APPROXIMATION OF TAIL PROBABILITIES USING THE $G_n^{(m)}$-TRANSFORM

by
J. McWilliams*, J. Balusek*, and H. L. Gray+

*Stephen F. Austin University, Nacogdoches Texas
+Southern Methodist University, Dallas Texas

Abstract: In Gray and Wang (1991) the deterministic version of the generalized jackknife, referred to as the $G_n^{(m)}$ transform, was shown to be a powerful tool for obtaining simple approximations functions for tail probabilities of most *pdfs*. These approximation functions are highly accurate in the tails of the distribution and are all of the form $f(x)R(x)$, where f is the *pdf* and R is a rational function. Gray and Wang were only able to give $R(x)$ for $G_n^{(1)}$ for $n = 1,2,3$ due to the extensive algebra required. Even so these approximations yielded relative errors typically in the 10^{-5} range. In this paper we review the generalized jackknife theory for this application and make use the computer algebra programs Maple and Mathematica to obtain approximation functions $f(x)R(x)$ for $n = 1,2\cdots7$ (up to $n = 10$ in the normal case). The resulting approximation functions have relative errors typically in the 10^{-10} range and in some cases 10^{-20} or better. Thus, for most practical purposes one can consider these approximations as good as closed form solutions in the tails of the distributions.

Keywords: Tail probabilities, Generalized jackknife, Rational approximations, G_m transform.

Introduction

Gray and Wang (1991) introduced the $G_n^{(m)}$-transformation as a general method for obtaining approximating functions for tail probabilities. The $G_n^{(m)}$ approximation functions have the unusual property that they actually improve in the tails. The need for approximation functions that are highly accurate in the extreme tails arises in a number of areas of statistics. For example, in clustering problems, it is often necessary to compute extremely small probabilities because of the large number of ways, $(2^{n+1} - 1)$, to separate $n + 2$ points into two groups. In reliability the need for such probabilities can arise from the desire for a highly reliable component that depends on other components in series; see Good (1986).

When viewed in the more general setting as introduced in Gray (1988), the $G_n^{(m)}$-transform is actually a "generalized jackknife." Moreover, in Gray and Wang (1991), it was demonstrated that these "jackknife approximations" are more accurate than other existing approximation functions. Additionally, the approximations are all of the form $f(x)R(x)$, where f is the pdf and R is a rational function. Unfortunately, the function $R(x)$ is not easily obtained for large values of m or n due to extensive algebra required. In Gray and Wang (1991), these functions were only given for $m = 1$ and $n \leq 3$. Even so, they achieved relative errors that are typically in the range 10^{-5}.

In this paper, we review the necessary generalized jackknife theory for this application and make use of the computer algebra program in *Mathematica* to obtain $f(x)R(x)$ for values of $n \leq 7$ (up to 10 in the normal case). The resulting approximations with relative errors generally in the range 10^{-10} and in some instances 10^{-20}.

Finally, in the appendix, we include tables giving the actual approximation functions for $n \leq 6$ (10 in the normal case). Additionally, we include the code that can be run in *Mathematica* to give more extensive approximations. Such approximations are also available by writing the authors.

The Generalized Jackknife

Let $\widehat{\theta}_1, \widehat{\theta}_2, \cdots, \widehat{\theta}_{k+1}$ be a collection of estimators and let c_j be a set of constants such that

$$(1) \qquad E\left[\widehat{\theta}_j - c_j\theta\right] = \sum_{i=1}^{\infty} a_{ij} b_i(\theta), j = 1, 2, \quad k+1$$

where $c_1 = 1$, the constants a_{ij} are given and the $b_i(\theta)$ are unknown functions of θ. The k^{th} order generalized jackknife is defined as follows:

$$(2) \qquad G(\theta_1, \theta_2, \cdots, \theta_{k+1}; a_{ij}) = \frac{H_{k+1}(\theta_j; a_{ij})}{H_{k+1}(c_j; a_{ij})}$$

where

$$H_{k+1}(z_j; a_{ij}) = \begin{vmatrix} z_1 & z_2 & & z_{k+1} \\ a_{11} & a_{12} & & a_{1,k+1} \\ \vdots & & & \cdot \\ a_{k,1} & a_{k,2} & \cdots & a_{k,k+1} \end{vmatrix}$$

In (2), the generalized jackknife is defined as an extension of the original jackknife definition given in Schucany, Gray and Owen (1971). The extension defined in (2) has been demonstrated as having significant value in numerical approximation in Gray (1988). If $a_{ij}b_i(\theta) = 0$ for $i > k$ in equation (1), then taking the expected value of both sides of equation (2) reveals that $G(\widehat{\theta}_1, \widehat{\theta}_2, \cdots, \widehat{\theta}_{k+1}; a_{ij})$ is an unbiased estimator for θ. If $a_{ij}b_i(\theta) \neq 0$ for $i > k$, the generalized jackknife is not unbiased, however, under general conditions it is of lower order bias.

The generalized jackknife is well-known as a method for reducing bias in estimators. Not as well-known is the fact that the generalized jackknife can provide a way of reducing error in numerical approximations; see Gray (1991). To elaborate, let $\hat{\theta}_j$ be an estimator which has its probability mass concentrated at a single point. This kind of point estimator is more commonly known as an approximation and $E\left[\hat{\theta}_j - \theta\right] = \hat{\theta}_j - \theta$, so the bias in this case is the numerical error. When the generalized jackknife is applied to a collection of approximations, it can be used as an approach for reducing error on these numerical approximations. This method of using the generalized jackknife as an approach to finding numerical approximations is discussed in Gray (1988). In addition, Gray pointed out that such well known numerical methods as Simpson's rule, trapezoidal rule, Romberg integration, Lagrange Interpolation, etc. can be viewed as generalized jackknives.

Use of Jackknife in Approximating Tail Probabilities

Let f be a pdf, and let

(4) $$F(x) = \int_a^x f(t)dt$$

and assume that $\lim_{x\to\infty} F(x) = S$. Note that here $F(x)$ is not a cdf, since in general $f(t) > 0$ for $t \le a$, as well as $t > a$. Now let $\varepsilon(x) = S - F(x)$ and

(5) $$U_k(x) = x^{l_k} \sum_{i=0}^{\infty} \frac{a_{k,i}}{x^i}$$

where l_k is an integer such that $l_k \le k$ and $a_{k,0} \ne 0$. Suppose further that m is the smallest possible integer such that the differential equation

$$U_m(x)\varepsilon^{(m)} + U_{m-1}(x)\varepsilon^{(m-1)} + \quad + U_1(x)\varepsilon' - \varepsilon = 0,$$

is satisfied by $\varepsilon(x)$ for some set of U_k's. The $a_{k,i}$ in (6) need not be known.

For example, suppose

$a_{1,i} = a_{2,i} = \cdots = a_{m,i} = 0$ for $i \ge n_1, n_2, \quad , n_m$, respectively, the sum in (5)

becomes $x^{l_k} \sum_{i=0}^{n_k-1} \frac{a_{k,i}}{x^i} = \sum_{i=0}^{n_k-1} a_{k,i} x^{l_k - i}$, and the differential equation (6) becomes

$$\sum_{i=0}^{n_m-1} \alpha_{m,i}x^{l_m-i}\varepsilon^{(m)}(x) + \sum_{i=0}^{n_{m-1}-1} \alpha_{m-1,i}x^{l_{m-1}-i}\varepsilon^{(m-1)}(x) + \cdots + \sum_{i=0}^{n_1-1} \alpha_{1,i}x^{l_1-i}\varepsilon'(x) - \quad \varepsilon(x) = 0$$

Then

$$\textbf{(7)} \quad F(x) = S + \sum_{i=0}^{n_m-1} \alpha_{m,i}x^{lm-i}f^{(m-1)}(x) + \cdots + \sum_{i=0}^{n_1-1} \alpha_{1,i}x^{l_1-i}f(x)$$

and

$$\textbf{(8)} \quad f^{(k)}(x) = \sum_{i=0}^{n_m-1} \alpha_{m,i}[x^{l_m-i}f^{(m-1)}(x)]^{(k+1)} + \cdots + \sum_{i=0}^{n_1-1} \alpha_{1,i}[x^{l_1-i}f(x)]^{(k+1)}$$

$$(k = 0, 1, 2, \cdot \quad, N-1; N = \sum_{i=1}^{m} n_i).$$

Equations (7) and (8) define a system of equations of the same form as (1) with $c_1 = 1$, and $c_j = 0$ for $2 \leq j \leq N+1$, and with the $a_{j,i}$ corresponding to the $b_i(\theta)$.

Clearly, this system can be solved for S and, therefore, the generalized jackknife defined by equations (7) and (8) is exact; that is, it gives an exact tail probability when applied to the N+1 functions $F^{(k)}(x)$ $(k = 0, 1, 2, \cdots, N)$ if $\varepsilon(x)$ satisfies (6).

Thus if $\varepsilon(x)$ satisfies (6) for $x \geq a$, then for any $x \geq a$

$$(9) \qquad G[F(x), F'(x), \cdots, F^{(N)}(x); a_{i,j}(x)] \equiv S. = \int_a^\infty f(x)dx$$

if the $a_{i,j}$ are properly defined by (7) and (8). Now note that (9) holds for $x \geq a$.

But $F(a) \equiv 0$, so we can take $x = a$ and no integration is required in (9).

To be more specific, suppose that

$m = 1, n_1 = n$, and $x = a$. Then $N = n$ and denoting $G(0, \quad F'(x),\cdots,F^{(n)}(x); a_{ij}(x))$ by $G_n^{(m)}(f(x); a_{ij}(x))$, we have

$$G_n^{(1)}(f(x);a_{ij}(x)) = \frac{\begin{vmatrix} 0 & f(x) & \cdots & f^{(n-1)}(x) \\ a_{11}(x) & a_{12}(x) & \cdots & a_{1,n+1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}(x) & a_{n,2}(x) & \cdots & a_{n,n+1}(x) \end{vmatrix}}{\begin{vmatrix} 1 & 0 & \cdots & 0 \\ a_{11}(x) & a_{12}(x) & \cdots & a_{1,n+1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}(x) & a_{n,2}(x) & \cdots & a_{n,n+1}(x) \end{vmatrix}}$$

$$a_{i,j}(x) = [x^{l_i-i+1}f(x)]^{(j-1)}$$

Thus, if (6) holds with $\alpha_{1,i} = \alpha_{2,i} = \ldots = 0$ for $i \geq n_1, n_2, \ldots n_m$, it follows that

(12) $$G_n^{(1)}(f(x);a_{i,j}(x)) = \int_x^{\infty} f(t)dt.$$

Now assume that $f(x)$ satisfies (6) for some set of $U_k(x)$ defined by (5) and

$$\lim_{x\to\infty} U_k^{(i-1)}(x)f^{(k-i)}(x) = 0 \quad (k = i,\ i+1, \quad ; i = 1,\ 2,\ \cdots).$$

Levin and Sidi (1981) showed that there exists $\alpha'_{k,i}$ such that

(13) $$U_m^*(x)\varepsilon^{(m)}(x) + U_{m-1}^*(x)\varepsilon^{(m-1)}(x) + \cdots + U_1^*(x)\varepsilon'(x) - \varepsilon(x) = O(x^{-n})$$

where

(14) $$U_k^*(x) = \sum_{i=0}^{n-1} \alpha'_{k,i} x^{l_k-i}.$$

From our observations in (12), it would, therefore, appear from (13) that $G_{(n)}^{(1)}$ should converge "super fast" to S. This is the case and it also is true for $m \geq 1$. See Levin and Sidi (1981). Equation (13) leads us to define the generalized jackknife tail probability function as follows:

Definition 1: Let $f(x)$ be a pdf, with infinite support and suppose that $f(x)$ satisfies the differential equation in (6) for some m and some collection of $U_k(x)$. Then we define the $G_n^{(m)}$-transformation of $f(x)$ as the generalized jackknife approximation of $\int_x^\infty f(t)dt$ corresponding to (13); that is,

$$(15) \qquad G_n^{(m)}[f(x); a_{ij}(x)] = G[0, f(x), \cdots, f^{(mn-1)}(x); a_{ij}(x)],$$

where

$$a_{ij}(x) = (x^{l_1-i+1}f(x))^{(j-1)}, \; i = 1, \cdots, n; \; j = 1, \cdots, mn+1,$$

$$= (x^{l_2-i+n+1}f'(x))^{(j-1)}, \; i = n+1, \cdots, 2n; \; j = \cdots, mn+1,$$

$$= (x^{l_m-i+(m-1)n+1}f^{(m-1)}(x))^{(j-1)}, \; i = (m-1)n+1, \cdots, mn;$$

$$j = 1, \cdots, mn+1$$

It was demonstrated in Levin and Sidi (1981) that the foregoing assumptions cover a wide class of integrands. In fact, it is difficult to think of a differentiable pdf that does not satisfy (or approximately satisfy) (6) for some set of $U_k(x)$ and some m. For instance, in the simple case in which all of the $U_k(x)$ are constants, that is, $U_k = c_k$, then for any m there exists an m^{th} order homogeneous differential equation with coefficients c_k whose solution is given by some linear combination of the elements of $S = \{e^{\beta_k x}, k = 1, 2, \cdots, m\}$, where $\beta_k = a_k + ib_k$, which would imply that if f can be represented by a Fourier series the method should be effective. In fact, under general differentiability conditions, it has been shown in Gray and Lewis (1971) that

$$\lim_{m\to\infty} G_1^{(m)}(f(x); a_{ij}(x)) = \int_x^\infty f(t)dt.$$

Of course, $G_n^{(m)}$ is much more general than $G_1^{(m)}$. Notice that if we take $U_1(x) = \frac{f(x)}{f'(x)}$, then f satisfies (6) with $m = 1$. Thus if $x^{-l_1}\frac{f(x)}{f'(x)}$ has a convergent Laurent series expansion about zero with only non-positive power terms,

$\lim_{n\to\infty} G$ $(f(x$ $(\cdot\,)$ $f(\cdot)dt$

ufficient condition for this to be that $\frac{f(x)}{f'(x}$ be analyti complex functi everywhere (including the point at neighborhood of zero There nothing magi about the point 'o; always translate the function about another point the complex plane Thus, th two simplest where or includes pdf' that statisti will encounter

It also be shown that $G_n^{(}$ $(f(\cdot)$ $(x))$ $f(t)dt$ for One should expect rapid convergence of $G_n^{(m)}(f(x$ $(\cdot))$ to $f(t)dt$ for large and of the being $O(x$ $)$ $))$. One important property of th $G_n^{(m}$ -transform that does not require any knowledge about or $_{k,i}$), contrast to as technique of integration by arts

Applications

In Gray and Wang (1991), the $G_n^{(m)}$-transformation was used to approximate tail probabilities for the standard normal, Student t, Inverse Gaussian, Pearson Type IV and, the ratio of a χ^2 and a log normal. In Gray and Lewis (1971), the Gamma distribution was used to demonstrate the power of another less general jackknife method called the B_n-transformation in getting approximations to tail probability. The B_n-transformation only assumes that the pdf satisfies an mth order constant coefficient homogeneous linear differential equation.

In this section we extend the results of Gray and Wang for the $G_n^{(1)}$-transformation for larger order transforms and introduce some new examples. Hereafter we will use G_n to mean $G_n^{(1)}$.

<u>Example 1</u>: The Normal Distribution

The standard normal pdf is $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$. Clearly, $x^{-1}f'(x) - f(x) = 0$, and therefore, $f(x)$ satisfies Equation 6 with $m = 1$. Then

$$U_1(x) = x^{l_1} \sum_{i=0}^{\infty} \frac{a_{1,i}}{x^i} = x^{l_1} \Rightarrow \ell_1 = -1, \text{ and } a_{1,i} \equiv 0, \text{ for } i > 0, \text{since } a_{10} \neq 0.$$

Then, for example

$$G_1(x) = f(x)\frac{x}{1+x^2}, \qquad G_2(x) = f(x)\frac{x(x^2+4)}{x^4+5x^2+2}, \qquad G_3(x) = f(x)\frac{x(x^4+11x^2+18)}{x^6+12x^4+27x^2+6},$$

$$G_4(x) = f(x)\frac{x(x^6+21x^4+104x^2+96)}{(x^8+22x^6+123x^4+168x^2+24)}, \qquad G_5(x) = f(x)\frac{x(x^8+34x^6+333x^4+1000x^2+600)}{x^{10}+35x^8+365x^6+1275x^4+1200x^2+120}$$

See the Appendix for a complete table of G_1 through G_{10}

The relative errors in estimating the tail probabilities for various values of x are listed in Table 1 for $G_1(x)$ through $G_8(x)$.

Table 1 Relative errors for the $G_n^{(1)}$-transforms of the normal

z	*true*	$E(G_1)$	$E(G_2)$	$E(G_3)$	$E(G_4)$	$E(G_5)$	$E(G_6)$	$E(G_7)$	$E(G_8)$
2	1.1507(1)	1.7(1)*	2.3(2)	3.0(4)	1.6(3)	9.3(4)	3.9(4)	1.4(4)	4.0(5)
..6	5.47993(2)	9.0(2)	5.1(3)	1.0(3)	5.9(4)	1.8(4)	4.4(5)	7.0(6)	3.1(7)
2	2.27501(2)	5.1(2)	7.5(4)	6.0(4)	1.7(4)	3.2(5)	3.7(6)	2.7(7)	3.5(7)
3	1.3499(3)	1.5(2)	3.2(4)	8.2(5)	7.7(6)	2.7(7)	6.8(8)	2.1(8)	3.7(9)
6	9.86588(10)		2.4(5)	5.0(7)	3.4(10)	3.5(10)	1.3(11)	5.7(15)	3.1(14)
10	7.61985(24)	[illegible]	1.6(6)	5.1(9)	2.6(11)	4.3(13)	5.8(16)	6.0(17)	8.5(19)
12	3.67097(51)	3.8(5)	1.6(7)	1.1(10)	3.6(13)	1.1(15)	2.0(18)	2.3(20)	9.9(21)

(* 1.1507(1) means 1.1507×10^{-1})

Example 2: The Gamma Distribution: $f(x) = \frac{1}{\Gamma(a)b^a}x^{a-1}e^{-\frac{x}{b}}$

Clearly $f(x)$ satisfies (6) with $m = 1$; that is, $(ax^{-1} - \frac{1}{b})^{-1}f'(x) - f(x) = 0$, and thus again, $m = 1$ and $U_1(x) = (ax^{-1} - \frac{1}{b})^{-1}$. In order to identify the l_1 parameter in equation (15) we find the Laurent series of $U_1(x)$ expanded about zero.

$U_1(x) = \frac{-b}{1-abx^{-1}} = -b[1 + \frac{ab}{x} + \frac{(ab)^2}{x^2} + \cdots]$. Hence the parameter $l_1 = 0$.

Now, for example, $a = 7$ and $b = 2$; that is, a χ^2 distribution with 14 degrees of freedom. In this case, the pdf becomes $f(x) = \frac{1}{92{,}160}x^6e^{-\frac{x}{2}}$, and $f(x)$ is the solution to some linear constant coefficient 7^{th} order homogeneous differential equation.

Therefore, $G_7^{(1)}[f(x); a_{ij}(x)] \equiv \int_x^\infty f(t)dt$. See Gray, Atchison and McWilliams (1971). For the given values of the parameters a and b,

$G_7(x) = 2e^{-x/2}(x^6 + 12x^5 + 120x^4 + 960x^3 + 5760x^2 + 23040x + 46080)$

as could have been obtained by repeated integration by parts. In fact, the Gamma pdf $f(x)$ is a solution of a homogeneous ODE with constant coefficients for any integer $a \geq 1$, so that $G_n^{(1)}[f(x); a_{ij}(x)]$ will be exact for $n = a$.

However, if a is not an integer $G_n(f(x))$ is still highly accurate in the tails. As an example we include a table of relative errors for parameter values $a = \frac{1}{2}$ and $b = 2$, and approximations G_1 through G_8.

Table 2 Relative errors for the $G_n^{(1)}$-transforms of the Gamma for $a = \frac{1}{2}$ and $b = 2$.

x	*true*	$E(G_1)$	$E(G_2)$	$E(G_3)$	$E(G_4)$	$E(G_5)$	$E(G_6)$	$E(G_7)$	$E(G_8)$
3	8.3265(2)*	7.4(2)	1.2(2)	2.9(3)	8.5(4)	2.8(4)	1.0(4)	4.0(5)	1.7(5)
6	1.4306(2)	2.8(2)	2.4(3)	3.3(4)	5.8(5)	1.2(5)	3.0(6)	8.0(7)	2.4(7)
12	5.3200(4)	9.4(3)	3.4(4)	2.2(5)	2.0(6)	2.4(7)	3.3(8)	5.4(9)	9.8(10)
25	5.7330(7)	2.6(3)	3.1(5)	7.7(7)	2.9(8)	1.5(9)	9.7(11)	7.6(12)	6.9(13)
35	3.2971(9)	1.4(3)	9.7(6)	1.4(7)	3.3(9)	1.1(10)	4.7(12)	2.5(13)	1.5(14)
50	1.5375(12)	7.2(4)	2.7(6)	2.2(8)	3.0(10)	5.9(12)	1.5(13)	5.0(25)	2.0(16)
75	1.7071(18)	3.3(4)	5.9(7)	2.4(9)	1.7(11)	1.8(13)	2.5(15)	4.5(17)	1.0(18)
120	6.3261(28)	1.3(4)	9.9(8)	1.7(10)	5.4(13)	2.5(15)	1.6(17)	1.3(19)	1.5(21)

(* 8.3265(2) means 8.3265×10^{-2})

The $G_n^{(1)}(f(x))$ for $n \in \{1,2, \;\cdot ,6\}$ and parameters a and b unspecified are listed in the appendix.

Example 3: The Student t Distribution

The pdf of the t distribution with parameter k degrees of freedom is

$$f(x) = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\sqrt{k\pi}}\left(1 + \frac{x^2}{k}\right)^{-\frac{k+1}{2}}$$

The Student t pdf satisfies the differential equation (6) with $m =$, as can be seen below:

$$x^1\left(-\frac{1}{k+1} - \frac{k}{(k+1)x^2}\right)f'(x) - f(x) = 0$$

Hence $U_1(x) = x^1\left(-\frac{1}{k+1} - \frac{k}{(k+1)x^2}\right)$, yielding $l_1 = 1$. Once again Definition 1 gives us the method for generating the $G_n(x)$-transforms for the Student t which are tabulated for $n \in \{1,2,3, \quad\}$ in the appendix. For example, if $k = 3$

$$G_1(f(x)) = f(x)\frac{x^3+3x}{3x^2-3x}, \qquad G_2(f(x)) = f(x)\frac{x^3+3x}{3x^2+3}$$

$$G_3(f(x)) = f(x)\frac{5x^5+30x^3+45x}{15x^4+54x^2+27}, \qquad G_4(f(x)) = f(x)\frac{30x^7+264x^5+774x^3+756x}{90x^6+576x^4+1026x^2+324}$$

Table 3 Relative errors for the G_n-transforms of the t distribution for various degrees of freedom k.

k	x	*True*	$E[G_1]$	$E[G_2]$	$E[G_3]$	$E[G_4]$	$E[G_5]$	$E[G_6]$	$E[G_7]$	$E[G_8]$
3	10.2145	1.0000(3)*	1.5(2)	3.8(3)	3.7(5)	5.6(6)	2.4(7)	2.2(8)	2.2(9)	1.1(10)
3	22.2037	1.0000(4)	3.3(3)	8.1(4)	1.7(6)	2.8(7)	2.4(9)	2.8(10)	5.1(12)	4.6(13)
4	7.1731	1.0000(3)	3.3(2)	6.6(3)	1.2(4)	1.4(5)	1.3(6)	6.1(8)	2.0(8)	3.4(10)
4	13.0336	1.0000(4)	9.9(3)	2.0(3)	1.1(5)	1.5(6)	4.1(8)	3.8(9)	2.3(10)	1.4(11)
5	5.8934	1.0000(3)	5.0(2)	8.5(3)	2.2(4)	1.8(5)	2.8(6)	1.2(8)	4.8(8)	4.5(9)
5	9.6775	1.0000(4)	1.8(2)	3.1(3)	3.1(5)	3.4(6)	1.7(7)	1.1(8)	1.5(9)	3.1(11)
8	4.5007	1.0000(3)	9.1(2)	1.1(2)	4.3(4)	1.5(5)	5.3(6)	4.1(7)	5.3(8)	2.0(8)
8	6.4420	1.0000(4)	4.4(2)	5.2(3)	9.9(5)	6.1(6)	7.4(7)	1.2(9)	8.1(9)	7.4(10)
11	4.0147	1.0000(3)	0.1(0)	1.2(2)	5.3(4)	6.4(6)	5.4(6)	6.5(7)	8.2(9)	2.0(8)
11	5.4527	1.0000(4)	6.3(2)	6.1(3)	1.4(4)	5.5(6)	1.0(6)	3.9(8)	8.4(9)	1.6(9)
20	3.5518	1.0000(3)	0.2(0)	1.3(2)	6.3(4)	6.5(6)	4.2(6)	7.2(7)	5.0(8)	7.6(9)
20	4.5385	1.0000(4)	9.4(2)	7.0(3)	1.9(4)	2.5(6)	1.0(6)	8.3(8)	1.6(9)	1.5(9)

* 1.0000(3)* means 1.0000×10^{-3}

Example 4: The Inverse Gaussian Distribution

The inverse Gaussian pdf is $f(x) = \sqrt{\frac{\lambda}{2\pi x^3}}\, e^{\frac{-\lambda(x-\mu)^2}{2\mu^2 x}}$, which satisfies the first order differential equation $\left(-2\frac{x^2}{3\mu^2 x+\lambda x^2-\lambda\mu^2}\mu^2\right)f'(x) - f(x) = 0$. The coefficient

$U_1(x) = \left(-2\frac{x^2}{3\mu^2 x+\lambda x^2-\lambda\mu^2}\mu^2\right)$ has the Laurent series expansion about zero:

$$U_1(x) = -2\frac{\mu^2}{\lambda} + 6\frac{\mu^4}{\lambda^2 x} - 2\frac{\mu^4+9\frac{\mu^6}{\lambda^2}}{\lambda x^2} + \cdots$$

from which we can see that $l_1 = 0$. Since $U_1(x)$ has the parameter $l_1 = 0$, then $G_n[f(x); a_{ij}(x)]$ has the form of equation (19), from which we can generate the $G_n^{(1)}$-transformation. Table 4 shows the relative errors for $\mu = \lambda = 1$ and

$n = 1, \ldots, 8$. The approximation is given for any λ and μ in the Appendix for $n \leq 5$

For example, by the Appendix, for $\mu = \lambda = 1$

$G_1(f(x)) = f(x)\frac{2x^2}{x^2+3x-1}$, $\quad G_2(f(x)) = f(x)\frac{2x^4+14x^3-2x^2}{x^4+10x^2+13x^2-6x+1}$,

$G_3(f(x)) = f(x)\frac{2x^6+36x^5+110x^4-20x^3+2x^2}{x^6+21x^5+102x^4+75x^3-42x^2+9x-1}$

Table 4 Relative errors for the G_n-transforms of the inverse Gaussian, $\lambda = \mu = 1$.

x	True	$E(G_1)$	$E(G_2)$	$E(G_3)$	$E(G_4)$	$E(G_5)$	$E(G_6)$	$E(G_7)$	$E(G_8)$
1.5	1.8923(1)*	1.7(1)	7.1(2)	2.8(2)	1.3(2)	6.2(3)	3.2(3)	1.7(3)	9.4(4)
2	1.1452(1)	1.5(1)	4.8(2)	1.7(2)	6.6(3)	2.8(3)	1.3(3)	6.3(4)	3.2(4)
3	4.6812(2)	1.1(1)	2.5(2)	7.0(3)	2.2(3)	7.9(4)	3.0(4)	1.2(4)	5.4(5)
4.5	1.4301(2)	7.4(2)	1.2(2)	2.5(3)	6.0(4)	1.7(4)	5.3(5)	1.8(5)	6.4(6)
6	4.8499(3)	5.3(2)	6.5(3)	1.1(3)	2.1(4)	4.9(5)	1.3(5)	3.6(6)	1.1(6)
10	9.4392(4)	2.8(2)	1.9(3)	1.9(4)	2.4(5)	3.7(6)	6.6(7)	1.3(7)	2.9(8)
16	9.4392(6)	1.4(2)	5.1(4)	2.9(5)	2.3(6)	2.3(7)	2.6(8)	3.5(9)	5.3(10)
32	1.2201(9)	4.4(3)	5.7(5)	1.3(6)	4.2(8)	1.8(9)	9.6(11)	6.2(12)	4.6(13)

* 1.8923(1) means 1.8923×10^{-1}

Example 5: The F distribution

The pdf of the F distribution is

$$f(x) = \frac{\Gamma(\frac{a+b}{2})}{\Gamma(\frac{a}{2})\Gamma(\frac{b}{2})}\left(\frac{a}{b}\right)^{\frac{a}{2}}\frac{x^{\frac{a-2}{2}}}{(1+(\frac{a}{b})x)^{\frac{a+b}{2}}}$$

which satisfies the first order pdf $\left(-2x\frac{b+ax}{-ab+2b+2ax+axb}\right)f'(x) - f(x) = 0$. Hence $U_1(x) = -2x\frac{b+ax}{-ab+2b+2ax+axb}$, which can be rewritten in a Laurent series of the form $a_{10}x + a_{11} + \frac{a_{12}}{x} + \cdots$, so $l_1 = 1$.

Since $U_1(x)$ has the parameter $l_1 = 1$, $G_n[f(x); a_{ij}(x)]$ has the form of equation (20), from which we can generate the $G_n^{(1)}$-transformation of $f(x)$. Table 6 shows the relative error for $n = 1, 2, \cdots 8$ and $a = 3, b = 4$. The approximation is given in the appendix for all a and b when $n = 1, 2, \cdots 5$.

When $a = 3, b = 4$

$$G_1(f(x)) = f(x)\frac{3x^2-4x}{2x-2}$$

$$G_2(f(x)) = f(x)\frac{27x^3+60x^2-16x}{108x^2-48x+24}$$

Table 5 Relative errors for the G_n-transforms of the F distribution:

x	*True*	$E(G_1)$	$E(G_2)$	$E(G_3)$	$E(G_4)$	$E(G_5)$	$E(G_6)$	$E(G_7)$	$E(G_8)$
4.19	1.000296(1)	1.2(2)	6.2(3)	1.3(4)	7.0(6)	5.4(7)	5.2(8)	5.7(9)	7.0(10)
6.59	5.00169(2)	6.3(3)	2.3(3)	3.3(5)	1.2(6)	6.2(8)	4.1(9)	3.1(10)	2.6(11)
9.98	2.49965(2)	3.7(3)	9.7(4)	9.2(6)	2.3(7)	8.2(9)	3.7(10)	2.0(11)	1.2(12)
16.7	9.99383(3)	2.0(3)	3.3(4)	1.9(6)	2.9(8)	6.6(10)	1.9(11)	6.1(13)	2.2(14)

Other Methods

There are, of course, other methods for obtaining approximation functions for tail probabilities. However, the only ones to date that are effective are of the form $f(x)R(x)$ where $R(x)$ is a rational function and f is the pdf. Besides the $G_n^{(m)}$ – *transform*, the only general method that is competitive with $G_n^{(m)}$ is the method of continued fractions. However, the only result available from that approach currently is for $f(x)$ a normal (0,1) pdf. In that particular case, $G_n^{(1)}$ and the continued fraction results are very similar in form and accuracy. Other results in the normal case of this rational function have been given by Hawkes (1982) and Lew (1981). Their 8th order approximations are comparable in complexity and accuracy with $G_4(f(x))$. Their methods do not extend to other distributions.

A Mathematica Program for Generating The $G_n^{(1)}$ – transform

The simple Mathematica program listed below was used to generate each of the $G_n^{(1)}$ – transforms in this paper. There are only three parameters needed: the pdf $f[x]$, the order of the transform n, and the parameter l_1 from equation (5) with $k = 1$. These are defined on the first line of the program.

The output consists of the rational expression $R[x]$ followed by the $G_n^{(1)}$ – transform $G[x] = f[x]R[x]$. Both $R[x]$ and $G[x]$ are then defined functions and can be evaluated at any value of x.

An example for the $G_8^{(1)}$ – transform of the standard normal

$$L = -1; n = 8; f[x\_] = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}};$$

$$Num = Table[0, \{i, 1, n + 1\}, \{j, 1, n + 1\}];$$

$$Num[[1, 1]] = 0;$$

$$For[j = 2, j <= n + 1, j + +, Num[[1, j]] = D[f[x], \{x, j - 2\}]]$$

$$For[i = 2, i <= n + 1, n + +,$$

$$For[j = 1, j <= n + 1, j + +, Num[[i, j]] = D[x\text{^}(L - i + 2) * f[x], \{x, j - 1\}]]]$$

$$Den = Table[0, \{i, 1, n + 1\}, \{j, 1, n + 1\}];$$

$$Den[[1, 1]] = 1;$$

$$For[j = 2, j <= n + 1, j + +, Den[[1, j]] = 0]$$

$$For[i = 2, i <= n + 1, n + +,$$

$$For[j = 1, j <= n + 1, j + +, Den[[i, j]] = D[x\text{^}(L - i + 2) * f[x], \{x, j - 1\}]]]$$

$$R[x\_] = \text{Simplify}\left[\frac{Det[Num]}{f[x] * Det[Den]}\right]$$

$$G[x\_] = f[x] * R[x]$$

The exact form of $G_8^{(1)}$ can be found in the appendix. Evaluations in Mathematica yielded $G[1.645] = 0.0499849$, $G[1.96] = 0.0249979$, and $G[2.326] = 0.0100093$.

Concluding Remarks

In this paper we have presented the theory of the $G_n^{(m)}$-transformation and the general methodology for finding easily evaluated functions that accurately approximate tail probabilities. Unlike most approximation functions which are distribution specific, the $G_n^{(m)}$-transformation is very general in application and can be applied to nearly any distribution that is differentiable. The examples presented in this paper are an extension of the results in Gray and Wang (1991) and were made possible using a computer algebra system that was not available in 1991. The examples presented should provide the reader with ample numerical evidence that when $\frac{x^{l_1} f(x)}{f'(x)}$ has a convergent Laurent expansion about zero with only non-positive power terms $G_n^{(1)}$ converges quite rapidly to $\int_x^{\infty} f(t)dt$. In fact, the approximation given here should be sufficiently accurate for virtually any application requiring highly accurate tail probabilities.

In our examples, we have used $m = 1$ and determined l_1 for each distribution, however $G_n^{(1)}$ is robust and in most cases accurate approximations can be obtained by simply letting $l_1 = 1$ for any relevant distribution and then increasing the order n until the desired approximation is reached. An example of this feature is presented for the ratio of a χ^2 and a lognormal in Gray and Wang (1991).

Finally, it should be mentioned that values of $m > 1$ are also of interest. For example, Gray and Wang (1993) have used $G_{(n)}^{(2)}$ to obtain tail probabilities approximation functions for the standard non-central distributions.

APPENDIX

$$\int_x^{\infty} f(t)dt \cong G_k[f(x)] = \frac{N_k(x)}{D_k(x)} f(x)$$

$$\text{Normal Distribution: } f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad -\infty < x < \infty$$

Coefficients of the x^k term

Term*	N_1	D_1	N_2	D_2	N_3	D_3	N_4	D_4	N_5	D_5
x^0		1		2		6		24		120
x^1	1		4		18		96		600	
x^2		1		5		27		168		1200
x^3			1		11		104		1000	
x^4				1		12		123		1275
x^5					1		21		333	
x^6						1		22		365
x^7							1		34	
x^8								1		35
x^9									1	
x^{10}										1

* An empty cell means that the coefficient of that term is zero.

Example: $G_3^{(1)}[f(x)] = f[x]\dfrac{x^5 + 11x^3 + 18x}{x^6 + 12x^4 + 27x^2 + 6}$

$\int_x^{\infty} f(t)dt \cong G_k[f(x)] = \frac{N_k(x)}{D_k(x)} f(x)$

Normal Distribution: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad -\infty < x < \infty$

Coefficients of the x^k term

Term*	N_6	D_6	N_7	D_7	N_8	D_8	N_9	D_9	N_{10}	D_{10}
x^0		720		5040		40320		362880		3628800
x^1	4320		35280		322560		3265920		36288000	
x^2		9720		88200		887040		9797760		117936000
x^3	10200		111720		1317120		16692480		226800000	
x^4		13950		163170		2046240		27488160		394632000
x^5	4938		73206		1114848		17654112		292299840	
x^6		5655		87465		1387680		22861440		393271200
x^7	807		1695		341760		6855840		139410720	
x^8		855		18480		383145		7901145		165082050
x^9	50		1655		46615		1222215		31123230	
x^{10}		51		1722		49476		1323189		34364925
x^{11}	1		69		3033		110136		3636045	
x^{12}		1		70		3122		115038		3854025
x^{13}			1		91		5124		232890	
x^{14}				1		92		5238		240750
x^{15}					1		116		8338	
x^{16}						1		117		8280
x^{17}							1		144	
x^{18}								1		145
x^{19}									1	
x^{20}										1

* An empty cell means that the coefficient of that term is zero

Example $G_6^{(1)}[f(x)] = f[x]\frac{x^{11} + 50x^9 + 807x^7 + 4938x^5 + 10200x^3 + 4320x}{x^{12} + 51x^{10} + 855x^8 + 5655x^6 + 13950x^4 + 9720x^2 + 720}$

II **Student *t*** $f(x) = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\sqrt{\pi k}}\left(\frac{1+x^2}{k}\right)^{-\frac{k+1}{2}}$

$$\int_x^{\infty} f(t)dt \cong G_k[f(x)] = \frac{N_k(x)}{D_k(x)} f(x)$$

Coefficients of the x^k term.

Term*	N_1	D_1	N_2	D_2	N_3	D_3	N_4	D_4
x^0				k		$3k^2$		$12k^3$
x^1	k	$-k$	k		$5k^2$		$28k^3$	
x^2		k				$6k^2$		$3k^2(13k-1)$
x^3	1		1	k	$k(k+7)$		$k^2(13k+47)$	
x^4						$k(k+2)$		$2k^2(7k+11)$
x^5					$k+2$		$k(k^2+18k+25)$	
x^6								$k(k+2)(k+3)$
x^7							$(k+2)(k+3)$	

Term*	N_5	D_5
x^0		$60k^4$
x^1	$180k^4$	
x^2		$15k^3(19k-3)$
x^3	$k^3(143k+329)$	
x^4		$15k^3(11k+13)$
x^5	$k^2(24k^2+241k+247)$	
x^6		$5k^2(k+2)(5k+11)$
x^7	$k(k^3+33k^2+124k+122)$	
x^8		$k(k+2)(k+3)(k+4)$
x^9	$(k+2)(k+3)(k+4)$	

Example: $G_3^{(1)}[f(x)] = f[x]\frac{(k+2)x^5 + k(k+7)x^3 + 5k^2x}{k(k+2)x^4 + 6k^2x^2 + 3k^2}$

II **Student t** $f(x) = \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})\sqrt{\pi k}}\left(\frac{1+x^2}{k}\right)^{-\frac{k+1}{2}}$

$$\int_x^\infty f(t)dt \cong G_k[f(x)] = \frac{N_k(x)}{D_k(x)} f(x)$$

Coefficients of the x^k term.

Term*	N_1	D_1	N_2	D_2	N_3	D_3	N_4	D_4
x^0				k		$3k^2$		$12k^3$
x^1	k	$-k$	k		$5k^2$		$28k^3$	
x^2		k				$6k^2$		$3k^2(13k-1)$
x^3	1		1	k	$k(k+7)$		$k^2(13k+47)$	
x^4						$k(k+2)$		$2k^2(7k+11)$
x^5					$k+2$		$k(k^2+18k+25)$	
x^6								$k(k+2)(k+3)$
x^7							$(k+2)(k+3)$	

Term*	N_5	D_5
x^0		$60k^4$
x^1	$180k^4$	
x^2		$15k^3(19k-3)$
x^3	$k^3(143k+329)$	
x^4		$15k^3(11k+13)$
x^5	$k^2(24k^2+241k+247)$	
x^6		$5k^2(k+2)(5k+11)$
x^7	$k(k^3+33k^2+124k+122)$	
x^8		$k(k+2)(k+3)(k+4)$
x^9	$(k+2)(k+3)(k+4)$	

Example: $G_3^{(1)}[f(x)] = f[x]\frac{(k+2)x^5 + k(k+7)x^3 + 5k^2x}{k(k+2)x^4 + 6k^2x^2 + 3k^2}$

II Student *t* $f(x) = \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})\sqrt{\pi k}}\left(\frac{1+x^2}{k}\right)^{-\frac{k+1}{2}}$ **(continued)**

$\int_x^{\infty} f(t)dt \cong G_k[f(x)] = \frac{N_k(x)}{D_k(x)} f(x)$

Coefficients of the x^k term.

Term*	N_6	D_6
x^0		$360k^5$
x^1	$1320k^5$	
x^2		$180k^4(13k-3)$
x^3	$12k^4(203+131k)$	
x^4		$45k^3(43k^2+36k+1)$
x^5	$3k^3(143k^2+996k+781)$	
x^6		$15k^3(101+108k+31k^2)$
x^7	$k^2(2158k+1833+723k^2+38k^3)$	
x^8		$3k^2(k+2)(k+3)(13k+37)$
x^9	$k(k+3)(k^3+49k^2+218k+242)$	
x^{10}		$k(k+2)(k+3)(k+4)(k+5)$
x^{11}	$(k+2)(k+3)(k+4)(k+5)$	

* An empty cell means that the coefficient of that term is zero.

III Gamma Distribution: $f(x) = \frac{1}{\Gamma(a)b^a} x^{a-1} e^{\frac{-x}{b}}$ $\int_x^\infty f(t)dt \cong G_k[f(x)] = \frac{N_k(x)}{D_k(x)} f(x)$

Coefficients of the x^k term

Term*	N_1	D_1	N_2	D_2	N_3	D_3
x^0		$-b(a-1)$		$b^2(a-1)(a-2)$		$b^3(a-1)(a-2)(a-3)$
x^1	b	1	$-b^2(a-3)$	$-2b(a-2)$	$-b^3(a^2-6a+11)$	$-3b^2(a-2)(a-3)$
x^2			b	1	$2b^2(a-4)$	$3b(a-3)$
x^3					$-b$	-1

Term*	N_4	D_4
x^0		$b^4(a-1)(a-2)(a-3)(a-4)$
x^1	$-b^4(a-5)(a^2-5a+10)$	$-4b^3(a-2)(a-3)(a-4)$
x^2	$b^3(3a^2+58-25a)$	$6b^2(a-3)(a-4)$
x^3	$-3b^2(a-5)$	$-4b(a-4)$
x^4	b	1

Term*	N_5	D_5
x^0		$b^5(a-1)(a-2)(a-3)(a-4)(a-5)$
x^1	$-b^5(85a^2-15a^3+a^4+274-225a)$	$-5b^4(a-2)(a-3)(a-4)(a-5)$
x^2	$2b^4(a-6)(2a^2-15a+37)$	$10b^3(a-3)(a-4)(a-5)$
x^3	$-3b^3(-21a+2a^2+59)$	$-10b^2(a-4)(a-5)$
x^4	$4b^2(a-6)$	$5b(a-5)$
x^5	$-b$	-1

Term*	N_6	D_6
x^0		$b^6(a-1)(a-2)(a-3)(a-4)(a-5)(a-6)$
x^1	$b^6(a-7)(a^4-14a^3+77a^2-196a+252)$	$6b^5(a-2)(a-3)(a-4)(a-5)(a-6)$
x^2	$b^5(-98a^3+5a^4+733a^2+3708-2548a)$	$15b^4(a-3)(a-4)(a-5)(a-6)$
x^3	$-2b^4(a-7)(5a^2-49a+144)$	$-20b^3(a-4)(a-5)(a-6)$
x^4	$2b^3(208-63a+5a^2)$	$15b^2(a-5)(a-6)$
x^5	$-5b^2(a-7)$	$-6b(a-6)$
x^6	b	1

* An empty cell means that the coefficient of that term is zero.

Example: $$G_3^{(1)}[f(x)] = f[x]\frac{-bx^3 + 2b^2(a-4)x^2 - b^3(a^2-6a+11)x}{-x^3 + 3b(a-3)x^2 - 2b^2(a-2)(a-3)x + b^3(a-1)(a-2)(a-3)}$$

$$\int_x^{\infty} f(t)dt \cong G_k[f(x)] = \frac{N_k(x)}{D_k(x)} f(x)$$

IV Inverse Gaussian Distribution: $f(x) = \sqrt{\frac{\lambda}{2\pi x^3}} e^{\frac{-\lambda(x-\mu)^2}{2\mu^2 x}}$

Coefficients of the x^k term

Term*	N_1	D_1	N_2	D_2	N_3	D_3
x^0		$-\lambda\mu^2$		$\lambda^2\mu^2$		$-\lambda^3\mu^6$
x^1		$3\mu^2$		$-6\lambda\mu^4$		$9\lambda^2\mu^6$
x^2	$2\mu^2$	λ	$-2\mu^4\lambda$	$-\mu^2(2\lambda^2-15\mu^2)$	$2\mu^6\lambda^2$	$-3\lambda\mu^4(-\lambda^2+15\mu^2)$
x^3			$14\mu^4$	$10\lambda\mu^2$	$-20\mu^6\lambda$	$15\mu^4(-2\lambda^2+7\mu^2)$
x^4			$2\mu^2\lambda$	λ^2	$2\mu^4(-2\lambda^2+57\mu^2)$	$3\mu^2\lambda(-\lambda^2+35\mu^2)$
x^5					$36\mu^4\lambda$	$21\lambda^2\mu^2$
x^6					$2\mu^2\lambda^2$	λ^3

Term*	N_4	D_4	N_5	D_5
x^0		$\lambda^4\mu^8$		$-\lambda^5\mu^{10}$
x^1		$-12\lambda^3\mu^8$		$15\lambda^4\mu^{10}$
x^2	$-2\mu^8\lambda^3$	$-2\lambda^2\mu^6(-45\mu^2+2\lambda^2)$	$2\mu^{10}\lambda^4$	$-5\lambda^3\mu^8(-\lambda^2+30\mu^2)$
x^3	$26\mu^8\lambda^2$	$60\lambda\mu^6(\lambda^2-7\mu^2)$	$-32\mu^{10}\lambda^3$	$50\mu^8\lambda^2(-2\lambda^2+21\mu^2)$
x^4	$6\lambda\mu^6(\lambda^2-35\mu^2)$	$3\mu^4(315\mu^4-140\lambda^2\mu^2+2\lambda^4)$	$8\mu^8\lambda^2(42\mu^2-\lambda^2)$	$-5\lambda\mu^6(2\lambda^4-210\lambda^2\mu^2+945\mu^4)$
x^5	$-2\mu^6(46\lambda^2-561\mu^2)$	$-84\lambda\mu^4(-15\mu^2+\lambda^2)$	$-168\mu^8\lambda(-\lambda^2+15\mu^2)$	$105\mu^6(2\lambda^4-60\lambda^2\mu^2+99\mu^4)$
x^6	$-6\lambda\mu^4(\lambda^2-95\mu^2)$	$-2\lambda^2\mu^2(2\lambda^2-189\mu^2)$	$2\mu^6(6\lambda^4-880\lambda^2\mu^2+6555\mu^4)$	$5\lambda\mu^4(2\lambda^4-378\lambda^2\mu^2+3465\mu^4)$
x^7	$66\mu^4\lambda^2$	$36\lambda^3\mu^2$	$240\mu^6\lambda(-\lambda^2+39\mu^2)$	$90\mu^4\lambda^2(-2\lambda^2+77\mu^2)$
x^8	$2\mu^2\lambda^3$	λ^4	$8\mu^4\lambda^2(-\lambda^2+210\mu^2)$	$5\mu^2\lambda^3(-\lambda^2+198\mu^2)$
x^9			$104\mu^4\lambda^3$	$55\lambda^4\mu^2$
x^{10}			$2\mu^2\lambda^4$	λ^5

* An empty cell means that the coefficient of that term is zero

Example

$$G_2^{(1)}[f(x)] = f[x]\frac{2\mu^2\lambda x^4 + 14\mu^4 x^3 - 2\mu^4\lambda x^2}{\lambda^2 x^4 + 10\lambda\mu^2 x^3 - \mu^2(2\lambda^2-15\mu^2)x^2 - 6\lambda\mu^4 x + \lambda^2\mu^2}$$

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