AN INVESTIGATION OF BIVARIATE NORMAL

LOGNORMAL DISTRIBUTIONS

by

Alfred E. Crofts, Jr.

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NORMAL LOGNORMAL DISTRIBUTIONS

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by

ALFRED E. CROFTS, JR.
(M.S., Mathematics, Southern Methodist University, 1958)

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Crofts, Alfred E., Jr.   B.S., University of Texas, 1955
                          M.S., Southern Methodist University, 1958

An Investigation of Bivariate Normal Lognormal Distributions

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A study is made of the distribution of (X, Y) for X normally distributed with mean μ_1 and variance σ_1^2, the conditional distribution of $\ln(Y - \alpha)$ given $X = x$ is normal with mean $\mu_2(x)$ and variance σ_2^2, and the form of $E(Y \mid X = x)$ is known. Properties of the distribution as well as a discussion of maximum likelihood estimation are given for $\alpha = 0$ and $E(Y \mid X = x) = e^{kx + c}$ for two situations. First, it is assumed that μ_1, σ_1^2, and σ_2^2 are known and k and c are to be estimated; next estimators are obtained when all five parameters are to be estimated.

Maximum likelihood estimation of μ_1, σ_1^2, k, c, σ_2^2, and α is then discussed for the case $E(Y \mid X = x) = e^{kx + c} + \alpha$.

Three appendixes include a summary of the lognormal distribution, numerical results obtained from application of the estimators obtained in the paper to several sets of data, and a FORTRAN program for estimation of μ_1, σ_1^2, k, c, σ_2^2, and α when $E(Y \mid X = x) = e^{kx + c} + \alpha$.

PREFACE

Both the normal and lognormal distributions are used to describe a great variety of phenomena. This paper is concerned with the bivariate distribution of (X, Y) that arises when one variate, say X, is distributed normally, the conditional distribution of the other, for a given X is lognormal, and the form of the regression $E(Y|X = x)$ is known. Two forms of the regression curve are investigated here; it is hoped that this method of approach will prove useful for other regression functions.

I would like to express my gratitude to the faculty members of the Statistics Department at Southern Methodist University who at all times have been willing to listen, answer, and argue. My particular thanks go to Dr. Paul Minton, who has helped me in more ways than I can begin to acknowledge, and to Dr. Donald B. Owen, my principal advisor whose knowledge of the literature of statistics and whose knack for encouragement when it is needed most has made the task of writing a dissertation a pleasant one. Finally, to Mr. W. R. Schucany of the Statistical Laboratory who wrote and ran all the computer programs used in this paper, and to Mrs. Martha Wells who has spent many hours typing it, my deepest thanks.

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CHAPTER I

DERIVATION OF THE BIVARIATE DISTRIBUTION

The following three assumptions are made throughout this paper:

1) The marginal distribution of X is normal with mean μ_1 and variance σ_1^2.

2) The conditional distribution of Y given X is lognormal with parameters $\mu_2(x)$ and σ_2^2.

3) The form of the regression curve of Y on X is known. In the first three chapters it is of the form:

$$E(Y|X = x) = e^{kx + c}$$

As an example of these assumptions suppose we consider the random variables X, intelligence quotient, and Y, income. We observe that for a random sample, X tends to be normally distributed while, according to Quensel (1944), for a given value of X, Y may be described by a lognormal distribution. Further, we believe that within the range of the variables, the expected value of Y for a given X is of the form:

$$E(Y|X = x) = e^{kx + c} \quad .$$

Now, if one variable is easy to obtain while the other is not, the problem becomes one of using a random sample to estimate the unknown parameters of the bivariate distribution so that one may make probability statements on the variable that is difficult to observe for a given value of the variable that is easily obtainable.

1

The marginal density function of x is given by

$$f_1(x) = \frac{1}{\sqrt{2\pi}\ \sigma_1}\ e^{-1/2\ \frac{(x-\mu_1)^2}{\sigma_1^2}}\ . \tag{1}$$

The conditional density function of y given x is

$$f_{2.1}(y|x) = \frac{1}{\sqrt{2\pi}\ \sigma_2 y}\ e^{-1/2\ \frac{(\ln y - \mu_2)^2}{\sigma_2^2}}\ . \tag{2}$$

By the properties of the lognormal distribution given in Appendix A we know that

$$E(Y|X = x) = e^{\mu_2 + \frac{\sigma_2^2}{2}} \tag{3}$$

$$\text{Var}(Y|X = x) = e^{2\mu_2 + \sigma_2^2}\left(e^{\sigma_2^2} - 1\right). \tag{4}$$

Since we require that

$$E(Y|X = x) = e^{kx + c}\ , \tag{5}$$

by equating (3) and (5) we obtain

$$\mu_2 = kx + c - \frac{\sigma_2^2}{2}\ . \tag{6}$$

The joint density function for X and Y is

$$f(x,y) = f_1(x)\ f_{2.1}(y|x)$$

$$= \frac{1}{2\pi\sigma_1\sigma_2 y}\ e^{-1/2\left[\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{\left(\ln y - (kx + c - \frac{\sigma_2^2}{2})\right)^2}{\sigma_2^2}\right]}\ . \tag{7}$$

Contours of constant probability for various values of the parameters are given in Figures 1 through 6.

At this time it should be noted that if we make the transformation

$$z = \ln y, \text{ we obtain}$$

$$f(x,z) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-1/2 \left[\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{\left(z - (kx + c - \frac{\sigma_2^2}{2}) \right)^2}{\sigma_2^2} \right]} \tag{8}$$

which is a bivariate normal distribution with

$$E(X) = \mu_1 \qquad E(Z) = k\mu_1 + c - \frac{\sigma_2^2}{2} . \tag{9}$$

The covariance matrix for X and Z is

$$\Sigma = \begin{pmatrix} \sigma_1^2 & k\sigma_1^2 \\ \\ k\sigma_1^2 & k^2\sigma_1^2 + \sigma_2^2 \end{pmatrix} . \tag{10}$$

The coefficient of correlation between X and Z is given by

$$\rho_{X,Z} = \frac{k\sigma_1}{\sqrt{k^2\sigma_1^2 + \sigma_2^2}} . \tag{11}$$

Reference to (9) and (10) shows that the marginal distribution for ln Y is normal with

Figure 1

Contours of constant
probability for (7)

$\mu_1 = 100$

$\sigma_1^2 = 225$

$c = 13.5$

$k = 0$

$\sigma_2^2 = 9$

Figure 2

Figure 3

Contours of constant
probability for (7)

$\mu_1 = 100$

$\sigma_1^2 = 225$

$c = 2.375$

$k = .1$

$\sigma_2^2 = 6.75$

Figure 4

Figure 5

Y

30,000

Contours of constant
probability for (7)

$\mu_1 = 100$

$\sigma_1^2 = 16$

$c = 2.375$

$k = .1$

$\sigma_2^2 = 6.75$

10

100

140

X

Y

30,000

Contours of constant

probability for (7)

$\mu_1 = 0$

$\sigma_1^2 = 225$

$c = 13.5$

$k = 0$

$\sigma_2^2 = 9$

X

-40

0

40
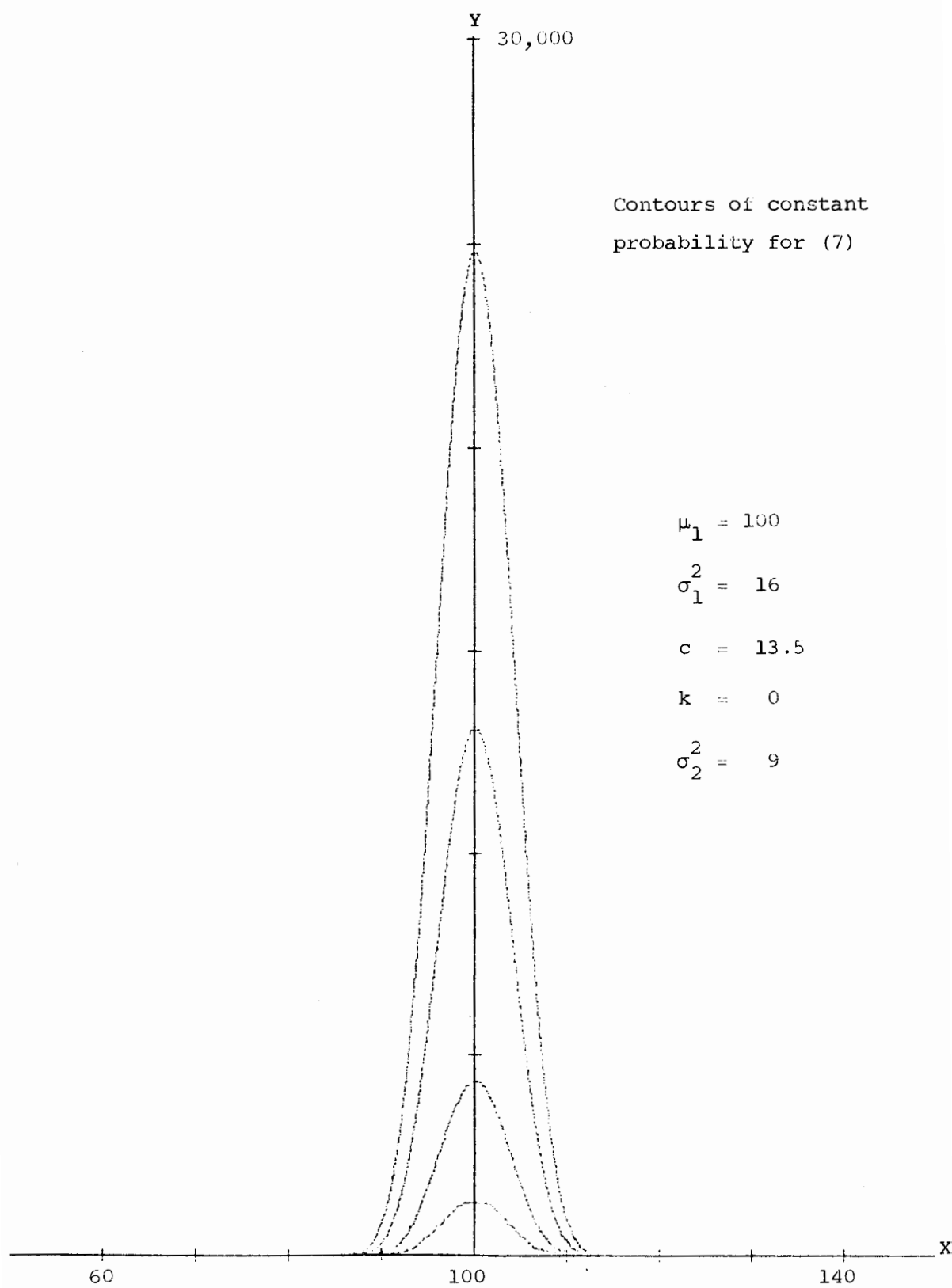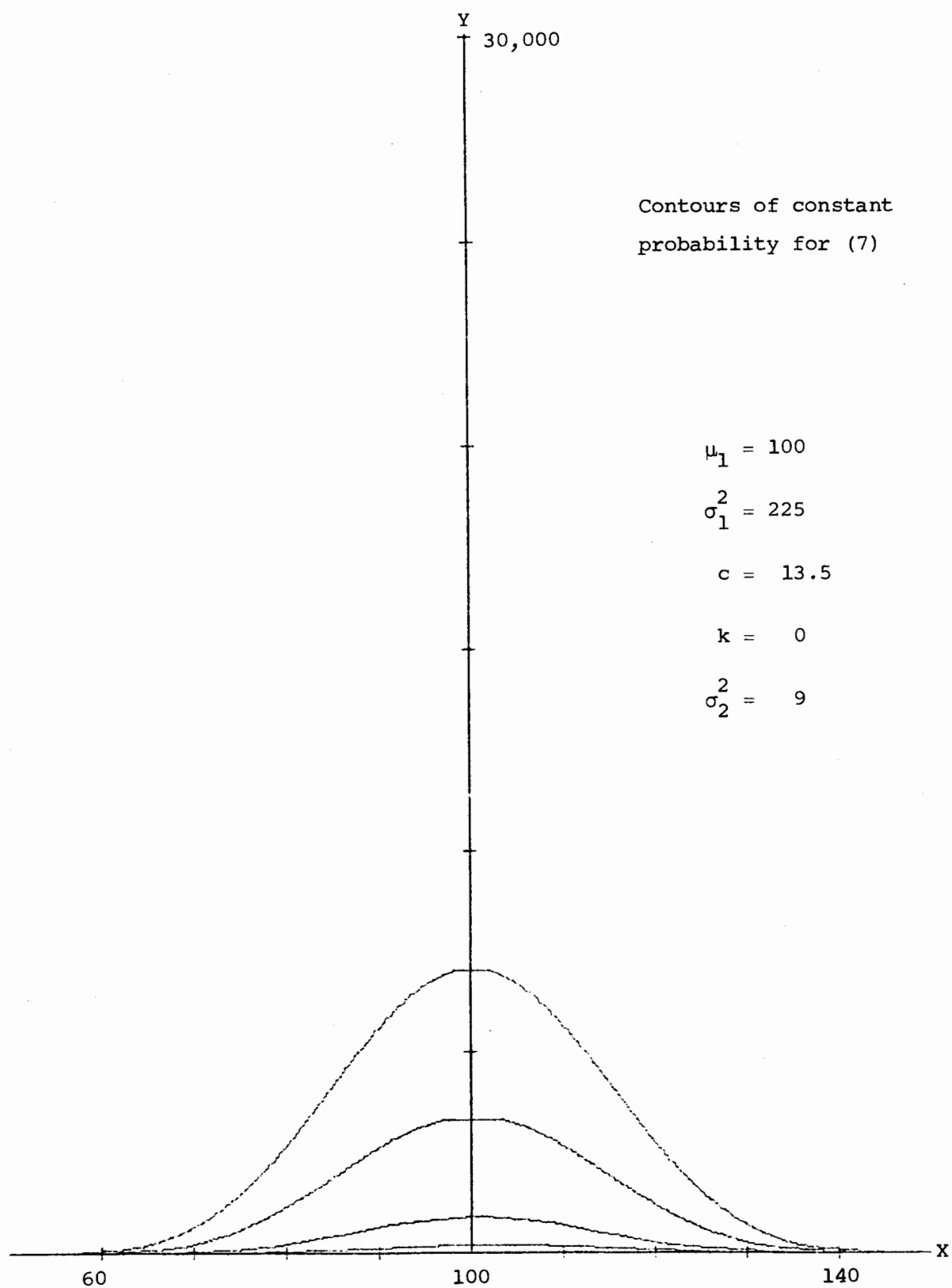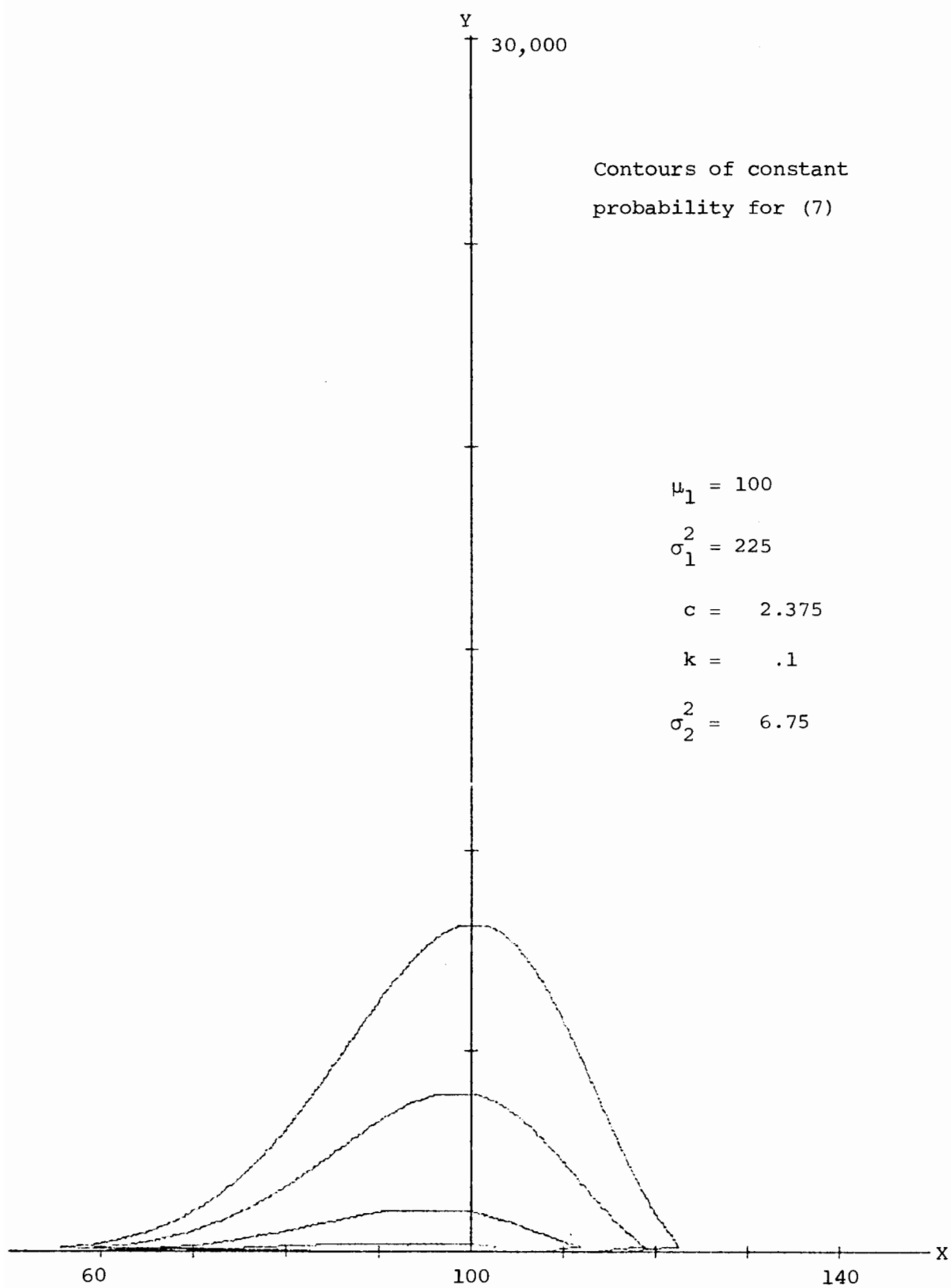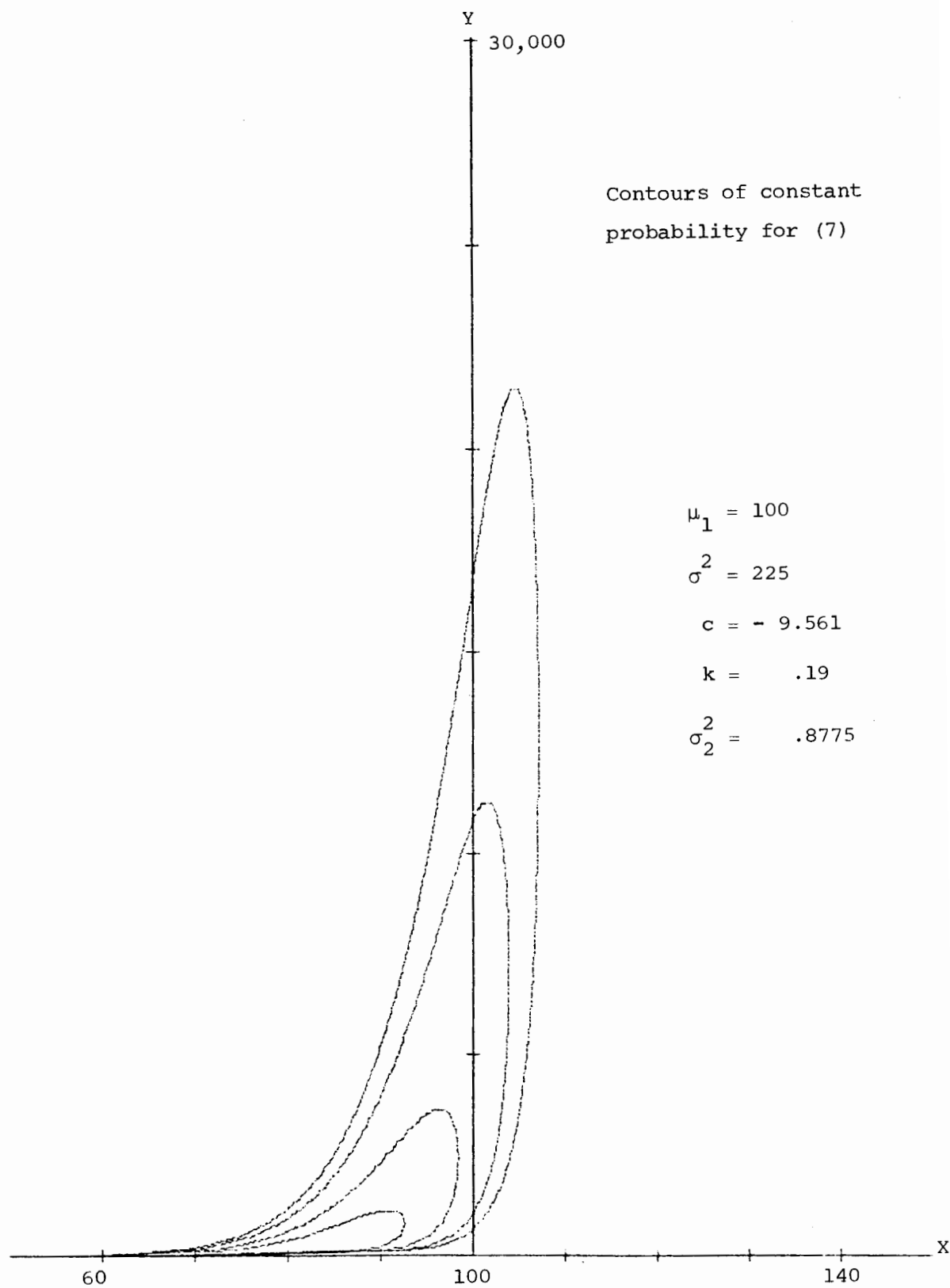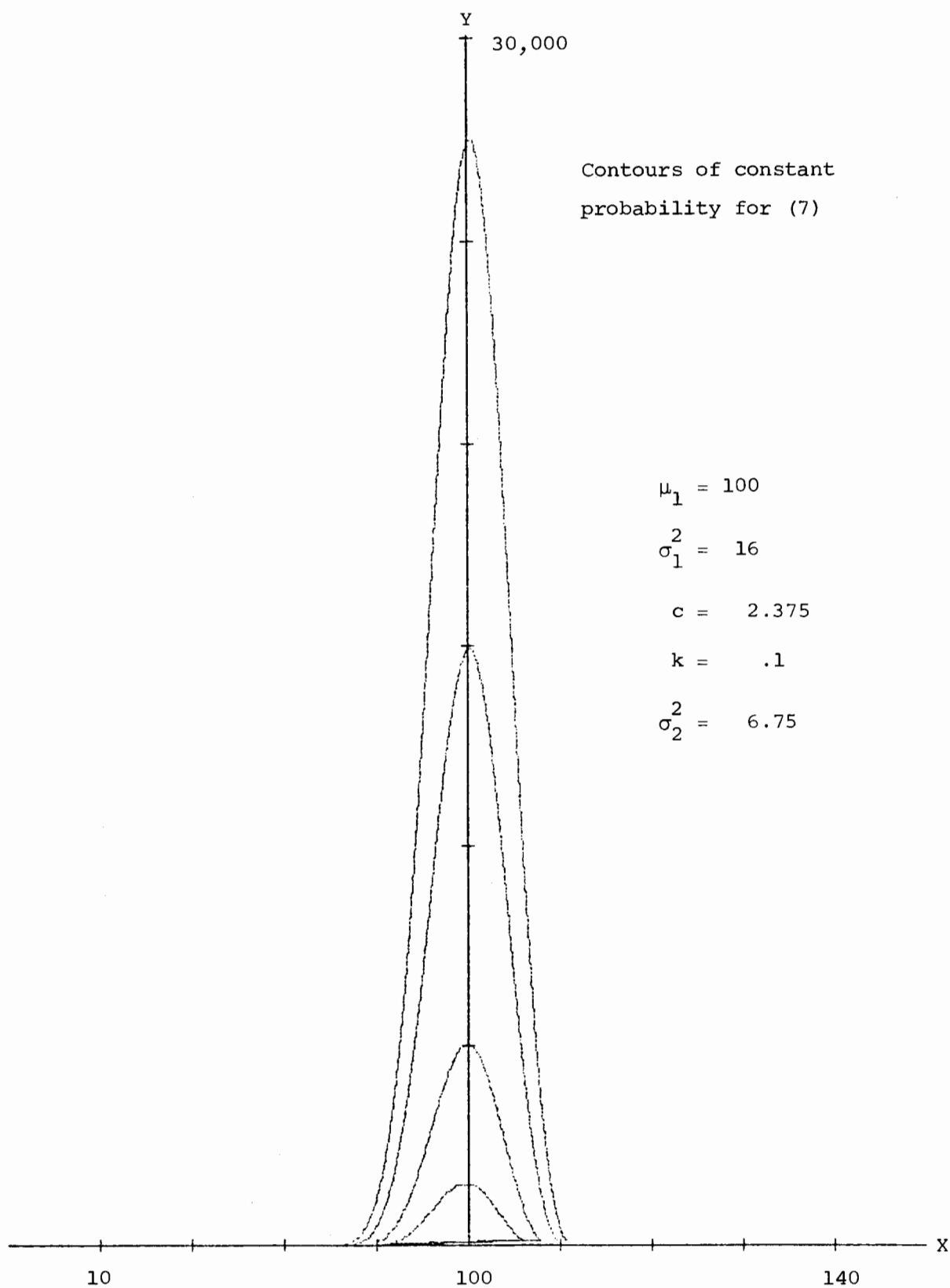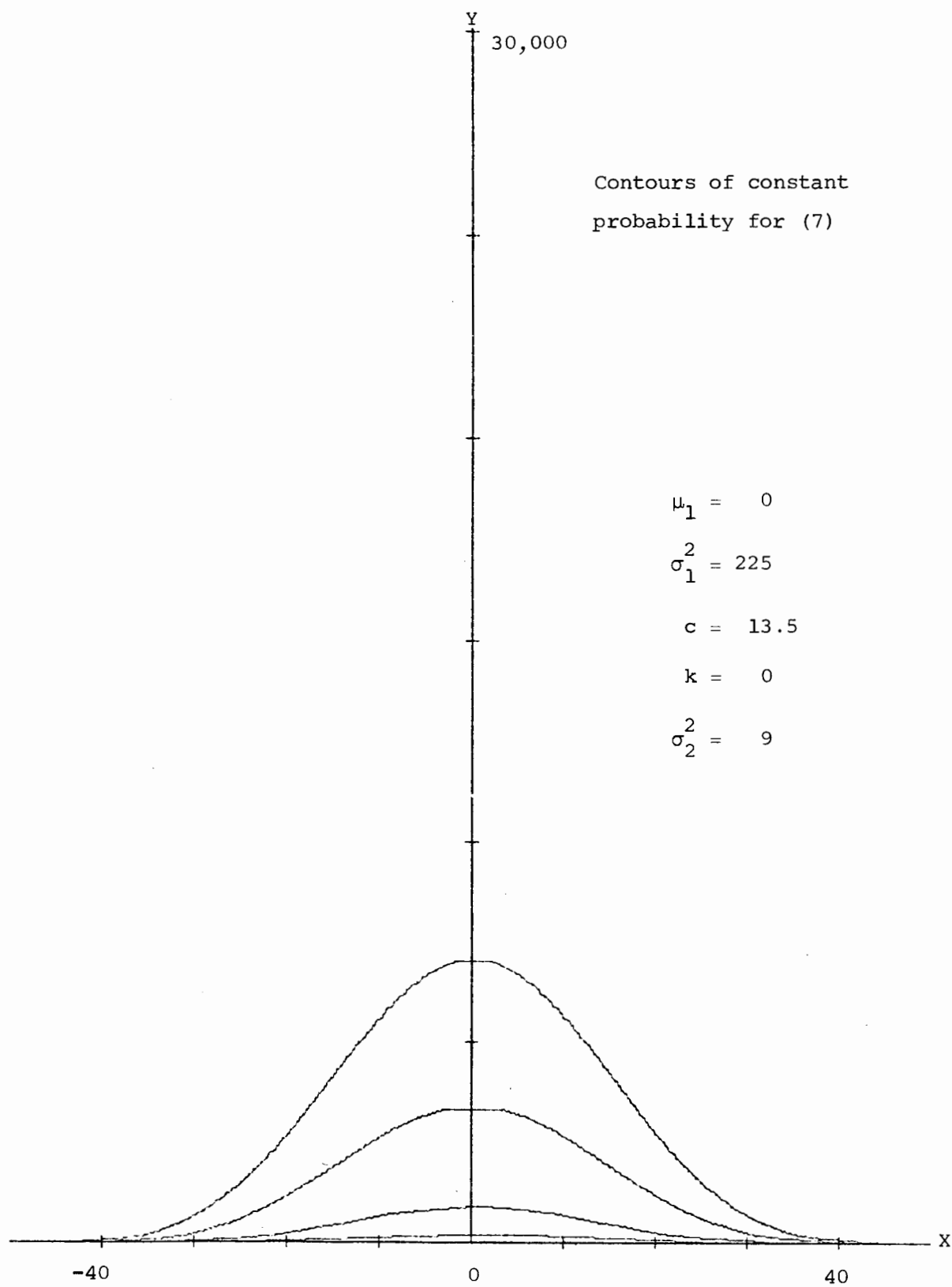
Figure 6

$$E(\ln\ Y)\ =\ k\mu_1\ +\ c\ -\ \frac{\sigma_2^2}{2} \tag{12}$$

and

$$Var(\ln\ Y)\ =\ k^2\sigma_1^2\ +\ \sigma_2^2\ . \tag{13}$$

Thus Y has a lognormal distribution with

$$E(Y)\ =\ e^{\left(k\mu_1\ +\ c\ -\ \frac{\sigma_2^2}{2}\right)\ +\ \left(\frac{k^2\sigma_1^2\ +\ \sigma_2^2}{2}\right)}$$

$$=\ e^{k\mu_1\ +\ c\ +\ \frac{k^2\sigma_1^2}{2}} \tag{14}$$

and

$$Var(Y)\ =\ e^{2k\mu_1\ +\ 2c\ +\ k^2\sigma_1^2}\left(e^{k^2\sigma_1^2\ +\ \sigma_2^2}\ -\ 1\right). \tag{15}$$

Again, reference to (9) and (10) shows the conditional density of X given Y = y to be normal with

$$E(X|Y\ =\ y)\ =\ \frac{\mu_1\sigma_2^2\ -\ \sigma_1^2 k\left(c\ -\ \frac{\sigma_2^2}{2}\ -\ \ln\ y\right)}{k^2\sigma_1^2\ +\ \sigma_2^2} \tag{16}$$

$$V(X|Y\ =\ y)\ =\ \frac{\sigma_1^2\sigma_2^2}{k^2\sigma_1^2\ +\ \sigma_2^2} \tag{17}$$

Product moments may be computed either directly or by use of the characteristic function which is here derived.

First, consider the univariate lognormal distribution with

$$f(y) = \frac{1}{\sqrt{2\pi}\,\sigma\,y} e^{-1/2\left(\frac{\ln y - A}{\sigma}\right)^2}.$$

(18)

It is shown in Appendix A that the rth moment about the origin for this distribution is given by

$$\mu_r{}' = e^{rA + \frac{r^2\sigma^2}{2}}$$

(19)

Heyde (1963) has shown that the moments of the lognormal distribution, even though they all exist, do not characterize the distribution. However by application of Theorem 2.3.3 in Lukacs (1960) we see that the characteristic function for the lognormal distribution (18) may be represented by

$$\varphi_X(t) = E(e^{itX})$$

$$= 1 + \sum_{j=1}^{n} \frac{1}{j!} e^{jA + \frac{1}{2}j^2\sigma^2} (it)^j + R_n(t)$$

(20)

where

$$R_n(t) = \frac{1}{n!}\left[\varphi^{(n)}(\theta t) - \mu_n' i^n\right]t^n, \quad 0 < \theta < 1.$$

(21)

Now, the joint characteristic function for the distribution (7) is given by

$$\Psi_{X,Y}(u,v) = E\left(e^{i(Xu + Yv)}\right)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1\sigma_2}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{1}{y}e^{ixu + iyv}\cdot$$

$$\left(e^{-1/2\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{\ln y - (kx + c - \frac{1}{2}\sigma_2^2)}{\sigma_2}\right)^2\right]}\right)dy\,dx$$

$$= \frac{1}{2\pi\,\sigma_1}\int_{-\infty}^{\infty}\left\{e^{ixu}e^{-1/2\left(\frac{x-\mu_1}{\sigma_1}\right)^2}\left(\frac{1}{\sqrt{2\pi}\,\sigma_2\,y}\right.\right.$$

$$\left.\left.\int_0^{\infty}\frac{1}{y}e^{ivy}e^{-1/2\left[\frac{\ln y - (kx + c - \frac{1}{2}\sigma_2^2)}{\sigma_2}\right]^2}dy\right)dx\right\}.$$

Note that the inner integral is simply the characteristic function for a univariate lognormal distribution with the A, σ^2, and t of (18) being replaced by $kx + c - \frac{1}{2}\sigma_2^2$, σ_2^2, and v respectively. Therefore (20) may be written

$$\frac{1}{\sqrt{2\pi}\ \sigma_1} \int_{-\infty}^{\infty} e^{ixu - \frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \left[1 + \right.$$

$$\left. \sum_{j=1}^{n} \left(\frac{1}{j!} \left(e^{j\left(kx + c - \frac{\sigma_2^2}{2}\right) + \frac{1}{2}\sigma_2^2 j^2} \right) (iv)^j \right) + R_n(v) \right] dx \qquad (23)$$

Now since

$$\frac{1}{\sqrt{2\pi}\ \sigma_1} \int_{-\infty}^{\infty} e^{i\ \theta\ w}\ e^{-1/2\left(\frac{w-\alpha}{\beta}\right)} dw$$

$$\qquad (24)$$

$$= e^{i\alpha\theta - \frac{1}{2}\theta^2\beta^2}$$

We may integrate (23) termwise to obtain

$$\Psi_{X,Y}(u,v) = e^{i\mu_1 u - \frac{1}{2}u^2\sigma_1^2}$$

$$\qquad (25)$$

$$+ \sum_{j=1}^{n} \frac{1}{j!} \left[e^{jc + \frac{\sigma_2^2}{2}(j^2 - j) + i\mu_1(u - ijk) - \frac{1}{2}(u - ijk)^2\sigma_1^2} (iv)^j \right]$$

$$+ \frac{1}{\sqrt{2\pi}\ \sigma_1} \int_{-\infty}^{\infty} \left\{ e^{iux} \frac{1}{n!} \left[\varphi^{(n)}(\theta v) - e^{n(kx + c - \frac{1}{2}\sigma_2^2) + \frac{n^2}{2}\sigma_2^2} i^n \right] v^n \right.$$

$$\left. e^{-1/2\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \right\} dx \quad .$$

This function may be used to compute $E(X^a Y^b)$ simply by choosing

$n = b + 1$. This is illustrated below for the computation of E (X Y).

First, write (25) with n = 2

$$\Psi_{X,Y}(u,v) = e^{iu\mu_1 - \frac{1}{2} u^2 \sigma_1^2}$$

$$+ e^{c + i\mu_1(u - ik) - \frac{1}{2}(u - ik)^2 \sigma_1^2} iv$$

$$- \frac{1}{2} e^{2c + \sigma_2^2 + i\mu_1(u - 2ik) - \frac{1}{2}(u - Zik)^2 \sigma_1^2} v^2$$

$$+ \frac{1}{2\sqrt{2\pi}\,\sigma_1} \int_{-\infty}^{\infty} \left(e^{iux} \left[\varphi^{(2)}(\theta v) + e^{2(kx + c - \frac{1}{2}\sigma_2^2) + 2\sigma_2^2} \right] v^2 \right.$$

$$\left. e^{-1/2\left(\frac{x - \mu_1}{\sigma_1}\right)^2} \right) dx$$

$$\frac{\partial \Psi_{X,Y}(u,v)}{\partial v} = i\, e^{c + i\mu_1(u - ik) - \frac{1}{2}(u - ik)^2 \sigma_1^2}$$

$$- v\, e^{2c + \sigma_2^2 + i\mu_1(u - 2ik) - \frac{1}{2}(u - 2ik)^2 \sigma_1^2}$$

$$+ \frac{1}{2\sqrt{2\pi}\,\sigma_1} \int_{-\infty}^{\infty} \left(e^{iux}\, e^{-1/2\left(\frac{x - \mu_1}{\sigma_1}\right)^2} \left[2v\left(\varphi^{(2)}(\theta v) \right.\right.\right.$$

$$\left.\left.\left. - e^{2(kx + c - \frac{1}{2}\sigma_2^2) + 2\sigma_2^2} \right) + v^2 \frac{\partial}{\partial v}\left(\varphi^{(2)}(\theta v) \right) \right] dx \right)$$

$$\frac{\partial^2 \Psi_{X,Y}(u,v)}{\partial u \ \partial v} = \left(ie^{c} + i\mu_1(u - ik) - \frac{1}{2}(u - ik)^2\sigma_1^2\right)$$

$$\left(i\mu_1 - (u - ik)\sigma_1^2\right) - v\left(e^{[2c + \sigma_1^2 + i\mu_1(u - 2ik) - \frac{1}{2} \cdot}\right.$$

$$\left.(u - 2ik)^2\sigma_1^2\right)\left(i\mu_1 - (u - 2ik)\sigma_1^2\right)$$

$$+ \frac{1}{2\sqrt{2\pi}\ \sigma_1} \int_{-\infty}^{\infty} ixe^{iux}\ e^{-1/2\left(\frac{x - \mu_1}{\sigma_1}\right)^2} \left[2v\left(\varphi^{(2)}\ (\theta v)\right.\right.$$

$$- e^{2(kx + c - \frac{1}{2}\sigma_2^2) + 2\sigma_2^2}\right) + v^2\ \frac{\partial}{\partial v}\left(\varphi^{(2)}\ (\theta v)\right)\right] dx$$

$$\frac{\partial^2 \Psi_{X,Y}(u,v)}{\partial u \ \partial v}\Bigg|_{\substack{u\,=\,0\\v\,=\,0}} = ie^{c + \mu_1 k + \frac{1}{2}k^2\sigma_1^2}\ (i\mu_1 + ik\sigma_1^2)$$

$$= -\ (\mu_1 + k\sigma_1^2)e^{c + \mu_1 k + \frac{1}{2}k^2\sigma_1^2}$$

and

$$E\ (XY) = (\mu_1 + k\sigma_1^2)e^{c + \mu_1 k + \frac{1}{2}k^2\sigma_1^2}\ . \tag{26}$$

We may now compute the covariances

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= e^{c + k\mu_1 + \frac{1}{2}k^2\sigma_1^2} \ (\mu_1 + k\sigma_1^2)$$

$$- \mu_1 e^{k\mu_1 + c + \frac{1}{2}k^2\sigma_1^2}$$

$$= k\sigma_1^2 \ e^{k\mu_1 + c + \frac{1}{2}k^2\sigma_1^2} \tag{27}$$

The coefficient of correlation between X and Y is then given by

$$\rho_{X,Y} = \frac{k\sigma_1^2 \ e^{k\mu_1 + c + \frac{1}{2}k^2\sigma_1^2}}{\sigma_1\sqrt{e^{2k\mu_1 + 2c + k^2\sigma_1^2}} \ (e^{k^2\sigma_1^2 + \sigma_2^2} - 1)} \tag{28}$$

$$= \frac{k\sigma_1}{\sqrt{e^{k^2\sigma_1^2 + \sigma_2^2} - 1}}$$

If we expand

$$e^{k^2\sigma_1^2 + \sigma_2^2} - 1 = (k^2\sigma_1^2 + \sigma_2^2) + \frac{1}{2}(k^2\sigma_1^2 + \sigma_2^2)^2 + \cdots$$

we see that

$$\rho_{x,y} = \frac{k\sigma_1}{\sqrt{(k^2\sigma_1^2 + \sigma_2^2) + \frac{1}{2!}(k^2\sigma_1^2 + \sigma_2^2)^2 + \ldots}}$$

so that
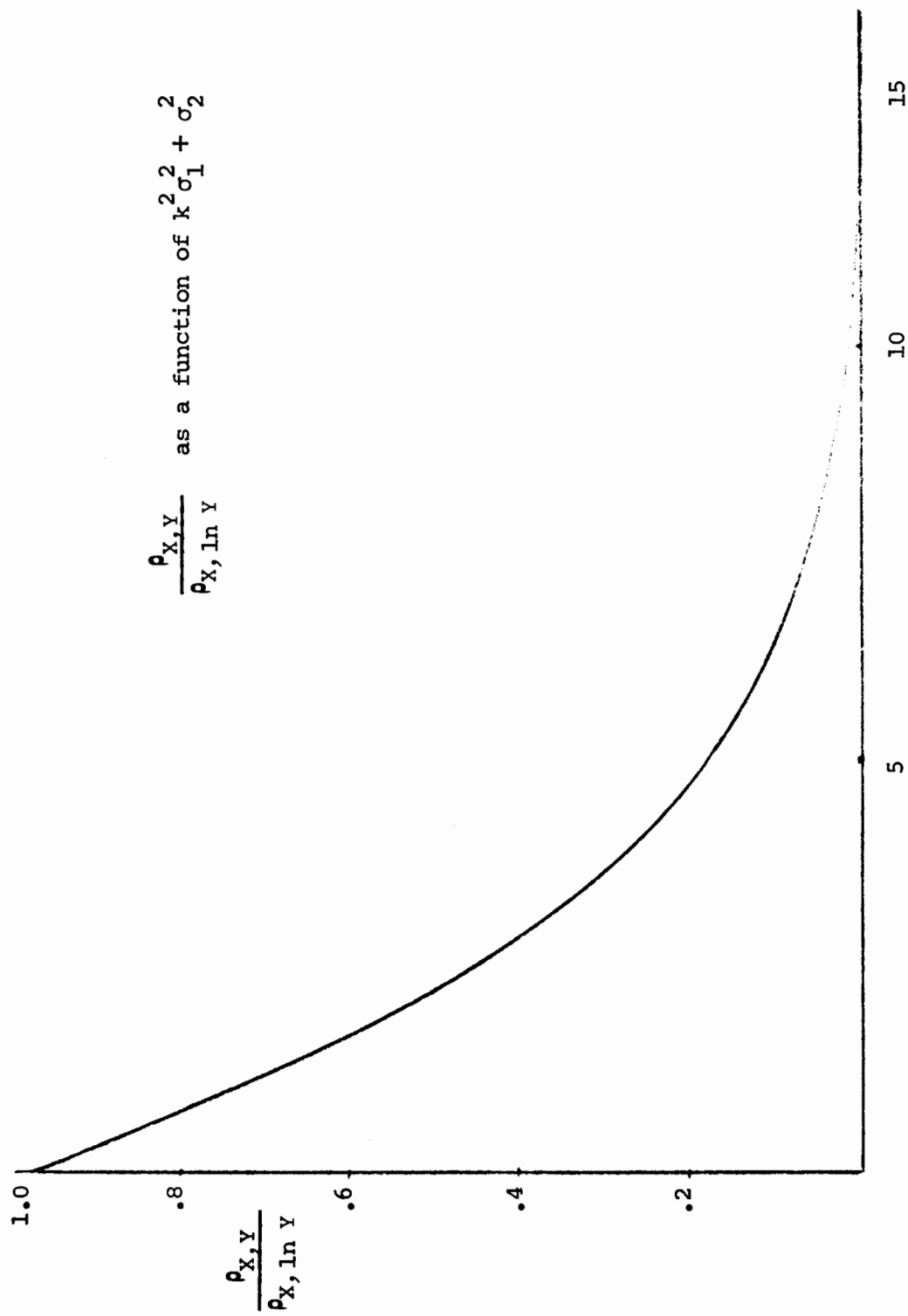
$$|\rho_{X,Y}| \le |\rho_{X,\ln Y}| \tag{29}$$

with equality holding only when $k = 0$.

Figure 7 shows $\dfrac{\rho_{x,y}}{\rho_{x,\ln y}}$ as a function of $k^2\sigma_1^2 + \sigma_2^2$.

If $\rho_{X,\ln Y} \to 1$ then $\dfrac{k\sigma_1}{\sqrt{k^2\sigma_1^2 + \sigma_2^2}} \to 1$ which implies $\sigma_2^2 \to 0$ and

$$\rho_{X,Y} \to \frac{k\sigma_1}{\sqrt{e^{k^2\sigma_1^2} - 1}} = \nu.$$ Figure 8 shows ν as a function of $k\sigma_1$.

$$\frac{\rho_{X,Y}}{\rho_{X,\ln Y}} \quad \text{as a function of } k^2\sigma_1^2 + \sigma_2^2$$

FIGURE 7

ν as a function of $k\sigma_1$
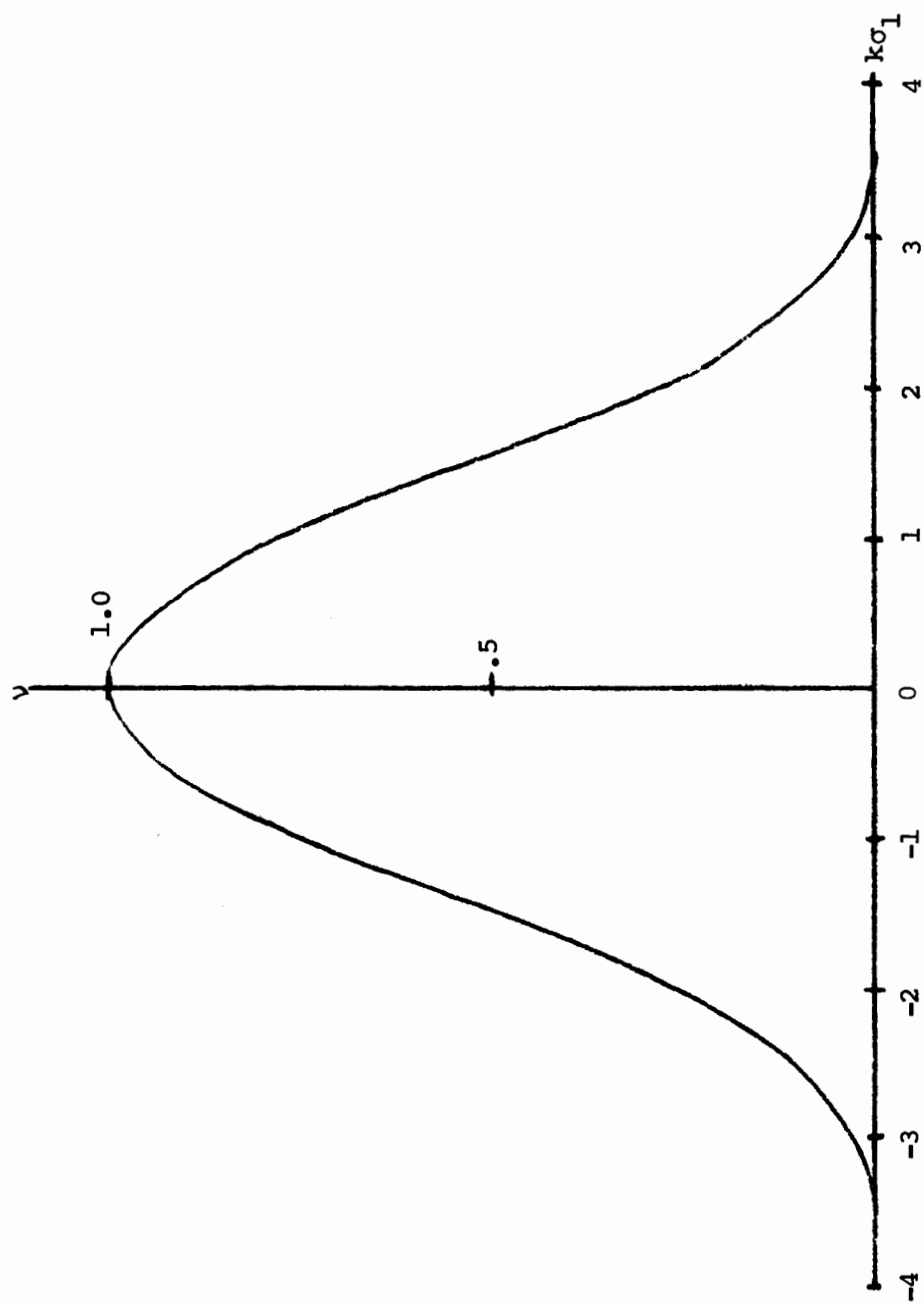
FIGURE 8

ESTIMATION OF PARAMETERS

Let (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n) be a sample of n independent observations drawn from the bivariate population with density given by (7). As there are 5 parameters to be estimated there are $2^5 - 1 = 31$ separate estimation problems, depending on which parameters are known. This paper will treat only 2 of these cases. It is felt that these typify the situations most likely to confront the statistician. The following cases are considered.

1. μ_1, σ_1^2, σ_2^2 are known.

2. All five parameters are unknown.

Case 1. μ_1, σ_1^2, σ_2^2 are known.

The likelihood function is given by

$$L = \frac{1}{(2\pi)^n \sigma_1^n \sigma_2^n \prod\limits_{i=1}^{n} y_i} e^{-1/2\left[\frac{\sum\limits_{i=1}^{n}(x_i-\mu_1)^2}{\sigma_1^2} + \frac{\sum\limits_{i=1}^{n}\left(\ln y_i - (kx_i + c - \frac{\sigma_2^2}{2})\right)^2}{\sigma_2^2}\right]} \tag{30}$$

and

$$\ln L = -n \ln 2\pi - \frac{n}{2} \ln \sigma_1^2 - \frac{n}{2} \ln \sigma_2^2 - \ln \prod y_i$$

$$- \frac{1}{2\sigma_1^2} \Sigma(x_i-\mu_1)^2 - \frac{1}{2\sigma_2^2} \Sigma\left(\ln y_i - kx_i - c + \frac{\sigma_2^2}{2}\right)^2 . \tag{31}$$

The derivatives of ln L with respect to k and c are

$$\frac{\partial \ln L}{\partial k} = \frac{1}{\sigma_2^2} \Sigma \left(\ln y_i - kx_i - c + \frac{\sigma_2^2}{2} \right) x_i$$

and

$$\frac{\partial \ln L}{\partial c} = \frac{1}{\sigma_2^2} \Sigma \left(\ln y_i - kx_i - c + \frac{\sigma_2^2}{2} \right) .$$

Setting these derivatives equal to zero it is easily seen that

$$\hat{k} = \frac{\Sigma (x_i - \bar{x}) \ln y_i}{\Sigma (x_i - \bar{x})^2} \tag{32}$$

$$\hat{c} = \frac{1}{n} \Sigma \ln y_i + \frac{\sigma_2^2}{2} - \hat{k} \bar{x} . \tag{33}$$

Investigation of these solutions by consideration of $\frac{\partial^2 L}{\partial k^2}$, $\frac{\partial^2 L}{\partial c^2}$,

and $\frac{\partial^2 L}{\partial c \partial k}$ show that they do maximize L.

We now check these estimators for bias

$$E(\hat{k}) = E \left(E(\hat{k} | \underline{X}) \right)$$

$$E(\hat{k} | \underline{X}) = \frac{\Sigma (x_i - \bar{x}) E(\ln Y_i | X)}{\Sigma (x_i - \bar{x})^2}$$

Reference to equation (6) shows that

$$E(\ln Y_i | \underline{X}) = kx_i + c - \frac{\sigma_2^2}{2} .$$

Thus

$$E(\hat{k}|\underline{x}) = \frac{\sum(x_i - \bar{x})\left([kx_i - k\bar{x}] + [k\bar{x} + c - \frac{\sigma_2^2}{2}]\right)}{\sum(x_i - \bar{x})^2}$$

$$= k$$

(34)

and

$$E\left(E(\hat{k}|\underline{x})\right) = k \quad .$$

(35)

Likewise, we show that $\hat{c}$ is unbiased.

$$E(\hat{c}) = E\left(E\left[\frac{1}{n} \sum \ln Y_i + \frac{\sigma_2^2}{2} - \hat{k}\bar{x} \,|\underline{x}\right]\right)$$

$$= E(\frac{1}{n} \sum (kx_i + c - \frac{\sigma_2^2}{2}) + \frac{\sigma_2^2}{2} - k\bar{x})$$

$$= \left(\frac{1}{n} kn\mu_1 + c - \frac{\sigma_2^2}{2} + \frac{\sigma_2^2}{2} - k\mu_1\right)$$

$$= c .$$

(36)

To find the variance of $\hat{k}$ we use the fact that

$$Var(\hat{k}) = Var\left(E(\hat{k}|\underline{x})\right) + E\left(Var(\hat{k}|\underline{x})\right)$$

(37)

Since by (34) $E(\hat{k}|\underline{x}) = k$, $Var\left(E(\hat{k}|\underline{x})\right) = 0$.

We see from (2) that $Var(\ln Y_i|\underline{x}) = \sigma_2^2$ so that

$$\text{Var}(\hat{k}|\underline{x}) = \frac{\sigma_2^2 \, \Sigma(x_i - \bar{x})^2}{[\Sigma(x_i - \bar{x})^2]^2}$$

$$= \frac{\sigma_2^2}{\Sigma(x_i - \bar{x})^2} \cdot \tag{38}$$

Suppose U has a chi-square distribution with ν degrees of freedom. Then

$$E\left(\frac{1}{U^p}\right) = \frac{1}{\Gamma(\frac{\nu}{2})2^{\nu/2}} \int_0^\infty \frac{1}{u^p} u^{\frac{\nu}{2}-1} e^{-\frac{u}{2}} du$$

$$= \frac{1}{\Gamma(\frac{\nu}{2})2^{\nu/2}} \Gamma(\frac{\nu}{2} - p)2^{\frac{\nu}{2} - p} \quad , \quad \nu > 2p \; . \tag{39}$$

We note two special cases

$$E\left(\frac{1}{U}\right) = \frac{1}{\nu - 2} \qquad \nu > 2 \tag{40}$$

$$E\left(\frac{1}{U^2}\right) = \frac{1}{(\nu - 2)(\nu - 4)} \cdot \quad \nu > 4 \tag{41}$$

Since $\dfrac{\Sigma(x_i - \bar{x})^2}{\sigma_1^2}$ has a chi-square distribution with n-1 degrees of

freedom it is readily seen that

$$E\left(\text{Var}(\hat{k}|\underline{X})\right) = \frac{\sigma_2^2}{(n-3)\sigma_1^2} \quad .$$

Thus from (29)

$$\text{Var}(\hat{k}) = \frac{\sigma_2^2}{(n-3)\sigma_1^2} \quad . \tag{42}$$

We will employ a similar technique for finding the variance of $\hat{c}$.

$$\text{Var}(\hat{c}) = \text{Var}\left(E(\hat{c}|\underline{X})\right) + E\left(\text{Var}(\hat{c}|\underline{X})\right) . \tag{43}$$

From the derivation of (36) we see that since $E(\hat{c}|\underline{X}) = c$

$$\text{Var } E(\hat{c}|x) = 0 \quad . \tag{44}$$

$$\text{Var}(\hat{c}|\underline{X}) = \text{Var}\left(\frac{1}{n} \Sigma \ln Y_i + \frac{\sigma_2^2}{2} - \hat{k}\,\bar{x}\,|\underline{X}\right)$$

$$= \frac{1}{n^2} \Sigma \text{Var}(\ln Y_i|\underline{X}) + \bar{x}^2 \text{Var}(\hat{k}|\underline{X})$$

$$- \frac{2\bar{x}}{n} \text{Cov}(\Sigma \ln Y_i, \hat{k}|\underline{X}) \quad . \tag{45}$$

From equation (2) we have

$$\frac{1}{n^2} \Sigma \text{Var}(\ln Y_i|\underline{X}) = \frac{\sigma_2^2}{n} \quad . \tag{46}$$

From (38)

$$\bar{x}^2 \text{Var}(\hat{k}|\underline{x}) = \frac{\bar{x}^2 \sigma_2^2}{\Sigma(x_i - \bar{x})^2} \cdot$$

(47)

To calculate $\text{Cov}(\Sigma \ln Y_j, \hat{k}|\underline{x})$ first we note that

$$E(\hat{k} \Sigma \ln Y_i |\underline{x})$$

$$= E\left(\Sigma \ln Y_i \frac{\Sigma(x_i - \bar{x})\ln Y_i}{\Sigma(x_i - \bar{x})^2} \Big| \underline{x}\right)$$

$$= \frac{1}{\Sigma(x_i - \bar{x})^2} E\left(\Sigma(x_i - \bar{x})\ln^2 Y_i + \underset{i \neq j}{\Sigma \Sigma} (x_i - \bar{x})\ln Y_i \ln Y_j \Big| \underline{x}\right).$$

(48)

Now, since given $\underline{x}$, $\ln Y_i$ has a normal distribution with mean

$kx_i + c - \dfrac{\sigma_2^2}{2}$ and variance σ_2^2 it follows that $\dfrac{1}{\sigma_2^2} \ln^2 Y_i$ has a noncentral

chi-square distribution with one degree of freedom and noncentrality

parameter

$$\lambda = \frac{1}{2\sigma_2^2} \left(kx_i + c - \frac{\sigma_2^2}{2}\right)^2 \cdot$$

Thus

$$E\left(\sum (x_i - \bar{x}) \ln^2 Y_i \,\big|\, \underline{X}\right)$$

$$= \sum (x_i - \bar{x})\left(\sigma_2^2 + (kx_i + c - \frac{\sigma_2^2}{2})^2\right)$$

$$= \sum (x_i - \bar{x})\left(kx_i + c - \frac{\sigma_2^2}{2}\right)^2 .$$

Further,

$$E\left(\sum_{i \neq j} \sum (x_i - \bar{x}) \ln Y_i \ln Y_j \,\big|\, \underline{X}\right)$$

$$= \sum_{i \neq j} \sum (x_i - \bar{x})\left(kx_i + c - \frac{\sigma_2^2}{2}\right)\left(kx_j + c - \frac{\sigma_2^2}{2}\right) .$$

$$E\left[\sum (x_i - \bar{x}) \ln^2 Y_i + \sum_{i \neq j} \sum (x_i - \bar{x}) \ln Y_i \ln Y_j \,\big|\, \underline{X}\right]$$

$$= \sum (x_i - \bar{x})\left(kx_i + c - \frac{\sigma_2^2}{2}\right)^2$$

$$+ \sum_{i \neq j} \sum (x_i - \bar{x})\left(kx_i + c - \frac{\sigma_2^2}{2}\right)\left(kx_j + c - \frac{\sigma_2^2}{2}\right)$$

$$= \left\{ \sum (x_i - \bar{x})\left[(kx_i - k\bar{x}) + \left(k\bar{x} + c - \frac{\sigma_2^2}{2}\right)\right] \right.$$

$$\left. \sum \left(kx_i + c - \frac{\sigma_2^2}{2}\right)\right\}$$

$$= k \, \Sigma(x_i - \bar{x})^2 \left(nk\bar{x} + n\left(c - \frac{\sigma_2^2}{2}\right)\right) .$$

So that by (48)

$$E(\hat{k} \, \Sigma \ln Y_i | \underline{X}) = k\left(nk\bar{x} + n\left(c - \frac{\sigma_2^2}{2}\right)\right) . \tag{49}$$

We know from (34) and (6) that

$$E(\hat{k} | \underline{X}) = k$$

$$E(\Sigma \ln Y_i | \underline{X}) = \Sigma\left(kx_i + c - \frac{\sigma_2^2}{2}\right)$$

$$= nk\bar{x} + n\left(c - \frac{\sigma_2^2}{2}\right) .$$

So that $\mathrm{Cov}(\hat{k}, \, \Sigma \ln Y_i | \underline{X}) = 0 .$ (50)

Using (46), (47) and (50) in equation (45) we find

$$\mathrm{Var}(\hat{c} | \underline{X}) = \frac{\sigma_2^2}{n} + \frac{\bar{x}^2 \, \sigma_2^2}{\Sigma(x_i - \bar{x})^2}$$

Now, since $\bar{x}$ and $\Sigma(x_i - \bar{x})^2$ are independent

$$E\left(\mathrm{Var}(\hat{c} | \underline{X})\right) = \frac{\sigma_2^2}{n} + \sigma_2^2 \, E(\bar{x}^2) \, E\left(\frac{1}{\Sigma(x_i - \bar{x})^2}\right) \tag{51}$$

From (40) we know that

$$E\left(\frac{1}{\sum_i (x_i - \bar{x})^2}\right) = \frac{1}{(n-3)\sigma_1^2} \quad .$$

Since $\bar{x}$ is distributed normally with mean μ_1 and variance σ_1^2/n we know

$$E(\bar{x}^2) = \frac{\sigma_1^2}{n} + \mu_1^2 \quad . \tag{52}$$

Finally from (40), (43), (44), (51) and (52) we see that

$$\begin{aligned}
\mathrm{Var}(\hat{c}) &= E\left(\mathrm{Var}(\hat{c}|\underline{x})\right) \\
&= \frac{\sigma_2^2}{n} + \frac{\sigma_2^2}{(n-3)\sigma_1^2}\left[\frac{\sigma_1^2}{n} + \mu_1^2\right] \quad .
\end{aligned} \tag{53}$$

To find the covariance between $\hat{k}$ and $\hat{c}$ we use

$$\mathrm{Cov}(\hat{k}, \hat{c}) = E(\hat{k}\hat{c}) - E(\hat{k})E(\hat{c}) \tag{54}$$

$$\begin{aligned}
E(\hat{k}\hat{c}) &= E\left(\hat{k}\left(\frac{1}{n}\sum \ln Y_i + \frac{\sigma_2^2}{2} - \hat{k}\,\bar{x}\right)\right) \\
&= \frac{1}{n}E(\hat{k}\sum \ln Y_i) + \frac{\sigma_2^2}{2}E(\hat{k}) - E(\hat{k}^2\bar{x})
\end{aligned}$$

From (49) we see that

$$E(\hat{k}\sum \ln Y_i|\underline{x}) = k\left(n\,k\,\bar{x} + n\left(c - \frac{\sigma_2^2}{2}\right)\right)$$

and

$$E\left(\hat{k} \sum \ln Y_i\right) = E\left(E(\hat{k} \ln Y_i \mid \underline{X})\right)$$

$$= k\left(nk\mu_1 + n\left(c - \frac{\sigma_2^2}{2}\right)\right) .$$

Since $\hat{k}$ is unbiased, $E(\hat{k}) = k$.

With the help of (30) we find

$$E(\hat{k}^2 \bar{x}) = E\left(\bar{x} E(\hat{k}^2 \mid \underline{X})\right)$$

$$= E\left(\bar{x}\left[\text{Var}(\hat{k} \mid \underline{X}) + E^2(\hat{k} \mid \underline{X})\right]\right)$$

$$= E\left(\bar{x}\left(\frac{\sigma_2^2}{\sum(x_i - \bar{x})^2} + k^2\right)\right)$$

$$= \frac{\sigma_2^2 \mu_1}{(n - 3)\sigma_1^2} + k^2\mu_1 .$$

Thus

$$\text{Cov}(\hat{k}, \hat{c}) = \frac{1}{n} kn\left(k\mu_1 + \left(c - \frac{\sigma_2^2}{2}\right)\right) + \frac{\sigma_2^2}{2} k$$

$$- \frac{\sigma_2^2 \mu_1}{(n - 3)\sigma_1^2} - k^2\mu_1 - kc$$

$$= - \frac{\sigma_2^2 \mu_1}{(n - 3)\sigma_1^2} . \tag{55}$$

If one is interested only in the large sample distribution of $\hat{k}$ and $\hat{c}$ he may verify by the usual properties of maximum likelihood estimation that asymptotically, $\begin{pmatrix} \hat{k} \\ \hat{c} \end{pmatrix}$ has a bivariate normal distribution with mean $\begin{pmatrix} k \\ c \end{pmatrix}$ and covariance matrix

$$
V = \begin{pmatrix} \dfrac{\sigma_2^2}{n\,\sigma_1^2} & -\dfrac{\mu_1 \sigma_2^2}{n\sigma_1^2} \\[20pt] -\dfrac{\mu_1 \sigma_2^2}{n\sigma_1^2} & \dfrac{\sigma_2^2}{n\sigma_1^2}(\sigma_1^2 + \mu_1^2) \end{pmatrix} . \tag{56}
$$

The techniques leading to this result may be found in Chapter 18 of Kendall and Stuart (1961).

Since we know μ_1 and σ_1^2 it seems reasonable to investigate estimators of k and c which involve these known values rather than the random variables $\bar{x}$ and $s_{\sim x}^2 = \dfrac{1}{n-1} \Sigma(x_{\sim i} - \bar{x})^2$ which estimate them.

From equation (32) we know the maximum likelihood estimator of k is

$$
\hat{k} = \frac{\Sigma x_i \ln y_i - \bar{x} \Sigma \ln y_i}{\Sigma(x_i - \bar{x})^2} .
$$

Suppose we construct an estimator similar to this, say

$$
\tilde{k} = \frac{\Sigma x_i \ln y_i - a\, \mu_1 \Sigma \ln y_i}{b\, \sigma_1^2} \tag{57}
$$

where a and b are constants that may be adjusted to eliminate bias.

$$E(\tilde{k}|\underline{x}) = \frac{\sum(x_i - a\mu_1)\left(\left[kx_i - ka\mu_1\right] + \left[ka\mu_1 + c - \frac{\sigma_2^2}{2}\right]\right)}{b\sigma_1^2} \tag{58}$$

$$E(\tilde{k}) = \frac{k}{b\sigma_1^2} E\left(\sum(x_i - a\mu_1)^2\right). \tag{59}$$

Now $\dfrac{1}{\sigma_1^2} \sum(x_i - a\mu_1)^2$ has a noncentral chi-square distribution with n

degrees of freedom and noncentrality parameter

$$\lambda = \frac{1}{2} \frac{\mu_1^2(1 - a)^2}{\sigma_1^2}.$$

Thus

$$E\left(\frac{\sum(x_i - a\mu_1)^2}{\sigma_1^2}\right) = n + \frac{\mu_1^2(1 - a)^2}{\sigma_1^2}$$

and

$$E(\tilde{k}) = \frac{k}{b}\left[n + \frac{\mu_1^2(1 - a)^2}{\sigma_1^2}\right]. \tag{60}$$

We may eliminate bias if a = 1 and b = n. Thus

$$\tilde{k} = \frac{\sum(x_i - \mu_1)\ln y_i}{n\sigma_1^2}. \tag{61}$$

$$\text{Var }(\tilde{k}) = E(\text{Var}(\tilde{k}|\underline{x})) + \text{Var}(E(\tilde{k}|\underline{x})). \tag{62}$$

$$\text{Var}(\tilde{k}|\underline{X}) = \frac{1}{n^2\sigma_1^4} \Sigma (x_i - \mu_1)^2 \sigma_2^2 .$$

$$E\left(\text{Var}(\tilde{k}|\underline{X})\right) = \frac{\sigma_2^2}{n^2\sigma_1^4} E\left(\Sigma(x_i - \mu_1)^2\right)$$

$$= \frac{\sigma_2^2}{n\sigma_1^2} .$$

(63)

We use (58) to find for $a = 1$, $b = n$

$$\text{Var}\left(E(\tilde{k}|\underline{X})\right) = \text{Var}\left[\frac{k \Sigma(x_i - \mu_1)^2 + \left(k\mu_1 + c - \frac{\sigma_2^2}{2}\right)\Sigma(x_i - \mu_1)}{n\sigma_1^2}\right]$$

$$= \frac{1}{n^2\sigma_1^4}\left[k^2 \text{Var}\left(\Sigma(x_i - \mu_1)^2\right)\right.$$

$$+ \left(k\mu_1 + c - \frac{\sigma_2^2}{2}\right)^2 \text{Var}\left(\Sigma(x_i - \mu_1)\right)$$

$$+ 2k\left(k\mu_1 + c - \frac{\sigma_2^2}{2}\right) \text{Cov}\left(\Sigma(x_i - \mu_1)^2 , \Sigma(x_i - \mu_1)\right)\bigg] .$$

$$\text{Var}\left(\Sigma(x_i - \mu_1)^2\right) = 2n\sigma_1^4$$

$$\text{Var}\left(\Sigma(x_i - \mu_1)\right) = n\sigma_1^2$$

$$\text{Cov}\left(\Sigma(x_i - \mu_1)^2 , \Sigma(x_i - \mu_1)\right) = E\left(\Sigma(x_i - \mu_1)^2 \Sigma(x_i - \mu_1)\right) - 0$$

$$= E\left(\Sigma(x_i - \mu_1)^3 + 2 \sum_{i \neq j} \sum (x_i - \mu_1)^2 (x_j - \mu_1)\right)$$

$$= 0 .$$

Further,

$$\text{Var}\left(E(\tilde{k}|\underline{X})\right) = \frac{1}{n^2 \sigma_1^4}\left[2k^2 n\sigma_1^4 + \left(k\mu_1 + c - \frac{\sigma_2^2}{2}\right)^2 n\sigma_1^2\right]$$

$$= \frac{2k^2}{n} + \frac{\left(k\mu_1 + c - \frac{\sigma_2^2}{2}\right)^2}{n\sigma_1^2} . \tag{64}$$

Applying equations (63) and (64) to (62) yields

$$\text{Var}(\tilde{k}) = \frac{\sigma_2^2}{n\sigma_1^2} + \frac{2k^2}{n} + \frac{\left(k\mu_1 + c - \frac{\sigma_2^2}{2}\right)^2}{n\sigma_1^2} . \tag{65}$$

Recall from (42) that

$$\text{Var}(\hat{k}) = \frac{\sigma_2^2}{(n - 3)\sigma_1^2} .$$

In order to decide whether, in a given situation, $\hat{k}$ or $\tilde{k}$ is the better estimator of k let us examine the relative efficiency of $\tilde{k}$ to $\hat{k}$.

$$E_{\tilde{k} \text{ to } \hat{k}} = \frac{\text{Var}(\hat{k})}{\text{Var}(\tilde{k})}$$

$$= \frac{n}{n - 3} \cdot \frac{1}{1 + \frac{2k^2\sigma_1^2}{\sigma_2^2} + \frac{1}{2\sigma_2^2}\left(k\mu_1 + c - \frac{\sigma_2^2}{2}\right)^2} \tag{66}$$

The above quantity exceeds unity when

$$2k^2\sigma_1^2 + \left(k\mu_1 + c - \frac{\sigma_2^2}{2}\right)^2 < \frac{3\sigma_2^2}{n-3} \; . \tag{67}$$

The quantity on the left will be zero, thus insuring the inequality, only if $k = 0$ and $c = \frac{\sigma_2^2}{2}$. If this is the case, $\tilde{k}$ is certainly the more efficient estimator for all finite n. If this is not the case, there will be some value for n above which the inequality will not hold and $\hat{k}$ will be the better estimator.

Although c and k are not known, it may be possible to place upper bounds on their values based on previous experience. If this is the case, one may use (67) as a criterion for choosing an estimator.

Recall that $k\mu_1 + c - \frac{\sigma_2^2}{2}$ is the expected value of ln Y. If, on the basis of past experience one is able to estimate the upper bound of E(ln Y) this may be useful in the application of (67) to the choice of estimators.

Let us investigate the efficiencies of $\hat{k}$ and $\tilde{k}$ relative to the Cramer-Rao minimum variance bound. These quantities may give us more insight into the proper choice of the estimator of k.

If we compute the Cramer-Rao bound for the variance of t, an unbiased estimator of k, we find, with the help of (56) that

$$\text{Var (t)} \geq \frac{\sigma_2^2}{n \, \sigma_1^2} \; . \tag{68}$$

We see that the efficiency of $\hat{k}$ relative to this bound is given by

$$E_{\hat{k}} = \frac{\dfrac{\sigma_2^2}{n\sigma_1^2}}{\dfrac{\sigma_2^2}{(n-3)\sigma_1^2}}$$

$$= \frac{n-3}{n} . \tag{69}$$

This estimator is asymptotically efficient.

The efficiency of $\tilde{k}$ relative to the minimum variance bound is given by

$$E_{\tilde{k}} = \frac{\dfrac{\sigma_2^2}{n\sigma_1^2}}{\dfrac{\sigma_2^2}{n\sigma_1^2} + \dfrac{2k^2\sigma_1^2}{n\sigma_1^2} + \dfrac{\left(k\mu_1 + c - \dfrac{\sigma_2^2}{2}\right)^2}{n\sigma_1^2}} \tag{70}$$

$$= \frac{1}{1 + \dfrac{2k^2\sigma_1^2}{\sigma_2^2} + \dfrac{1}{\sigma_2^2}\left(k\mu_1 + c - \dfrac{\sigma_2^2}{2}\right)^2}$$

$\tilde{k}$ is efficient only when $k = 0$ and $c = \dfrac{\sigma_2^2}{2}$. If these properties do not hold, there will be some value of n above which $\hat{k}$ will be a better

estimator of k than $\tilde{k}$.

To summarize, in a given situation in which k and c are not known, but k can not be considered zero, i.e. x and y are not independent, we see from (66) that $\hat{k}$ is asymptotically relatively more efficient than $\tilde{k}$ and probably would be chosen. If, on the other hand we have some information about k and c, inequality (67) might indicate the use of $\tilde{k}$ for a particular n.

Let us now consider a similar modification for $\hat{c}$, our ML estimator for c.

Recall that

$$\hat{c} = \frac{1}{n} \Sigma \ln y_i + \frac{\sigma_2^2}{2} - \hat{k}\bar{x} \quad .$$

Since $\bar{x}$ is an estimator of μ_1 we shall investigate

$$\tilde{c} = \frac{1}{n} \Sigma \ln y_i + \frac{\sigma_2^2}{2} - \tilde{k}\mu_1 \quad . \tag{71}$$

where $\tilde{k}$ is the quantity discussed in the foregoing section whose formula is given by (61).

Using equations (13) and (60) we see that

$$E(\tilde{c}) = \frac{1}{n} \Sigma \left(k\mu_1 + c - \frac{\sigma_2^2}{2} \right) + \frac{\sigma_2^2}{2} - k\mu_1$$

$$= c \tag{72}$$

$$\text{Var}(\tilde{c}) = \frac{1}{n^2} \text{Var}(\Sigma \ln y_i) + \mu_1^2 \text{Var}(\tilde{k})$$

$$- \frac{2\mu_1}{n} \text{Cov}(\tilde{k}, \Sigma \ln y_i) \; .$$

We know from (13) and (65) that

$$\text{Var}(\Sigma \ln y_i) = n\left(k^2 \sigma_1^2 + \sigma_2^2\right)$$

$$\text{Var}(\tilde{k}) = \frac{\sigma_2^2}{n\sigma_1^2} + \frac{2k^2}{n} + \frac{\left(k\mu_1 + c - \frac{\sigma_2^2}{2}\right)^2}{n\sigma_1^2} \; .$$

We need to find $\text{Cov}(\tilde{k}, \Sigma \ln Y_i)$

$$\text{Cov}(\tilde{k}, \Sigma \ln Y_i) = E(\tilde{k} \Sigma \ln Y_i) - E(\tilde{k}) E(\Sigma \ln Y_i) \; .$$

$$E(\tilde{k} \Sigma \ln Y_i) = E\left(\frac{1}{n\sigma_1^2} \Sigma(x_i - \mu_1) \ln Y_i \Sigma \ln Y_i\right)$$

$$= \frac{1}{n\sigma_1^2} E\left[\Sigma(x_i - \mu_1) \ln^2 Y_i + \underset{i \neq j}{\Sigma \Sigma} (x_i - \mu_1) \ln Y_i \ln Y_j\right] \; .$$

$$E(\tilde{k} \, \Sigma \ln Y_i | \underline{X}) = \frac{1}{n\sigma_1^2} \left[\Sigma(x_i - \mu_1)\left(\sigma_2^2 + \left(kx_i + c - \frac{\sigma_2^2}{2}\right)^2\right) \right.$$

$$\left. + \Sigma\Sigma_{i\neq j}(x_i - \mu_1)\left(kx_i + c - \frac{\sigma_2^2}{2}\right)\left(kx_j + c - \frac{\sigma_2^2}{2}\right) \right]$$

$$= \frac{1}{n\sigma_1^2} \left[\Sigma(x_i - \mu_1)\sigma_2^2 \right.$$

$$\left. + \Sigma(x_i - \mu_1)\left(kx_i + c - \frac{\sigma_2^2}{2}\right) \Sigma\left(kx_i + c - \frac{\sigma_2^2}{2}\right) \right] .$$

$$E(\tilde{k} \, \Sigma \ln Y_i) = k\left(k\mu_1 + c - \frac{\sigma_2^2}{2}\right)(n + 1) .$$

Thus

$$\text{Cov}(\tilde{k}, \, \Sigma \ln y_i) = k\left(k\mu_1 + c - \frac{\sigma_2^2}{2}\right) .$$

We see that

$$\text{Var}(\tilde{c}) = \frac{1}{n}\left[k^2\sigma_1^2 + \sigma_2^2\right] \tag{73}$$

$$+ \mu_1^2 \left[\frac{\sigma_2^2}{n\sigma_1^2} + \frac{2k^2}{n} + \frac{\left(k\mu_1 + c - \frac{\sigma_2^2}{2}\right)^2}{n\sigma_1^2} \right]$$

$$- \frac{2\mu_1}{n} k\left(k\mu_1 + c - \frac{\sigma_2^2}{2}\right) .$$

This is a complicated expression which will be difficult to examine for efficiency. Recall, however, that since μ_1, the expected value for X is known, we may make the transformation

$$Z = X - \mu_1 . \tag{74}$$

Our $\tilde{c}$ or $\hat{c}$ now estimates the exponent of the regression function at $x = \mu_1$, but our transformation allows us to replace μ_1 by 0. Thus, let us work with

$$\hat{c}_T = \frac{1}{n} \Sigma \ln y_i + \frac{\sigma_2^2}{2} - \hat{k} \bar{z} \tag{75}$$

$$\tilde{c}_T = \frac{1}{n} \Sigma \ln y_i + \frac{\sigma_2^2}{2} . \tag{76}$$

Note that $\hat{k}$ is unaffected by the transformation.

Now, by (73)

$$\text{Var } (\tilde{c}_T) = \frac{1}{n} [k^2 \sigma_1^2 + \sigma_2^2] . \tag{77}$$

We may see from (53) that

$$\text{Var } (\hat{c}_T) = \frac{\sigma_2^2}{n} + \frac{\sigma_2^2}{n(n-3)} . \tag{78}$$

The relative efficiency of $\tilde{c}_T$ to $\hat{c}_T$ is given by

$$E_{\tilde{c} \text{ to } \hat{c}} = \frac{\dfrac{\sigma_2^2}{n} + \dfrac{\sigma_2^2}{n(n-3)}}{\dfrac{\sigma_2^2}{n} + \dfrac{k^2 \sigma_1^2}{n}} . \tag{79}$$

Clearly $\widetilde{c}_T$ is the more efficient estimator of c when

$$\frac{\sigma_2^2}{n(n-3)} > \frac{k^2\sigma_1^2}{n}$$

that is

$$\frac{\sigma_2^2}{(n-3)\sigma_1^2} > k^2 \, . \tag{80}$$

Again we see that if k cannot be considered quite small we would probably choose $\hat{c}$ since it is asymptotically relatively more efficient than $\widetilde{c}$.

Let us consider, as was done with k, the individual efficiencies. The Cramer-Rao minimum variance bound for t, an unbiased estimator for c_T is given by

$$\mathrm{Var}(t) \geq \frac{\sigma_2^2}{n} \, . \tag{81}$$

Using this, together with (77) and (78) we may compute the efficiencies of $\hat{c}_T$ and $\widetilde{c}_T$ relative to this bound, to be, respectively

$$E_{\hat{c}_T} = \frac{n-3}{n-2} \tag{82}$$

$$E_{\widetilde{c}_T} = \frac{\sigma_2^2}{k^2\sigma_1^2 + \sigma_2^2} \tag{83}$$

Again, as was the case with estimators of k we would probably be inclined to use $\hat{c}_T$, especially for large n unless some extra knowledge

concerning k led us to believe that inequality (80) holds for some particular sample size.

Reference to Appendix B will show a comparison of calculated value of $\hat{c}$, $\tilde{c}$, $\hat{c}_T$, $\tilde{c}_T$, $\hat{k}$, and $\tilde{k}$ with actual values of these parameters for data generated from bivariate normal lognormal distributions.