_2) f(x_1, x_2) dx_1 dx_2}{\pi_{rs}}$$

$$= \frac{\int\int_{\mathcal{X}} f(x_1, x_2, y) p(r, s|x_1, x_2) dx_1 dx_2}{\pi_{rs}}, \tag{4}$$

since $f(y|x_1, x_2, r, s) = f(y|x_1, x_2)$. In the spirit of David et al. (1977), it can be shown that

$$p(r, s|x_1, x_2) = \sum_{k=\mathcal{L}}^{\mathcal{U}} C_k \theta_1^k \theta_2^{r-1-k} \theta_3^{s-1-k} \theta_4^{H-r-s+1+k}, \tag{5}$$

yielding the numerator of (1). Similarly, we can show that its denominator, the bivariate rank distribution, is

$$\begin{aligned} \pi_{rs} &= \int\int_{\mathcal{X}} p(r, s|x_1, x_2) f(x_1, x_2) dx_1 dx_2 \\ &= \sum_{k=\mathcal{L}}^{\mathcal{U}} C_k \int\int_{\mathcal{X}} \theta_1^k \theta_2^{r-1-k} \theta_3^{s-1-k} \theta_4^{H-r-s+1+k} f(x_1, x_2) dx_1 dx_2 \end{aligned} \tag{6}$$

Further for the mean of $Y_{[r,s]}$, we have

$$\mu_{[r,s]} = \int_{\mathcal{Y}} y\, f_{[r,s]}(y) \mathrm{d}y = \int\int_{\mathcal{X}} m(x_1, x_2) f_{(r,s)}(x_1, x_2) dx_1 dx_2 = E\left[m(X_{1(r,s)}, X_{2(r,s)})\right]$$

where $\mathcal{Y}$ is the support of the distribution of Y, and $f_{(r,s)}(x_1, x_2)$ is the joint pdf of $(X_{1(r,s)}, X_{2(r,s)})$, i.e., $f(x_1, x_2|R = r, S = s)$. Similarly, the variance of $Y_{[r,s]}$ can be written as (3). □

Remark 1. If (X_1, X_2) and Y are independent, i.e., $f(x_1, x_2, y) = f(x_1, x_2)f(y)$, then it follows from (1) immediately that $f_{[r,s]}(y) = f(y)$.

Remark 2. Suppose there exists a monotonic function $\psi(\cdot)$ such that $X_2 = \psi(X_1)$. In this case, $f(x_1, x_2) = f(x_1) I\left(x_2 = \psi(x_1)\right)$ and $f(x_1, x_2, y) = f(x_1, y) I\left(x_2 = \psi(x_1)\right)$, where $I(\cdot)$ is the indicator function. Based on (1), it is easy to verify that both π_{rs} and $f_{[r,s]}(y)$ degenerate to the univariate case. When $\psi(\cdot)$ is increasing (or decreasing), if $r = s$ (or $r = H + 1 - s$), then $\pi_{rs} = 1/H$ and

$$f_{[r,s]}(y) = \int_{\mathcal{X}_1} f(y|x_1) f_{(r)}(x_1) \mathrm{d}x_1 = f_{[r,\bullet]}(y),$$

where $f_{[r,\bullet]}(y)$ is the distribution for $Y_{[r,\bullet]}$, the concomitant of the rth order statistic of X_1 while $f_{(r)}(x_1)$ is the distribution for $X_{1(r)}$, the rth order statistic of X_1; otherwise, $\pi_{rs} = 0$ and $f_{[r,s]}(y)$ does not exist.

Consider the application of concomitants to ranking and selection of employees. Remarks 1 and 2 formalize the intuitive notion that if the screening tests are unrelated to the performance measure, then using them for selection is of no benefit, or if the screening tests are identical, the second one is of no marginal benefit.

Example 1. Suppose $U_1, U_2, Y \sim$Uniform(0,1) and iid. Let $X_i = (Y + U_i)/2$ for $i = 1, 2$. We illustrate Theorem 1 by deriving $f_{[1,1]}(y)$ and $f_{[1,2]}(y)$ for $H = 2$, where we condition on the ranks of $\mathbf{X} = (X_1, X_2)$. The theorem requires both joint $f(x_1, x_2, y)$ and marginal $f(x_1, x_2)$ densities. The former can be determined to be uniform over the region $J : 0 \leq y \leq 1$ and $y/2 \leq x_1, x_2 \leq (y+1)/2$; i.e.,

$$f(x_1, x_2, y) = 4I_J(x_1, x_2, y). \tag{7}$$

The marginal density is found by integrating the joint density over the appropriate region to obtain:

$$f(x_1, x_2) = \begin{cases} 8x_2 & \text{Area } A \\ 8x_1 & \text{Area } D \\ 4(2x_2 - 2x_1 + 1) & \text{Area } B \\ 4(2x_1 - 2x_2 + 1) & \text{Area } E \\ 8(1 - x_1) & \text{Area } C \\ 8(1 - x_2) & \text{Area } F \end{cases} \tag{8}$$

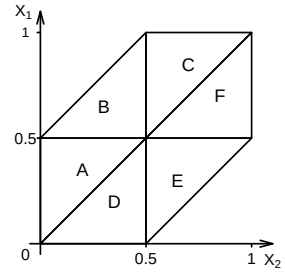
where areas A through F are shown in Figure 1. To find $f_{[1,1]}(y)$, first compute from (6) $\pi_{11} = \int\int_{\mathcal{X}} \theta_4 f(x_1, x_2) dx_1 dx_2$, where θ_4 must be determined separately for each area of Figure 1 (for example, in Area A, $\theta_4 = 1 - 2x_1^2 - 2x_2^2 + 4x_1x_2^2 - 4x_2^3/3$). The result is that $\pi_{11} = 1/3$. The numerator of (1) becomes

$$f(y, R = 1, S = 1) = \int_{\frac{y}{2}}^{\frac{y+1}{2}} \int_{\frac{y}{2}}^{\frac{y+1}{2}} 4\theta_4 dx_1 dx_2,$$

which, after some calculation, can be shown to be

$$f_{[1,1]}(y) = \frac{1}{20}(43 - 45y - 30y^2 + 50y^3 - 15y^4) \tag{9}$$

Figure 1: The Sampling Space of $f(x_1, x_2)$



for $0 \leq y \leq 1$. Noting that $\pi_{12} = 1/2 - \pi_{11} = 1/6$, a similar calculation yields

$$f_{[1,2]}(y) = \frac{1}{10}(7 + 15y - 30y^3 + 15y^4). \tag{10}$$

Suppose that (X_1, X_2, Y), with joint distribution (7), denote scores on two screening tests and a performance measure for an employee. The advantage in performance expected from an employee who performs best (in this case, the lowest value, as for speed tests) on one screening test can be measured by the selection differential, which Nagaraja (1982) defined as

$$\eta_{[1]} = \frac{\mu_{[1,\bullet]} - \mu_y}{\sigma_y}, \tag{11}$$

where $\mu_y = E(Y)$, $\sigma_y^2 = Var(Y)$ and $\mu_{[1,\bullet]} = E(Y_{[1,\bullet]})$. From (9) and (10),

$$f_{[1,\bullet]}(y) = \frac{\pi_{11} f_{[1,1]}(y) + \pi_{12} f_{[1,2]}(y)}{\pi_{11} + \pi_{12}} = \frac{1}{3}(5 - 3y - 3y^2 + 2y^3),$$

for $0 \leq y \leq 1$, yielding $\mu_{[1,\bullet]} = 23/60$ and

$$\eta_{[1]} = \left(\frac{23}{60} - \frac{1}{2}\right) / \sqrt{\frac{1}{12}} \approx -0.40.$$

Generalizing the selection differential (11) to the bivariate concomitant, we compute

$$\eta_{[1,1]} = \left(\frac{13}{40} - \frac{1}{2}\right) / \sqrt{\frac{1}{12}} \approx -0.60$$

from (9). Comparing these two shows that the additional screening test improves selectivity by about 50%.

It is sometimes straightforward to calculate moments of the concomitant directly from the density (1), as in Example 1. In other cases, calculation is easier using (2) and (3), an example of which will be given in Section 3.

2.2 Simplifying Properties

Computing densities of concomitants using Theorem 1 is tedious. However, symmetry in the distribution of (X_1, X_2, Y) can be exploited to reduce the number of calculations required to obtain densities and moments for the set of all concomitants. In this section, we present some useful results for that purpose.

First we make some observations about the rank distribution π_{rs}. For convenience, let $\bar{r} \equiv H + 1 - r$ and $\bar{s} \equiv H + 1 - s$.

Property 1: A monotonically increasing transformation on X_1 or X_2 does not change π_{rs}. A monotonically decreasing transformation on X_1 leads to $\pi'_{rs} = \pi_{\bar{r}s}$, and a monotonically decreasing transformation on X_2 leads to $\pi'_{rs} = \pi_{r\bar{s}}$, where π'_{rs} is the bivariate rank distribution based on the transformed variables.

Property 2: If the joint pdf $f(x_1, x_2)$ of X_1 and X_2 is symmetric, *i.e.*, $f(x_1, x_2) = f(x_2, x_1)$, then $\pi_{rs} = \pi_{sr}$.

Property 3: If $f(x_1, x_2) = f(-x_1, -x_2)$, then $\pi_{rs} = \pi_{\bar{r}\bar{s}}$.

Property 1 is obvious from observing that the rank of any observation is invariant to a monotonically increasing transformation. The other two are proved in David et al. (1977).

Example 2. Example 1 (con't.) Properties 1–3 can be used to calculate π_{21} and π_{22}. Observe from (8) that $f(x_1, x_2) = f(x_2, x_1)$; thus $\pi_{21} = \pi_{12} = 1/6$ from Property 2. Define $Z_i = X_i - 1/2$ for $i = 1, 2$. According to Property 1, (Z_1, Z_2) has the same rank distribution as (X_1, X_2). Since the joint density of Z_1 and Z_2 satisfies $g(z_1, z_2) = g(-z_1, -z_2)$, Property 3 yields $\pi_{22} = \pi_{11} = 1/3$.

Next, we observe some properties of the concomitant distribution that follow directly from Theorem 1.

Corollary 1. Suppose there exist monotonic functions $\psi_1(\cdot)$ and $\psi_2(\cdot)$ such that $Z_1 = \psi_1(X_1)$, $Z_2 = \psi_2(X_2)$, and the joint pdf of Z_1, Z_2, Y satisfies $g(z_1, z_2, y) = g(z_2, z_1, y)$. Then (1) if both $\psi_1 \uparrow$(increasing) and $\psi_2 \uparrow$ or both $\psi_1 \downarrow$ (decreasing) and $\psi_2 \downarrow$, then $f_{[r,s]}(y) = f_{[s,r]}(y)$; (2) if $\psi_1 \uparrow$ and $\psi_2 \downarrow$ or $\psi_1 \downarrow$ and $\psi_2 \uparrow$, then $f_{[r,s]}(y) = f_{[\bar{s},\bar{r}]}(y)$.

Corollary 2. Suppose there exist monotonic functions $\psi_1(\cdot)$ and $\psi_2(\cdot)$ such that $Z_1 = \psi_1(X_1)$, $Z_2 = \psi_2(X_2)$. Then (1) if the joint pdf of Z_1, Z_2, Y satisfies $g(z_1, z_2, y) = g(-z_1, -z_2, y)$, then $f_{[r,s]}(y) = f_{[\bar{r},\bar{s}]}(y)$; (2) if $g(z_1, z_2, \mu_y + d) = g(-z_1, -z_2, \mu_y - d)$, then $f_{[r,s]}(\mu_y + d) = f_{[\bar{r},\bar{s}]}(\mu_y - d)$.

Example 3. Example 1 (con't.) From (7), $f(x_1, x_2, y) = f(x_2, x_1, y)$. Thus Corollary 1 yields $f_{[2,1]}(y) = f_{[1,2]}(y)$. Since the joint density of Z_1, Z_2 and Y satisfies $g(z_1, z_2, 1/2+d) = g(-z_1, -z_2, 1/2-d)$, Corollary 2 yields $f_{[2,2]}(1/2+d) = f_{[1,1]}(1/2-d)$. Let $y = 1/2+d$, then $f_{[2,2]}(y) = f_{[1,1]}(1-y) = (3 + 15y + 30y^2 + 10y^3 - 15y^4)/20$.

In the following theorem, we establish properties of the mean $\mu_{[r,s]}$ and variance $\sigma^2_{[r,s]}$ of a concomitant, where the distribution of Y is not required to be symmetric.

Theorem 2. *Suppose there exist monotonic functions $\psi_1(\cdot)$ and $\psi_2(\cdot)$ such that (1) $Z_1 = \psi_1(X_1)$, $Z_2 = \psi_2(X_2)$ and their joint pdf is symmetric about 0, i.e., $g(z_1, z_2) = g(-z_1, -z_2)$; and (2) $E(Y|Z_1 = z_1, Z_2 = z_2)$ is a linear function of z_1 and z_2. Then the mean of the concomitant of bivariate order statistics of (X_1, X_2) satisfies*

$$\mu_{[r,s]} + \mu_{[\bar{r},\bar{s}]} = 2\mu_y \tag{12}$$

for $r \in \{1, \cdots H\}$ and $s \in \{1, \cdots H\}$. Furthermore, if $Var(Y|z_1, z_2) = Var(Y| - z_1, -z_2)$, then the variance of the concomitant satisfies

$$\sigma^2_{[r,s]} = \sigma^2_{[\bar{r},\bar{s}]}. \tag{13}$$

Proof. See Appendix A. □

Note that in Theorem 2, if H is odd, then $\mu_{[(H+1)/2,(H+1)/2]} = \mu_y$.

Example 4. Consider a type of regression setup: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \epsilon$ where ϵ is independent of X_1 and X_2 and follows a distribution with mean 0. Also, assume X_1 and X_2 can be linearly transformed so that their joint pdf after transformation is symmetric about 0. Then (12) and (13) hold.

More results about $\mu_{[r,s]}$ and $\sigma^2_{[r,s]}$ can be obtained easily from Corollary 1 and 2. For example, equations (12) and (13) follow directly from the second part of Corollary 2.

3 The Normal Case

Here we discuss the special case of the concomitant of the order statistics of a bivariate normal random vector. Let (X_1, X_2, Y) be trivariate normal with means μ_1, μ_2 and μ_y, variances σ_1^2, σ_2^2 and σ_y^2, and correlations ρ_{12}, ρ_{1y} and ρ_{2y}. Properties of the normal distribution allow the conditional mean and variance of Y given $\mathbf{X} = (x_1, x_2)$ to be written as

$$\begin{aligned} m(x_1, x_2) &= \mu_y + (\tau_1 z_1 + \tau_2 z_2)\sigma_y \\ v(x_1, x_2) &= (1 - \tau_1\rho_{1y} - \tau_2\rho_{2y})\sigma_y^2, \end{aligned}$$

where $\tau_1 = (\rho_{1y} - \rho_{2y}\rho_{12})/(1-\rho_{12}^2)$, $\tau_2 = (\rho_{2y} - \rho_{1y}\rho_{12})/(1-\rho_{12}^2)$, and $z_j = (x_j - \mu_j)/\sigma_j$, $j = 1, 2$. From (2),

$$\begin{aligned} \mu_{[r,s]} &= \mu_y + \sigma_y \left[\tau_1 E(Z_{1(r,s)}) + \tau_2 E(Z_{2(r,s)})\right] \\ &= \mu_y + \sigma_y \left[\tau_1 E(Z_{1(r,s)}) + \tau_2 E(Z_{1(s,r)})\right] \end{aligned} \tag{14}$$

where $(Z_1, Z_2)^T$ has the standard bivariate normal distribution with correlation ρ_{12}, and $(Z_{1(r,s)}, Z_{2(r,s)})$ are the bivariate order statistics of (Z_1, Z_2) with joint density $g_{(r,s)}(z_1, z_2)$. The second line follows since $E(Z_{2(r,s)}) = E(Z_{1(s,r)})$. From (3),

$$\sigma^2_{[r,s]} = \left[\tau_1^2 Var(Z_{1(r,s)}) + \tau_2^2 Var(Z_{1(s,r)}) + 2\tau_1\tau_2 Cov(Z_{1(r,s)}, Z_{2(r,s)}) + 1 - \tau_1\rho_{1y} - \tau_2\rho_{2y}\right]\sigma_y^2 \tag{15}$$

Noting that τ_1 and τ_2 are the standardized regression coefficients, (14) and (15) suggest that the concomitant and related order statistics retain the same linearity as in multiple regression.

To calculate $\mu_{[r,s]}$ and $\sigma^2_{[r,s]}$ using (14) and (15), values of the means, variances, and covariances of the bivariate standard normal order statistics are needed. Tables of these moments for $H = 2$, 3 and 4 are available in Wang and Stokes (2005). The method we used to obtain these tables is briefly outlined here. To calculate the mean, one must evaluate

$$E(Z_{1(r,s)}) = \int\int_{\mathcal{R}^2} z_1 g_{(r,s)}(z_1, z_2) dz_1 dz_2 = \frac{\int\int_{\mathcal{R}^2} z_1 p(r, s|z_1, z_2)\phi(z_1, z_2) dz_1 dz_2}{\pi_{rs}}$$

$$= \frac{\frac{\sqrt{2}}{\pi}\sum_{k=\mathcal{L}}^{\mathcal{U}} C_k \int\int_{\mathcal{R}^2} u_1 \theta_1^k \theta_2^{r-1-k} \theta_3^{s-1-k} \theta_4^{H-r-s+1+k} e^{-(u_1^2+u_2^2)} du_1 du_2}{\pi_{rs}}, \tag{16}$$

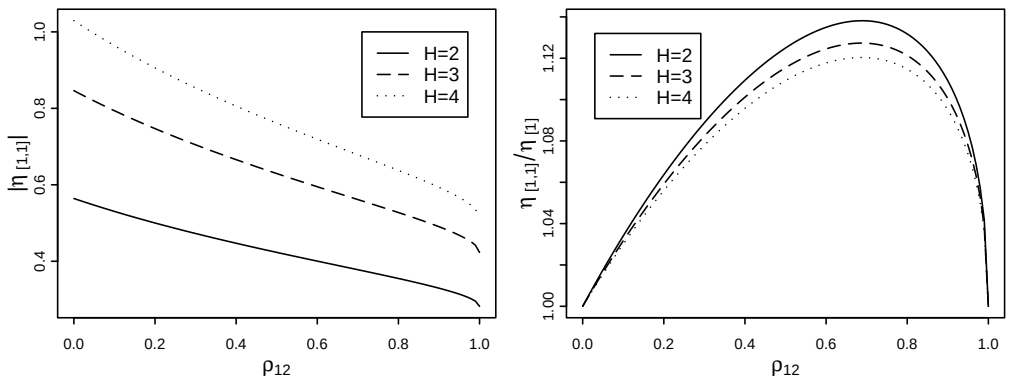
where $\phi(\cdot,\cdot)$ is the joint pdf of the standard bivariate normal distribution, and $\theta_i \equiv \theta_i(\sqrt{2}\mu_1, \sqrt{2}\rho_{12}\mu_1 + \sqrt{2(1-\rho_{12}^2)}\mu_2)$, for $i = 1, ..., 4$. The second line follows from (5) after the change of variables $z_1 = \sqrt{2}u_1$ and $z_2 = \sqrt{2}\rho_{12}u_1 + \sqrt{2(1-\rho_{12}^2)}u_2$. Similar expressions can be written for π_{rs}, $E(Z^2_{1(r,s)})$ and $E(Z_{1(r,s)}Z_{2(r,s)})$. These were all evaluated numerically using Gaussian quadrature. For example, the numerator of (16) was approximated by

$$\frac{\sqrt{2}}{\pi}\sum_{k=\mathcal{L}}^{\mathcal{U}} C_k \sum_{j=1}^{M}\sum_{i=1}^{M} \left\{\omega_i\omega_j t_i \theta_1^k \theta_2^{r-1-k} \theta_3^{s-1-k} \theta_4^{H-r-s+1+k}\right\}$$

where $\theta_l \equiv \theta_l(\sqrt{2}t_i, \sqrt{2}\rho_{12}t_i + \sqrt{2(1-\rho_{12}^2)}t_j)$ for $l = 1, ..., 4$, t_i is the $i-$th zero of the Hermite polynomial $H_M(t)$, and ω_i is the $i-$th weight factor. Tables of t_i and ω_i are available for $M = 1$ to 20 in Salzer et al. (1952).

Due to the symmetry of the normal density, the properties in Section 2.2 can be used to reduce the number of numerical evaluations needed to obtain $\mu_{[r,s]}$ and $\sigma^2_{[r,s]}$. Property 1 shows (by standardizing X_1 and X_2) that π_{rs} is related only to ρ_{12}, so can be calculated based on Z_1 and Z_2. From Properties 2 and 3, $\pi_{rs} = \pi_{sr} = \pi_{\bar{r}\bar{s}}$. Theorem 2 shows that $E(Z_{1(r,s)}) + E(Z_{1(\bar{r},\bar{s})}) = 0$, so one need only calculate elements in the upper triangular matrix of $\left[E(Z_{1(r,s)})\right]_{H\times H}$. The number of covariance (variance) calcula-

Figure 2: An example for the normal case; the left panel shows selection differential for pairs of moderately efficient ($\rho_{1y} = \rho_{2y} = 0.5$) screening tests; the right panel shows ratio of selection differential for one and two screeners, when one test is perfect.



tions needed is reduced by observing that $Cov(Z_{1(r,s)}, Z_{2(r,s)}) = Cov(Z_{1(s,r)}, Z_{2(s,r)})$ and $Cov(Z_{k(r,s)}, Z_{l(r,s)}) = Cov(Z_{k(\bar{r},\bar{s})}, Z_{l(\bar{r},\bar{s})})$ for any $k, l = 1, 2$, the latter of which is an intermediate result in the proof of Theorem 2.

Example 5. Suppose that (X_1, X_2, Y), whose joint distribution is trivariate normal, denote scores on two screening tests and a performance measure for an employee. The advantage in performance expected from an employee who performs best (in this case, the lowest value, as for speed tests) on both screeners among H =2, 3, or 4 competitors can be measured by the selection differential. It can be written, using (14), as

$$\eta_{[1,1]} = \frac{\mu_{[1,1]} - \mu_y}{\sigma_y} = \frac{\rho_{1y} + \rho_{2y}}{1 + \rho_{12}} E(Z_{1(1,1)}). \tag{17}$$

This expression shows that the selection differential increases in magnitude as screening tests grow more effective (larger values of ρ_{1y}, ρ_{2y}) and as the number of competitors increases (since $E(Z_{1(1,1)})$ is an increasing function of H). One would also expect less advantage as screening tests become more similar (ρ_{12} increases), but this is not clear from (17) since $\left|E(Z_{1(1,1)})\right|$ increases in ρ_{12}. The left panel of Figure 2 displays $\left|\eta_{[1,1]}\right|$ for $H = 2, 3$, and 4 as functions of ρ_{12} for two moderately effective screening tests ($\rho_{1y} = \rho_{2y} = .5$). It confirms that the second test is less useful for selection as it becomes more similar to the first.

One might expect that if the first screening test were perfect ($\rho_{1y} = 1$), then the second

one would provide no advantage in the selection process. This is incorrect. To see why, note first that $\rho_{2y} = \rho_{12}$ in this case. Then from (17), $\eta_{[1,1]} = E(Z_{1(1,1)})$, while $\eta_{[1]} = E(Z_{(1)})$, where $Z_{(1)}$ is the first order statistic of a standard normal random variable. The right panel of Figure 2 shows the ratio $\eta_{[1,1]}/\eta_{[1]}$ as a function of ρ_{12} for $H = 2, 3, 4$. The ratio is always larger than 1, but the advantage is greatest when the second screener has a correlation of around .70 with the performance measure. Its advantage diminishes to 0 as ρ_{12} $(=\rho_{2y})$ increases to 1. An intuitive explanation is that even perfect ranking information does not provide complete information about the mean. The second ranking variable, to the extent that its information differs from that of the first, can still improve estimation. Note also that the finer are the perfect ranker's post-strata (larger H), the less additional information remains for the second ranker to provide.

4 Extension to the Multivariate Case

In the previous sections, we have investigated the concomitant of bivariate order statistics. We now seek analytic expressions for the general case, the concomitant of multivariate order statistics where the number of X variables ≥ 2.

Let $(\mathbf{X}_h, Y_h)_{h=1}^H$ be an iid random sample from a multivariate distribution with a joint pdf $f(\mathbf{x}, y)$, where $\mathbf{X}_h^T$ is an absolutely continuous vector of length m. Denote the order of X_{hi} among $X_{1i}, \ldots, X_{Hi}$ by R_{hi}, and the rank vector associated with $\mathbf{X}_h$ by $\mathbf{R}_h^T = (R_{hi})_{i=1}^m$. Given a fixed H, we consider the concomitant of multivariate order statistics of $\mathbf{X}_h$, i.e., the random variable Y_h given $\mathbf{R}_h$. To obtain its density, (4) can be generalized as

$$f_{[\mathbf{r}]}(y) = \frac{\int_{\mathcal{X}} f(\mathbf{x}, y) p(\mathbf{r}|\mathbf{x}) d\mathbf{x}}{\pi_{\mathbf{r}}}, \tag{18}$$

where $\pi_{\mathbf{r}} = \int_{\mathcal{X}} p(\mathbf{r}|\mathbf{x}) f(\mathbf{x}) d\mathbf{x}$. Only $p(\mathbf{r}|\mathbf{x})$ is needed, which can be computed by recursion. To illustrate the idea, we describe the method for deriving $p(\mathbf{r}|\mathbf{x})$ for $m = 3$ from that for $m = 2$.

As in David et al. (1977), we represent the ways in which the compound event $R_{h1} = r_1$ and $R_{h2} = r_2$ given $X_{h1} = x_1$ and $X_{h2} = x_2$ can occur in the following 2×2 table,

	$X_{h'2} < x_2$	$X_{h'2} > x_2$	
$X_{h'1} < x_1$	k	$r_1 - 1 - k$	$r_1 - 1$
$X_{h'1} > x_1$	$r_2 - 1 - k$	$H - r_1 - r_2 + 1 + k$	$H - r_1$
	$r_2 - 1$	$H - r_2$	$H - 1$

Then $p(r_1, r_2 | \, x_1, x_2)$, can be obtained from the multinomial distribution with the four outcomes defined by the cells of the table. We split each of the four cells further by a third variable, as shown below.

	$X_{h'2} < x_2$		$X_{h'2} > x_2$		
	$X_{h'3} < x_3$	$X_{h'3} > x_3$	$X_{h'3} < x_3$	$X_{h'3} > x_3$	
$X_{h'1} < x_1$	l_0	$k - l_0$	l_1	$r_1 - 1 - k - l_1$	$r_1 - 1$
$X_{h'1} > x_1$	l_2	$r_2 - 1 -$ $k - l_2$	$r_3 - 1 -$ $(l_0 + l_1 + l_2)$	$H - r_1 - r_2 - r_3 +$ $2 + (k + l_0 + l_1 + l_2)$	$H - r_1$
	$r_2 - 1$		$H - r_2$		$H - 1$

After splitting, label the eight cells 1,...,8 and denote the number of observations in the $j-$th cell by t_j, $1 \leq j \leq 8$ (i.e., $t_1 = l_0$, $t_2 = k - l_0$, $t_3 = l_1$,...,$t_8 = H - r_1 - r_2 - r_3 + 2 + k + l_0 + l_1 + l_2$). Thus,

$$p(r_1, r_2, r_3 | \, x_1, x_2, x_3) = \sum_{k, l_0, l_1, l_2 \in \mathcal{A}} \left\{ C_{k, l_0, l_1, l_2} \prod_{j=1}^{8} [\theta_j(x_1, x_2, x_3)]^{t_j} \right\}$$

where $\mathcal{A}$ is an integer set $\{k, l_0, l_1, l_2 | \, k \geq 0; l_0 \geq 0; l_1 \geq 0; l_2 \geq 0; \, t_j \geq 0, \text{ for } 1 \leq j \leq 8\}$, $C_{k, l_0, l_1, l_2} \equiv (H-1)!/\{\prod_{j=1}^{8} t_j!\}$. Define $\theta_j(x_1, x_2, x_3)$ as the corresponding probability in the $j-$th cell; that is, $\theta_1(x_1, x_2, x_3) \equiv Pr(X_1 < x_1, X_2 < x_2, X_3 < x_3)$, $\theta_2(x_1, x_2, x_3) \equiv Pr(X_1 < x_1, X_2 < x_2, X_3 > x_3)$, etc.

Similarly, $p(\mathbf{r}|\mathbf{x}, x_{m+1})$ can be derived from $p(\mathbf{r}|\mathbf{x})$ by partitioning each of the 2^m cells into 2 subcells based on the value of x_{m+1} and then applying the multinomial distribution with 2^{m+1} possible outcomes. From (18), we can obtain an analytic expression for $f_{[\mathbf{r}]}(y)$

when the length of $\mathbf{x} > 2$ by recursion. With trivial modifications, the properties discussed in Section 2 can be generalized to $m > 2$.

Consider the normal case, where

$$(\mathbf{X}, Y)^T \sim N\left(\begin{pmatrix} \boldsymbol{\mu}_{\mathbf{x}} \\ \mu_y \end{pmatrix}, diag\left(\boldsymbol{\sigma}_{\mathbf{x}}, \sigma_y\right) \begin{pmatrix} \boldsymbol{\rho}_{\mathbf{x}} & \boldsymbol{\rho}_{\mathbf{x}y} \\ \boldsymbol{\rho}_{\mathbf{x}y}^T & 1 \end{pmatrix} diag\left(\boldsymbol{\sigma}_{\mathbf{x}}, \sigma_y\right)\right). \tag{19}$$

The mean and variance of the concomitants can be expressed as generalizations of (14) and (15), as

$$\mu_{[\mathbf{r}]} = \mu_y + \sigma_y \boldsymbol{\rho}_{\mathbf{x}y}^T \boldsymbol{\rho}_{\mathbf{x}}^{-1} \mathbf{E}\left(\mathbf{Z}_{(\mathbf{r})}\right) \tag{20}$$

$$\sigma_{[\mathbf{r}]}^2 = \sigma_y^2 + \sigma_y^2 \boldsymbol{\rho}_{\mathbf{x}y}^T \boldsymbol{\rho}_{\mathbf{x}}^{-1} \left[\mathbf{Cov}\left(\mathbf{Z}_{(\mathbf{r})}\right) \boldsymbol{\rho}_{\mathbf{x}}^{-1} - \mathbf{I}\right] \boldsymbol{\rho}_{\mathbf{x}y} \tag{21}$$

where $\mathbf{Z}_{(\mathbf{r})}$ is a vector of multivariate order statistics of $\mathbf{Z}$ that follows the standard multivariate normal distribution with the correlation matrix $\boldsymbol{\rho}_{\mathbf{x}}$. Calculation of the moments of the standard normal multivariate order statistic involves $p(\mathbf{r}|\mathbf{z})$, for example

$$E(Z_{1(\mathbf{r})}) = \frac{\int_{\mathcal{R}^m} z_1 p(\mathbf{r}|\mathbf{z}) \phi(\mathbf{z}) d\mathbf{z}}{\pi_{\mathbf{r}}},$$

which can be evaluated numerically using Gaussian quadrature, similar to the method used in the bivariate case.

5 Application to Judgement Post-stratification

Here we apply the theory of the previous sections to suggest new estimators of the mean from judgment post-stratified samples, and to examine their properties.

5.1 Background

MacEachern et al. (2004) introduced JP-S sampling as an alternative to ranked set sampling for estimating the mean of Y, which is expensive to quantify, but relatively cheap to rank by judgement. To obtain a JP-S sample, one first draws an SRS of n units from a population

and records the value of Y for each, denoted y_i, $1 \leq i \leq n$. For each measured unit i, an additional sample of size $H-1$ is chosen at random and the order of y_i among the H units is assessed by some inexpensive, and likely imperfect, ranking method not requiring measurement of the $H-1$ units. This rank information is used to classify the n measured units into H post-strata. MacEachern et al. (2004) proposed as an estimator of μ_y

$$\hat{\mu}_y = \frac{1}{H} \sum_{h=1}^{H} \frac{\sum_{i=1}^{n} y_i I_{ih}}{\sum_{i=1}^{n} I_{ih}} \tag{22}$$

where $I_{ih} = 1$ if y_i is assigned into the stratum of rank h, otherwise $I_{ih} = 0$.

Ranked set sampling differs from JP-S sampling in that in the former, judgment ranking of the group of H sample units occurs first, and then a specified rank is designated for measurement from the group. Ranking of groups of size H continues until some specified number of units of each judgement rank are quantified. The unweighted mean of such a sample can be shown to be unbiased for μ_y and to have smaller variance than an SRS of an equal number of measured observations (Dell and Clutter 1972). MacEachern et al. (2004) show that the asymptotic relative efficiency of $\hat{\mu}_y$ to this RSS estimator is 1.

Although RSS is described as using subjective judgement in ranking, applications have often used the rank of an easily observed auxiliary variable as a proxy for the rank of Y. But what if information about the rank of more than one auxiliary variable should be available? It is difficult to use this information in RSS, since one cannot guarantee that any particular vector of ranks will occur, so prespecifying sample sizes from strata defined by joint ranks is infeasible unless a multiple-layer design of RSS is used (Chen and Shen 2003). By contrast, JP-S uses rank information only for estimation, not for sample design. Our goal is to use such rank information along with the measured y_i to estimate μ_y. An example of such data is discussed in Chen (2002) for estimating mean age of a population of fish. Aging a fish is expensive; but the rank of its length and width, which are correlated with age, among a group of H fish can be easily obtained.

MacEachern et al. (2004) also cite the ability to use more than one ranker as an advantage

for JP-S over RSS. In the case that assessments of ranks are available from m rankers (or auxiliary variables) they propose as an estimator of μ_y

$$\hat{\mu}_M^{(m)} = \frac{1}{H} \sum_{h=1}^{H} \frac{\sum_{i=1}^{n} y_i \hat{p}_{ih}}{\sum_{i=1}^{n} \hat{p}_{ih}}, \tag{23}$$

where $\hat{p}_{ih}$ is the proportion of rankers who classify y_i as having rank h. That is, they prorate the measured value among the post-strata receiving any "votes" from a ranker. This estimator requires no distributional assumptions for its justification. When $m = 1$, $\hat{\mu}_M^{(m)}$ degenerates to (22).

5.2 New Estimators of Mean

In this section, we propose several new estimators of μ_y based on data from a JP-S sample, where post-strata are defined on ranks of m auxiliary variables. Our proposed estimators are designed to take advantage of knowledge of the distribution of the concomitant. Here, we restrict attention to the most tractable yet important case, the multivariate normal distribution. We first assume that σ_y, $\boldsymbol{\rho}_{\mathbf{x}y}$ and $\boldsymbol{\rho}_{\mathbf{x}}$ in (19) are known, and then examine the performance of the estimators in the practical case in which they are estimated. Methods for extension to the nonnormal case (with some mild distributional assumptions) are discussed in Section 6.

JP-S again begins with selection of a random sample of n units on which Y is measured. In addition, m related and easily ranked characteristics $\mathbf{X}$ are available on each unit. For each i, an additional $H-1$ units are randomly selected and the ranks of the components of $\mathbf{X}_i$ among its H comparison units are determined. The vector of ranks is denoted by $\mathbf{R}_i = (R_{i1}, \cdots, R_{im})$. There are thus H^m post-strata jointly grouped by the ranks $\mathbf{R} = (\mathbf{R}_i)_{i=1}^n$. Let $\mathrm{PS}_{\mathbf{r}}$ denote the post-stratum in which $\mathbf{R}_i = \mathbf{r}$, and $\pi_{\mathbf{r}}$, $n_{\mathbf{r}}$ and $\bar{Y}_{[\mathbf{r}]}$ denote the probability, number and sample mean of observations falling in $\mathrm{PS}_{\mathbf{r}}$. Let $\mu_{[\mathbf{r}]}$ and $\sigma^2_{[\mathbf{r}]}$ denote the mean and variance within the post-stratum; that is $\mu_{[\mathbf{r}]} = E(Y_i|\mathbf{R}_i = \mathbf{r})$, $\sigma^2_{[\mathbf{r}]} = Var(Y_i|\mathbf{R}_i = \mathbf{r})$. Define $\delta_{[\mathbf{r}]}$ as the difference between $\mu_{[\mathbf{r}]}$ and μ_y, i.e., $\delta_{[\mathbf{r}]} \equiv \mu_{[\mathbf{r}]} - \mu_y$. Finally, let $I_{i\mathbf{r}}$ be the

indicator variable such that $I_{i\mathbf{r}} = 1$ if $\mathbf{R}_i = \mathbf{r}$; otherwise $I_{i\mathbf{r}} = 0$.

We consider a class of linear JP-S estimators of the form

$$\hat{\mu}^{(m)} = \sum_{\mathbf{r}} w_{\mathbf{r}}(\mathbf{n}) \left(\bar{Y}_{[\mathbf{r}]} - c_{\mathbf{r}}\right), \tag{24}$$

where the summation is over all H^m realizations of the rank vector $\mathbf{R}_i$, $\mathbf{n}$ is the random vector containing the counts of Y in the H^m post-strata, $w_{\mathbf{r}}(\cdot)$ is a weight associated with $\mathrm{PS}_{\mathbf{r}}$ that can be a function of $\mathbf{n}$; and $c_{\mathbf{r}}$ is a constant associated with $\mathrm{PS}_{\mathbf{r}}$ that can be used for bias correction. This class, denoted by $\mathcal{E}$, contains familiar members, as well as useful new ones. The SRS estimator $\bar{Y}$ is in $\mathcal{E}$, with $w_{\mathbf{r}} = n_r/n$ and $c_{\mathbf{r}} = 0$. It obviously makes use of neither auxiliary nor distributional information. A version of the classical post-stratified estimator that does use distributional knowledge is $\hat{\mu}_S^{(m)} = \sum_{\mathbf{r}} \pi_{\mathbf{r}} \bar{Y}_{[\mathbf{r}]} \in \mathcal{E}$. The $\pi_{\mathbf{r}}$'s can be calculated for normal data. A nonparametric variant of $\hat{\mu}_S^{(m)}$that is also a member of the class is $\hat{\mu}_{vS}^{(m)} = \sum_{\mathbf{r}} \hat{\pi}_{\mathbf{r}}(\mathbf{n}) \bar{Y}_{[\mathbf{r}]}$ where $\hat{\pi}_{\mathbf{r}}(\cdot)$ is an estimate of the cell probability $\pi_{\mathbf{r}}$ based on $\mathbf{n}$. The cell probabilities can be estimated by the raking procedure (Deming and Stephan 1940), since the marginal probability for each auxiliary variable rank is known to be $1/H$ due to characteristics of order statistics. The estimator of MacEachern et al. (2004) is also a member of $\mathcal{E}$, since we can rewrite (23) as

$$\hat{\mu}_M^{(m)} = \sum_{\mathbf{r}} a_{\mathbf{r}}(\mathbf{n}) \bar{Y}_{[\mathbf{r}]}; \quad a_{\mathbf{r}}(\mathbf{n}) = \frac{1}{H} \sum_{i=1}^{H} \frac{b_{\mathbf{r}i} n_{\mathbf{r}}}{\sum_{\mathbf{r}'} b_{\mathbf{r}'i} n_{\mathbf{r}'}}$$

where $b_{\mathbf{r}i}$ is the count of rank i in the row vector $\mathbf{r}$. Now we examine several new estimators in this class suggested by commonly-used estimation methods. Each of them makes use of the distributional knowledge through the structure of the moments of the concomitant of multivariate order statistics.

We first consider the ordinary least squares estimator of μ_y, denoted $\hat{\mu}_{oLS}^{(m)}$, and defined as the estimator minimizing the sum of squared distances from each y_i to the mean of its post stratum; namely

$$\min_{\mu_y} \sum_{i=1}^{n} \sum_{\mathbf{r}} I_{i\mathbf{r}} \left[y_i - (\mu_y + \delta_{[\mathbf{r}]})\right]^2. \tag{25}$$

Under the normality assumption, we have from (20) that $\delta_{[\mathbf{r}]} = \sigma_y \boldsymbol{\rho}_{\mathbf{x}y}^T \boldsymbol{\rho}_{\mathbf{x}}^{-1} \mathbf{E}\left(\mathbf{Z}_{(\mathbf{r})}\right)$. Solving

(25) yields

$$\hat{\mu}_{oLS}^{(m)} = \sum_{\mathbf{r}} \frac{n_{\mathbf{r}}}{n} \left(\bar{Y}_{[\mathbf{r}]} - \delta_{[\mathbf{r}]}\right).$$

Since $E(\hat{\mu}_{oLS}^{(m)}|\mathbf{n}) = \mu_y$ and $Var(\hat{\mu}_{oLS}^{(m)}|\mathbf{n}) = \sum_{\mathbf{r}} n_{\mathbf{r}}\sigma_{[\mathbf{r}]}^2/n^2$, we have $E\left(\hat{\mu}_{oLS}^{(m)}\right) = \mu_y$ and

$$Var\left(\hat{\mu}_{oLS}^{(m)}\right) = Var\left[E\left(\hat{\mu}_{oLS}^{(m)}\middle|\,\mathbf{n}\right)\right] + E\left[Var\left(\hat{\mu}_{oLS}^{(m)}\middle|\,\mathbf{n}\right)\right] = \frac{1}{n}\sum_{\mathbf{r}} \pi_{\mathbf{r}}\sigma_{[\mathbf{r}]}^2. \tag{26}$$

Next consider the weighed least squares estimator, denoted $\hat{\mu}_{wLS}^{(m)}$, which minimizes the sum of the weighted squared distances to the post-strata means, namely

$$\min_{\mu_y} \sum_{i=1}^{n} \sum_{\mathbf{r}} I_{i\mathbf{r}} \left[\frac{y_i - (\mu_y + \delta_{[\mathbf{r}]})}{\sigma_{[\mathbf{r}]}}\right]^2. \tag{27}$$

Solving (27) yields

$$\hat{\mu}_{wLS}^{(m)} = \sum_{\mathbf{r}} \frac{n_{\mathbf{r}}/\sigma_{[\mathbf{r}]}^2}{\sum_{\mathbf{r}} n_{\mathbf{r}}/\sigma_{[\mathbf{r}]}^2} \left(\bar{Y}_{[\mathbf{r}]} - \delta_{[\mathbf{r}]}\right), \tag{28}$$

where in the normal case $\sigma_{[\mathbf{r}]}^2$ is given by (21). It is easy to show that $\hat{\mu}_{wLS}^{(m)}$ is unbiased, and

$$Var\left(\hat{\mu}_{wLS}^{(m)}\right) = E\left[\left(\sum_{\mathbf{r}} n_{\mathbf{r}}/\sigma_{[\mathbf{r}]}^2\right)^{-1}\right], \tag{29}$$

where the $n_{\mathbf{r}}$ is multinomial with parameters n and $\pi_{\mathbf{r}}$ for all H^m possible $\mathbf{r}$.

In addition, one might naturally think of the best linear unbiased estimator, whose weights are *constant* (i.e., not functions of random variables). In our JP-S setting, this estimator, denoted $\hat{\mu}_{BLU}^{(m)}$, minimizes the variance of a subclass of $\mathcal{E}$, the unbiased estimators of the form $\sum_{\mathbf{r}} w_{\mathbf{r}}(\bar{Y}_{[\mathbf{r}]} - \delta_{[\mathbf{r}]})$, where the weights $w_{\mathbf{r}}$'s are restricted to be *constant* and sum to 1. $\hat{\mu}_{BLU}^{(m)}$ has the form

$$\hat{\mu}_{BLU}^{(m)} = \sum_{\mathbf{r}} \frac{\frac{1}{\sigma_{[\mathbf{r}]}^2 E(1/n_{\mathbf{r}})}}{\sum_{\mathbf{r}} \frac{1}{\sigma_{[\mathbf{r}]}^2 E(1/n_{\mathbf{r}})}} \left(\bar{Y}_{[\mathbf{r}]} - \delta_{[\mathbf{r}]}\right)$$

with

$$Var\left(\hat{\mu}_{BLU}^{(m)}\right) = \left[\sum_{\mathbf{r}} \frac{1}{\sigma_{[\mathbf{r}]}^2 E(1/n_{\mathbf{r}})}\right]^{-1}.$$

Now we proceed to compare the three unbiased estimators, $\hat{\mu}_{oLS}^{(m)}$, $\hat{\mu}_{wLS}^{(m)}$ and $\hat{\mu}_{BLU}^{(m)}$ that are all in $\mathcal{E}$. First, $\hat{\mu}_{wLS}^{(m)}$ has the smallest variance among the three, as will be seen in Theorem 3.

Second, $\hat{\mu}_{oLS}^{(m)}$ is easier to compute, especially when the number of post-strata H^m is large, since it does not require the variances $\sigma^2_{[\mathbf{r}]}$. Last, $\hat{\mu}_{wLS}^{(m)}$ and $\hat{\mu}_{BLU}^{(m)}$ have similar expressions as $\hat{\mu}_{BLU}^{(m)}$ can be obtained by replacing $1/n_{\mathbf{r}}$ by $E(1/n_{\mathbf{r}})$ in (28); they also have the same asymptotic variance $(n\sum_{\mathbf{r}}\pi_{\mathbf{r}}/\sigma^2_{[\mathbf{r}]})^{-1}$. However, $\hat{\mu}_{BLU}^{(m)}$ is not quite satisfactory. It is not applicable when the sample size n is small compared to the number of strata H^m. In this case, there are many empty cells with inestimable means and this would cause trouble since $\hat{\mu}_{BLU}^{(m)}$ assigns a prespecified nonzero weight to each cell. In contrast, $\hat{\mu}_{oLS}^{(m)}$ and $\hat{\mu}_{wLS}^{(m)}$ are both data-adaptive by assigning nonzero weights to nonempty cells only. Even if no empty cell occurs, the performance of $\hat{\mu}_{BLU}^{(m)}$ is very sensitive to n and is much worse than that of $\hat{\mu}_{oLS}^{(m)}$ and $\hat{\mu}_{wLS}^{(m)}$, as will be shown in our simulation.

The following theorem establishes an optimal property for $\hat{\mu}_{wLS}^{(m)}$.

Theorem 3. *$\hat{\mu}_{wLS}^{(m)}$ has the least mean square error (MSE) among the class of estimators of the form (24).*

Proof. It is obvious that $\hat{\mu}_{wLS}^{(m)}$'s weights $w^*_{\mathbf{r}}(\mathbf{n}) = (n_{\mathbf{r}}/\sigma^2_{[\mathbf{r}]})/\sum_{\mathbf{r}} n_{\mathbf{r}}/\sigma^2_{[\mathbf{r}]}$ minimize $Var(\hat{\mu}^{(m)}|\mathbf{n}) = \sum_{\mathbf{r}} w^2_{\mathbf{r}}(\mathbf{n})\sigma^2_{[\mathbf{r}]}/n_{\mathbf{r}}$. Since $E(\hat{\mu}_{wLS}^{(m)}|\mathbf{n}) = \mu_y$, we have $Var[E(\hat{\mu}_{wLS}^{(m)}|\mathbf{n})] = 0$. Thus $Var(\hat{\mu}^{(m)}) \geq E[Var(\hat{\mu}^{(m)}|\mathbf{n})] \geq E[Var(\hat{\mu}_{wLS}^{(m)}|\mathbf{n})] = Var(\hat{\mu}_{wLS}^{(m)})$ and $MSE(\hat{\mu}^{(m)}) \geq MSE(\hat{\mu}_{wLS}^{(m)})$, since $\hat{\mu}_{wLS}^{(m)}$ is unbiased. □

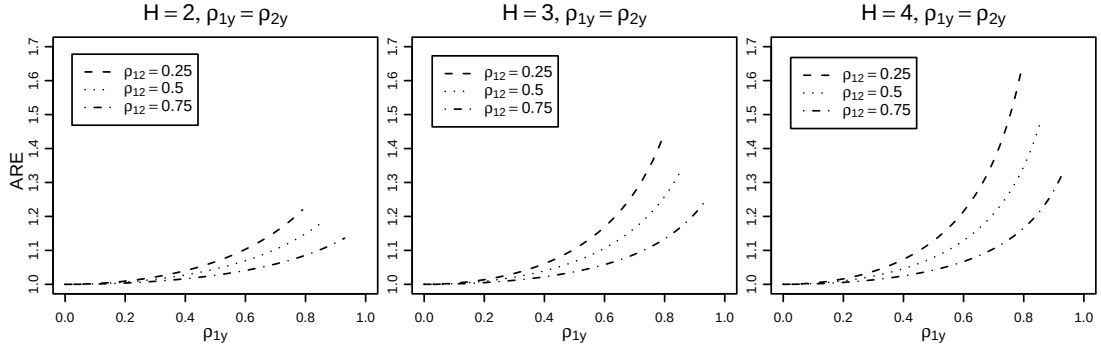
This theorem assures us that $\hat{\mu}_{wLS}^{(m)}$ is the most efficient among the estimators discussed, not limited to $\hat{\mu}_{oLS}^{(m)}$, $\hat{\mu}_{wLS}^{(m)}$ and $\hat{\mu}_{BLU}^{(m)}$. However, its variance (29) is not expressed in a closed form so is hard to compute. In the following corollary, we provide an upper bound by comparing it with $\hat{\mu}_{oLS}^{(m)}$ and also a lower bound by considering its asymptotic variance.

Corollary 3. Lower and upper bound for the variance of $\hat{\mu}_{wLS}^{(m)}$ are, i.e.,

$$\frac{1}{n}\left(\sum_{\mathbf{r}}\pi_{\mathbf{r}}/\sigma^2_{[\mathbf{r}]}\right)^{-1} \leq Var\left(\hat{\mu}_{wLS}^{(m)}\right) \leq \frac{1}{n}\sum_{\mathbf{r}}\pi_{\mathbf{r}}\sigma^2_{[\mathbf{r}]}. \tag{30}$$

The lower bound (i.e., the asymptotic variance) provides a good approximation to the variance of $\hat{\mu}_{wLS}^{(m)}$ when n is reasonably large. It also works well for small n if the difference

Figure 3: Asymptotic efficiency of $\hat{\mu}_{wLS}^{(2)}$ over $\hat{\mu}_{wLS}^{(1)}$ for pairs of equally effective rankers



between the upper and lower bound is small, which occurs for normal data in many cases.

Finally, the following theorem justifies our intuition that for both $\hat{\mu}_{wLS}^{(m)}$ and $\hat{\mu}_{oLS}^{(m)}$, adding an extra ranking variable improves estimation efficiency, at least in an asymptotic sense.

Theorem 4. *Suppose $\hat{\mu}_{wLS}^{(m+1)}$($\hat{\mu}_{oLS}^{(m+1)}$) is the weighted (ordinary) least squares estimator with ranking variables $(\mathbf{X}, X_{m+1})$; $\hat{\mu}_{wLS}^{(m)}$ ($\hat{\mu}_{oLS}^{(m)}$) is the corresponding estimator using the first m ranking variables $\mathbf{X}$ only. Then $\hat{\mu}_{wLS}^{(m+1)}$ is more efficient than $\hat{\mu}_{wLS}^{(m)}$ for large n; and $\hat{\mu}_{oLS}^{(m+1)}$ is more efficient than $\hat{\mu}_{oLS}^{(m)}$ for any n (see Appendix B for the proof).*

Though the theorem establishes that the addition of ranking variables is helpful, a practical question is just how much gain can be expected. We investigated this for the special case of adding a second auxiliary variable to the first. The asymptotic relative efficiency $ARE = \lim_{n\to+\infty}\left[Var(\hat{\mu}_{wLS}^{(1)})/Var(\hat{\mu}_{wLS}^{(2)})\right]$ was computed from the lower bound in (30), (15) and Eqn (6.8.3b) of David and Nagaraja (2003, Chapter 6.8) for $\rho_{12} = 0.25, 0.5, 0.75$ and $H = 2, 3, 4$, and a range of values of ρ_{1y} and ρ_{2y}. Figure 3 shows the results for two equally effective rankers (i.e, $\rho_{1y} = \rho_{2y}$) and the three values each of ρ_{12} and H. We see that the gain from the second ranker can be substantial; it increases as either the ranking quality or the number of ranking classes increases, and decreases as the two rankers become more similar. We also computed the RE for $\hat{\mu}_{oLS}^{(2)}$ over $\hat{\mu}_{oLS}^{(1)}$, in which we observed the tightness of the two bounds in (30). As a result, the values of the RE were virtually identical to those of the ARE for the weighted one, so are not shown in Figure 3.

5.3 Simulation

We have demonstrated that under the normality assumption with known σ_y, $\boldsymbol{\rho}_{\mathbf{x}y}$ and $\boldsymbol{\rho}_{\mathbf{x}}$, the weighted least squares estimator $\hat{\mu}_{wLS}^{(m)}$ is the most efficient among the class $\mathcal{E}$ including members $\bar{Y}$, $\hat{\mu}_M^{(m)}$, $\hat{\mu}_S^{(m)}$, $\hat{\mu}_{vS}^{(m)}$, $\hat{\mu}_{wLS}^{(m)}$, $\hat{\mu}_{oLS}^{(m)}$ and $\hat{\mu}_{BLU}^{(m)}$. In practice, however, these parameters will not be known, and the distributional assumptions may not hold exactly. Thus we designed a simulation study for two purposes: (1) to compare the performance of the estimators when σ_y, $\boldsymbol{\rho}_{\mathbf{x}y}$ and $\boldsymbol{\rho}_{\mathbf{x}}$ are unknown and must be estimated from the data; (2) to test their robustness when the normality assumption is violated. In our preliminary simulations, we found that the MacEachern et al. (2004) estimator performed consistently best among the three "sampling" estimators ($\hat{\mu}_M^{(m)}$, $\hat{\mu}_S^{(m)}$, $\hat{\mu}_{vS}^{(m)}$). Hence, we included only $\hat{\mu}_M^{(m)}$, $\hat{\mu}_{wLS}^{(m)}$, $\hat{\mu}_{oLS}^{(m)}$ and $\hat{\mu}_{BLU}^{(m)}$ in the full study presented here, along with $\bar{Y}$ as a benchmark estimator.

In our first experiment, we simulated JP-S samples from the standard multivariate normal distribution for Y and two auxiliary variables (X_1, X_2) for four sets of $(\rho_{1y}, \rho_{2y}, \rho_{12})$: $(0.9, 0.9, 0.65)$, $(0.8, 0.8, 0.5)$, $(0.8, 0.5, 0.5)$ and $(0.5, 0.5, 0.5)$. We set H to be 2, 4 or 10 and n to be 10, 20, 50 or 100. Since m is fixed at 2, we omit the superscripts in the discussion below. When calculating $\hat{\mu}_{wLS}$, $\hat{\mu}_{oLS}$ and $\hat{\mu}_{BLU}$ from each sample, we substituted estimates for σ_y and the ρ's, computed using standard methods. Table 1 reports the simulated relative efficiency of the four JP-S estimators to the SRS estimator $\bar{Y}$ for each setting. Here, efficiency is defined as the ratio of the variance of $\bar{Y}$ to MSE of each JP-S estimator, where MSE is estimated from 20,000 replicates.

The results in Table 1 show that the two least squares estimators outperform the other two in all cases, even though they use estimates of σ_y and the ρ's. The performance of $\hat{\mu}_{wLS}$ is at most only slightly better than that of $\hat{\mu}_{oLS}$. Both estimators perform well even for small n. The improvement from the two parametric estimators over the nonparametric one $\hat{\mu}_M$ is considerable, especially when n is small and H large, as long as the ranking variables are effective. By contrast, $\hat{\mu}_{BLU}$ performs poorly overall. Its performance is sensitive to sample size and is not applicable when empty cells occur. Hence, we do not consider $\hat{\mu}_{BLU}$ further.

Table 1: Comparing efficiency of the JP-S Estimators with estimated parameters. Note that due to the empty-cell problems, $\hat{\mu}_{BLU}$ is not applicable when n is small compared to H^2.

	Mean	H=2 Sample Size				H=4 Sample Size				H=10 Sample Size			
$(\rho_{1y}, \rho_{2y}, \rho_{12})$	Est.	10	20	50	100	10	20	50	100	10	20	50	100
	$\hat{\mu}_{wLS}$	1.55	1.55	1.56	1.56	2.51	2.55	2.63	2.67	4.82	5.32	5.53	5.57
	$\hat{\mu}_{oLS}$	1.54	1.54	1.55	1.56	2.48	2.51	2.57	2.61	4.62	5.04	5.28	5.24
(0.9,0.9,0.65)	$\hat{\mu}_M$	1.38	1.44	1.47	1.48	1.80	2.11	2.29	2.35	1.82	2.74	3.94	4.21
	$\hat{\mu}_{BLU}$	1.28	1.35	1.44	1.51	–	–	2.09	2.24	–	–	–	–
	$\hat{\mu}_{wLS}$	1.42	1.41	1.43	1.44	2.01	2.06	2.09	2.14	2.90	3.22	3.35	3.39
	$\hat{\mu}_{oLS}$	1.41	1.41	1.43	1.44	2.00	2.04	2.07	2.12	2.87	3.17	3.29	3.33
(0.8,0.8,0.5)	$\hat{\mu}_M$	1.29	1.32	1.35	1.36	1.56	1.74	1.84	1.90	1.55	2.00	2.56	2.70
	$\hat{\mu}_{BLU}$	1.15	1.19	1.34	1.39	–	–	1.57	1.80	–	–	–	–
	$\hat{\mu}_{wLS}$	1.24	1.28	1.29	1.29	1.50	1.57	1.63	1.63	1.79	1.93	2.07	2.08
	$\hat{\mu}_{oLS}$	1.24	1.28	1.29	1.29	1.50	1.56	1.62	1.62	1.78	1.92	2.06	2.07
(0.8,0.5,0.5)	$\hat{\mu}_M$	1.15	1.20	1.22	1.22	1.23	1.34	1.42	1.43	1.19	1.34	1.61	1.67
	$\hat{\mu}_{BLU}$	1.03	1.08	1.21	1.25	–	–	1.29	1.35	–	–	–	–
	$\hat{\mu}_{wLS}$	1.10	1.10	1.13	1.16	1.12	1.21	1.24	1.24	1.13	1.28	1.34	1.37
	$\hat{\mu}_{oLS}$	1.10	1.10	1.13	1.16	1.12	1.21	1.24	1.24	1.14	1.28	1.34	1.37
(0.5,0.5,0.5)	$\hat{\mu}_M$	1.08	1.08	1.11	1.14	1.08	1.15	1.20	1.21	1.03	1.12	1.25	1.30
	$\hat{\mu}_{BLU}$	0.90	0.94	1.07	1.13	–	–	0.94	1.04	–	–	–	–

Table 2: Theoretical values of (asymptotic) relative efficiency of $\hat{\mu}_{wLS}$ and $\hat{\mu}_{oLS}$

Theoretical	(0.9,0.9,0.65)			(0.8,0.8,0.5)			(0.8,0.5,0.5)			(0.5,0.5,0.5)		
Value	H=2	H=4	H=10	H=2	H=4	H=10	H=2	H=4	H=10	H=2	H=4	H=10
$ARE(\hat{\mu}_{wLS}, \bar{Y})$	1.56	2.63	5.56	1.44	2.14	3.41	1.29	1.65	2.13	1.14	1.26	1.38
$RE(\hat{\mu}_{oLS}, \bar{Y})$	1.55	2.58	5.27	1.44	2.12	3.35	1.29	1.64	2.12	1.14	1.26	1.38

To examine the effect of estimation of the unknown correlations and variance more closely, we computed asymptotic efficiency for $\hat{\mu}_{wLS}$ over $\bar{Y}$ using the lower bound in (30) and efficiency for $\hat{\mu}_{oLS}$ over $\bar{Y}$ using (26). Their theoretical values are reported in Table 2. Comparing the simulated values in Table 1 to these, we observe that estimating these parameters has a negligible effect when $n \geq 50$. For smaller sample sizes, both $\hat{\mu}_{wLS}$ and $\hat{\mu}_{oLS}$ lose some efficiency by doing so, though they still perform better than $\hat{\mu}_M$.

In the second experiment, we examine the performance of the JP-S estimators when the normality assumption is violated. We considered three cases: (1) $(\log X_1, \log X_2, Y)$ are generated from the standard normal distributions with the four sets of correlations as before; (2) $(\log X_1, \log X_2, \log Y)$ are generated from the same distributions as in (1); (3) (X_1, X_2, Y) follows the multivariate uniform distribution described in Example 1. Here, we set $H = 4$ and generated 20,000 JP-S samples for each setting, and calculated $\hat{\mu}_{wLS}$, $\hat{\mu}_{oLS}$ and $\hat{\mu}_M$ from each. The former two estimators were computed as if (X_1, X_2, Y) were multivariate normal,

Table 3: Comparing efficiency of the JP-S Estimators with estimated parameters ($H = 4$ only) when the normality assumption is violated

Mean Est.	$(\rho'_{1y}, \rho'_{2y}, \rho'_{12})$	$(\log X_1, \log X_2, Y) \sim MVN$ Sample Size 10	20	50	100	$(\log X_1, \log X_2, \log Y) \sim MVN$ Sample Size 10	20	50	100	Multivariate Uniform Sample Size 10	20	50	100
$\hat{\mu}_{wLS}$	(0.9,0.9,0.65)	2.30	2.49	2.50	2.51	1.73	1.61	1.55	1.45	1.49	1.57	1.62	1.64
$\hat{\mu}_{oLS}$		2.30	2.45	2.45	2.47	1.47	1.32	1.33	1.33	1.53	1.62	1.67	1.68
$\hat{\mu}_M$		1.81	2.11	2.26	2.30	1.33	1.36	1.49	1.44	1.34	1.49	1.57	1.59
$\hat{\mu}_{wLS}$	(0.8,0.8,0.5)	1.91	2.02	2.06	2.11	1.52	1.46	1.43	1.42				
$\hat{\mu}_{oLS}$		1.91	2.01	2.05	2.09	1.33	1.25	1.24	1.26				
$\hat{\mu}_M$		1.56	1.76	1.85	1.92	1.25	1.31	1.33	1.33				
$\hat{\mu}_{wLS}$	(0.8,0.5,0.5)	1.45	1.54	1.60	1.59	1.26	1.30	1.28	1.32				
$\hat{\mu}_{oLS}$		1.45	1.54	1.60	1.58	1.14	1.16	1.16	1.21				
$\hat{\mu}_M$		1.24	1.35	1.44	1.42	1.12	1.16	1.17	1.23				
$\hat{\mu}_{wLS}$	(0.5,0.5,0.5)	1.07	1.19	1.23	1.23	1.06	1.07	1.14	1.12				
$\hat{\mu}_{oLS}$		1.08	1.19	1.23	1.23	1.01	1.01	1.11	1.09				
$\hat{\mu}_M$		1.04	1.16	1.20	1.20	1.00	1.07	1.11	1.11				

but using estimated values for σ_y and ρ's. Table 3 reports the simulated efficiency.

Several observations can be made from Table 3. When the ranking variables violate the normality assumption but Y is still normal, $\hat{\mu}_{wLS}$ and $\hat{\mu}_{oLS}$ perform comparably and have efficiencies similar to those in the normal case. When both $\mathbf{X}$ and Y are log-normal, $\hat{\mu}_{wLS}$ outperforms $\hat{\mu}_{oLS}$, while the situation is reversed when (X_1, X_2, Y) follow the multivariate uniform distribution. As expected, the least squares estimators are less efficient than in the normal case when Y is no longer normally distributed. Surprisingly, $\hat{\mu}_M$ does not perform as well as $\hat{\mu}_{wLS}$ in any of the cases considered, nor as well as $\hat{\mu}_{oLS}$ except when Y is heavily tailed. This leads to our belief that even with moderate deviation from normality, $\hat{\mu}_{wLS}$ and $\hat{\mu}_{oLS}$ may still achieve better performance than $\hat{\mu}_M$, especially for small n.

5.4 An Empirical Study: Human Teeth Width

This section uses a real dataset to compare the JP-S estimators of the mean. To examine their performance in both infinite and finite population settings, samples were selected with and without replacement from a small population containing measurements on teeth widths for 295 subjects in a health survey conducted in Seoul, Korea (Lee et al. 2006). All teeth were measured by digital Vernier calipers, a process that requires three-week training to master. Here, our goal is to estimate the mean width of teeth in the back of the mouth,

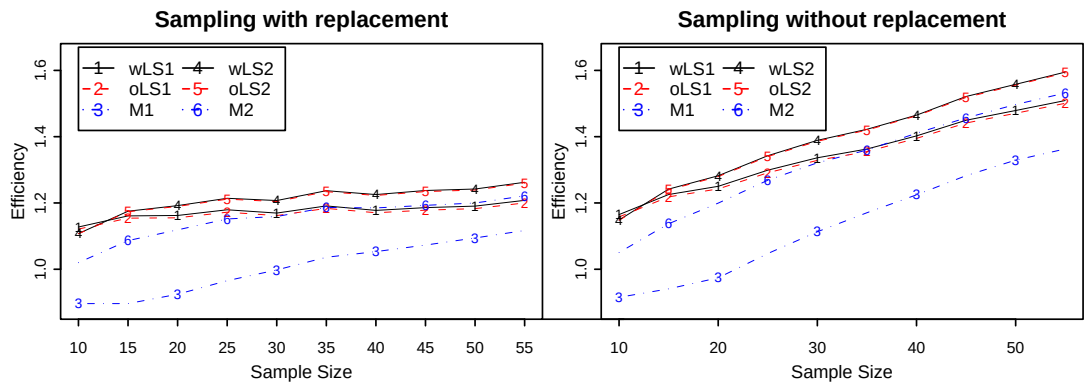
using the middle ones as ranking variables. A practical justification for doing so is that a tooth close to the center is much easier to order than a tooth farther back.

The widths of the first upper incisors U_1 (the first tooth from the center) and the first upper canines U_3 (the third tooth from the center) were used as ranking variables. Selection of JP-S samples was simulated and estimates of the mean width of the first lower molars L_6 (the sixth tooth from the center) were calculated. The 295 subjects were treated as the "true" population, and parameter estimates computed from their data were taken as the true population parameters. These included $\rho_{16} = 0.503$, $\rho_{36} = 0.540$, $\rho_{13} = 0.576$, $\mu_{L6} = 10.95$ and $\sigma^2_{L6} = 0.319$. Standard diagnostic checking, performed on (U_1, U_3, L_6) through the macro %MULTNORM in SAS, did not reveal any gross violation of normality.

In this simulation, we set $H = 5$ and sample sizes $n = 10, 15 \cdots, 55$. To obtain a JP-S sample of size n with replacement, we repeated the following procedure n times: a set of five subjects were randomly selected from all 295 subjects and bivariate ranking was done based on U_1 and U_3 within the set; then one of the five subjects was randomly selected to enter the sample. By contrast, to obtain a JP-S sample of size n without replacement, the set of five subjects selected on each draw were excluded from the data set so they were not available for the next selection. For each JP-S sample of size n, we calculated $\hat{\mu}^{(2)}_{wLS}$, $\hat{\mu}^{(2)}_{oLS}$ and $\hat{\mu}^{(2)}_{M}$ with post-strata determined by ranks of U_1 and U_3, and $\hat{\mu}^{(1)}_{wLS}$, $\hat{\mu}^{(1)}_{oLS}$ and $\hat{\mu}^{(1)}_{M}$ with post-strata determined by U_3 only. All least squares estimators were computed using σ_y and ρ's estimated from the sample.

Figure 4 shows values of the simulated relative efficiency of the six JP-S estimators to $\bar{Y}$ for each sample size n. Here, MSE is estimated from 100,000 replicates. The figure shows that no matter whether sampling is with or without replacement, the least squares estimators have almost identical performance and with the two ranking variables they have the best performance among all. In addition, the results show that the benefit from using a second ranking variable when sampling without replacement is larger than from with replacement samples. So it may be safe to use the least squares estimators for small populations.

Figure 4: An empirical study – Human Teeth Width



6 Discussion

We have defined concomitants of multivariate order statistics and provided analytical expressions for their densities and moments. We have also illustrated use of the theory by providing new estimators that use ranking information from more than one auxiliary variable for improving estimation of the mean.

We note that the proposed least squares estimators do not require normality. They are available when certain distributional assumptions about the data can be made: development of $\hat{\mu}_{oLS}$ requires that $\delta_{[\mathbf{r}]}$ is not a function of μ_y (say, $\delta_{[\mathbf{r}]} \perp \mu_y$) for each post-stratum; and $\hat{\mu}_{wLS}$ requires that both $\delta_{[\mathbf{r}]} \perp \mu_y$ and $\sigma^2_{[\mathbf{r}]} \perp \mu_y$. Suppose for each ranking variable X_i ($1 \leq i \leq m$), there exists a monotonic function $g_i(\cdot)$ such that $Z_i = g_i(X_i)$ and $\mathbf{Z} = (Z_1, \cdots, Z_m)$ has a joint distribution $f(\mathbf{z}; \Theta)$ with the parameter set $\Theta \perp \mu_y$ (e.g., a special case is that each X_i is from a location-scale distribution family). Let $m(\mathbf{z}; \Theta_m) \equiv E(Y - \mu_y | \mathbf{Z} = \mathbf{z})$ that is a function of $\mathbf{z}$ and a set of distributional parameters Θ_m; let $v(\mathbf{z}; \Theta_v) \equiv Var(Y | \mathbf{Z} = \mathbf{z})$ that is a function of $\mathbf{z}$ and a set of distributional parameters Θ_v. Then a sufficient condition for $\delta_{[\mathbf{r}]} \perp \mu_y$ is $\Theta_m \perp \mu_y$ and a sufficient condition for $\sigma^2_{[\mathbf{r}]} \perp \mu_y$ is $\Theta_v \perp \mu_y$, which follow directly from Theorem 1 and its higher-dimensional generalization. These sufficient conditions may be milder than those assumptions in most regression setups as they do not require linearity or any other functional form for the conditional expectations.

In fact, the result in Theorem 3 (i.e., the optimality of $\hat{\mu}_{wLS}$ among linear estimators)

is rather general. Under the assumptions discussed above, it can be directly extended to situations where the sampling space can be partitioned to strata, whether through sampling design or post-stratification. However, obtaining the bias correction term $\delta_{[\mathbf{r}]}$ and the variance $\sigma^2_{[\mathbf{r}]}$ for each stratum is nontrivial. In our JP-S applications, these can be derived through the theoretical developments in Sections 2–4, which greatly facilitate our computations of the most efficient linear estimator.

There are other examples in the concomitant literature in which a single ranked variable is used to improve estimation of some parameter. The methods we have developed here could be used in those applications as well. For example, Barnett et al. (1976) for obtaining linear estimates of correlation coefficients can be directly adapted when information is available from two or more ranking variables, using the moment expressions (20) and (21).

Other useful applications will require additional theoretical development. For example, the properties of concomitants of extreme order statistics have been a topic of study for its use in ranking and selection (Yeo and David 1984; Arnold and Beyer 2005). The notion of "extreme" order statistics of a vector of ranking variables can be defined in a variety of ways, with the best way undoubtedly depending on its purpose.

Finally, we note that we have assumed that the number of ranking classes is the same for all ranking variables. There are applications in which a generalization to the case in which one ranking variable allows H classes while another allows H' may be needed. For example, consider the employee selection problem in which not every candidate had the complete battery of screening tests. In that case, it would be useful to have a way to examine properties of the concomitant of multivariate order statistics, some of which are partially ranked.

Appendix A: Proof of Theorem 2

For notational clarity, let $Y_{[r,s];z}$ explicitly denote the concomitant of the rth order statistics of Z_1 and the sth order statistics of Z_2 with mean $\mu_{[r,s];z}$ and variance $\sigma^2_{[r,s];z}$; let q_{rs} denote the bivariate rank distribution of Z_1 and Z_2; and let $(Z_{1(r,s)}, Z_{2(r,s)})$ be the bivariate order statistics of (Z_1, Z_2) with the joint density $g_{(r,s)}(z_1, z_2)$.

Since ψ_1 and ψ_2 are monotonic, to show (12) and (13), it is equivalent to show for $r \in \{1, \cdots H\}$ and $s \in \{1, \cdots H\}$,

$$\mu_{[r,s];z} + \mu_{[\bar{r},\bar{s}];z} = 2\mu_y \tag{A.1}$$

$$\sigma^2_{[r,s];z} = \sigma^2_{[\bar{r},\bar{s}];z}. \tag{A.2}$$

Under the conditions that (1) $E(Y|z_1, z_2)$ is a linear function of z_1 and z_2, and (2) $g(z_1, z_2) = g(-z_1, -z_2)$, we can obtain

$$\mu_{[r,s];z} = \mu_y + \beta_1 E(Z_{1(r,s)}) + \beta_2 E(Z_{2(r,s)}); \tag{A.3}$$

$$\begin{aligned} \sigma^2_{[r,s];z} &= \int\int_{\mathcal{Z}} Var(Y|z_1, z_2) g_{(r,s)}(z_1, z_2) dz_1 dz_2 + \\ &\quad \beta_1^2 Var(Z_{1(r,s)}) + 2\beta_1\beta_2 Cov(Z_{1(r,s)}, Z_{2(r,s)}) + \beta_2^2 Var(Z_{2(r,s)}) \end{aligned} \tag{A.4}$$

where β_1 and β_2 are constants, and $\mathcal{Z}$ is the support of the distribution of (Z_1, Z_2).

Now consider $E(Z_{1(r,s)})$, which can be expressed by

$$E(Z_{1(r,s)}) = \int\int_{\mathcal{Z}} z_1 g_{(r,s)}(z_1, z_2) dz_1 dz_2.$$

Define $z_1^* = -z_1$ and $z_2^* = -z_2$. Then

$$E(Z_{1(r,s)}) = -\int\int_{\mathcal{Z}} z_1^* g_{(r,s)}(z_1, z_2) dz_1^* dz_2^*. \tag{A.5}$$

Since $g(z_1, z_2) = g(z_1^*, z_2^*)$, $q_{rs} = q_{\bar{r}\bar{s}}$ so that

$$g_{(r,s)}(z_1, z_2) = \frac{q(r, s|z_1, z_2) g(z_1^*, z_2^*)}{q_{\bar{r}\bar{s}}} . z_2^*. \tag{A.6}$$

From (5),

$$q(r, s|z_1, z_2) = q(\bar{r}, \bar{s}|z_1^*, z_2^*) \tag{A.7}$$

by noticing that $\theta_1(z_1, z_2) = \theta_4(z_1^*, z_2^*)$, $\theta_2(z_1, z_2) = \theta_3(z_1^*, z_2^*)$, $\theta_3(z_1, z_2) = \theta_2(z_1^*, z_2^*)$, and $\theta_4(z_1, z_2) = \theta_1(z_1^*, z_2^*)$. Inserting (A.7) in (A.6) and (A.8) in (A.5) yields

$$g_{(r,s)}(z_1, z_2) = g_{(\bar{r},\bar{s})}(z_1^*, z_2^*) \tag{A.8}$$

$$E(Z_{1(r,s)}) + E(Z_{1(\bar{r},\bar{s})}) = 0, \tag{A.9}$$

respectively. Similarly, from (A.8) is obtained

$$\begin{gathered} E(Z_{2(r,s)}) + E(Z_{2(\bar{r},\bar{s})}) = 0, \\ \int\int_{\mathcal{Z}} Var(Y|z_1, z_2) g_{(r,s)}(z_1, z_2) dz_1 dz_2 = \int\int_{\mathcal{Z}} Var(Y|z_1^*, z_2^*) g_{(\bar{r},\bar{s})}(z_1^*, z_2^*) dz_1^* dz_2^*, \\ Var(Z_{1(r,s)}) = Var(Z_{1(\bar{r},\bar{s})}), \quad Var(Z_{2(r,s)}) = Var(Z_{2(\bar{r},\bar{s})}) \\ Cov(Z_{1(r,s)}, Z_{2(r,s)}) = Cov(Z_{1(\bar{r},\bar{s})}, Z_{2(\bar{r},\bar{s})}) \end{gathered} \tag{A.10}$$

Finally, combining (A.9), (A.10) with (A.3) and (A.4) yields (A.1) and (A.2), completing the proof of (12) and (13).

Appendix B: Proof of Theorem 4

Since the weighted (ordinary) least squares estimators are unbiased, we only need to compare their variances. We want to show for $\hat{\mu}_{oLS}^{(m+1)}$and $\hat{\mu}_{oLS}^{(m)}$,

$$\sum_{\mathbf{r}} \pi_{\mathbf{r}} \sigma_{[\mathbf{r}]}^2 \geq \sum_{\mathbf{r}} \sum_{s=1}^{H} \pi_{\mathbf{r}s} \sigma_{[\mathbf{r}s]}^2; \tag{B.1}$$

and for $\hat{\mu}_{wLS}^{(m+1)}$and $\hat{\mu}_{wLS}^{(m)}$

$$\left(\sum_{\mathbf{r}} \pi_{\mathbf{r}} / \sigma_{[\mathbf{r}]}^2 \right)^{-1} \geq \left(\sum_{\mathbf{r}} \sum_{s=1}^{H} \pi_{\mathbf{r}s} / \sigma_{[\mathbf{r}s]}^2 \right)^{-1} \tag{B.2}$$

where s denotes the rank of the extra ranking variable X_{m+1}. Noting

$$\sigma_{[\mathbf{r}]}^2 = \sum_{s=1}^{H} \frac{\pi_{\mathbf{r}s}}{\pi_{\mathbf{r}}} \sigma_{[\mathbf{r}s]}^2 + \left[\sum_{s=1}^{H} \frac{\pi_{\mathbf{r}s}}{\pi_{\mathbf{r}}} \mu_{[\mathbf{r}s]}^2 - \left(\sum_{s=1}^{H} \frac{\pi_{\mathbf{r}s}}{\pi_{\mathbf{r}}} \mu_{[\mathbf{r}s]} \right)^2 \right] \geq \sum_{s=1}^{H} \frac{\pi_{\mathbf{r}s}}{\pi_{\mathbf{r}}} \sigma_{[\mathbf{r}s]}^2 \tag{B.3}$$

yields (B.1). Now noting that $\{\sum_{s=1}^{H} \pi_{\mathbf{r}s}\sigma^2_{[\mathbf{r}s]}/\pi_{\mathbf{r}}\}\{\sum_{s=1}^{H} \pi_{\mathbf{r}s}/(\pi_{\mathbf{r}}\sigma^2_{[\mathbf{r}s]})\} \geq 1$ combined with (B.3), we have $\sum_{s=1}^{H} \pi_{\mathbf{r}s}/\sigma^2_{[\mathbf{r}s]} \geq \pi_{\mathbf{r}}/\sigma^2_{[\mathbf{r}]}$ that leads to (B.2).

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