CHAPTER III

ESTIMATION OF PARAMETERS (CONTINUED)

Case 2. All 5 Parameters Unknown

In this chapter we consider that all the parameters of the distribution (7) are to be estimated on the basis of n independent observations, $(x_1, y_1), \cdots (x_n, y_n)$. The likelihood function, as given by (30) is

$$L = \frac{1}{(2\pi)^n \sigma_1^n \sigma_2^n \prod_{i=1}^{n} y_i} e^{-1/2\left[\frac{\sum_{i=1}^{n}(x_i - \mu_1)^2}{\sigma_1^2} + \frac{\sum_{i=1}^{n}\left(\ln y_i - \left(kx_i + c - \frac{\sigma_2^2}{2}\right)\right)^2}{\sigma_2^2}\right]}$$

and the log likelihood function as given by (31) is

$$\ln L = - n \ln 2\pi - \frac{n}{2} \ln \sigma_1^2 - \frac{n}{2} \ln \sigma_2^2$$

$$- \ln \prod y_i - \frac{1}{2\sigma_1^2} \Sigma(x_i - \mu_1)^2$$

$$- \frac{1}{2\sigma_2^2} \Sigma\left(\ln y_i - \left(kx_i + c - \frac{\sigma_2^2}{2}\right)\right)^2 \quad .$$

To find $\hat{\mu}_1$ and $\hat{\sigma}_1^2$, note that

$$\frac{\partial \ln L}{\partial \mu_1} = \frac{1}{\sigma_1^2} \Sigma(x_i - \mu_1) \tag{84}$$

$$\frac{\partial \ln L}{\partial \sigma_1^2} = - \frac{n}{2\sigma_1^2} + \frac{1}{2\sigma_1^4} \Sigma(x_i - \mu_1)^2 \ . \tag{85}$$

Simultaneous solution of the two equations

$$\frac{\partial \ln L}{\partial \mu_1} = 0 \qquad\qquad \frac{\partial \ln L}{\partial \sigma_1^2} = 0$$

gives the expected result

$$\hat{\mu}_1 \ = \ \frac{1}{n} \ \Sigma x_i$$

$$= \ \bar{x} \tag{86}$$

$$\hat{\sigma}_1^2 \ = \ \frac{1}{n} \ \Sigma(x_i - \bar{x})^2$$

$$= \ \frac{n-1}{n} \ s_x^2 \ . \tag{87}$$

where $s_x^2 = \frac{1}{n-1} \Sigma(x_i - \bar{x})^2$ is the unbiased estimator for σ_1^2 obtained by adjusting the ML estimator.

$$\frac{\partial \ln L}{\partial c} = \frac{1}{\sigma_2^2} \ \Sigma\left(\ln y_i - kx_i - c + \frac{\sigma_2^2}{2}\right) \tag{88}$$

$$\frac{\partial \ln L}{\partial k} = \frac{1}{\sigma_2^2} \ \Sigma\left(x_i \ln y_i - kx_i^2 - cx_i + \frac{\sigma_2^2}{2} x_i\right) \tag{89}$$

$$\frac{\partial \ln L}{\partial \sigma_2^2} = -\frac{n}{2\sigma_2^2} - \frac{1}{2\sigma_2^2} \Sigma \left(\ln y_i - kx_i - c + \frac{\sigma_2^2}{2} \right) \tag{90}$$

$$+ \frac{1}{2\sigma_2^4} \Sigma \left(\ln y_i - kx_i - c + \frac{\sigma_2^2}{2} \right)^2 \quad .$$

These lead to the equations

$$\Sigma \ln y_i - k\Sigma x_i - nc + \frac{n}{2} \sigma_2^2 = 0 \tag{91}$$

$$\Sigma x_i \ln y_i - k\Sigma x_i^2 - c\Sigma x_i + \frac{\sigma_2^2}{2} \Sigma x_i = 0 \tag{92}$$

$$- \frac{n}{4} \sigma_2^4 - n\sigma_2^2 + \Sigma \ln^2 y_i + k^2 \Sigma x_i^2$$

$$+ nc^2 - 2k\Sigma x_i \ln y_i - 2c\Sigma \ln y_i$$

$$+ 2kc\Sigma x_i^2 = 0 \quad . \tag{93}$$

We may solve (91) and (92) immediately for $\hat{k}$ to obtain

$$\hat{k} = \frac{\Sigma(x_i - \bar{x}) \ln y_i}{\Sigma(x_i - \bar{x})^2} \tag{94}$$

which is the same as the $\hat{k}$ obtained in Chapter II.

Using this result in (91) we obtain

$$\hat{c} = \frac{1}{n} \Sigma \ln y_i - \hat{k}\bar{x} + \frac{\hat{\sigma}_2^2}{2} . \tag{95}$$

which is just (33) with σ_2^2 replaced by $\hat{\sigma}_2^2$.

Substituting this equation in (93) and solving for $\hat{\sigma}_2^2$ we obtain

$$\hat{\sigma}_2^2 = \frac{1}{n} \left[\hat{k}^2 \Sigma(x_i - \bar{x})^2 - 2\hat{k} \Sigma(x_i - \bar{x}) \ln y_i \right.$$

$$\left. + \left(\Sigma \ln^2 y_i - \frac{1}{n}(\Sigma \ln y_i)^2 \right) \right] \tag{96}$$

$$= \frac{1}{n} \left[\Sigma \ln^2 y_i - \frac{1}{n}(\Sigma \ln y_i)^2 - \hat{k}^2 \Sigma(x_i - \bar{x})^2 \right] . \tag{97}$$

This result may be used in (95) to obtain the ML estimate for c.

Let us now examine our estimators for bias. Certainly $\bar{x}$ and s_x^2 are unbiased estimators of μ_1 and σ_1^2 respectively.

Now consider

$$E(\hat{k}) = E\left(E(\hat{k}|\underline{x}) \right)$$

$$= E\left(\frac{\Sigma(x_i - \bar{x}) \left[\left(kx_i - k\bar{x} \right) + \left(k\bar{x} + c - \frac{\sigma_2^2}{2} \right) \right]}{\Sigma(x_i - \bar{x})^2} \right)$$

$$= E(k)$$

$$= k \; . \tag{98}$$

In order to find $E(\hat{\sigma}_2^2)$, first note that

$$\text{Var}(\hat{k}|\underline{x}) \;\; = \;\; \frac{\sigma_2^2}{\Sigma(x_i - \bar{x})^2}$$

so that

$$E[\hat{k}^2|\underline{x}] \;\; = \;\; \frac{\sigma_2^2}{\Sigma(x_i - \bar{x})^2} \; + \; k^2$$

and we know from (13) that

$$E\left(\Sigma \ln^2 y - \frac{1}{n}(\Sigma \ln y_i)^2\right) = (n - 1)(k^2\sigma_1^2 + \sigma_2^2) \; .$$

Referring to (96)

$$E\left[\hat{k}^2\Sigma(x_i - \bar{x})^2\right] \;\; = \;\; E\left[\Sigma(x_i - \bar{x})^2 \; E(\hat{k}^2|\underline{x})\right]$$

$$= \; E\left[\sigma_2^2 + k^2\Sigma(x_i - \bar{x})^2\right]$$

$$= \; \sigma_2^2 + k^2(n - 1)\sigma_1^2 \; . \tag{99}$$

Thus

$$E(\hat{\sigma}_2^2) \;\; = \;\; \frac{1}{n} \left[(n - 2)\sigma_2^2\right] \; . \tag{100}$$

We may easily adjust $\hat{\sigma}_2^2$ for bias

$$\hat{\hat{\sigma}}_2^2 = \frac{n}{n-2} \hat{\sigma}_2^2$$

$$= \frac{1}{n-2} \left[\Sigma \ln^2 y_i - \frac{1}{n} (\Sigma \ln y_i)^2 - \hat{k}^2 \Sigma (x_i - \bar{x})^2 \right] . \tag{101}$$

Let us redefine our estimator of c in terms of $\hat{\hat{\sigma}}_2^2$

$$\hat{\hat{c}} = \frac{1}{n} \Sigma \ln y_i - \hat{k}\bar{x} + \frac{\hat{\hat{\sigma}}_2^2}{2}$$

$$E(\hat{\hat{c}}) = \frac{1}{n} \Sigma \left(k\mu_1 + c - \frac{\sigma_2^2}{2} \right) - k\mu_1 + \frac{\sigma_2^2}{2}$$

$$= c . \tag{102}$$

We will now examine the covariance matrix for the estimators. It is well known that

$$\text{Var}(\bar{x}) = \frac{\sigma_1^2}{n} \tag{103}$$

$$\text{Var}(s_x^2) = \frac{2\sigma_1^4}{(n-1)} . \tag{104}$$

We may use a similar argument to that which was used for (42) to show

$$\text{Var}(\hat{k}) = \frac{\sigma_2^2}{(n-3)\sigma_1^2} . \tag{105}$$

We know that

$$\mathrm{Var}\,(\hat{\hat{\sigma}}_2^2) \;\; = \;\; \mathrm{Var}\!\left(\mathrm{E}\,(\hat{\hat{\sigma}}_2^2|\mathrm{X})\right) \; + \; \mathrm{E}\!\left(\mathrm{Var}\,(\hat{\hat{\sigma}}_2^2|\underline{\mathrm{X}})\right)\;.$$

Using facts obtained in the derivation of (49) and (99)

$$\mathrm{E}\,(\hat{\hat{\sigma}}_2^2\lfloor\underline{\mathrm{X}}) \;=\; \frac{1}{n-2}\; \left[\; \Sigma\!\left(\sigma_2^2 \;+\; \left(kx_i \;+\; c \;-\; \frac{\sigma_2^2}{2}\right)^2\right)\right.$$

$$-\; \left(\sigma_2^2 \;+\; \frac{1}{n}\left[\Sigma\!\left(kx_i \;+\; c \;-\; \frac{\sigma_2^2}{2}\right)\right]^2\right)$$

$$-\; \left(\sigma_2^2 \;+\; k^2\;\Sigma(x_i \;-\; \bar{x})^2\right)\Bigg]$$

$$=\; \frac{1}{n-2}\left[\; (n-2)\sigma_2^2 \;+\; \Sigma\!\left[\left(kx_i \;-\; k\bar{x}\right) \;+\; \left(k\bar{x} \;+\; c \;-\; \frac{\sigma_2^2}{2}\right)\right]^2\right.$$

$$-\; \frac{1}{n}\left[\Sigma\!\left(\left(kx_i \;-\; k\bar{x}\right) \;+\; \left(k\bar{x} \;+\; c \;-\; \frac{\sigma_2^2}{2}\right)\right)\right]^2$$

$$-\; k^2\Sigma(x_i \;-\; \bar{x})^2\Bigg]$$

$$=\; \sigma_2^2 \;.$$

$$\tag{106}$$

Thus

$$\mathrm{Var}\;\left(\mathrm{E}\,(\hat{\hat{\sigma}}_2^2|\underline{\mathrm{X}})\right) \;\; = \;\; 0\;.$$

$$\tag{107}$$

$$\text{Var}(\hat{\sigma}_2^2 | \underline{X}) = \frac{1}{(n-2)^2} [\text{Var}(\Sigma \ln^2 y_i | \underline{X})$$

$$+ \frac{1}{n^2} \text{Var}\left((\Sigma \ln y_i)^2 | \underline{X}\right) + \left(\Sigma(x_i - \bar{x})^2\right)^2 \text{Var}(\hat{k}^2 | \underline{X})$$

$$- \frac{2}{n} \text{Cov}\left(\Sigma \ln^2 y_i, (\Sigma \ln y_i)^2 | \underline{X}\right)$$

$$- 2 \Sigma(x_i - \bar{x}) \text{Cov}(\Sigma \ln^2 y_i, \hat{k}^2 | \underline{X})$$

$$+ \frac{2\Sigma(x_i - \bar{x})^2}{n} \text{Cov}\left((\Sigma \ln y_i)^2, \hat{k}^2 | \underline{X}\right). \tag{108}$$

From the derivation of (49) we know that

$$\text{Var}(\Sigma \ln^2 Y_i | \underline{X}) = 2n\sigma_2^4 + 4\sigma_2^2 \Sigma\left(kx_i + c - \frac{\sigma_2^2}{2}\right)^2 \tag{109}$$

and

$$\text{Var}\left((\Sigma \ln Y_i)^2 | \underline{X}\right) = 2n^2\sigma_2^4 + 4n\sigma_2^2 \left[\Sigma\left(kx_i + c - \frac{\sigma_2^2}{2}\right)\right]^2 . \tag{110}$$

$$\text{Var}(\hat{k}^2 | \underline{X}) = \text{Var}\left[\left(\frac{\Sigma(x_i - \bar{x}) \ln Y_i}{\Sigma(x_i - \bar{x})^2}\right)^2 \Bigg| \underline{X}\right]$$

$$= \frac{1}{[\Sigma(x_i - \bar{x})^2]^4} \text{Var}\left(\left(\Sigma(x_i - \bar{x}) \ln Y_i\right)^2 | \underline{X}\right) .$$

Now, $\Sigma(x_i - \bar{x}) \ln Y_i$ with given X has a normal distribution with

mean

$$E\left(\Sigma(x_i - \bar{x}) \ln Y_i \Big| \underline{X}\right) = \Sigma(x_i - \bar{x})\left(kx_i + c - \frac{\sigma_2^2}{2}\right)$$

$$= k \; \Sigma(x_i - \bar{x})^2$$

and variance

$$\text{Var}\left(\Sigma(x_i - \bar{x}) \ln Y_i \Big| \underline{X}\right) = \sigma_2^2 \; \Sigma(x_i - \bar{x})^2 \quad .$$

Therefore, for a given $\underline{X}$

$$\frac{1}{\sigma_2^2 \; \Sigma(x_i - \bar{x})^2}\left[\Sigma(x_i - \bar{x}) \ln Y_i\right]^2 \text{ has a noncentral chi square}$$

distribution with one degree of freedom and noncentrality parameter

$$\lambda = \frac{k^2}{2}\left[\Sigma(x_i - \bar{x})^2\right]^2 \cdot \frac{1}{\sigma_2^2 \; \Sigma(x_i - \bar{x})^2}$$

$$= \frac{k^2 \; \Sigma(x_i - \bar{x})^2}{2\sigma_2^2}$$

and

$$\text{Var}\left(\frac{\left[\Sigma(x_i - \bar{x}) \ln Y_i\right]^2}{\sigma_2^2 \; \Sigma(x_i - \bar{x})^2} \Big| \underline{X}\right) = 2 + \frac{4k^2 \; \Sigma(x_i - \bar{x})^2}{\sigma_2^2} \quad .$$

Thus

$$\text{Var}(\hat{k}^2 | \underline{x}) = \frac{2\sigma_2^4}{[\Sigma(x_i - \bar{x})^2]^2} + \frac{4k^2\sigma_2^2}{\Sigma(x_i - \bar{x})^2} \ . \tag{111}$$

To find the covariance terms of (108) we shall first find $\text{Cov}\left(\Sigma z_i^2, (\Sigma a_i z_i)^2\right)$ for Z_i distributed NID (u_i, σ^2).

Please note that where multiple sums are indicated in the following arguments the notation $i \neq j \neq \ell$ with reference to indices is understood to mean that i, j, and ℓ are all three different. The notation $i \neq j \neq \ell \neq m$ is likewise interpreted to mean that i, j, ℓ, and m are all different.

$$\text{Cov}\left(\Sigma\ z_i^2\ ,\ (\Sigma\ a_i z_i)^2\right)$$

$$= E\left(\Sigma\ z_i^2\ (\Sigma\ a_i z_i)^2\right) - E(\Sigma\ z_i^2)E\left((\Sigma\ a_i z_i)^2\right)$$

$$= \Sigma\ a_i^2 (u_i^4 + 6u_i^2\sigma^2 + 3\sigma^4)$$

$$+ \underset{i \neq j}{\Sigma\ \Sigma}\ a_i^2 (\sigma^2 + u_i^2)(\sigma^2 + u_j^2)$$

$$+ 2 \underset{i \neq j}{\Sigma\ \Sigma}\ a_i a_j (u_i^3 + 3\sigma^2 u_i)u_j$$

$$+ \underset{i \neq j \neq \ell}{\Sigma\ \Sigma\ \Sigma}\ a_i a_j u_i u_j (\sigma^2 + u_\ell^2)$$

$$- (n\sigma^2 + \Sigma\ u_i^2)(\Sigma\ a_i^2 u_i^2 + \sigma^2\ \Sigma\ a_i^2 + \underset{i \neq j}{\Sigma\ \Sigma}\ a_i a_j u_i u_j)$$

$$= 4\sigma^2\ \Sigma\ a_i^2 u_i^2 + 2\sigma^4\ \Sigma\ a_i^2$$

$$+ 4\sigma^2 \underset{i \neq j}{\Sigma\ \Sigma}\ a_i a_j u_i u_j$$

$$= 4\sigma^2 [\Sigma\ a_i u_i]^2 + 2\sigma^4\ \Sigma\ a_i^2\ . \tag{112}$$

Applying (112) to (108) we see that

$$\text{Cov}\left(\Sigma \ln^2 Y_i, \ (\Sigma \ln Y_i)^2 \mid \underline{x}\right)$$

$$= 4\sigma_2^2\left[\Sigma\left(kx_i + c - \frac{\sigma_2^2}{2}\right)\right]^2 + 2n\sigma_2^4 \ . \tag{113}$$

$$\text{Cov}\ (\Sigma \ln^2 Y_i, \ \hat{k}^2 \mid \underline{x})$$

$$= \text{Cov}\left(\Sigma \ln^2 Y_i \ , \ \left[\frac{\Sigma(x_i - \bar{x}) \ln Y_i}{\Sigma(x_i - \bar{x})^2}\right]^2 \ \Bigg| \ \underline{x}\right)$$

$$= 4\sigma_2^2\left[\frac{\Sigma(x_i - \bar{x})\left(kx_i + c - \frac{\sigma_2^2}{2}\right)}{\Sigma(x_i - \bar{x})^2}\right]^2$$

$$+ \ \frac{2\sigma_2^4}{\Sigma(x_i - \bar{x})^2} \ . \tag{114}$$

We now need to find an expression for $\text{Cov}\left((\Sigma a_i Z_i)^2 \ , \ (\Sigma b_i Z_i)^2\right)$ where, as above, Z_i is distributed NID (u_i, σ^2)

$$\text{Cov}\left((\Sigma a_i Z_i)^2 \ , \ (\Sigma b_i Z_i)^2\right)$$

$$= E\left[(\Sigma a_i Z_i)^2 (\Sigma b_i Z_i)^2\right]$$

$$- E\left[(\Sigma a_i Z_i)^2\right] E\left[(\Sigma b_i Z_i)^2\right]$$

$$= \Sigma \, a_i^{\,2} b_i^{\,2} (u_i^{\,4} + 6u_i^{\,2}\sigma^2 + 3\sigma^4)$$

$$+ \sum_{i \neq j} \Sigma \, a_i^{\,2} b_j^{\,2} (u_i^{\,2} + \sigma^2)(u_j^{\,2} + \sigma^2)$$

$$+ 2\sum_{i \neq j} \Sigma \, a_i a_j b_i b_j (u_i^{\,2} + \sigma^2)(u_j^{\,2} + \sigma^2)$$

$$+ 2\sum_{i \neq j} \Sigma \, a_i^{\,2} b_i b_j (u_i^{\,3} + 3u_i \sigma^2) u_j$$

$$+ 2\sum_{i \neq j} \Sigma \, a_i a_j b_i^{\,2} (u_i^{\,3} + 3u_i \sigma^2) u_j$$

$$+ \sum_{i \neq j \neq \ell} \Sigma \, \Sigma \, a_i a_j b_\ell^{\,2} u_i u_j (\sigma^2 + u_\ell^{\,2})$$

$$+ 4\sum_{i \neq j \neq \ell} \Sigma \, \Sigma \, a_i a_j b_i b_\ell (u_i^{\,2} + \sigma^2) u_j u_\ell$$

$$+ \sum_{i \neq j \neq \ell} \Sigma \, \Sigma \, a_i^{\,2} b_j b_\ell (u_i^{\,2} + \sigma^2) u_j u_\ell$$

$$+ \sum_{i \neq j \neq \ell \neq m} \Sigma \, \Sigma \, \Sigma \, a_i a_j b_\ell b_m u_i u_j u_\ell u_m$$

$$- \sigma_2^{\,4} \, \Sigma \, a_i^{\,2} \Sigma \, b_i^{\,2} - \sigma^2 \Sigma \, a_i^{\,2} \; (\Sigma \, b_i^{\,2} u_i^{\,2}$$

$$+ \sum_{i \neq j} \Sigma \, b_i b_j u_i u_j) - \sigma^2 \Sigma \, b_i^{\,2} \; (\Sigma \, a_i^{\,2} u_i^{\,2}$$

$$+ \sum_{i \neq j} \Sigma \, a_i a_j u_i u_j) - (\Sigma \, a_i u_i)^2 (\Sigma \, b_i u_i)^2$$

$$= 2\sigma^4 (\Sigma \, a_i b_i)^2 + 4\sigma^2 \left[\Sigma \, a_i^{\,2} b_i^{\,2} u_i^{\,2} \right.$$

$$+ \sum_{i \neq j} \Sigma \, a_i a_j b_i b_j u_i^{\,2} + \sum_{i \neq j} \Sigma \, a_i^{\,2} b_i b_j u_i u_j$$

$$+ \sum_{i \neq j} \Sigma \, a_i a_j b_i^{\,2} u_i u_j$$

$$+ \left. \sum_{i \neq j \neq \ell} \Sigma \, \Sigma \, a_i a_j b_i b_\ell u_j u_\ell \right]$$

$$= 2\sigma^4 \left(\Sigma \, a_i b_i\right)^2 + 4\sigma^2 \, \Sigma \, a_i b_i \, \Sigma \, a_i u_i \, \Sigma \, b_i u_i \; . \tag{115}$$

Reference to (108) shows that in our case $a_i = 1$, $i = 1, 2, \cdots, n$, and

$$\mathrm{Cov}\left(\left(\Sigma \, z_i\right)^2, \; \left(\Sigma \, b_i z_i\right)^2\right)$$

$$= 2\sigma^4 \left(\Sigma \, b_i\right)^2$$

$$+ 4\sigma^2 \, \sum_{i=1}^{n} b_i \, \sum_{i=1}^{n} u_i \, \sum_{i=1}^{n} b_i u_i \; . \tag{116}$$

Now,

$$\mathrm{Cov}\left(\left(\Sigma \, \ln Y_i\right)^2, \; \hat{k}^2 \,\big|\, \underline{x}\right)$$

$$= \mathrm{Cov}\left(\left(\Sigma \, \ln Y_i\right)^2 \, , \; \left[\frac{\Sigma (x_i - \bar{x}) \, \ln Y_i}{\Sigma (x_i - \bar{x})^2}\right]^2 \,\Big|\, \underline{x}\right) \; .$$

Here,
$$b_i = \frac{x_i - \bar{x}}{\Sigma (x_i - \bar{x})^2} \; , \qquad \Sigma \, b_i = 0 \; .$$

Therefore

$$\mathrm{Cov}\left(\left(\Sigma \, \ln Y_i\right)^2 \, , \; \hat{k}^2 \,\big|\, \underline{x}\right) = 0 \; . \tag{117}$$

Applying equations (109), (110), (111), (113), (114), and (117) to (108) we find

$$E\left(\text{Var}(\hat{\hat{\sigma}}_2^2|X)\right) = \frac{2\sigma_2^4}{n-2} \quad . \tag{118}$$

Using this equation in conjunction with (107) we have

$$\text{Var}(\hat{\hat{\sigma}}_2^2) = \frac{2\sigma_2^4}{n-2} \quad . \tag{119}$$

Let us now calculate $\text{Var}(\hat{\hat{c}})$

$$\text{Var}(\hat{\hat{c}}) = \text{Var}\left(E(\hat{\hat{c}}|\underline{X})\right) + E\left(\text{Var}(\hat{\hat{c}}|\underline{X})\right) \quad . \tag{120}$$

$$E(\hat{\hat{c}}|\underline{X}) = \frac{1}{n} \sum\left(kx_i + c - \frac{\sigma_2^2}{2}\right) - \bar{x}E(\hat{k}|\underline{x})$$

$$+ \frac{1}{2} E(\hat{\hat{\sigma}}_2^2|\underline{X}) \quad .$$

Using the derivations of equations (98) and (99) we see immediately that

$$E(\hat{\hat{c}}|\underline{X}) = c$$

and

$$\text{Var}\left(E(\hat{\hat{c}}|\underline{X})\right) = 0 \quad . \tag{121}$$

Returning to (120) we now wish to find

$$\text{Var}(\hat{\hat{c}}|X) = \text{Var}\left(\frac{1}{n} \sum \ln Y_i - \hat{k}\bar{x} + \frac{\hat{\hat{\sigma}}_2^2}{2} \mid \underline{x}\right)$$

$$= \frac{1}{n^2} \text{Var}(\sum \ln Y_i|\underline{X}) + \bar{x}^2 \text{Var}(\hat{k}|\underline{X})$$

$$+ \frac{1}{4} \text{Var}(\hat{\hat{\sigma}}_2^2|\underline{X}) - \frac{2\bar{x}}{n} \text{Cov}(\sum \ln Y_i, \hat{k}|\underline{X})$$

$$+ \frac{2}{n} \text{Cov}(\Sigma \ln Y_i, \hat{\hat{\sigma}}_2^2 | \underline{x})$$

$$- \bar{x} \text{Cov}(\hat{k}, \hat{\hat{\sigma}}_2^2 | \underline{x}) \; . \tag{122}$$

We know from equations (2), (38), (118), and (50) respectively, that

$$\text{Var}(\Sigma \ln Y_i | \underline{x}) = n\sigma_2^2$$

$$\text{Var}(\hat{k} | \underline{x}) = \frac{\sigma_2^2}{\Sigma(x_i - \bar{x})^2}$$

$$\text{Var}(\hat{\hat{\sigma}}_2^2 | \underline{x}) = \frac{2\sigma_2^4}{n - 2}$$

$$\text{Cov}(\Sigma \ln Y_i, \hat{k} | \underline{x}) = 0 \; .$$

We need to find $\text{Cov}(\Sigma \ln Y_i, \hat{\hat{\sigma}}_2^2 | \underline{x})$;

$$\text{Cov}(\Sigma \ln Y_i, \hat{\hat{\sigma}}_2^2 | \underline{x})$$

$$= \text{Cov}\left(\Sigma \ln Y_i, \frac{1}{n-2}\left[\Sigma \ln^2 Y_i - \frac{1}{n}(\Sigma \ln Y_i)^2 - \hat{k}^2 \Sigma(x_i - \bar{x})^2\right] \Big| \underline{x}\right)$$

$$= \frac{1}{n-2}\left[\text{Cov}(\Sigma \ln Y_i, \Sigma \ln^2 Y_i | \underline{x}) \right. \tag{123}$$

$$- \frac{1}{n} \text{Cov}(\Sigma \ln Y_i, (\Sigma \ln Y_i)^2 | \underline{x})$$

$$\left. + \Sigma(x_i - \bar{x})^2 \text{Cov}(\Sigma \ln Y_i, \hat{k}^2 | \underline{x})\right] \; .$$

In order to calculate these covariances let us suppose as we did to derive (112) and (115) that Z_i is distributed $NID(u_i, \sigma^2)$ and find

$$Cov(\Sigma \, a_i Z_i, \, \Sigma \, b_i Z_i^2) = E(\Sigma \, a_i b_i Z_i^3$$

$$+ \sum_{i \neq j} \sum a_i b_j Z_i Z_j^2) - E(\Sigma \, a_i Z_i) E(\Sigma \, b_i Z_i^2)$$

$$= \Sigma \, a_i b_i (u_i^3 + 3u_i \sigma^2) + \sum_{i \neq j} \sum a_i b_j (\sigma^2 + u_j^2) u_i$$

$$- \Sigma \, a_i u_i \, \Sigma \, b_i (u_i^2 + \sigma^2)$$

$$= 2\sigma^2 \, \Sigma \, a_i b_i u_i \, . \tag{124}$$

$$Cov\left(\Sigma \, a_i Z_i, \, (\Sigma \, b_i Z_i)^2\right) =$$

$$E\left[\Sigma \, a_i b_i^2 Z_i^3 + \sum_{i \neq j} \sum a_i b_j^2 Z_i Z_j^2 \right.$$

$$\left. + 2 \sum_{i \neq j} \sum a_i b_i b_j Z_i^2 Z_j + \sum_{i \neq j \neq \ell} \sum \sum a_i b_j b_\ell Z_i Z_j Z_\ell \right]$$

$$- E[\Sigma \, a_i Z_i] E[(\Sigma \, b_i Z_i)^2]$$

$$= \Sigma \, a_i b_i^2 (u_i^3 + 3u_i \sigma^2) + \sum_{i \neq j} \sum a_i b_j^2 u_i (\sigma^2 + u_j^2)$$

$$+ 2 \sum_{i \neq j} \sum a_i b_i b_j (u_i^2 + \sigma^2) u_j$$

$$+ \sum_{i \neq j \neq \ell} \sum \sum a_i b_j b_\ell u_i u_j u_\ell$$

$$- \Sigma \, a_i u_i \left(\Sigma \, b_i^2 (u_i^2 + \sigma^2) + \sum_{i \neq j} \sum b_i b_j u_i u_j \right)$$

$$= 2\sigma^2 (\Sigma \, a_i b_i) \, \Sigma \, b_i u_i \, . \tag{125}$$

Applying (124) and (125) to (123) we find

$$\text{Cov}(\Sigma \ln Y_i, \ \Sigma \ln^2 Y_i | \underline{x})$$

$$= 2\sigma_2^2 \ \Sigma\left(kx_i + c - \frac{\sigma_2^2}{2}\right) \ . \tag{126}$$

$$\text{Cov}\left(\Sigma \ln Y_i, \ (\Sigma \ln Y_i)^2 | \underline{x}\right)$$

$$= 2\sigma_2^2 n \ \Sigma\left(kx_i + c - \frac{\sigma_2^2}{2}\right) .$$

$$\text{Cov}(\Sigma \ln Y_i, \ \hat{k}^2 | \underline{x})$$

$$= 2\sigma_2^2\left[\frac{\Sigma(x_i - \bar{x})}{\Sigma(x_i - \bar{x})^2}\right] \ \Sigma\left(kx_i + c - \frac{\sigma_2^2}{2}\right)$$

$$= 0 \quad . \tag{127}$$

Equation (127) is what we might have expected since $\text{Cov}(\Sigma \ln Y_i, \ k) = 0$.

$$\text{Cov}(\Sigma \ln Y_i, \ \hat{\hat{\sigma}}_2^2 | \underline{x})$$

$$= \frac{1}{n-2} \left[2\sigma_2^2 \ \Sigma\left(kx_i + c - \frac{\sigma_2^2}{2}\right) - 2\sigma_2^2 \ \Sigma\left(kx_i + c - \frac{\sigma_2^2}{2}\right)\right]$$

$$= 0 \quad . \tag{128}$$

Finally, in order to evaluate (122) we need

$$\text{Cov}(\hat{k}, \ \hat{\hat{\sigma}}_2^2 | \underline{x})$$

$$= \frac{1}{n-2} \left[\text{Cov}(\hat{k}, \ \Sigma \ln^2 Y_i | \underline{x}) \right.$$

$$- \frac{1}{n} \text{Cov}(\hat{k}, \ (\Sigma \ln Y_i)^2 | \underline{x})$$

$$\left. - \Sigma(x_i - \bar{x})^2 \ \text{Cov}(\hat{k}, \ \hat{k}^2 | \underline{x}) \right] \ . \tag{129}$$

We may use (124) and (125) to find these covariances.

$$\text{Cov}(\hat{k}, \ \Sigma \ln^2 Y_i | \underline{x})$$

$$= 2\sigma^2 \ \frac{1}{\Sigma(x_i - \bar{x})^2} \ \Sigma(x_i - \bar{x}) \left(kx_i + c - \frac{\sigma_2^2}{2} \right) \ . \tag{130}$$

$$\text{Cov}\left(\hat{k}, \ (\Sigma \ln Y_i)^2 | \underline{x}\right)$$

$$= 2\sigma_2^2 \ \frac{1}{\Sigma(x_i - \bar{x})^2} \ \Sigma(x_i - \bar{x}) \ \Sigma\left(kx_i + c - \frac{\sigma_2^2}{2} \right)$$

$$= 0 \ . \tag{131}$$

$$\text{Cov}(\hat{k}, \ \hat{k}^2 | \underline{x})$$

$$= \frac{1}{[\Sigma(x_i - \bar{x})^2]^3} \ \text{Cov}\left(\Sigma(x_i - \bar{x}) \ln Y_i, \ \left(\Sigma(x_i - \bar{x}) \ln Y_i \right)^2 | \underline{x} \right)$$

$$= \frac{2\sigma_2^2}{[\Sigma(x_i - \bar{x})^2]^3} \ \Sigma(x_i - \bar{x})^2 \ \Sigma(x_i - \bar{x}) \left(kx_i + c - \frac{\sigma_2^2}{2} \right)$$

$$= \frac{2\sigma_2^2}{[\Sigma(x_i - \bar{x})^2]^2} \Sigma(x_i - \bar{x})\left(kx_i + c - \frac{\sigma_2^2}{2}\right) \quad . \tag{132}$$

Thus, referring to (129) we find

$$\text{Cov}(\hat{k}, \hat{\sigma}_2^2 | \underline{X}) = \frac{2\sigma_2^2}{n - 2}\left[\frac{\Sigma(x_i - \bar{x})\left(kx_i + c - \frac{\sigma_2^2}{2}\right)}{\Sigma(x_i - \bar{x})^2}\right.$$

$$\left. - \frac{\Sigma(x_i - \bar{x})\left(kx_i + c - \frac{\sigma_2^2}{2}\right)}{\Sigma(x_i - \bar{x}L^2}\right]$$

$$= 0 \quad . \tag{133}$$

Using equations (128) and (132) in (122) we find

$$\text{Var}(\hat{\hat{c}} | \underline{X}) = \frac{\sigma_2^2}{n} + \frac{\bar{x}^2\sigma_2^2}{\Sigma(x_i - \bar{x})^2} + \frac{\sigma_2^4}{2(n - 2)} \quad . \tag{134}$$

Now, we know that $\bar{x}$ and $\Sigma(x_i - \bar{x})^2$ are independent; thus, recalling (41) we find, upon applying (134) and (121) to (120)

$$\text{Var}(\hat{\hat{c}}) = \frac{\sigma_2^2}{n} + \frac{\sigma_2^4}{2(n - 2)} + \sigma_2^2 \, E(\bar{x}^2) \, E\left(\frac{1}{\Sigma(x_i - \bar{x})^2}\right)$$

$$= \frac{\sigma_2^2}{n} + \frac{\sigma_2^4}{2(n - 2)} + \left(\mu_1^2 + \frac{\sigma_1^2}{n}\right) \frac{\sigma_2^2}{(n - 3)\sigma_1^2} \quad . \tag{135}$$

We now wish to complete our covariance matrix. We know

$$\text{Cov}(\bar{x}, \; s_x^2) = 0.$$

$$\begin{aligned}
\text{Cov}(\bar{x}, \; \hat{k}) &= E\left(\bar{x} \; E(\hat{k}|\underline{x})\right) - E(\bar{x})E(\hat{k}) \\
&= \mu_1 k - \mu_1 k \\
&= 0 \; .
\end{aligned}$$

(136)

Using the derivation of (100) we find

$$\begin{aligned}
\text{Cov}(\bar{x}, \; \hat{\sigma}_2^2) &= E\left(\bar{x} \; E(\hat{\sigma}_2^2|\underline{x})\right) - E(\bar{x})E(\hat{\sigma}_2^2) \\
&= \mu_1 \sigma_2^2 - \mu_1 \sigma_2^2 \\
&= 0 \; .
\end{aligned}$$

(137)

Using the derivation of (102) we find

$$\begin{aligned}
\text{Cov}(\bar{x}, \; \hat{\hat{c}}) &= E\left(\bar{x} \; E(\hat{\hat{c}}|\underline{x})\right) - E(\bar{x})E(\hat{\hat{c}}) \\
&= \mu_1 c - \mu_1 c \\
&= 0
\end{aligned}$$

(138)

$$\begin{aligned}
\text{Cov}(s_x^2, \; \hat{k}) &= E\left(s_x^2 \; E(\hat{k}|\underline{x})\right) - E(s_x^2)E(\hat{k}) \\
&= \sigma_1^2 k - \sigma_1^2 k \\
&= 0 \; .
\end{aligned}$$

(139)

By similar arguments it can be seen that

$$\text{Cov}(s_x^{\ 2}, \hat{\hat{\sigma}}_2^2) = 0 \tag{140}$$

$$\text{Cov}(s_x^{\ 2}, \hat{\hat{c}}) = 0 \ . \tag{141}$$

We may use equation (133) to show

$$\text{Cov}(\hat{k}, \hat{\hat{\sigma}}_2^2) = E\left(E(\hat{k}\hat{\hat{\sigma}}_2^2|\underline{x})\right) - E(\hat{k})E(\hat{\hat{\sigma}}_2^2)$$

$$= k\sigma_2^2 - k\sigma_2^2$$

$$= 0 \ . \tag{142}$$

$$\text{Cov}(\hat{k}, \hat{\hat{c}}) = \text{Cov}\left(\hat{k}, \frac{1}{n} \Sigma \ln Y_i - \hat{k}\bar{x} + \frac{\hat{\hat{\sigma}}_2^2}{2}\right)$$

$$= \frac{1}{n} \text{Cov}(\hat{k}, \Sigma \ln Y_i) - \text{Cov}(\hat{k}, \hat{k}\bar{x}) + \frac{1}{2} \text{Cov}(\hat{k}, \hat{\hat{\sigma}}_2^2) \ .$$

Now, from (50) we know

$$E(\hat{k} \Sigma \ln Y_i|\underline{x}) = k \Sigma\left(kx_i + c - \frac{\sigma_2^2}{2}\right)$$

so that

$$\text{Cov}(\hat{k}, \Sigma \ln Y_i) = E\left(E(\hat{k} \Sigma \ln Y_i|\underline{x})\right) - E(\hat{k})E(\Sigma \ln Y_i)$$

$$= k \Sigma\left(k\mu_1 + c - \frac{\sigma_2^2}{2}\right) - k \Sigma\left(k\mu_1 + c - \frac{\sigma_2^2}{2}\right)$$

$$= 0 \ . \tag{143}$$

We may use the derivation of (55) to obtain

$$\text{Cov}(\hat{k}, \ \hat{k}\bar{x}) \ = \ E\left(\bar{x}E(\hat{k}^2|\underline{X})\right) \ - \ E(\hat{k})E(\hat{k}\bar{x})$$

$$= \ E\left(\bar{x}\left(\frac{\sigma_2^2}{\Sigma(x_i - \bar{x})^2} \ + \ k^2\right)\right) \ - \ k^2\mu_1$$

$$= \ \frac{\sigma_2^2\mu_1}{(n-3)\sigma_1^2} \ . \tag{144}$$

Therefore

$$\text{Cov}(\hat{k}, \ \hat{c}) \ = \ \frac{\sigma_2^2\mu_1}{(n-3)\sigma_1^2} \ . \tag{145}$$

Now, to complete our covariance matrix we need

$$\text{Cov}(\hat{\hat{\sigma}}_2^2, \ \hat{\hat{c}}) \ = \ \text{Cov}\left(\hat{\hat{\sigma}}_2^2, \ \frac{1}{n} \ \ln Y_i \ - \ \hat{k}\bar{x} \ + \ \frac{\hat{\hat{\sigma}}_2^2}{2}\right)$$

$$= \ \frac{1}{n}\text{Cov}(\hat{\hat{\sigma}}_2^2, \ \Sigma \ \ln Y_i) \ - \ \text{Cov}(\hat{\hat{\sigma}}_2^2, \ \hat{k}\bar{x})$$

$$+ \ \frac{1}{2} \ \text{Var}(\hat{\hat{\sigma}}_2^2) \ .$$

We may use (128) to show

$$\text{Cov}(\hat{\hat{\sigma}}_2^2, \; \Sigma \ln Y_i) \;\; = \;\; E\left(E(\hat{\hat{\sigma}}_2^2 \; \Sigma \ln Y_i \,|\, \underline{X})\right) - E(\hat{\hat{\sigma}}_2^2) E(\Sigma \ln Y_i)$$

$$= \;\; E\left[\Sigma\left(kx_i + c - \frac{\sigma_2^2}{2}\right)\hat{\hat{\sigma}}_2^2\right] - \sigma_2^2 \; \Sigma\left(k\mu_1 + c - \frac{\sigma_2^2}{2}\right)$$

$$= \;\; 0 \;\; . \tag{146}$$

We may use equations (133) and (136) to show

$$\text{Cov}(\hat{\hat{\sigma}}_2^2, \; \hat{k}\bar{x}) \;\; = \;\; E\left(\bar{x}E(\hat{\hat{\sigma}}_2^2 \hat{k} \,|\, \underline{X})\right) \; - \; E(\hat{\hat{\sigma}}_2^2) E(\hat{k}\bar{x})$$

$$= \;\; E(\bar{x}\sigma_2^2 k) \; - \; \sigma_2^2 k\mu_1$$

$$= \;\; \sigma_2^2 k\mu_1 \; - \; \sigma_2^2 k\mu_1$$

$$= \;\; 0 \;\; . \tag{147}$$

Thus, using (119) we see that

$$\text{Cov}(\hat{\hat{\sigma}}_2^2, \; \hat{c}) \;\; = \;\; \frac{\sigma_2^4}{n-2} \;\; .$$

To summarize, the covariance matrix for $\bar{x}$, s_x^2, $\hat{k}$, $\hat{\hat{\sigma}}_2^2$ and $\hat{\hat{c}}$ is given by

$$V = \begin{bmatrix} \dfrac{\sigma_1^2}{n} & 0 & 0 & 0 & 0 \\[2ex] 0 & \dfrac{2\sigma_1^4}{n-1} & 0 & 0 & 0 \\[2ex] 0 & 0 & \dfrac{\sigma_2^2}{(n-3)\sigma_1^2} & 0 & -\dfrac{\sigma_2^2\mu_1}{(n-3)\sigma_1^2} \\[2ex] 0 & 0 & 0 & \dfrac{2\sigma_2^4}{n-2} & \dfrac{\sigma_2^4}{n-2} \\[2ex] 0 & 0 & -\dfrac{\sigma_2^2\mu_1}{(n-3)\sigma_1^2} & \dfrac{\sigma_2^4}{n-2} & \sigma_{\hat{c}}^2 \end{bmatrix} \quad (149)$$

where $\sigma_{\hat{c}}^2 = \dfrac{\sigma_2^4}{2(n-2)} + \dfrac{\sigma_2^2}{n} + \left(\mu_1^2 + \dfrac{\sigma_1^2}{n}\right)\dfrac{\sigma_2^2}{(n-3)\sigma_1^2}$.

If we use the methods outlined in Chapter 18 of Kendall and Stuart we find that the corresponding large-sample covariance matrix is given by

$$V_{\ell s} = \begin{bmatrix} \dfrac{\sigma_1^2}{n} & 0 & 0 & 0 & 0 \\[2ex] 0 & \dfrac{2\sigma_1^4}{n} & 0 & 0 & 0 \\[2ex] 0 & 0 & \dfrac{\sigma_2^2}{n\sigma_1^2} & 0 & -\dfrac{\mu_1\sigma_2^2}{n\sigma_1^2} \\[2ex] 0 & 0 & 0 & \dfrac{2\sigma_2^4}{n} & \dfrac{\sigma_2^4}{n} \\[2ex] 0 & 0 & -\dfrac{\mu_1\sigma_2^2}{n\sigma_1^2} & \dfrac{\sigma_2^4}{n} & \dfrac{\sigma_2^4}{2n} + \dfrac{\sigma_2^2}{n} + \dfrac{\sigma_2^2\mu_1^2}{n\sigma_1^2} \end{bmatrix} \quad (150)$$

Let us now investigate the efficiencies of our estimators using the fact that if t is an unbiased estimator of $T(\theta_1, \theta_2, \cdots, \theta_K)$, then

$$\text{Var } (t) \geq \sum_{j=1}^{k} \sum_{i=1}^{k} \frac{\partial J}{\partial \theta_i} \frac{\partial J}{\partial \theta_j} I_{ij}^{-1} \tag{151}$$

where

$$I_{ij} = E\left(\frac{1}{L} \frac{\partial L}{\partial \theta_i} \frac{1}{L} \frac{\partial L}{\partial \theta_j}\right) \quad .$$

In our case $\theta_1 = \mu_1$, $\theta_2 = \sigma_1^2$, $\theta_3 = k$, $\theta_4 = \sigma_2^2$, $\theta_5 = c$, and the elements of I_{ij}^{-1} are given by (150).

Using (151) we see immediately that

$$\text{Var } (\hat{\mu}_1) \geq \frac{\sigma_1^2}{n}$$

$$\text{Var } (\hat{\sigma}_1^2) \geq \frac{2\sigma_1^4}{n}$$

$$\text{Var } (\hat{k}) \geq \frac{\sigma_2^2}{n\sigma_1^2}$$

$$\text{Var } (\hat{\sigma}_2^2) \geq \frac{2\sigma_2^4}{n}$$

$$\text{Var } (\hat{c}) \geq \frac{\sigma_2^4}{2n} + \frac{\sigma_2^2}{n} + \frac{\sigma_2^2 \mu_1^2}{n\sigma_1^2} \quad .$$

Thus, we may compute the efficiencies of our estimators relative to these minimum variance bounds.

$$\text{Eff }(\bar{x}) = \frac{\dfrac{\sigma^2}{n}}{\text{Var }(\bar{\bar{x}})}$$

$$= \frac{\dfrac{\sigma^2}{n}}{\dfrac{\sigma^2}{n}}$$

$$= \quad 1 \ . \tag{152}$$

$$\text{Eff }(s_x^2) = \frac{\dfrac{2\sigma^4}{n}}{\dfrac{2\sigma^4}{n-1}}$$

$$= \frac{n-1}{n} \ . \tag{153}$$

$$\text{Eff }(\hat{k}) = \frac{\dfrac{\sigma_2^2}{n\sigma_1^2}}{\dfrac{\sigma_2^2}{(n-3)\sigma_1^2}}$$

$$= \frac{n-3}{n} \ . \tag{154}$$

$$\text{Eff }(\hat{\hat{\sigma}}_2^2) = \frac{\dfrac{2\sigma_2^4}{n}}{\dfrac{2\sigma_2^4}{n-2}}$$

$$= \frac{n-2}{n} \ . \tag{155}$$

$$\text{Eff } (\hat{\hat{c}}) = \cfrac{\dfrac{\sigma_2^4}{2n} + \dfrac{\sigma_2^2}{n} + \dfrac{\sigma_2^2 \mu_1^2}{n\sigma_1^2}}{\dfrac{\sigma_2^4}{2(n-2)} + \dfrac{\sigma_2^2}{n} + \left(\mu_1^2 + \dfrac{\sigma_1^2}{n}\right)\dfrac{\sigma_2^2}{(n-3)\sigma_1^2}}$$

$$= \cfrac{1 + \dfrac{\sigma_2^2}{2} + \dfrac{\mu_1^2}{\sigma_1^2}}{1 + \dfrac{1}{n-3}\ \dfrac{n}{2(n-2)}\sigma_2^2 + \dfrac{n}{n-3}\ \dfrac{\mu_1^2}{\sigma_1^2}} \ . \qquad (156)$$

Suppose in (156) we let $u = \sigma_2^2$ and $v = \dfrac{\mu_1^2}{\sigma_1^2}$, then, if $E\hat{\hat{c}}$ is efficiency, we may write

$$\frac{n(E\hat{\hat{c}} - 1) + 2}{2(n-2)}u + \frac{n(E\hat{\hat{c}} - 1) + 3}{n-3}v$$

$$= \frac{(n-3)(1 - E\hat{\hat{c}}) - E\hat{\hat{c}}}{n-3} \ . \qquad (157)$$

It is now quite easy for a given value of n to draw contours of constant efficiency for $\hat{\hat{c}}$. Furthermore since u and v both must be positive it is easy to find bounds for $E\hat{\hat{c}}$ for a given n.

Note that for a contour to exist for some $E\hat{\hat{c}}$, the terms containing n and $E\hat{\hat{c}}$ in (157) must all have the same algebraic sign. This implies

that

$$\frac{n-2}{n} \leq E\hat{c} \leq \frac{n-3}{n-2} \; .$$

Finally, we may use the results of Finney (1941) to give minimum variance unbiased estimators for $E(Y)$ and $Var(Y)$

$$\widehat{E(Y)} = e^{\frac{1}{n} \Sigma \ln y_i} \; \Psi\left(\frac{1}{2} s_z^2\right) \tag{159}$$

where $\varphi(t)$ is the series given in Appendix A, and

$$s_z^2 = \frac{1}{n-1}\left[\Sigma \ln^2 y_i - \frac{1}{n}\left(\Sigma \ln y_i\right)^2\right] \tag{160}$$

$$\widehat{(Var \; Y)} = e^{\frac{2}{n} \Sigma \ln y_i} \left(\Psi(2s_z^2) - \Psi\left(\frac{n-2}{n-1} s_z^2\right)\right) \; . \tag{161}$$

Numerical results of application of the methods of this chapter to generated data are shown in Appendix B.

ESTIMATION OF PARAMETERS IN THE CASE

$$E(Y \mid X = x) = e^{kx + c} + \alpha$$

In this case we assume, as before, that X is normally distributed with mean μ_1 and variance σ_1^2. But now, as x approaches $-\infty$, $E(Y \mid X)$ approaches α, so it seems reasonable to let

$$f_{2.1}(y \mid X = x) = \frac{1}{\sqrt{2\pi}\, \sigma_2\, (y - \alpha)}\, e^{-\frac{1}{2\sigma_2^2}\, [\ln\, (y - \alpha)\, -\, \mu_2]} \tag{162}$$

To find μ_2, note from the properties of the moment generating function for the normal distribution that

$$E\left((Y - \alpha)^j \mid \underline{X}\right) = e^{j\mu_2 + \frac{j^2}{2}\, \sigma_2^2} \tag{163}$$

so that

$$E(Y \mid \underline{X}) = e^{\mu_2 + \frac{\sigma_2^2}{2}} + \alpha \tag{164}$$

We therefore have

$$e^{\mu_2 + \frac{\sigma_2^2}{2}} + \alpha = e^{kx + c} + \alpha$$

and

$$\mu_2 = kx + c - \frac{\sigma_2^2}{2} \tag{165}$$

The central moments for (162) are the same as those for (2) since we have merely translated the density function by an amount, α.

The bivariate distribution of X and Y now has the density function

$$f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2(y - \alpha)} e^{-\frac{1}{2}\left[\left(\frac{x - \mu_1}{\sigma_1}\right)^2 + \left(\frac{\ln(y - \alpha) - \left(kx + c - \frac{\sigma_2^2}{2}\right)}{\sigma_2}\right)^2\right]}$$

$$-\infty < x < \infty$$
$$y > \alpha \tag{166}$$

An example of the application of this type of distribution is given by Yuan (1933) who investigates the heights and weights of 11,382 school boys between the ages of 5 and 14 years. The data first appeared in a paper by L. Isserli (1915). Yuan estimates parameters by the method of moments whereas this paper will deal with maximum likelihood estimation.

If α is known, the estimation procedure is the same as that of the previous chapter. If α is not known, the solution of the likelihood equations is more complicated; we shall use a method similar to one employed by A. C. Cohen (1951) for the univariate case.

The likelihood function is

$$L = \left\{ \frac{1}{(2\pi)^n (\sigma_1^2)^{\frac{n}{2}} (\sigma_2^2)^{\frac{n}{2}} \Pi (y_i - \alpha)} \right\} \times$$

$$\left\{ e^{-\frac{1}{2}\left[\Sigma \left(\frac{x_i - \mu_1}{\sigma_1} \right)^2 + \Sigma \left(\frac{\ln \left(y_i - \alpha \right) - \left(kx_i + c - \frac{\sigma_2^2}{2} \right)}{\sigma_2} \right)^2 \right]} \right\}$$

and

$$\ln L = - n \ln 2\pi - \frac{n}{2} \ln \sigma_1^2 - \frac{n}{2} \ln \sigma_2^2 \qquad (167)$$

$$- \ln \Pi (y_i - \alpha) - \frac{1}{2\sigma_1^2} \Sigma (x_i - \mu_1)^2$$

$$- \frac{1}{2\sigma_2^2} \Sigma \left(\ln \left(y_i - \alpha \right) - \left(kx_i + c - \frac{\sigma_2^2}{2} \right) \right)^2$$

It is easily seen, as before, that $\bar{x}$ and s_x^2 are unbiased estimates of μ_1 and σ_1^2 respectively. Further

$$\frac{\partial \ln L}{\partial k} = \frac{1}{\sigma_2^2} \Sigma \left(\ln \left(y_i - \alpha \right) - \left(kx_i + c - \frac{\sigma_2^2}{2} \right) \right) x_i \qquad (168)$$

$$\frac{\partial \ln L}{\partial c} = \frac{1}{\sigma_2^2} \Sigma \left(\ln \left(y_i - \alpha \right) - \left(kx_i + c - \frac{\sigma_2^2}{2} \right) \right) \qquad (169)$$

$$\frac{\partial \ln L}{\partial \sigma_2^2} = - \frac{n}{2\sigma_2^2} - \frac{1}{2\sigma_2^2} \Sigma\left(\ln\left(y_i - \alpha\right) - \left(kx_i + c - \frac{\sigma_2^2}{2}\right)\right)$$

$$+ \frac{1}{2\sigma_2^4} \Sigma\left(\ln\left(y_i - \alpha\right) - \left(kx_i + c - \frac{\sigma_2^2}{2}\right)\right)^2 \tag{170}$$

$$\frac{\partial \ln L}{\partial \alpha} = \Sigma \frac{1}{y_i - \alpha} + \frac{1}{\sigma_2^2} \Sigma\left[\left(\ln\left(y_i - \alpha\right) - \left(kx_i + c - \frac{\sigma_2^2}{2}\right)\right)\frac{1}{y_i - \alpha}\right] \tag{171}$$

Setting (168), (169) and (170) equal to zero we find

$$\hat{k} = \frac{\Sigma \ln(y_i - \hat{\alpha})(x_i - \bar{x})}{\Sigma(x_i - \bar{x})^2} \tag{172}$$

$$\hat{c} = \frac{1}{n} \Sigma \ln(y_i - \hat{\alpha}) - \hat{k}\bar{x} + \frac{\hat{\sigma}_2^2}{2} \tag{173}$$

$$\hat{\sigma}_2^2 = \frac{1}{n}\left[\Sigma \ln^2(y_i - \hat{\alpha}) - \frac{1}{n}\left[\Sigma \ln(y_i - \hat{\alpha})\right]^2 - \hat{k}^2 \Sigma(x_i - \bar{x})^2\right]$$

Using these quantities in (171) and simplifying we obtain

$$\lambda(\alpha) = \frac{1}{n}\left[\Sigma \ln^2(y_i - \hat{\alpha}) - \frac{1}{n}\left(\Sigma \ln(y_i - \alpha) \right)^2 \right.$$

$$\left. - \frac{\left(\Sigma \ln(y_i - \hat{\alpha})(x_i - \bar{x}) \right)^2}{\Sigma(x_i - \bar{x})^2} - \frac{1}{n} \Sigma \ln(y_i - \alpha) \right] \Sigma \frac{1}{y_i - \alpha}$$

$$+ \Sigma \frac{\ln(y_i - \alpha)}{y_i - \alpha} - \frac{\Sigma \ln(y_i - \alpha)(x_i - \bar{x})}{\Sigma(x_i - \bar{x})^2} \Sigma \frac{x_i - \bar{x}}{y_i - \alpha}$$

$$= 0 \qquad\qquad\qquad (175)$$

One may first solve (175) numerically, then use $\hat{\alpha}$ to find $\hat{k}$, $\hat{c}$, and $\hat{\sigma}_2^2$. A FORTRAN program for finding these parameters has been prepared by W. R. Schucany of the Southern Methodist University Statistical Laboratory and is included in Appendix C. This program was used to analyze the data on heights and weights of school boys mentioned above. A comparison of these results with those of Yuan is given in Appendix B.

Although it is not possible to calculate the expected values and variances of $\hat{k}$, $\hat{c}$, $\hat{\sigma}_2^2$, and $\hat{\alpha}$, we may note that the large sample properties of maximum likelihood estimators imply consistency and allow us to find the covariance matrix for the estimators of the six unknown parameters. It may be that under certain conditions the likelihood equations do not have a real solution. Futher study is needed to define these conditions.

Although a full derivation of this matrix is not given in this paper, two interesting expected-value properties for the distribution whose density function is given by (166) were found in its derivation and are presented here.

First, since X is distributed normally with mean μ_1 and variance σ_1^2, integration by parts, used in conjunction with the moment generating function shows that

$$E(Xe^{Xt}) = e^{\mu_1 t + \frac{\sigma_1^2}{2} t^2} (\sigma^2 t + \mu) \tag{176}$$

If $Y - \alpha$ has a lognormal distribution with $E\left(\ln(Y - \alpha)\right) = \mu_2$ and $\text{Var}\left(\ln(Y - \alpha)\right) = \sigma_2^2$, a similar argument leads to the conclusion

$$E\left(\frac{\ln(Y - \alpha)}{(Y - \alpha)^\nu}\right) = e^{-\nu\mu_2 + \frac{\nu^2}{2}\sigma_2^2}(\mu_2 - \nu\sigma_2^2) \tag{177}$$

These properties, along with others used in Chapters II and III enable us to find the large sample covariance matrix for $\bar{x}$, s_x^2, $\hat{k}$, $\hat{\sigma}_2^2$, $\hat{c}$, and $\hat{\alpha}$.

The Fisher information matrix is given by (181) and its inverse the large sample covariance matrix by (182). Since the covariance matrix, V, is rather unwieldly, its elements are listed separately. To simplify let

$$\omega = \sigma_2^2 - c - k\mu_1 + \frac{k^2}{2}\sigma_1^2 \tag{178}$$

$$z = (1 - 3\sigma_2^2)e^{2\omega} + k^2\sigma_1^2 + \sigma_2^2 \tag{179}$$

$$u = \frac{\sigma_1^2}{2\sigma_2^2}\left(z - e^{2\omega}(1 + k^2\sigma_1^2)\right) \tag{180}$$

where u is the determinant of the information matrix.

$$
I =
\begin{bmatrix}
\dfrac{n}{2\sigma_1^2} & 0 & 0 & 0 & 0 & 0 \\[2ex]
0 & \dfrac{n}{2\sigma_1^4} & 0 & 0 & 0 & 0 \\[2ex]
0 & 0 & \dfrac{n(\mu_1^2+\sigma_1^2)}{\sigma_2^2} & -\dfrac{n\mu_1}{2\sigma_2^2} & \dfrac{n\mu_1}{\sigma_2^2} & \dfrac{n}{2}\dfrac{(\mu_1-k\sigma_1^2)e^\omega}{\sigma_2^2} \\[2ex]
0 & 0 & -\dfrac{n\mu_1}{2\sigma_2^2} & \dfrac{n}{4\sigma_2^2}+\dfrac{n}{2\sigma_2^4} & -\dfrac{n}{2\sigma_2^2} & -\dfrac{n}{2}\dfrac{e^\omega}{\sigma_2^2} \\[2ex]
0 & 0 & \dfrac{n\mu_1}{\sigma_2^2} & -\dfrac{n}{2\sigma_2^2} & \dfrac{n}{\sigma_2^2} & \dfrac{n}{2}\dfrac{e^\omega}{\sigma_2^2} \\[2ex]
0 & 0 & \dfrac{n}{2}\dfrac{(\mu_1-k\sigma_1^2)e^\omega}{\sigma_2^2} & -\dfrac{n}{2}\dfrac{e^\omega}{\sigma_2^2} & \dfrac{n}{2}\dfrac{e^\omega}{\sigma_2^2} & \dfrac{n}{2}\dfrac{1-3\sigma_2^2}{\sigma_2^2}e^{2\omega+k^2\sigma_1^2+\sigma_2^2}
\end{bmatrix}
\tag{181}
$$

The symmetric large sample covariance matrix V_1 has elements

$$V_{11} = \frac{\sigma_1^2}{n}$$

$$V_{12} = V_{13} = V_{14} = V_{15} = V_{16} = 0$$

$$V_{22} = \frac{2\sigma_1^4}{n}$$

$$V_{23} = V_{24} = V_{25} = V_{26} = 0$$

$$V_{33} = \frac{1}{2nu} (z - e^{2\omega})$$

$$V_{34} = 0$$

$$V_{35} = -\frac{1}{2nu} [\mu_1 z - e^{2\omega}(\mu_1 - k\sigma_1^2)]$$

$$V_{36} = \frac{k\sigma_1^2 e^\omega}{2nu}$$

$$V_{44} = \frac{\sigma_1^2 \sigma_2^2}{nu} [z - e^{2\omega}(1 + k^2\sigma_1^2)]$$

$$V_{45} = \frac{\sigma_1^2 \sigma_2^2}{2nu} [z - e^{2\omega}(1 + k^2\sigma_1^2)]$$

$$V_{46} = 0$$

$$V_{55} = \frac{1}{nu} \left\{ \left[\sigma_1^2 \left(\frac{\sigma_2^2 + 2}{4} \right) + \frac{\mu_1^2}{2} z \right] - \frac{1}{4} e^{2\omega} \left[(\sigma_1^2 + \mu_1 k) \sigma_2^2 + 2\mu_1 (\mu_1 - k\sigma_1^2) \right. \right.$$

$$\left. \left. - k\sigma_1^2 (\mu_1 - k\sigma_1^2)(\sigma_2^2 + 2) \right] \right\}$$

$$V_{56} = \frac{1}{2nu} e^{\omega} \sigma_1^2 (1 + k)$$

$$V_{66} = \frac{\sigma_1^2}{2nu}$$

The above covariance matrix is only valid for $\sigma_2^2 < \frac{1}{3}$. For greater values, Cramer's (1966) conditions for asymptotic normality are not met and other methods of estimation must be employed. This restriction on σ_2^2 may not cause practical difficulty in many cases. Recall that the variance of ln Y is $k^2 \sigma_1^2 + \sigma_2^2$. Aitchison and Brown (1958) have pointed out that frequently in practice this variance is in the neighborhood of 0.5 ; thus, if the variables are not independent and the variance of X is of reasonable magnitude, one would expect σ_2^2 to be less than one third.

APPENDIX A

THE LOGNORMAL DISTRIBUTION

Suppose that $Y = \ln X$ has a normal distribution with mean μ, and variance σ^2, then we say that X has a lognormal distribution whose density function is

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma\,x}\,e^{-\frac{1}{2}\left[\frac{\ln x - \mu}{\sigma}\right]^2} \;,\; x > 0 \; .$$

We may abbreviate this by writing X is $\Lambda(\mu,\,\sigma^2)$.

The jth moment about the mean is easily found by using the moment generating function for the normal distribution

$$E(X^j) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_0^\infty x^j \frac{1}{x}\, e^{-\frac{1}{2}\left[\frac{\ln x - \mu}{\sigma}\right]^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^\infty e^{jz}\, e^{-\frac{1}{2}\left[\frac{z - \mu}{\sigma}\right]^2} dz$$

$$= e^{j\mu + \frac{1}{2} j^2 \sigma^2} \; .$$

Thus,

$$E(X) = e^{\mu + \frac{\sigma^2}{2}}$$

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and

$$\text{Var}(x) = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1)$$

Suppose ξ_p is the quantile of order p from a lognormal distribution with parameters μ and σ^2, and η_p is the quantile of order p from a standard normal distribution then

$$\xi_p = e^{\mu + \eta_p \sigma} .$$

Thus if X is $\Lambda(\mu, \sigma^2)$ the above relation shows that the median is at

$$\xi_{.5} = e^{\mu} .$$

Differentiation shows that the mode occurs at

$$x = e^{\mu - \sigma^2} .$$

The following theorems on the reproductive properties of lognormal distributions are quoted from Chapter 2 of Aitchison and Brown (1957).

Theorem 1

If X is $\Lambda(\mu, \sigma^2)$ and b and c are constants where $c > 0$ (say $c = e^a$) then cX^b is $\Lambda(a + b\mu, b^2\sigma^2)$.

Theorem 2

If $\{x_j\}$ is a sequence of independent Λ-variates where X_j is $\Lambda(\mu_j, \sigma_j^2)$, $\{b_j\}$ a sequence of constants and $c = e^a$ a positive constant, then provided $\sum_j b_j \mu_j$ and $\sum_j b_j^2 \sigma_j^2$ both converge, the product $c\Pi X_j^{b_j}$ is

$\Lambda (a + \Sigma b_j \mu_j, \ \Sigma b_j^2 \sigma_j^2)$.

Now consider the random vector $\underline{X} = (X_1, X_2, \cdots, X_n)'$. We will say that $\underline{X}$ has a multivariate lognormal distribution if the vector $\underline{Y} = [\ln X_1, \ln X_2, \cdots, \ln X_n]$ has a multivariate normal distribution with mean $\underline{\mu}$ and covariance matrix V.

Theorem 3

If $\underline{X}$ is a multivariate lognormal and $\underline{b}$ is a (column) vector of constants with transpose b', then the product

$$c \prod_{j=1}^{n} X_j^{b_j}$$

is $\quad \Lambda (a + \underline{b}'\underline{\mu}, \ \underline{b}'V\underline{b})$

The following central limit theorem is also quoted from Aitchison and Brown.

Theorem 4

If $\{X_i\}$ is a sequence of independent, positive variates having the same probability distribution such that

$$E[\ln X_j] = \mu$$

and

$$\text{Var}(\ln X_j) = \sigma^2$$

both exist, then the product $\prod_{j=1}^{n} X_j$ is asymptotically distributed as $\Lambda(n\mu, \ n\sigma^2)$.

In their text, Aitchison and Brown consider several methods for the estimation of μ and σ^2 and conclude that maximum likelihood estimators

tend to be more efficient for these parameters. They point out that for large samples, if computing machinery is not readily available one might use estimators obtained by the method of quantiles which, although not as efficient as ML estimators are easier to compute, and are reasonably efficient.

If one tries to employ maximum likelihood methods to estimate $E(X)$ and $Var(X)$ where X is $\Lambda(\mu, \sigma^2)$ he finds the equations have no explicit solution. D. J. Finney (1941) devised a method to obtain unbiased, minimum-variance estimators for these quantiles. His method makes use of the fact that if Y is $N(\mu, \sigma^2)$ then $\bar{y}$ and s_y^2 are jointly sufficient statistics for μ and σ^2.

If $(x_1, x_2, \cdots, x_n)$ is a sample of n independent observations from a $\Lambda(\mu, \sigma^2)$ population and $y_i = \ln x_i$,

$$\bar{y} = \frac{1}{n} \Sigma \ln y_i$$

and

$$s_y^2 = \frac{1}{n-1} \Sigma(y_i - \bar{y})^2$$

then

$$a_1 = e^{\bar{y}} \Psi_n(\frac{1}{2} s_y^2)$$

and

$$b_1^2 = e^{2\bar{y}} X_n(s_y^2)$$

are unbiased minimum variance estimators of $E(X)$ and $Var(X)$ respectively where

$$\Psi_n(t) = 1 + \frac{n-1}{n}t + \frac{(n-1)^3}{n^2(n+1)}\frac{t^2}{2!}$$

$$+ \frac{(n-1)^5}{n^3(n+1)(n+3)}\frac{t^3}{3!} + \cdots$$

and

$$X_n(t) = \Psi_n(2t) - \Psi_n\left(\frac{n-2}{n-1}t\right) .$$

Tables of these functions may be found in Aitchison and Brown.

If, instead of having its lower bound at 0, X is bounded below by α, and $Y = \ln(X - \alpha)$ is normally distributed with parameters μ and σ^2 we obtain the three parameters lognormal distribution whose density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma(x-\alpha)}e^{-\frac{1}{2}\left[\frac{\ln(x-\alpha)-\mu}{\sigma}\right]^2} \quad x > \alpha$$

We may describe this by writing X is $\Lambda(\alpha,\ \mu,\ \sigma^2)$.

The moments about α are given by

$$E(|X - \alpha|^j) = e^{j\mu + \frac{1}{2}j^2\sigma^2}$$

Since the distribution has been displaced by an amount α, but its shape unchanged, it follows that the mean, median, and mode are, respectively

$e^{u + \frac{\sigma^2}{2}} + \alpha$, $e^\mu + \alpha$, and $e^{\mu - \sigma^2} + \alpha$, λ_p, the quantile of order p is given

by $\xi_p + \alpha$, where ξ_p is the corresponding quantile of $\Lambda(\mu, \sigma^2)$, and the central moments are unchanged.

None of the reproductive properties listed in Theorems 1, 2, and 3 hold if X is $\Lambda(\alpha, \mu, \sigma^2)$.

If α is known, no new estimation problems arise, however if α is unknown the estimation problem is increased considerably. The maximum likelihood equations have no explicit solution and one is forced to employ numerical methods. A. C. Cohen (1951) devised a method of solution and it is his solution upon which the solution of the likelihood equations in Chapter IV of this paper are based. Aitchison and Brown devote a full chapter to this problem and compare different methods of estimation.

Suppose $\alpha < X < \beta$ and $X' = \dfrac{X - \alpha}{\beta - X}$ is $\Lambda(\mu, \sigma^2)$. We now have a four parameter lognormal distribution. It is not further considered in this paper.

An extensive bibliography for the lognormal distribution is included in Aitchison and Brown.

APPENDIX B

NUMERICAL RESULTS

1. Ten bivariate normal lognormal random samples of size 100 were generated for various values of k, c, σ_2^2 with μ_1 = 100 and σ_1^2 = 225. The methods of Chapter II and Chapter III were then applied to estimate the unknown parameters.

In the case of Chapter II where μ_1, σ_1^2, and σ_2^2 were assumed known, we have, from equations (32), (33), (62), (71), (75), and (76)

$$\hat{k} = \frac{\Sigma(x_i - \bar{x})\ln y_i}{\Sigma(x_i - \bar{x})^2}$$

$$\hat{c} = \frac{1}{n} \Sigma \ln y_i + \frac{\sigma_2^2}{2} - \hat{k}\bar{x}$$

$$\tilde{k} = \frac{\Sigma(x_i - \mu_1)\ln y_i}{n\sigma_1^2}$$

$$\tilde{c} = \frac{1}{n} \Sigma \ln y_i + \frac{\sigma_2^2}{2} - \tilde{k}\mu_1$$

$$\hat{c}_T = \frac{1}{n} \Sigma \ln y_i + \frac{\sigma_2^2}{2} - \hat{k}\bar{z}$$

$$\tilde{c}_T = \frac{1}{n} \Sigma \ln y_i + \frac{\sigma_2^2}{2}$$

where

$$\bar{z} = \frac{1}{n} \Sigma(x_i - \mu_1)$$

$$= \bar{x} - \mu_1$$

The results of application of these equations to the generated data are given in Table 1.

In Chapter III it was assumed that all parameters were unknown. We found from equations (86), (87), (94), (101), and (102) that

$$\hat{\mu}_1 = \bar{x}$$

$$\hat{\sigma}_1^2 = s_x^2$$

$$= \frac{1}{n - 1} \Sigma(x_i - \bar{x})^2$$

$$\hat{k} = \frac{\Sigma(x_i - \bar{x}) \ln y_i}{\Sigma(x_i - \bar{x})^2}$$

$$\hat{\sigma}_2^2 = \frac{1}{n - 2} \left[\Sigma \ln^2 y_i - \frac{1}{n}(\Sigma \ln y_i)^2 - \hat{k}^2 \Sigma(x_i - \bar{x})^2 \right]$$

$$\hat{c} = \frac{1}{n} \Sigma \ln y_i - \hat{k} \bar{x} + \frac{\hat{\sigma}_2^2}{2}$$

Table 2 shows the results of application of these formulas to the generated data.

TABLE 1

Estimation of k, c, and c_T for

Ten Random Samples of Size 100

with $\mu_1 = 100$, $\sigma_1^2 = 225$ and known σ_2^2

σ_2^2	k	$\hat{k}$	$\tilde{k}$	c	$\hat{c}$	$\tilde{c}$	c_T	$\hat{c}_T$	$\tilde{c}_T$
9	0	.011	-.105	13.5	12.775	24.34	13.5	13.875	13.84
8.999	.002	.015	.021	13.300	12.273	11.636	13.500	13.753	13.756
8.978	.01	.017	.073	12.489	11.944	6.351	13.489	13.637	13.661
8.910	.02	.030	.090	11.455	10.437	4.441	13.455	13.413	13.445
8.64	.04	.022	-.010	9.320	10.971	14.164	13.320	13.191	13.174
6.75	.1	.106	.193	2.375	1.497	-6.95	12.375	12.070	12.321
4.79	.14	.111	.010	-2.705	.186	10.106	11.295	11.333	11.083
1.79	.18	.178	.205	-8.145	-7.976	-10.566	9.855	9.780	9.974
.878	.19	.192	.268	-9.561	-9.848	-17.197	9.439	9.369	9.633
.179	.198	.196	.100	-10.711	-10.500	-1.297	9.090	9.135	8.743

TABLE 2

Estimation of μ_1, σ_1^2, k, c, and σ_2^2

for Ten Random Samples of Size 100

μ_1	$\bar{x}$	σ_1^2	s_x^2	k	$\hat{k}$	σ_2^2	$\hat{\sigma}_2^2$	c	$\overset{\ll}{c}$
100	97.222	225	194.233	0	.011	9	8.594	13.5	12.672
100	100.131	225	243.414	.002	.015	8.999	13.379	13.300	14.464
100	101.451	225	186.764	.01	.017	8.978	12.971	12.489	13.941
100	101.568	225	208.626	.02	.030	8.910	8.864	11.455	10.399
100	99.202	225	212.525	.04	.022	8.64	9.493	9.320	11.398
100	102.360	225	212.555	.1	.106	6.75	7.373	2.375	1.809
100	97.762	225	198.172	.14	.112	4.79	3.991	- 2.705	2.295
100	100.976	225	212.162	.18	.178	1.79	1.548	- 8.145	- 8.037
100	101.833	225	228.939	.19	.192	.878	1.085	- 9.561	- 9.832
100	98.003	225	205.164	.198	.196	.179	.185	- 10.711	- 10.496

2. In Chapter II it was noted that if $k = 0$ and $c = \dfrac{\sigma_2^2}{2}$ one would expect $\tilde{k}$ to be a more efficient estimator of k than $\hat{k}$. To see if an actual difference could be observed, random samples of different sizes were generated with $\mu_1 = 100$, $\sigma_1^2 = 225$, $k = 0$, and $c = \dfrac{\sigma_2^2}{2} = 4.5$. Table 3 presents a comparison of $\hat{k}$ with $\tilde{k}$ for these samples. It will be noted that no appreciable difference exists between $\hat{k}$ and $\tilde{k}$ for these data.

TABLE 3

Comparison of $\hat{k}$ and $\tilde{k}$

for Samples of Various Sizes when k = 0 and c = $\dfrac{\sigma_2^2}{2}$

n	$\hat{k}$	$\tilde{k}$
10	.04243	.04896
10	− .03509	− .06619
10	− .03932	− .04407
10	.02007	.02838
10	.01432	.02926
50	.00682	.00812
50	.012172	.00446
50	.02804	.02757
50	.01856	.01907
50	.00066	.00060
100	.01631	.01550
100	− .01755	− .01612
100	.00688	.00291
100	− .03974	− .03606
100	− .00647	− .00663
500	.01521	.01254
500	.01164	.01147
500	.01733	.01914
500	.00253	.00286
500	.00945	.00971

3. Table 4 gives the results of measurements of heights and
weights of 11,382 school boys between the ages of 5 and 14 years. These
data were presented by Isserlis (1915) and used by Yuan (1933) as a
sample from a bivariate distribution of the type discussed in Chapter IV
with the marginal distribution of heights being normal with mean μ_1 and
σ_1^2 and the marginal distribution of weights being a three parameter log-
normal distribution, $\Lambda(\alpha, \mu_2, \sigma_2^2)$.

Yuan computed the values of the parameters by the method of moments
whereas this paper uses maximum likelihood methods. The FORTRAN program
which was used to solve the likelihood equations appears in Appendix
C.

A comparison of results appears in Table 5.

$$E(X) = \mu_1$$

$$Var(X) = \sigma_1^2$$

$$E(\ln Y) = kx + c$$

TABLE 4

Heights and Weights of School Boys

Height

Weight – Pounds	30 32	33 35	36 38	39 41	42 44	45 47	48 50	51 53	54 56	57 59	60 62	63 65	Total
24–28	4	9	2			1							16
29–33	3	42	62	25	3	1							136
34–38		16	220	414	72	6							728
39–43	1	3	51	617	697	95	11	1					1476
44–48		1	7	122	875	603	38	8	1				1655
49–53			4	12	249	988	411	33	5	4			1706
54–58		1	3	1	17	436	905	171	11	4	3		1552
59–63			1		1	39	630	568	51	6	1		1297
64–68				1		8	161	621	206	3	2	2	1004
69–73				1			35	374	340	24	2		776
74–78							3	106	335	76	5		525
79–83							2	22	120	93	4	1	242
84–88						1		8	32	87	8	2	138
89–93								1	10	36	18	1	66
94–98									3	23	9	2	37
99–103										5	11	3	19
104–108									1		5	1	7
109–113											1		1
114–118													0
119–123												1	1
Total	8	72	350	1193	1914	2178	2196	1913	1115	361	69	13	11,382

TABLE 5

Comparison of Values of Parameters
Obtained by the Method of Moments with Those
Obtained by Maximum Likelihood Estimation

Parameter	Method of Moments	Maximum Likelihood
μ_1	47.4644	47.4644
σ_1^2	27.3456	28.7275
α	$-$ 11.25	$-$ 4.1034
k	.02146	.003712
σ_2^2	.02462	.007376
c	3.1705	2.2970

APPENDIX C

A FORTRAN PROGRAM FOR COMPUTING THE MAXIMUM LIKELIHOOD ESTIMATORS

FOR THE PARAMETERS OF THE DISTRIBUTION OF CHAPTER IV

The main program solves for $\hat{\alpha}$ by the method of false positions.
The only subprogram necessary is the FUNCTION F.

The FUNCTION F (ALPHA) evaluates $\lambda(\alpha)$ of equation (175). If we let

$$A = \sum_{i=1}^{n} \ln^2(y_i - \alpha)$$

$$B = \sum_{i=1}^{n} \ln(y_i - \alpha)$$

$$C = \sum_{i=1}^{n} \ln(y_i - \alpha)(x_i - \bar{x})$$

$$D = \sum_{i=1}^{n} \frac{\ln(y_i - \alpha)}{y_i - \alpha}$$

$$E = \sum_{i=1}^{n} \frac{1}{y_i - \alpha}$$

$$G = \sum_{i=1}^{n} \frac{x_i - \bar{x}}{y_i - \alpha}$$

$$S = \sum_{i=1}^{n} (x_i - \bar{x})^2$$

then

$$F = \frac{E}{n}\left[A - \frac{B^2}{n} - \frac{C^2}{S} - B\right] + D - \frac{CG}{S} \quad .$$

```
C         MAXIMUM LIKELIHOOD ESTIMATES
C      FINDS ROOT OF THE DERIVATIVE OF LOG L W.R.T. ALPHA
C          BY THE METHOD OF FALSE POSITION
       COMMON X(150),Y(150),XYN(150),XBAR,TOT,S,N,
     *        A,B,C,D,E,G
 1000 XBAR=0.0
       TOT=0.0
       WRITE(6,11)
   11 FORMAT(10X,1HX,11X,1HY,11X,1HN/)
       READS(5,1) N,ALFO,DALF
    1 FORMAT(15,2E10.4)
C
C       N=NO. OF X,Y PAIRS (MAXIMUM OF 150)
C       ALFO=INITIAL ESTIMATE OF ALPHA (LESS THAN ALPHA HAT)
C       DALF=STEP SIZE OF ALPHA IN THE SEARCH FOR AXIS CROSSING
C
       YMIN=1.0E10
       DO 100 I=1,N
       READ(5,2) X(I),Y(I),XYN(I)
    2 FORMAT(3E10.2)
C
C  X NORMAL, Y SHIFTED LOGNORMAL, XYN=NO. OF PTS.
C          IN THE CLASS INTERVAL
       IF(Y(I).LT.YMIN) YMIN=Y(I)
       TOT=TOT+XYN(I)
       WRITE(6,3) X(I),Y(I),XYN(I)
    3 FORMAT(3F12.1)
  100 XBAR=XBAR+X(I)*XYN(I)
       XBAR=XBAR/TOT
       S=0.0
       DO 110 I=1,N
  110 S=S+XYN(I)*(X(I)-XBAR)**2
       WRITE(6,10) XBAR,S
   10 FORMAT(40X,5HXBAR=,E17.8,5X,2HS=,E17.8/)
       FO=F(ALFO)
       ALF1=ALFO+DALF
       F1=F(ALF1)
       WRITE(6,5) ALFO,FO,ALF1,F1
    5 FORMAT(2E20.8)
C           SEARCH FOR AXIS CROSSING
  115 IF(F1) 120,400,120
  120 IF(FO/F1) 150,410,140
  140 ALFO=ALF1
       FO=F1
       ALF1=ALF1+DALF
  141 IF(ALF1.GE.YMIN) GO TO 142
       GO TO 144
```

```
142 ALF1=(ALFO+ALF1)*0.5
    GO TO 141
144 F1=F(ALF1)
    WRITE(6,5) ALF1,F1
    GO TO 115
C        ALPHA HAT BRACKETED, REGULA FALSI
150 ALF=(F1*ALFO-FO*ALF1)/(F1-FO)
    FN=F(ALF)
    IF(FN/F1) 155,420,160
155 ALFO=ALF
    FO=FN
    GO TO 180

160 ALF1=ALF
    F1=FN
180 WRITE(6,4) ALF,FN
  4 FORMAT(5X,6HALPHA=,E15.7,5X,9HF(ALPHA)=,E15.7)
    IF(ABS(FN).GT.1.OE-03) GO TO 150
    GO TO 420
400 ALF=ALF1
    GO TO 420
410 ALF=ALFO
420 HATK=C/S
    SIG22=(A-B*B/TOT-HATK*HATK*S)/TOT
    CHAT=B/TOT-HATK*XBAR+0.5*SIG22
    WRITE(6,6) ALF,HATK,SIG22,CHAT
  6 FORMAT(1H1,10X,6HALPHA=,E16.8//14X,2HK=,E16.8//,
  * 10X,6HSIG22=,E16.8//,14X,2HC=,E16.8)
    WRITE(6,7)
  7 FORMAT(1H1)
    GO TO 1000
    END
```

```
      FUNCTION F(ALF)
      COMMON X(150),Y(150),XYN(150),XBAR,TOT,S,N,
     *       A,B,C,D,E,G
      A=0.0
      B=0.0
      C=0.0
      D=0.0
      E=0.0
      G=0.0
      DO 100 I=1,N
      YLOG=ALOG(Y(I)-ALF)
      A=A+XYN(I)*YLOG*YLOG
      B=B+XYN(I)*YLOG
      C=C+XYN(I)*YLOG*(X(I)-XBAR)
      D=D+YLOG/(Y(I)-ALF)*XYN(I)
      E=E+XYN(I)/(Y(I)-ALF)
  100 G=G+XYN(I)*(X(I)-XBAR)/(Y(I)-ALF)
      F=(A-B*B/TOT-C*C/S)/TOT*E+D-C*G/S-B*E/TOT
      RETURN
      END
```

BIBLIOGRAPHY

Aitchison, J. and Brown, J. A. C. (1957). <u>The Lognormal Distribution</u>, Cambridge University Press, Cambridge.

Backer, Frederick Jr. (1961). <u>Estimation of Parameters in a Truncated Log Normal Distribution with a Type of Incomplete Data</u>, Unpublished Masters Thesis, Department of Mathematics, Southern Methodist University.

Cohen, A. C. (1951). "Estimating parameters of logarithmic normal distributions by maximum likelihood," <u>Journal of the American Statistical Association</u>, <u>24</u>, 202-206.

Cramér, H. (1966). <u>Mathematical Methods of Statistics</u>, Princeton University Press, Princeton.

Feinleib, M. (1960). "A method of analyzing log-normally distributed survival data with incomplete follow-up," <u>Journal of the American Statistical Association</u>.

Finney, D. J. (1941). "On the distribution of a variate whose logarithm is normally distributed," <u>Journal of the Royal Statistical Society</u>, Series B, <u>7</u>, 155-161.

Heyde, C. C. (1963). "On a property of the lognormal distribution," <u>Journal of the Royal Statistical Society</u>, Series B, <u>25</u>, 392-393.

Isserlis, L. (1915). "Partial Correlation Ratio, Part II," <u>Biometrika</u>, <u>11</u>, 50-66.

Kendall, M. G. and Stuart, A. (1961). <u>The Advanced Theory of Statistics</u>, Vols. 1 and 2, Hafner Publishing Company, New York.

Koopmans, L. H., Owen, D. B., and Rosenblatt, J. I. (1964). "Confidence intervals for the coefficient of variation for the normal and log normal distributions," <u>Biometrika</u>, <u>51</u>, <u>1</u> and <u>2</u>, 25-32.

Laurent, A. G. (1963). "The log normal distribution and the translation method," <u>Journal of the American Statistical Association</u>, <u>58</u>, 231-235.

Lukacs, E. (1960). <u>Characteristic Functions</u>, Hafner Publishing Company, New York.

McCulloch, A. J. and Walsh, J. E. (1967). "Life testing results based on a few heterogeneous lognormal observations," Journal of the American Statistical Association, 62, 45-47.

Pearson, E. S. (1962). Frequency Surfaces. Statistical Techniques Research Group Technical Report No. 49, Princeton University Department of Mathematics.

Quensel, C. E. (1944). Inkomstfördelning och Skatterych. Sveriges Industriforbund, Stokholm.

_____ (1945). "On the logarithmic normal distribution," Skandinavisk Aktuarietidskrift, 28, 141-153.

Shannon, B. S. (1963). Some Properties of Dynamic Censored Samples from Four Distributions Used in Survival Time Analysis. Unpublished Masters Thesis, Department of Statistics, Southern Methodist University.

Wicksell, S. D. (1917). "On the genetic theory of frequency," Arkiv für Matematik, Astronomi, och Fysik, 12, number 20.

_____ (1917). On Logarithmic Correlation with an Application to the Distribution of Ages at First Marriage. Meddelande från Lunds Astronomiska Observatorium, No. 84.

Yuan, P. T. (1933). "On the lagarithmic frequency distribution and the semi-logarithmic correlation surface," Annals of Mathematical Statistics, 4, 30-74.

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13 ABSTRACT

A study is made of the distribution of (X, Y) for X normally distributed with mean μ_1 and variance σ_1^2, the conditional distribution of $\ln(Y - \alpha)$ given $X = x$ is normal with mean $\mu_2(x)$ and variance σ_2^2, and the form of $E(Y|X = x)$ is known. Properties of the distribution as well as a discussion of maximum likelihood estimation are given for $\alpha = 0$ and $E(Y|X = x) = e^{kx + c}$ for two situations. First, it is assumed that μ_1, σ_1^2, and σ_2^2 are known and k and c are to be estimated; next estimators are obtained when all five parameters are to be estimated.

Maximum likelihood estimation μ_1, σ_1^2, k, c, σ_2^2, and α is then discussed for the case $E(Y|X = x) = e^{kx + c} + \alpha$.

Three appendixes include a summary of the lognormal distribution, numerical results obtained from application of the estimators obtained in the paper to several sets of data, and a FORTRAN program for estimation of μ_1, σ_1^2, k, c, σ_2^2 and α when $E(Y|X = x) = e^{kx + c} + \alpha$.

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