# EXPLORATORY GRAPHICAL METHODS FOR PERMUTATIONS DATA VIA POLYTOPES

Georgia L. Thompson

Technical Report No. SMU/DS/TR/251
Department of Statistical Science

July 1991

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G. L. Thompson

Department of Statistical Sciences
Southern Methodist University
Dallas, Texas 75275

**Abstract**

Exploratory graphical methods to display fully and partially ranked data, that is, to display data on the symmetric group of permutations or on a coset space of the symmetric group, are critically needed. Commonly used graphical methods such as histograms and bar graphs are inappropriate because the symmetric group and its coset spaces do not have natural linear orderings. In this paper, graphical methods are developed to display frequency distributions of fully and partially ranked data with or without pseudoranks. Simply, the frequencies for each full or partial (pseudo)ranking are plotted on the vertices of a generalized permutation polytope that preserves a natural partial ordering as well as the metric Spearman's $\rho$ on full and partial rankings, and the metric Kendall's $\tau$ on full rankings. The proposed graphical method also induces a very natural extension of Kendall's $\tau$ to partial rankings.

A permutation polytope is the convex hull the n! points in $\mathbb{R}^n$ whose coordinates are the permutations of n distinct numbers. This paper proposes a generalized definition of permutation polytopes, and proves the necessary properties of generalized permutation polytopes by considering polytopes as the intersection of a set of linear inequalities. The issues of visualizing a plot on a polytope in $\mathbb{R}^n$ are addressed by developing the theory needed to define all faces, especially the three and four dimensional faces, of any generalized permutation polytope. The proposed graphical methods are illustrated with five different data sets.

Key Words: Partial Ranking, Full Ranking, Permutation, Symmetric Group, Polytope, Exploratory Data Analysis, Graphs

This research was supported by NSF Grants DMS 8918318 and DMS 9002914. The author is very grateful to Pat Gerard for his computational help.

## 1. Introduction

Exploratory graphical methods are critically needed to display frequency distributions for fully and partially ranked data. Fully ranked data occur, for example, when a set of judges are each asked to rank (possibly, with pseudoranks) n items in order of preference. Each observation is a permutation of n distinct pseudoranks, and the resulting set of frequencies is a function on $S_n$, the symmetric group of n elements. In partially ranked data, the judges are asked for an incomplete ranking of n items which can be represented as a permutation of nondistinct pseudoranks. A set of frequencies of partial rankings is a function on a coset space of $S_n$. Because neither the elements of $S_n$, nor the cosets of $S_n$, have a natural linear ordering, graphical methods such as histograms and bar graphs cannot be used in a reasonable manner to display frequency distributions for full or partial rankings. Other graphical methods for representing rankings include multidimensional scaling, minimal spanning trees, and nearest neighbor graphs as discussed by Diaconis (1988) for fully ranked data and by Critchlow (1985) for partially ranked data. Cohen (1990) presents additional exploratory data techniques for both full and partial rankings, and Cohen and Mallows (1980) propose graphical procedures based on multidimensional scaling and biplots.

In this paper, graphical techniques are developed to display frequency distributions of fully and partial pseudorankings by using a generalized definition of permutation polytopes. A polytope is the convex hull of a finite set of points in $\mathbb{R}^n$. Yemelichev, Kovalev, and Kravtstov (1984) define a permutation polytope to be the convex hull of the n! points in $\mathbb{R}^n$ whose coordinates are the permutations of n distinct numbers. In this paper, we generalize this definition and define a permutation polytope to be the convex hull of the points in $\mathbb{R}^n$ whose coordinates are the distinct permutations of the n (not necessarily distinct) pseudoranks. Then, to graphically display a set of full or partial rankings, the frequencies with which each permutation of pseudoranks is chosen are plotted, not on a line as is done with histograms, but on the vertices of the permutation polytope.

The resulting graphical displays are especially useful as diagnostic tools because they are compatible with the two most commonly used metrics on $S_n$: Kendall's $\tau$ and Spearman's $\rho$. For fully ranked data, both the $\tau$- and $\rho$- distances are easily interpretable on the permutation polytope. Kendall's $\tau$ is the minimum number of edges that must be traversed to get from one vertex to another; and Spearman's $\rho$ is proportional to the straight line distance between vertices. If pseudoranks are used, the straight line distance between two vertices is proportional to the fixed vector extension of Spearman's $\rho$. And the minimum number of edges that must be traversed to get from one vertex to another induces a new extension of Kendall's $\tau$ on partially ranked data that is equivalent neither to the Haussdorf metric of Critchlow (1985) nor to the metric suggested by Diaconis (1988).

The permutation polytope on which the frequencies are displayed is inscribed in a sphere in an $n-1$ dimensional subspace of $\mathbb{R}^n$. (This is related to the observation by McCullagh (1990) that the n! elements of $S_n$ lie on the surface of a sphere in $\mathbb{R}^{n-1}$ in such a way as to be compatible with both Kendall's $\tau$ and Spearman's $\rho$.) Hence, for $n > 4$, the problem of visualization of points on a polytope in higher dimensions must be addressed. One approach to this problem is to explore a higher dimensional polytope by examining its three dimensional faces and portions of its four dimensional faces. An algorithm is derived to determine the defining characteristics all of the faces of any permutation polytope. It depends on first defining a permutation polytope as the solution to a finite set of linear inequalities, and then extending the results of Yemelichev, Kovalev, and Kravtstov (1984), which apply only to permutation polytopes with distinct values, to our more general definition of a permutation polytope. In particular, for any full or partial ranking it is shown that any two-dimensional face is combinatorily equivalent to either a triangle, a square or a hexagon, and any three dimensional face is combinatorily equivalent to one of the following 8 Archimedian solids: truncated tetrahedron, triangular prism, octahedron, tetrahedron, truncated octahedron, cube, cuboctahedron, or hexagonal prism. For fully ranked data, all two dimensional faces are combinatorily equivalent to either squares or hexagons, and all three-dimensional faces to either truncated octahedrons, cubes, or hexagonal prisms.

In Section 2, the proposed graphical techniques are illustrated for $n = 3$ and $n = 4$ with ordinary ranks. Section 3 contains the theory needed to develop the proposed graphics for $n > 4$ and for partially ranked data with pseudoranks. A number of examples are contained in Section 4. Section 5 concludes with proofs.

## 2. Permutation Polytopes for Fully Ranked Data with n=3,4

Before developing the concepts needed to support the proposed graphical methods for $n > 4$, for pseudoranks, or for partial rankings, the proposed graphical technique is illustrated with ordinary full rankings for n=3 and $n = 4$. In fully ranked data, a judge can express his preferences for n items either as an ordering or as a ranking. Orderings are denoted by permutations of the n item labels, bracketed by $< >$. Items are frequently labeled with the integers 1 through n, but in this section, items will be labeled with letters to avoid confusion between rankings and orderings. For example, $<$ B,C,A,D $>$ means that item B is ranked first, item C second, item A third, and item D is ranked last. A ranking is a permutation of n values that are written as a vector $\underset{\sim}{\pi} = (\pi_1, \ldots, \pi_n)$ where $\pi_1$ is the rank of item A, $\pi_2$ is the rank of item B, etc. The ranking corresponding to the ordering $<$ B,C,A,D $>$ is (3,1,2,4).

Figure 1 contains the orderings and corresponding rankings of the 6 elements of $S_3$. Note that two points are adjacent and connected by an edge if their orderings differ by a pairwise adjacent transposition, or equivalently, if their rankings differ by an inversion of two consecutive values. Hence, the minimum number of edges that must be transversed on the hexagon to get from one vertex to another is equal to Kendall's $\tau$. Formally, if $\underset{\sim}{\pi}$ and $\underset{\sim}{\sigma}$ are two full rankings, then $\tau(\underset{\sim}{\pi},\underset{\sim}{\sigma})$ is defined to be the number of pairs (i,j) such that $\pi_i < \pi_j$ and $\sigma_i > \sigma_j$. This is equivalent to the minimum number of pairwise adjacent transpositions needed to change the ordering corresponding to $\underset{\sim}{\pi}$ into the ordering corresponding to $\underset{\sim}{\sigma}$. The placement of the vertices in Figure 1 is also compatible with Spearman's $\rho$ which is defined as $\rho(\underset{\sim}{\pi},\underset{\sim}{\sigma}) = \left( \sum_{i=1}^{n} (\pi_i - \sigma_i)^2 \right)^{1/2}$. If the edges of the regular hexagon are all of length

$\sqrt{2}$, then Spearman's $\rho$ is the Euclidian distance between two vertices. Also note that the two vertices of an edge of the hexagon either have the same item ranked first or the same item ranked last.

These ideas [see, for example, Knuth (1981), McCullaugh (1990), Thompson (1991)] can be extended to n=4 by placing the 24 permutations at the vertices of a truncated octahedron, as shown in Figure 2. The truncated octahedron is an Archimedian solid with 8 hexagonal faces and 6 square faces. In Figure 2, as in Figure 1, $\tau$ is the minimum number of edges that must be traversed to get from one vertex to another, and $\rho$ is the Euclidean distance between two vertices if each edge is of length $\sqrt{2}$ (see Schulman (1979) for a proof). Examination of the two-dimensional faces of the truncated octahedron in Figure 2 shows that the four vertices of any square have the same two items ranked in the first two positions and the remaining two items ranked in the last two positions. Similarly, the six vertices on any hexagon all have either the same item ranked first or the same item ranked last. There are 24 one-dimensional faces which are edges. The two vertices of any edge agree on the ranking of the first and last choice if the edge is between two hexagons, and the vertices agree on either the first two choices or the last two choices if the edge is between a square and a hexagon. This idea that each face of a permutational polytope has a "defining property" is instrumental in the development of the proposed graphical methods for $n > 4$.

To illustrate the proposed graphical techniques with $n = 3$, consider the data in Table 1 from Duncan and Brody (1982) in which 1439 people were asked to rank city, suburban, and rural living in order of preference. The respondent's current residence is recorded as a covariate, and within each covariate, the relative frequencies for each permutation are calculated. In Figure 3 these relative frequencies are plotted on the corresponding vertices of 3 hexagons. Each hexagon corresponds to one of the three covariates, and the sizes of the circles at the vertices indicate the relative values of the relative frequencies. It is immediately obvious that rural and suburban residents are fairly similar to each other, and both are different from city dwellers. Those who prefer the city as their first choice seem to live in the city. Relatively few rural and suburban dwellers prefer their current location least, while many city dwellers would rather be anyplace else. In the case of $n = 3$, this proposed graphical technique is similar to the plots of Cohen and Mallows (1980) in which circles with areas proportional to the frequencies are placed at the ends of 6 vectors radiating from the origin.

To illustrate the effectiveness of plotting ranked data with $n = 4$ on truncated octahedrons, consider the following example. At the beginning of a course in literary criticism, 38 high school students read the short story by Faulkner and ranked 4 different styles of literary criticism in order of their preference. At the conclusion of the course, they read another short story by Faulkner and again ranked the same four styles of literary criticism. The 4 styles were authorial (a), comparative (c), personal (p), and textual (t); and the question of interest was whether or not the post-course rankings had moved in the direction of the teacher's own preferred ordering < p,c,a,t >. Table 2 contains the pre- and post-course rankings. The frequencies of the 38 pre-course rankings are shown in Figure 4a and the 38 post-course rankings are shown in Figure 4b. Although the bivariate nature of the data is lost, valuable insight into this data is gained from the plots. Most obviously, the frequencies do change a great deal between the two sets of rankings. First, there seems to be a notable increase in the frequencies at the vertices of the hexagon corresponding to the 6 orderings that begin with c. The post-course ranking do not seem to have moved *toward* the teacher's preferred ranking, <p,c,a,t>, but as suggested by the conclusions of Critchlow and Verducci (1990), they appear to be closer to <p,c,a,t> than are the pre-course rankings. One might hypothesize that the orderings have moved toward <c,p,t,a> because almost half of the post-rankings lie either on < c,p,t,a > or on one of the three vertices within on edge (pairwise transposition) of < c,p,t,a >. McCullaugh and Ye (1990) reach a similar conclusion which they illustrate by plotting the vectors of the average pre- and post-course ranking on a truncated octahedron. Other observations that can be drawn from Figure 4 include 1) the frequencies at the 6 vertices corresponding to the ordering that end in c decrease; 2) style a is rarely chosen as either a first or second choice after the course is completed; and 3) the incidence of style t as a first choice decreases.

To make the plots perceptually accurate, the areas of the circles in Figures 3 and 4 are based on Steven's Law which says that a person's perceived scale, p, of the size of an area is

$$p \propto (\text{area})^{.7}$$

(Cleveland, 1985). Hence, the areas of the circles are calculated as

$$\text{area} \propto f^{10/7},$$

where f is the value of the frequency. If the area is proportional to the frequency, area $\propto$ f, then small circles appear too large and large circles appear too small. Conversely, if the radius of the circle is proportional to the frequency, that is, area $\propto f^2$, then large values are magnified and small values are minimized.

## 3. Permutation Polytopes with Arbitrary Pseudoranks

Many applications, including partial rankings, lend themselves to the use of pseudoranks. With pseudoranks, a ranking is a vector whose elements are a permutation of n not necessarily distinct numbers (pseudoranks), and an ordering is a permutation of the items such that the first item is assigned the smallest pseudorank, the second item is assigned the second smallest pseudorank, etc. Unless otherwise stated, it is assumed without loss of generality that the pseudoranks are $0 < a_1 \leq a_2 \leq \dots \leq a_n$. Ordinary full ranks correspond to $a_i = i$ . In partially ranked data, the n items are partitioned into r groups of prespecified sizes $n_1$, $n_2$, ..., $n_r$ such that $\sum_{i=1}^{r} n_i = n$ . The $i^{th}$ group contains the judges $n_i$ $i^{th}$ favorite items. If r = n and $n_1=n_2= \dots =n_r=1$, then the data is fully ranked. The judges preferences are specified as rankings via the pseudoranks

(1) $$0 < a_1 = \dots = a_{n_1} < a_{n_1+1} = \dots = a_{n_1+n_2} < \dots < a_{n-n_r+1} = \dots = a_n$$

Equivalently, pseudorankings can be described with the very useful notation of multisets [see Stanley (1986)]. The multiset corresponding to (1) is $\{w_1^{n_1}, w_2^{n_2}, \dots, w_r^{n_r}\}$ where $w_1 < w_2 < \dots < w_r$ are the r distinct values of the pseudoranks and $w_i$ occurs $n_i$ times. Let M denote the multiset $M = \{1^{n_1}, 2^{n_2}, \dots, r^{n_r}\}$, and let $\mathfrak{S}(M)$ denote the set of $\binom{n}{n_1\ n_2\ n_3\ \dots\ n_r}$ distinct permutations of the multiset M. If $n_i = 1$, $i=1,2,\dots,n$, then $S_n = \mathfrak{S}(M)$. A permutation of (nondistinct) pseudoranks is denoted by $w_{\underset{\sim}{\sigma}} = (w_{\sigma_1}, w_{\sigma_2}, \dots, w_{\sigma_n})$ where $\underset{\sim}{\sigma} \in \mathfrak{S}(M)$. Define the set $^*\underset{\sim}{\sigma} = \{\underset{\sim}{\pi} \in S_n$ such that $w_{\underset{\sim}{\sigma}} = a_{\underset{\sim}{\pi}}\}$, and define $^*\underset{\sim}{\sigma}^{-1}$ to be the set of permutations in $S_n$ that are inverses of elements in $^*\underset{\sim}{\sigma}$. Note that $\underset{\sim}{\pi}$ is in

${}^*\underset{\sim}{\sigma}^{-1}$ if and only if $x_{\pi_k} = a_k$ implies $x_k = w_{\sigma_k}$. We will now define the pseudo-inverse of $\underset{\sim}{\pi} \in S_n$ to be the element $\underset{\sim}{\sigma} \in \mathfrak{S}(M)$ such that $\underset{\sim}{\pi} \in {}^*\underset{\sim}{\sigma}^{-1}$. This pseudo-inverse is unique and well defined, but more than one element of $S_n$ can have the same pseudo-inverse. To denote a partial ranking as an ordering, we will adopt the notation of Critchlow (1985) in which the $n_i$ items in the $i^{th}$ group are enclosed in parentheses. An ordering, with items labeled with numbers, corresponding to any pseudoranking can be obtained by simply inserting the proper notation into any element of ${}^*\underset{\sim}{\sigma}^{-1}$.

To illustrate these definitions, suppose that $n_1=2$, $n_2=2$, $n_3=1$, and $n=5$. This means that the judges state a pair of first choices, a pair of second choices, and a last choice. The pseudoranks are $0<a_1=a_2<a_3=a_4<a_5$, and the multiset is $\{w_1^2, w_2^3, w_3\}$ where $w_1=a_1$, $w_2=a_3$, and $w_3=a_5$. Further suppose that some judge ranks items 1 and 4 first and ranks item 5 last. Then the ranking is $(w_1,w_2,w_2,w_1,w_3)$ so that $\underset{\sim}{\sigma}=(1,2,2,1,3)$. Then we have ${}^*\underset{\sim}{\sigma} = \{(1,3,4,2,5), (1,4,3,2,5), (2,3,4,1,5), (2,4,3,1,5)\}$ and ${}^*\underset{\sim}{\sigma}^{-1} = \{(1,4,2,3,5), (4,1,2,3,5), (1,4,3,2,5), (4,1,3,2,5)\}$. By inserting the proper parentheses, any element of ${}^*\underset{\sim}{\sigma}^{-1}$ immediately gives the partial ordering $<(1,4),(2,3),5>$. Also, $\underset{\sim}{\sigma}$ is the pseudo-inverse of each of the four elements of ${}^*\underset{\sim}{\sigma}^{-1}$.

To extend Spearman's $\rho$ to pseudoranks, let $a_{\underset{\sim}{\pi}}$ and $a_{\underset{\sim}{\sigma}}$ be two rankings where $\underset{\sim}{\pi}$ and $\underset{\sim}{\sigma}$ are in $S_n$. Then, we define $\rho(a_{\underset{\sim}{\pi}}, a_{\underset{\sim}{\sigma}}) = \left( \sum_{i=1}^{n} (a_{\pi_i} - a_{\sigma_i})^2 \right)^{1/2}$. Critchlow (1985) shows that this extension of Spearman's $\rho$ to partially ranked data is equivalent to the fixed vector metric, $F_{fv}$. The $\rho$-distance between any two points that differ by the inversion of two consecutive pseudoranks, $a_i$ and $a_{i+1}$, is $\rho = \sqrt{2}\,|a_i - a_{i+1}|$. If the pseudoranks are chosen so that $w_i = i$, $1 \le i \le r$, then this distance reduces to $\rho = \sqrt{2}$ so that the $\rho$-distance between all "adjacent" points is constant. On the other hand, if the pseudoranks are tied ranks, then the distance between adjacent points is not constant, but instead it still holds that $\sum_{i=1}^{n} a_i = N(N+1)/2$. Both of these methods of assigning pseudoranks are of interest in the subsequent development of graphical techniques for partially ranked data via permutation polytopes.

In extending the graphical techniques of Section 2 to arbitrary pseudoranks,

$0 < a_1 \le a_2 \le \dots \le a_n$, we first consider the $\binom{n}{n_1\ n_2\ n_3\ \dots\ n_r}$ permutations, $a_{\underline{\pi}}$, as points in $\mathbf{R}^n$ where $\underline{\pi}$ is a permutation of the multiset $\{1^{n_1}, 2^{n_2}, \dots, r^{n_r}\}$. These points lie in the intersection of the sphere

$$\sum_{i=1}^{n} (x_i - \bar{a})^2 = \sum_{i=1}^{n} (a_i - \bar{a})^2 = \sum_{i=1}^{r} n_i (w_i - \bar{a})^2$$

and the $n-1$ dimensional hyperplane

$$\sum_{i=1}^{n} x_i = n\,\bar{a}$$

where $\bar{a} = n^{-1} \sum_{i=1}^{n} a_i = n^{-1} \sum_{i=1}^{r} n_i w_i$ . We define a permutation polytope as the convex hull of these points. When $a_i = i$, i=1,2,3, Figure 5 shows that the permutation polytope is a hexagon (plus its interior) inscribed in a circle in the plane $x_1 + x_2 + x_3 = 6$. When $a_3 = a_2 = 2.5$ and $a_1 = 1$, the permutation polytope is a triangle in the same plane. A permutation polytope can be mapped into $\mathbf{R}^{n-1}$ by shifting $\sum_{i=1}^{n} x_i = n\bar{a}$ to $\sum_{i=1}^{n} x_i = 0$ and then mapping the hyperplane $\sum_{i=1}^{n} x_i = 0$ onto the hyperplane $x_n = 0$ via a transformation equivalent to the Helmert transformation. In particular, the mapping is accomplished with the $n \times n$ matrix whose $n^{th}$ row is $\frac{1}{\sqrt{n}}(1, 1, 1, \dots, 1)$; and whose $i^{th}$ row, $\frac{1}{\sqrt{i^2+i}}(1, 1, \dots, 1, -\sqrt{i}, 0, \dots, 0)$, has i ones, followed by $-\sqrt{i}$ , followed by $n-i-1$ zeros, $1 \le i \le n-1$ . Because this transformation is orthonormal, distances and angles are unchanged by the mapping, so that the polytope is still inscribed in a sphere in $\mathbf{R}^{n-1}$ and Spearman's $\rho$ (the Euclidean distance between two points) is preserved.

With straightforward computations, we can now determine all possible permutation polytopes for $n = 3$ and $n = 4$. If $n = 3$ and if the three pseudoranks are distinct, then the permutation polytope is a 6-sided convex polygon, i.e., it is combinatorily equivalent to a hexagon. Unless it is a regular hexagon, it has 3 short sides alternating with three long sides. If $n = 3$ and two of the pseudoranks are equal (i.e., either just the first choice or just the last choice is specified), then the polytope is an equilateral triangle.

When n=4 and the four pseudoranks are distinct, then the polytope is combinatorily equivalent to a truncated octahedron. If $a_i = i$ (i.e., ordinary ranks), the resulting polytope, when mapped into $\mathbf{R}^3$, is the regular truncated octahedron shown in Figure 2. Its vertices are the points corresponding to the permutations and whose edges are all of length $\sqrt{2}$. In addition to the truncated octahedron, there are four other permutation polytopes for n=4. They correspond to the partial rankings. If $a_1 < a_2 < a_3 = a_4$ or $a_1 = a_2 < a_3 < a_4$, then the permutation polytope is combinatorily equivalent to a truncated tetrahedron. Figure 6 shows the regular truncated tetrahedron which corresponds to $w_i = i$, $i = 1,2,3$. Each of the 4 triangular faces corresponds to the partial rankings in which the same item is ranked first; each of the 4 hexagonal faces correspond to the partial rankings in which the same item is ranked last. In the next case, suppose that the judge specifies a first and fourth choice, but does not differentiate between the middle two items: $a_1 < a_2 = a_3 < a_4$. The resulting permutation polytope has twelve vertices and is combinatorily equivalent to the cuboctahedron in Figure 7. The regular cuboctahedron in Figure 7 is obtained from $w_i = i$, $i=1,2,3$. If three of the four pseudoranks are equal (that is, if only the first choice is specified or only the last choice is specified), then the resulting polytope is a regular tetrahedron (see Figure 8). And lastly, if $n_1 = n_2 = 2$ so that $a_1 = a_2 < a_3 = a_4$, then the permutation polytope is a regular octahedron shown in Figure 9.

A set of partially ranked data with $n = 4$ is then graphed on one of the above polytopes by placing circles whose radii are determined by

$$\text{radius} \propto f^{5/7}$$

at each appropriate vertices. If $w_i = i$, then the polytope is a regular Archimedian solid. On the other hand, if tied ranks are used, then the polytope is not necessarily regular, but it can be inscribed in a truncated octahedron by placing the vertex corresponding to a partial ranking at the centroid of the set of compatible full rankings. This offers a promising method for visually examining data sets, such as that APA voting data [Diaconis (1988)], that contain both full and partial rankings.

These permutation polytopes induce a the natural extension of Kendall's $\tau$ to partial rankings for $n = 3$ and $n = 4$. We define the extension of Kendall's $\tau$ to be equal to the minimum number of edges that must be traversed to get from one point to another. This extension clearly satisfies all of the properties of a metric, and by the symmetries of the above polytopes, is right invariant. As shown in Table 3 for $a_1<a_2=a_3<a_4$, this graphically induced metric for partially ranked data is different from the induced Hausdorff metric, $T^*$, discussed in detail by Critchlow (1985). It also differs from the

"metric" $I(\underset{\sim}{\pi},\underset{\sim}{\sigma})$ proposed by Diaconis (1988, p. 127). Although $I(\underset{\sim}{\pi},\underset{\sim}{\sigma})$ has beautiful combinatoric properties, it is not a metric because it is not symmetric in its two arguments. To obtain a counterexample, consider the simplest partial ranking in which $n = 3$, $n_1 = 1$, and $n_2 = 2$; and denote the identity rankings as $id = (1,2,2)$. By Theorem 2 of Diaconis (1988), p.127, it follows that $\sum_{\underset{\sim}{\eta}} q^{I(id,\underset{\sim}{\eta})} = q^2+q+1$ where $\underset{\sim}{\eta}$ takes the values (1,2,2), (2,1,2), and (2,2,1). This means that there is exactly one point, $\underset{\sim}{\pi}$, such that $I(id,\underset{\sim}{\pi}) = 1$, and one other point, $\underset{\sim}{\sigma}$, such that $I(id,\underset{\sim}{\sigma}) = 2$. Also, by the right invariance of I, there is exactly one point a distance 1 from $\underset{\sim}{\pi}$ and one point a distance 2 from $\underset{\sim}{\pi}$. If I is symmetric, then $I(id,\underset{\sim}{\pi}) = I(\underset{\sim}{\pi},id) = 1$, and $I(\underset{\sim}{\pi},\underset{\sim}{\sigma}) = I(\underset{\sim}{\sigma},\underset{\sim}{\pi}) = 2$. Hence, there are two points a distance 2 from $\underset{\sim}{\sigma}$ which violates the fact that $\sum_{\underset{\sim}{\eta}} q^{I(\underset{\sim}{\sigma},\underset{\sim}{\eta})} = q^2+q+1$ .

To define the relationship between the above proposed extension of Kendall's $\tau$ and arbitrary permutation polytopes for $n > 4$, we first write a permutation polytope as the solution to a system of linear inequalities and characterize all of its faces. Let $N_n$ be the set $\{1,2, ..., n\}$. For notational convenience in the following discussion, items are labeled with the integers 1 through n instead of with letters. Yemelichev et. al. [(1984), Chapter 5, Theorem 3.1] show that a permutation polytope is equivalent to the intersection of the following system of constraints:

$$\sum_{i \in \omega} x_i \le \sum_{i=1}^{|\omega|} a_{n+1-i} \quad \text{for all } \omega \subseteq N_n \tag{2}$$

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} a_i \, . \tag{3}$$

Although Yemelichev et. al. (1984) define permutation polytopes only for distinct values (i.e., distinct pseudoranks), their proof of this equivalence does not require the pseudoranks to be distinct. Also, Yemelichev et. al. (1984) use decreasing values (i.e, $0 < a_n < a_{n-1} < ... < a_1$), but without loss of generality we will use increasing values because they are far more natural in the context of rankings. The above definition of a polytope is illustrated in Figure 10. When $a_i = i$, i=1,2,3, equations (2) and (3) are

$$x_1 \le 3,$$
$$x_2 \le 3,$$
$$x_3 \le 3,$$
$$x_1+x_2 \le 5,$$
$$x_1+x_3 \le 5,$$
$$x_2+x_3 \le 5,$$
$$x_1+x_2+x_3 = 6\ .$$

On the other hand, if $a_3 = a_2 = 2.5$ and $a_1 = 1$, then equations (2) and (3) are

$$x_1 \le 2.5,$$
$$x_2 \le 2.5,$$
$$x_3 \le 2.5,$$
$$x_1+x_2 \le 5,$$
$$x_1+x_3 \le 5,$$
$$x_2+x_3 \le 5,$$
$$x_1+x_2+x_3 = 6.$$

Note that not all of the equations are needed to define the triangle.

Yemelichev et. al. (1984) then proceed in Theorem 3.4 of Chapter 5 to characterize all of the faces of a permutation polytope for distinct $0 < a_1 < a_2 < \dots < a_n$. They prove that the set of solutions to (2) and (3) is an i-dimensional face (i-face), $0 \le i \le n\text{-}2$, if and only if the inequalities in (2) are satisfied as equalities for subsets $\omega_1, \omega_2, \dots, \omega_{n-i-1}$ of $N_n$ such that $\omega_1 \subset \omega_2 \subset \dots \subset \omega_{n-i-1} \subset \omega_{n-i} = N_n$. To be able to graph partially ranked data, we extend these results to nondistinct pseudoranks. The proof of the following theorem is in Section 5.

**Theorem 1.** For $0 \le i \le n-2$, let $\omega_1, \dots, \omega_{n-i-1}$ be non-empty subsets of $N_n$, let $\omega_o = \emptyset$, and let $\omega_{n-i} = N_n$ . Then any set of solutions to

$$\sum_{i \in \omega} x_i \le \sum_{i=1}^{|\omega|} a_{n-i+1} \quad \text{for all } \omega \subseteq N_n \tag{4}$$

$$\sum_{i \in \omega_k} x_i = \sum_{i=1}^{|\omega_k|} a_{n-i+1} \quad \text{for } k = 1,2, \dots, n\text{-}i \tag{5}$$

is an i-face of the permutation polytope if and only if

1) $\omega_1 \subset \omega_2 \subset \ldots \subset \omega_{n-i-1} \subset \omega_{n-i} = N_n$, and

2) if $|\omega_j \Delta \omega_{j-1}| \geq 2$, then $a_{n-|\omega_{j-1}|} > a_{n-|\omega_j|+1}$ .

The major difference between Theorem 1 and the corresponding theorem of Yemelichev et. al. (1984) is the condition that if $|\omega_j \Delta \omega_{j-1}| \geq 2$, then $a_{n-|\omega_{j-1}|} > a_{n-|\omega_j|+1}$. This condition is satisfied trivially if all the pseudoranks are distinct. If this condition is omitted in Theorem 1, then the resulting set of solutions is a face of the polytope, but its dimension may be less than i.

To use Theorem 1, it is useful to define $Q_k = \omega_k \backslash \omega_{k-1}$, $1 \leq k \leq n\text{-}i$ , and to rephrase Theorem 1 as follows: the set of solutions to (4) and to

$$\sum_{i \in Q_k} x_i = \sum_{i=i_k}^{|Q_k|} a_{n-i+1} \text{ for } k = 1,2, \ldots, n\text{-}i \,, \tag{6}$$

where $i_k = |\omega_{k-1}|+1$, is an i-face if and only if 1) $Q_1, Q_2, \ldots, Q_{n-1}$ are disjoint with $\bigcup_{j=1}^{n-1} Q_j = \{1,2, \ldots, n\}$, and 2) if $|Q_j| \geq 2$, then $a_{n-|\omega_j|+1} < a_{n-|\omega_{j-1}|}$ . We can use Theorem 1 to characterize all of the 0-faces (i.e., vertices). For any 0-face, each of the n sets $Q_k$, $1 \leq k \leq n$, contains exactly one element (item label) which induces a permutation $\underset{\sim}{\pi}$ defined by $Q_k = \{\pi_k\}$. Then, equations (4) and (5) reduce to $x_{\pi_k} = a_{n-k+1}$ which is equivalent to $x_k = w_{\sigma_k}$ where $\underset{\sim}{\sigma}$ is the pseudo-inverse of $(\pi_n, \pi_{n-1}, \ldots, \pi_1)$. This proves that the 0-faces (vertices) of the permutation polytope are exactly the $\binom{n}{n_1\ n_2\ n_3\ \ldots\ n_r}$ points whose elements are distinct permutations of the pseudoranks. Note that $w_{\underset{\sim}{\sigma}}$, which is a vector in $\mathbf{R}^n$, is the ranking. The corresponding ordering, with items labeled as numbers, is obtained by inserting parentheses into $<\pi_n, \pi_{n-1}, \ldots, \pi_1>$ as required by the shape of the partial ranking, $n_1$, $n_2$, ..., $n_r$ .

Next, define $S(Q_1, Q_2, \ldots, Q_{n-i})$ to be the set of all possible permutations $\underset{\sim}{\pi} \in S_n$ such that $Q_k = \{\pi_{|\omega_{k-1}|+1}, \pi_{|\omega_{k-1}|+2}, \ldots, \pi_{|\omega_k|}\}$, and define $S^{-1}(Q_1, Q_2, \ldots, Q_{n-i})$ to be the set of pseudo-inverses of the permutations $(\pi_n, \pi_{n-1}, ..., \pi_1)$ where $\underset{\sim}{\pi} \in S(Q_1, Q_2, ..., Q_{n-i})$. By using pseudo-inverses, we can prove the following generalization of Corollary 3.8, Section 5, of Yemelichev et. al. (1984). The proof is in Section 5.

**Corollary 2.** The face corresponding to a partition $Q_1, Q_2, ..., Q_{n-i}$ of $\{1,2, ..., n\}$ is generated by the set of points $w_{\underset{\sim}{\sigma}}$ where $\underset{\sim}{\sigma} \in S^{-1}(Q_1, Q_2, ..., Q_{n-i})$.

For any 1-face, Theorem 1 implies that there is an integer $j \in \{1,2,...n\}$ and a permutation $\underset{\sim}{\pi}$ such that $Q_k = \{\pi_k\}$, $1 \leq k < j$; $Q_j = \{\pi_j, \pi_{j+1}\}$, $Q_k = \{\pi_{k+1}\}$, $j<k \leq n-1$; and $a_{n-j+1} > a_{n-j}$. Note that $|\omega_{j-1}| = j-1$ and $|\omega_j| = j+1$. Then, $S(Q_1, Q_2, ..., Q_{n-1})$ contains only $\underset{\sim}{\pi}$ and $\underset{\sim}{\pi}_o = (\pi_1,\pi_2, ..., \pi_{j-1},\pi_{j+1},\pi_j,\pi_{j+2},\pi_{j+3}, ...,\pi_n)$, from which it follows that the elements of $S^{-1}(Q_1, Q_2, ..., Q_{n-1})$ differ only by the inversion of two consecutive integers. By Corollary 2, it follows that 2 vertices of a permutation polytope are adjacent (on the same 1-face) if and only if they differ by a single inversion of $a_k$ and $a_{k+1}$, for some $1 \leq k \leq r-1$, where $a_k < a_{k+1}$. Equivalently, the orderings, $<\pi_n,\pi_{n-1}, ..., \pi_{n-j+2},\pi_{n-j},\pi_{n-j+1},\pi_{n-j-1},\pi_{n-j-2}, ...,\pi_1>$ and $<\pi_n,\pi_{n-1}, ..., \pi_1>$ (with extra parentheses as needed) differ only by the transposition of items $\pi_{n-j}$ and $\pi_{n-j+1}$ which do not have the same pseudorank. This extends Corollary 3.9, Section 5, of Yemelichev et. al. (1984) to nondistinct values. For ordinary ranks, it follows immediately that Kendall's $\tau$ is equal to the minimum number of edges (1-faces) that must be traversed to get from one point to another. Similarly, for partially ranked data, the above proposed extension of Kendall's $\tau$ can now be defined for $n > 4$ as the minimum number of 1-faces that must be traversed to get from one partial ranking to another. Because this extension of Kendall's $\tau$ has such appealing graphical properties, it merits further study.

By using Corollary 2, we can characterize all of the possible the 2- and 3-faces of permutation polytopes. For any 2-face, there exists a permutation $\underset{\sim}{\pi}$ of the n item labels such that one of the three cases in Table 4 holds. Case 1 is combinatorily equivalent to a hexagon because $n-3$ of the pseudoranks are assigned to $n-3$ of the items, and the other three pseudoranks, which are distinct, are permuted among the remaining three items. Case 2 is combinatorily equivalent to a triangle because

$n-3$ of the pseudoranks are fixed, and the remaining three pseudoranks, two of which are equal, are permuted among the remaining three items. And Case 3 is combinatorily equivalent to a square: $n-4$ of the pseudoranks are fixed, and of the remaining 4 pseudoranks, the two smaller are permuted and the two larger are permuted. On the squares, it is allowable that $j = i-1$, and also that $a_{n-j} = a_{n-i}$.

All 3-dimensional faces of any permutation polytope can be similarly characterized. Clearly, each 3-face is combinatorily equivalent to some Archimedian solid whose 2-faces are either triangles, squares, or hexagons. For any 3-face, we can use the same methods used above for 2-faces to show that there exists a permutation $\underset{\sim}{\pi}$ of the n item labels such that one of the eight cases shown in Table 5 holds. If the data is fully ranked, then the only possible 3-faces are combinatorily equivalent to truncated octahedrons, hexagonal prisms, or cubes. Triangular prisms and hexagonal prisms require that $n \geq 5$, and cubes require that $n \geq 6$. Note that triangular prisms and hexagonal prisms can also occur when $Q_i$ has two elements and $Q_j$ has three elements with $i > j$.

## 4. Examples

Data sets in which the observations are partial rankings with $n > 4$ can be illustrated by a set of 3-dimensional polytopes in which the frequencies are plotted on the appropriate vertices. Frequently, it is useful to also plot portions of the 4-dimensional polytopes. The surface of a permutation polytope in four dimensions can be drawn with distortion in three dimensions just as the surface of a globe can be drawn an a planar map.

To determine all of the i-faces most easily in practice, the sequence of pseudoranks is first written down with appropriate equal or less than signs. Then, Theorem 1 is used to determine all of the possible values for $|\omega_k|$, k=1,...,n-1, such that $\omega_1 \subset \omega_2 \subset \dots \subset \omega_{n-i-1} \subset \omega_{n-i} = S_n$, and such that if $|\omega_j| - |\omega_{j-1}| \geq 2$, then $a_{n-|\omega_{j-1}|} > a_{n-|\omega_j|+1}$. This determines the sizes of the sets $Q_k$, k=1, ...,n-1. Then, for an arbitrary, fixed $\underset{\sim}{\pi}$, and for each set of possible values of $|Q_k|$, the sets $Q_k$ are written as $\{\pi_{|\omega_{k-1}|+1}, \pi_{|\omega_{k-1}|+2}, \dots, \pi_{|\omega_k|}\}$, k=1,...,n-1. It then follows from Corollary 2 that the set of vertices of the i-face correspond exactly to the orderings $<\pi_n, \pi_{n-1}, \ldots, \pi_1>$ (with parentheses inserted in the required places) which are compatible with $Q_k=\{\pi_{|\omega_{k-1}|+1}, \pi_{|\omega_{k-1}|+2}, \cdots, \pi_{|\omega_k|}\}$, $k=1,\ldots,n\text{-}i$ . Then, the number of faces are counted by letting $\underline{\pi}$ range over $S_n$. This is illustrated in the following examples.

Example 1: In a university department, 5 job candidates, named A, B, C, D, and E, were being evaluated by the seven faculty members to determine which one should be invited for an interview. The chairman asked each faculty member to name his first and second choice. The data are shown in Table 6, both as orderings and as rankings with $a_1 = 1$, $a_2 = 2$, $a_3 = a_4 = a_5 = 3$ . In this case, we have $a_1<a_2<a_3=a_4=a_5$. There are 20 possible partial rankings. The questions of interest are whether there is a "most popular" candidate to invite and whether there are any "outliers" among the faculty. In determining the 3-faces, it follows from Theorem 1 that the only possible 3-faces occur when $|\omega_1| = 1$ or $|\omega_1| = 4$ . It is not possible to have $|\omega_1|=2$ because $a_5=a_4$, and it is not possible to have $|\omega_1|=3$ because $a_5=a_3$; in neither case is Condition 2 of Theorem 1 satisfied. Hence, for each 3-face we have either $Q_1 = \{\pi_1\pi_2,\pi_3,\pi_4\}$ and $Q_2 = \{\pi_5\}$, or $Q_1 = \{\pi_1\}$ and $Q_2 = \{\pi_2,\pi_3,\pi_4,\pi_5\}$. In the first case, the orderings of the points on the vertices are $<\pi_5,\pi_4,(\pi_3,\pi_2,\pi_1)>$, $<\pi_5,\pi_3,(\pi_4,\pi_2,\pi_1)>$, $<\pi_5,\pi_2,(\pi_3,\pi_4,\pi_1)>$, and $<\pi_5,\pi_1,(\pi_3,\pi_2,\pi_4)>$. The resulting figure is a tetrahedron in which all points have the same first choice, and the four vertices correspond to the four possible second choices. There are 5 different 3-faces that are tetrahedrons, each corresponding to a different first choice. In the second case, the figure is a truncated tetrahedron in which each vertex has the same candidate ranked among the last three. There are five 3-faces that are tetrahedrons. As shown in Figure 11, all of the data are contained on two adjoining faces, the tetrahedron in which candidate A is ranked first, and the truncated tetrahedron in which candidate E is ranked last. The distances between the points on two different 3-faces of a four dimensional permutation polytope can be somewhat distorted when plotted in three dimensionals, but much information is still preserved. It is immediately seen from Figure 11 that candidate A is most popular. Also, there is an outlier (who is, coincidentally, the chairman) at 32133. Better intuition about Figure 11 can be obtained by noting that the maximum number of edges that must be traversed to get from any one point to another is 3. That is equivalent to saying that the extended version of Kendall's $\tau$ (proposed above) takes the values 0,1,2, and 3. In fact, each point has 4 points that are adjacent, 6 points that are two edges away, and 9 points that are 3 edges away.

Example 2. Table 7 contains the orderings for partially ranked data in which 16 mothers and 22 preschool boys were asked to taste 5 crackers, A = animal crackers, R = Ritz crackers, S = Saltine crackers, C = cheese crackers, and G = Graham crackers. [See Critchlow (1985).] Each mother and preschooler named their first three choices, but did not differentiate between their last two choices, so that $n = 5$, $n_1 = 1$, $n_2 = 1$, $n_3 = 1$, and $n_4 = 2$. The pseudoranks are $a_1<a_2<a_3<a_4=a_5$. From Theorem 1, the 3-faces correspond to $|\omega_2| = 5$, and either $|\omega_1| = 4$, $|\omega_1| = 3$, or $|\omega_1| = 1$. Hence, we have one of the following three cases

1) $Q_1 = \{\pi_1\pi_2,\pi_3,\pi_4\}$, $Q_2 = \{\pi_5\}$,

2) $Q_1 = \{\pi_1\pi_2,\pi_3\}$, $Q_2 = \{\pi_4,\pi_5\}$,

3) $Q_1 = \{\pi_1\}$, $Q_2 = \{\pi_2,\pi_3,\pi_4,\pi_5\}$.

Condition (2) of Theorem 1 is not satisfied if $|\omega_1|=2$ because $a_4=a_5$ . In the first case, the polytope is a truncated tetrahedron corresponding to the cracker labeled $\pi_5$ being ranked first. In the second case, the polytope is a triangular prism. The vertices of the triangular prism are $<\pi_5,\pi_4,\pi_3,(\pi_2,\pi_1)>$, $<\pi_5,\pi_4,\pi_1,(\pi_3,\pi_2)>$, $<\pi_4,\pi_5,\pi_3,(\pi_2,\pi_1)>$, $<\pi_4,\pi_5,\pi_2,(\pi_1,\pi_3)>$, $<\pi_4,\pi_5,\pi_1,(\pi_2,\pi_3)>$, and $<\pi_5,\pi_4,\pi_2,(\pi_1,\pi_3)>$. In the third case, the polytope is a truncated octahedron corresponding to the cracker labeled $\pi_1$ being ranked among the last. The resulting polytope, which is inscribed in a sphere in four dimensions, has 60 vertices. Its three dimensional faces consist of 5 truncated octahedrons, 5 truncated tetrahdrons, and 10 triangular prisms. The maximum number of edges that must be traversed to get from one point to another is 6, and each point is adjacent to 4 other points.

Figures 12a and 12b show plots of the partial rankings for the mothers and boys, respectively, on a portion of the four dimensional polytope. Clearly visible in both figures are 4 truncated tetrahedrons and three triangular prisms. The figure has some distortion: the dashed lines are all of length 1, and form 3 more triangular prisms. The missing truncated tetrahedron, which has vertices all beginning with R, as well as the other four triangular prisms, are behind the middle of the figure and are not drawn. Only two of the mothers and one of the preschool boys chose partial ranking beginning with R which would be plotted on the omitted truncated tetrahedron. The differences between the mothers and the boys are readily apparent: the boys tend to prefer animal crackers and the mothers tend to prefer saltines. In these drawings the truncated octahedrons are distorted and difficult to see. They can be plotted separately, but little additional insight into the data is obtained.

Example 3. In a major city, Catholic Charities mailed a survey to a sampling of the members of the local diocese asking each person to rank from 1 to 3 the top three services needed in the community as they saw them. The list of possible choices were

I = Intensive therapy for emotionally troubled youth and their families
E = Employment assistance for the unemployed
F= Food and financial assistance for families in crisis
L = Legal assistance for immigrants and families
M = Marriage and family counseling
D = Day care for low income families
A = Adoption
O = Outreach to refugees arriving in Dallas
S = Alcohol and substance abuse treatment for adolescents
P = Prepared meal and health services for low-income senior citizens
H = Housing for low and moderate income families

Table 8 shows the frequencies with which each partial ordering was chosen. Altogether there were 576 respondents to the survey who listed a first, second, and third choice; and 284 of the possible 990 partial rankings were chosen.

In this example the pseudoranks are $a_1<a_2<a_3<a_4=a_5=a_6=a_7=a_8=a_9=a_{10}=a_{11}$. Using Theorem 1, it follows that there are four different possibilities for the sizes of $\omega_k$:

1) $|\omega_k|=1$, $k=1, \ldots, 7$; $|\omega_8|=11$

2) $|\omega_k|=1$, $k=1, \ldots, 6$; $|\omega_7|=10$; $|\omega_8|=11$

3) $|\omega_k|=1$, $k=1, \ldots, 6$; $|\omega_7|=9$; $|\omega_8|=11$

4) $|\omega_k|=1$, $k=1, \ldots, 5$; $|\omega_6|=9$; $|\omega_7|=10$; $|\omega_8|=11$ .

The resulting sets $Q_k$ are

1) $Q_k=\{\pi_k\}$, $k=1, \ldots, 7$; $Q_8=\{\pi_8,\pi_9,\pi_{10},\pi_{11}\}$

2) $Q_k=\{\pi_k\}$, $k=1, \ldots, 6$; $Q_7=\{\pi_7,\pi_8,\pi_9,\pi_{10}\}$; $Q_8=\{\pi_{11}\}$

3) $Q_k=\{\pi_k\}$, $k=1, \ldots, 6$; $Q_7=\{\pi_7,\pi_8,\pi_9\}$; $Q_8=\{\pi_{10},\pi_{11}\}$

4) $Q_k=\{\pi_k\}$, $k=1, \ldots, 5$; $Q_6=\{\pi_6,\pi_7,\pi_8,\pi_9\}$; $Q_7=\{\pi_{10}\}$; $Q_8=\{\pi_{11}\}$ .

In the first case, the figure is a truncated octahedron in which items labeled $\pi_1$ through $\pi_7$ are ranked among the last 8, and the remaining 4 items are permuted among the first, second, third and fourth places. In the second case, the figure is a truncated tetrahedron in which item $\pi_{11}$ is ranked first, and items $\pi_7$, $\pi_8$, $\pi_9$, and $\pi_{10}$ are permuted among second, third and two fourth places. In the third case, the figure is a triangular prism in which items $\pi_{10}$ and $\pi_{11}$ are permuted between first and second place, and $\pi_7$, $\pi_8$, and $\pi_9$ are permuted among the third and two of the fourth places. And in the fourth case, item $\pi_{11}$ is ranked first, item $\pi_{12}$ is ranked second, and items $\pi_6$, $\pi_7$, $\pi_8$, and $\pi_9$ are permuted among the third and three of the fourth places. The resulting permutation polytope is inscribed in a sphere in $\mathbb{R}^{10}$ and has 990 vertices. Its three dimensional faces consist of 330 truncated octahedrons, 1050 truncated tetrahedrons, 4620 triangular prisms, and 13,860 tetrahedrons.

In spite of the large number of 3-faces, a great deal of the data can be illustrated by choosing the 3- and 4- faces that contain the largest percentages of the data. For example, if interest is restricted to any 5 items, then the points can be plotted on the figure used in Example 2. This is done in Figure 13 for F,E,D,H, and P, with the truncated octahedron corresponding to rankings beginning with P, which is hidden behind the figure, shown at the bottom of Figure 13. The most striking feature of the 4-face in Figure 13 is that although it has only 20 3-faces, it contains almost 1/3 of the data. The 12 points on the tetrahedron in which F is ranked first, contain 92 respondents, or 16 percent of the data. Also interesting is the fact that the points of this truncated tetrahedron are almost uniformly distributed except for the vertices FDH and FDP. The frequency distributions on the other

4 truncated octahedrons are not all that different from each other, but overall their frequencies are considerably smaller than the frequencies of the truncated octahedron beginning with F. These observations are useful in guiding the choice of a model to fit the data.

Faces can also be chosen to answer specific question of interest. In this case, there is interest in the relationship between choices I and S. Figure 14a contains the truncated tetrahedron whose vertices correspond to the partial rankings in which the first, second and third choices are chosen from F, S, P, and I. On the right side of the figure S precedes I; on the left side, I precedes S. Clearly, S precedes I more frequently on this particular 3-face. As a contrast, Figure 14b contains a hexagons with all the permutations of I, S, and M. In this case, I precedes S. These observations can be interpreted in light of the qualitative differences between the choices F and P, and the choice M. Other faces to illustrate the relationship between S and I contain too few points to be of interest.

## 5. Proofs

**Proof of Theorem 1.** First, we prove that the system of equations (4) and (5) determines an i-dimensional face when Conditions 1 and 2 hold. Clearly, the points satisfying (4) and (5) are a subset of the points satisfying (2) and (3) in the definition of the permutation polytope. Because the system is consistent, it determines a face. The rank of the system is $n-i$ . To show that the face is of dimension i, we use Proposition 4.3 of Yemelichev et. al. (1984), p. 36, and show that the system has exactly $n-i$ linearly independent constraints by showing that none of the inequalities in (4) are rigid. That is, we must show that all of the inequalities in (4) can be solved as strict inequalities. Let $\omega$ be a subset of $N_n$ such that $\omega \neq \omega_k$ , k=0,1,2 ..., $n-i$ . From (4) we have that

$$\sum_{i \in \omega} x_i \leq \sum_{i=1}^{|\omega|} a_{n-i+1} \, .$$

There exist an integer $p \in \{0,1,2,\ldots,n\text{-}i\text{-}1\}$ such that $\omega_p \subset \omega$ and $\omega_{p+1} \not\subset \omega$ . Because $|\omega_p| < |\omega|$, we can write

$$\sum_{i \in \omega} x_i = \sum_{i \in \omega_p} x_i + \sum_{i \in \omega \backslash \omega_p} x_i = \sum_{i=1}^{|\omega_p|} a_{n-i+1} + \sum_{i \in \omega \backslash \omega_p} x_i \,.$$

Similarly, there exists an integer $q \in \{1,2,...,n\text{-}i\}$ such that $\omega \subset \omega_q$ and $\omega \not\subset \omega_{q-1}$ ; and because $|\omega| < |\omega_q|$, we can write

$$\sum_{i \in \omega_q} x_i = \sum_{i \in \omega} x_i + \sum_{i \in \omega_q \backslash \omega} x_i = \sum_{i=1}^{|\omega_q|} a_{n-i+1} \,.$$

This implies that

$$\sum_{i \in \omega \backslash \omega_p} x_i \le \sum_{i=|\omega_p|+1}^{|\omega|} a_{n-i+1} \,; \quad \sum_{i \in \omega_q \backslash \omega} x_i \ge \sum_{i=|\omega|+1}^{|\omega_q|} a_{n-i+1} \,; \text{ and } \sum_{i \in \omega_q \backslash \omega_p} x_i = \sum_{i=|\omega_p|+1}^{|\omega_q|} a_{n-i+1} \,.$$

Because $\omega_p \subset \omega \subset \omega_q$ , $\omega_{p+1} \not\subset \omega$ , and $\omega \not\subset \omega_{q-1}$ , we have that $|Q_k| \ge 2$ for each integer k such that $p < k \le q$ . This implies that $a_{n-|\omega_k|+1} < a_{n-|\omega_{k-1}|}$ for $p < k \le q$ , from which it follows that the above pair of inequalities can be satisfied as a strict inequalities which in turn implies that the face has dimension i.

Conversely, suppose that we have an i-face of the permutation polytope satisfying equations (4) and (5). Without loss of generality, assume that $|\omega_j| < |\omega_k|$ implies $j < k$ , j,k=1,2, ..., n-i. Assume that Condition 2 does not hold so that for some integer j we have $|\omega_j \Delta \omega_{j-1}| \ge 2$ and $a_{n-|\omega_j|+1} = a_{n-|\omega_{j-1}|}$ . Straightforward calculations show that the system of equations in (4) and (5) is equivalent to the system of equations defined by (4) and (5) augmented with

$$\sum_{i \in \omega_k} x_i = \sum_{i=1}^{|\omega_*|} a_{n-i+1} \,,$$

where $\omega_* = \omega_{j-1} \cup \{x_*\}$ where $x_* \in Q_j$ . Hence, the face is of dimension less than i which is a contradiction. Next, suppose that the inclusions in Condition 1 of the theorem do not hold. Then there is a pair of sets, $\omega_p$ and$\omega_q$, such that neither is a subset of the other. Without loss of generality, we can assume that $p = q - 1$ and that $|\omega_p| \le |\omega_q|$ . Then for any point x on the i-face, we have

$$\sum_{i \in \omega_p} x_i + \sum_{i \in \omega_q} x_i = \sum_{i=1}^{|\omega_p|} a_{n-i+1} + \sum_{i=1}^{|\omega_q|} a_{n-i+1}$$

$$= \sum_{i \in \omega_p \cup \omega_q} x_i + \sum_{i \in \omega_p \cap \omega_q} x_i \leq \sum_{i=1}^{|\omega_p \cup \omega_q|} a_{n-i+1} + \sum_{i=1}^{|\omega_p \cap \omega_q|} a_{n-i+1} .$$

Because neither $\omega_q$ or $\omega_p$ is a subset of the other, we have that $|\omega_p \Delta \omega_q| \geq 2$, which in turn implies that $a_{n-|\omega_q|+1} < a_{n-|\omega_p|}$ . Because the pseudoranks are nondecreasing, it follows that

$$\sum_{i=1}^{|\omega_p \cup \omega_q|} a_{n-i+1} + \sum_{i=1}^{|\omega_p \cap \omega_q|} a_{n-i+1} < \sum_{i=1}^{|\omega_q|} a_{n-i+1} + \sum_{i=1}^{|\omega_p|} a_{n-i+1}$$

This contradiction implies that the inclusions in Condition 1 must hold.

**Proof of Corollary 2.** Corresponding to any i-face is a partition $Q_1, Q_2, ..., Q_{n-i}$ and a set of permutations $S(Q_1, Q_2, ..., Q_{n-i})$ such that $Q_k = \{\pi_{|\omega_{k-1}|+1}, \pi_{|\omega_{k-1}|+2}, \cdots, \pi_{|\omega_k|}\}$ for any $\underset{\sim}{\pi} \in S(Q_1, Q_2, ..., Q_{n-i})$ . First, suppose that $w_{\underset{\sim}{\sigma}}$ is a vertex such that $\underset{\sim}{\sigma} \in S^{-1}(Q_1, Q_2, ..., Q_{n-i})$ and let $\underset{\sim}{x} = w_{\underset{\sim}{\sigma}}$ . Then, by definition of $S^{-1}(Q_1, Q_2, ..., Q_{n-i})$, we have $x_{\pi_k} = a_{n-k+1}$ , k=1,2,...,n where $\underset{\sim}{\pi} \in S(Q_1, Q_2, ..., Q_{n-i})$ . Hence,

$$\text{(7)} \qquad \sum_{j \in Q_k} x_j = \sum_{j=j_k}^{|Q_k|} x_{\pi_j} = \sum_{j=j_k}^{|Q_k|} a_{n-j+1} \text{ for } k = 1,2, ..., n\text{-}i ,$$

where $j_k = |\omega_{k-1}|+1$, which shows that $a_{\underset{\sim}{\sigma}}$ is on the face determined by $Q_1, Q_2, ..., Q_{n-i}$ . Conversely, suppose that $w_{\underset{\sim}{\eta}}$ is not on the face generated by $Q_1, Q_2, ..., Q_{n-i}$; that is, $\underset{\sim}{\eta} \notin S^{-1}(Q_1, Q_2, ..., Q_{n-i})$. Let $\underset{\sim}{x} = w_{\underset{\sim}{\eta}}$ . Then, because $a_{n-j+1}$ is decreasing in j, equations (7) hold at $\underset{\sim}{x} = w_{\underset{\sim}{\eta}}$ only if $\{x_{\pi_{j_k}}, x_{\pi_{j_k+1}}, ..., x_{\pi_{|Q_k|}}\} = \{a_{n-j_k+1}, a_{n-j_k+2}, ..., a_{n-|Q_k|+1}\}$ for each $k=1,2,...,n-i$ . This implies that $\underset{\sim}{\eta} \in S^{-1}(Q_1, Q_2, ..., Q_{n-i})$, which is a contradiction. □

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Table 1: Data Set

n = 3 Residence Types with a Covariate
1439 Survey Respondents

| | | Frequencies | | | | Relative Frequencies | | |
|---|---|---|---|---|---|---|---|---|
| Orderings | Rankings | City | Suburb | Rural | Total | City | Suburb | Rural |
| <C,S,R> | (1,2,3) | 210 | 22 | 10 | 242 | .330 | .044 | .033 |
| <C,R,S> | (1,3,2) | 23 | 4 | 1 | 28 | .036 | .008 | .003 |
| <S,C,R> | (2,1,3) | 111 | 45 | 14 | 170 | .174 | .090 | .046 |
| <S,R,C> | (3,1,2) | 204 | 299 | 125 | 628 | .320 | .598 | .414 |
| <R,C,S> | (2,3,1) | 8 | 4 | 0 | 12 | .012 | .008 | .000 |
| <R,S,C> | (3,2,1) | 81 | 126 | 152 | 359 | .127 | .252 | .503 |

Table 2: Data Set

n = 4 Types of Literary Criticism
38 Students

| | Frequencies | |
|---|---|---|
| Ordering | before | after |
| ACPT | 0 | 0 |
| ACTP | 0 | 1 |
| ATCP | 0 | 0 |
| ATPC | 0 | 1 |
| APTC | 4 | 1 |
| APCT | 1 | 0 |
| CAPT | 0 | 1 |
| CATP | 0 | 2 |
| CTPA | 2 | 3 |
| CTAP | 1 | 4 |
| CPAT | 1 | 5 |
| CPTA | 1 | 4 |
| PCAT | 3 | 2 |
| PCTA | 2 | 4 |
| PTAC | 2 | 0 |
| PTCA | 2 | 2 |
| PATC | 2 | 2 |
| PACT | 3 | 1 |
| TACP | 1 | 1 |
| TAPC | 0 | 0 |
| TCPA | 4 | 2 |
| TCAP | 2 | 0 |
| TPAC | 2 | 0 |
| TPCA | 5 | 2 |

Teacher's preference: PCAT

Table 3: Haussdorf Metric $T^*$ vs. Proposed Extension of Kendall's $\tau$: Distance From <a,(b,c),d>

| Ordering | $T^*$ | $\tau$ |
|---|---|---|
| <a,(b,c),d> | 0 | 0 |
| <a,(b,d),c> | 2 | 1 |
| <a,(d,c),b> | 2 | 1 |
| <b,(a,c),d> | 2 | 1 |
| <b,(a,d),c> | 3 | 2 |
| <b,(d,c),b> | 4 | 2 |
| <c,(a,d),b> | 3 | 2 |
| <c,(a,b),d> | 2 | 1 |
| <c,(b,d),a> | 4 | 2 |
| <d,(b,c),a> | 5 | 3 |
| <d,(a,c),b> | 4 | 2 |
| <d,(b,a),c> | 4 | 2 |

Table 4: 2-Faces of Permutation Polytopes

| | | |
|---|---|---|
| 1) $Q_k=\{\pi_k\},\ 1 \le k<j$<br>$Q_j=\{\pi_j,\pi_{j+1},\pi_{j+2}\}$<br>$Q_k=\{\pi_{k+2}\},\ j<k \le n-2$ | $a_{n-j+1}>a_{n-j}>a_{n-j-1}$ | hexagon |
| 2) $Q_k=\{\pi_k\},\ 1 \le k<j$<br>$Q_j=\{\pi_j,\pi_{j+1},\pi_{j+2}\}$<br>$Q_k=\{\pi_{k+2}\},\ j<k \le n-2$ | $a_{n-j+1}=a_{n-j}>a_{n-j-1}$ or<br>$a_{n-j+1}>a_{n-j}=a_{n-j-1}$ | triangle |
| 3) $Q_k=\{\pi_k\},\ 1 \le k<j$<br>$Q_j=\{\pi_j,\pi_{j+1}\}$<br>$Q_k=\{\pi_{k+1}\},\ j<k<i$<br>$Q_i=\{\pi_{i+1},\pi_{i+2}\}$<br>$Q_k=\{\pi_{k+3}\},\ i<k \le n-2$ . | $a_{n-j+1}>a_{n-j}$ and $a_{n-i}>a_{n-i-1}$ | square |

Table 5: 3-Faces of Permutation Polytopes

| | | |
|---|---|---|
| 1) $Q_k=\{\pi_k\},\ 1 \leq k<j$<br>$Q_j=\{\pi_j,\pi_{j+1},\pi_{j+2},\pi_{j+3}\}$<br>$Q_k=\{\pi_{k+3}\},\ j<k \leq n-3$ | $a_{n-j+1}<a_{n-j}<a_{n-j-1}<a_{n-j-2}$ | truncated octahedron |
| 2) $Q_k=\{\pi_k\},\ 1 \leq k<j$<br>$Q_j=\{\pi_j,\pi_{j+1},\pi_{j+2},\pi_{j+3}\}$<br>$Q_k=\{\pi_{k+3}\},\ j<k \leq n-3$ | $a_{n-j+1}=a_{n-j}<a_{n-j-1}<a_{n-j-2}$; or<br>$a_{n-j+1}<a_{n-j}<a_{n-j-1}=a_{n-j-2}$ | truncated tetrahedron |
| 3) $Q_k=\{\pi_k\},\ 1 \leq k<j$<br>$Q_j=\{\pi_j,\pi_{j+1},\pi_{j+2},\pi_{j+3}\}$<br>$Q_k=\{\pi_{k+3}\},\ j<k \leq n-3$ | $a_{n-j+1}<a_{n-j}=a_{n-j-1}<a_{n-j-2}$ | cuboctahedron |
| 4) $Q_k=\{\pi_k\},\ 1 \leq k<j$<br>$Q_j=\{\pi_j,\pi_{j+1},\pi_{j+2},\pi_{j+3}\}$<br>$Q_k=\{\pi_{k+3}\},\ j<k \leq n-3$ | $a_{n-j+1}=a_{n-j}=a_{n-j-1}<a_{n-j-2}$; or<br>$a_{n-j+1}<a_{n-j}=a_{n-j-1}=a_{n-j-2}$ | tetrahedron |
| 5) $Q_k=\{\pi_k\},\ 1 \leq k<j$<br>$Q_j=\{\pi_j,\pi_{j+1},\pi_{j+2},\pi_{j+3}\}$<br>$Q_k=\{\pi_{k+3}\},\ j<k \leq n-3$; | $a_{n-j+1}=a_{n-j}<a_{n-j-1}=a_{n-j-2}$; or<br>$a_{n-j+1}=a_{n-j}<a_{n-j-1}=a_{n-j-2}$ | octahedron |
| 6) $Q_k=\{\pi_k\},\ 1 \leq k<j$<br>$Q_j=\{\pi_j,\pi_{j+1}\}$<br>$Q_k=\{\pi_{k+1}\},\ j<k<i$<br>$Q_i=\{\pi_{i+1},\pi_{i+2},\pi_{i+3}\}$<br>$Q_k=\{\pi_{k+4}\};\ i<k \leq n-3$ | $a_{n-j+1}<a_{n-j}$ and<br>$a_{n-i}<a_{n-i-1}<a_{n-i-2}$ | hexagonal prism |
| 7) $Q_k=\{\pi_k\},\ 1 \leq k<j$<br>$Q_j=\{\pi_j,\pi_{j+1}\}$<br>$Q_k=\{\pi_{k+1}\},\ j<k<i$<br>$Q_i=\{\pi_{i+1},\pi_{i+2},\pi_{i+3}\}$,<br>$Q_k=\{\pi_{k+4}\},\ i<k \leq n-3$ or | $a_{n-j+1}<a_{n-j}$ ; and<br>$a_{n-i}<a_{n-i-1}=a_{n-i-2}$ or<br>$a_{n-i}=a_{n-i-1}<a_{n-i-2}$ | triangular prism. |
| 8) $Q_k=\{\pi_k\},\ 1 \leq k<j$<br>$Q_j=\{\pi_j,\pi_{j+1}\}$<br>$Q_k=\{\pi_{k+1}\},\ j<k<i$<br>$Q_i=\{\pi_{i+1},\pi_{i+2}\}$<br>$Q_k=\{\pi_{k+4}\},i<k<m$<br>$Q_m=\{\pi_{m+2},\pi_{m+3}\}$<br>$Q_k=\{\pi_{k+4}\},\ j<k<n\text{-}3$ | $a_{n-j+1}<a_{n-j}$ and<br>$a_{n-i}<a_{n-i-1}$ and<br>$a_{n-m-1}<a_{m-m-2}$ | cube |

Table 6: Data Set

n = 5 Candidates, A, B, C, D, E
7 Rankers

| Orderings | Rankings |
|---|---|
| < A,E,(B,C,D) > | (1,3,3,3,2) |
| < A,D,(B,C,E) > | (1,3,3,2,3) |
| < D,A,(B,C,E) > | (2,3,3,1,3) |
| < A,B,(C,D,E) > | (1,2,3,3,3) |
| < B,A,(C,D,E) > | (2,1,3,3,3) |
| < C,B,(A,D,E) > | (3,2,1,3,3) |
| < A,C,(B,D,E) > | (1,3,2,3,3) |

Table 7: Data Set

n = 5 Types of Crackers
16 Mothers and 22 Preschool Boys

| Orderings | |
|---|---|
| Boys | Mothers |
| ACS | CRA |
| GCA | SRG |
| ACG | CSA |
| CAG | CSA |
| CGA | SRA |
| ARC | SCR |
| CSA | SCG |
| SCR | GAR |
| AGC | SAR |
| ARG | CSA |
| AGC | RSC |
| ACS | RAG |
| GRA | SCG |
| CGA | SAR |
| ACS | GAS |
| CGS | SCA |
| ARC | |
| ACG | |
| RAC | |
| AGC | |
| ACG | |
| CAG | |

Table 8: Data Set

n = 11 Choices
576 Survey Respondents

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ADH | 1 | DSI | 1 | FDP | 2 | FPS | 7 | IEP | 3 | MID | 1 | PME | 3 |
| ADS | 1 | DSM | 2 | FDS | 1 | FSD | 1 | IES | 2 | MIE | 1 | PMF | 1 |
| AED | 1 | EAF | 1 | FEA | 1 | FSE | 1 | IFO | 1 | MIF | 3 | PMO | 1 |
| AEF | 1 | EDF | 3 | FED | 10 | FSH | 1 | IFS | 2 | MIL | 1 | SAD | 2 |
| AEL | 1 | EDH | 3 | FEH | 11 | FSI | 2 | IMA | 1 | MIO | 1 | SAE | 1 |
| AEO | 1 | EDM | 2 | FEI | 6 | FSL | 1 | IMD | 1 | MIS | 3 | SDE | 2 |
| AFP | 2 | EDP | 3 | FEM | 6 | FSM | 1 | IMF | 1 | MLA | 1 | SDF | 1 |
| AHS | 1 | EFA | 2 | FEP | 10 | FSO | 2 | IML | 1 | MPA | 2 | SDI | 1 |
| AIS | 1 | EFD | 4 | FES | 2 | FSP | 4 | IMP | 2 | MPF | 2 | SED | 2 |
| ALH | 1 | EFH | 5 | FHD | 9 | HAD | 1 | IMS | 4 | MSE | 3 | SEF | 1 |
| AMI | 1 | EFI | 2 | FHE | 10 | HDE | 2 | IOA | 1 | MSF | 1 | SEL | 1 |
| APD | 1 | EFL | 2 | FHI | 2 | HDF | 2 | IPD | 1 | OFE | 1 | SEP | 2 |
| APM | 1 | EFM | 3 | FHM | 2 | HDI | 3 | IPE | 1 | OIS | 1 | SFD | 2 |
| ASM | 1 | EFO | 1 | FHO | 2 | HDL | 1 | IPF | 2 | OLF | 1 | SFE | 2 |
| DAL | 1 | EFP | 4 | FHP | 9 | HDM | 2 | IPH | 1 | OME | 1 | SFH | 1 |
| DAS | 1 | EFS | 1 | FHS | 4 | HDO | 1 | IPM | 1 | OMF | 1 | SFI | 3 |
| DEH | 2 | EHD | 2 | FIE | 4 | HDP | 2 | IPO | 1 | OPF | 1 | SFL | 1 |
| DEP | 1 | EHF | 2 | FIH | 2 | HDS | 1 | ISD | 2 | OPL | 1 | SFM | 1 |
| DFE | 2 | EHI | 1 | FIP | 1 | HED | 2 | ISE | 1 | PAF | 3 | SFP | 2 |
| DFH | 2 | EHL | 1 | FIS | 3 | HEF | 1 | ISF | 1 | PDE | 1 | SHE | 2 |
| DFI | 1 | EIF | 2 | FLH | 1 | HEP | 1 | ISM | 3 | PDF | 2 | SHL | 1 |
| DFM | 1 | ELA | 1 | FLS | 1 | HFD | 1 | ISP | 1 | PDI | 2 | SHP | 1 |
| DFO | 1 | ELI | 1 | FMA | 1 | HFE | 9 | LMD | 1 | PDL | 1 | SIA | 1 |
| DFP | 2 | EMS | 1 | FMD | 2 | HFM | 1 | MAD | 2 | PED | 2 | SIF | 3 |
| DFS | 1 | EPF | 2 | FME | 1 | HFO | 1 | MAE | 2 | PEF | 2 | SIH | 1 |
| DHE | 1 | EPH | 1 | FMH | 1 | HFP | 6 | MAI | 1 | PEI | 1 | SIL | 1 |
| DHF | 3 | EPM | 2 | FMI | 1 | HFS | 2 | MAP | 1 | PEM | 1 | SIM | 3 |
| DHO | 1 | ESH | 1 | FML | 1 | HIE | 1 | MAS | 1 | PFA | 2 | SIP | 5 |
| DHP | 2 | ESI | 1 | FMP | 1 | HLA | 1 | MDP | 2 | PFD | 5 | SLE | 1 |
| DIF | 1 | ESM | 1 | FMS | 2 | HOF | 1 | MED | 1 | PFE | 7 | SMD | 1 |
| DIP | 1 | ESP | 3 | FOD | 2 | HPF | 3 | MEF | 1 | PFH | 4 | SME | 1 |
| DMF | 1 | FAD | 1 | FOE | 2 | HPI | 4 | MEH | 1 | PFI | 1 | SMP | 1 |
| DMI | 1 | FAM | 2 | FOL | 1 | HPL | 1 | MEP | 1 | PFL | 1 | SOD | 1 |
| DMS | 1 | FAP | 2 | FOP | 1 | HPS | 1 | MFA | 1 | PFO | 1 | SPE | 4 |
| DOI | 1 | FAS | 1 | FPA | 2 | HSE | 1 | MFE | 2 | PFS | 6 | SPF | 1 |
| DPF | 4 | FDA | 1 | FPD | 11 | HSP | 1 | MFH | 3 | PHD | 1 | SPH | 1 |
| DPH | 1 | FDE | 9 | FPE | 7 | IAS | 1 | MFI | 2 | PHF | 3 | SPM | 1 |
| DPM | 2 | FDH | 2 | FPH | 12 | IDE | 1 | MFL | 1 | PHS | 2 | SPO | 1 |
| DPS | 2 | FDI | 3 | FPI | 5 | IDS | 1 | MFP | 2 | PID | 1 | | |
| DSE | 1 | FDL | 1 | FPM | 1 | IED | 1 | MFS | 2 | PIE | 1 | | |
| DSH | 1 | FDM | 2 | FPO | 5 | IEF | 2 | MHE | 1 | PIF | 1 | | |

All other orderings were not chosen

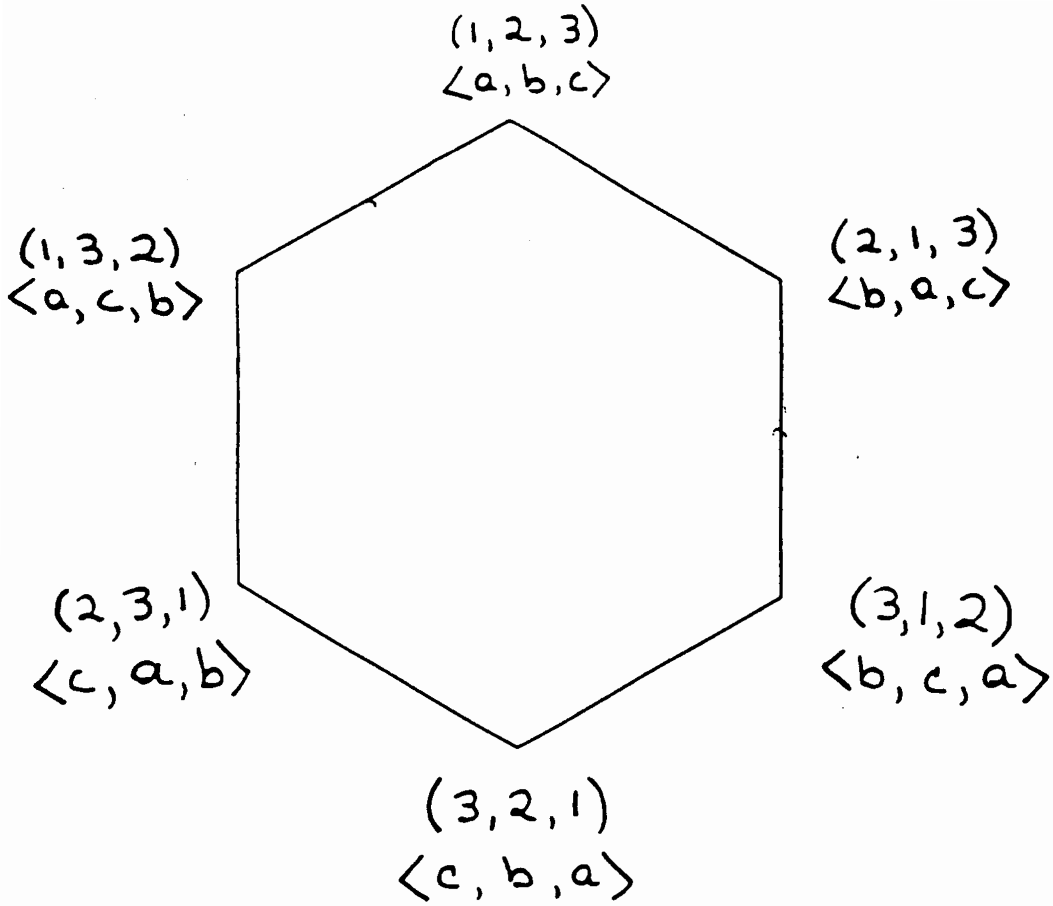
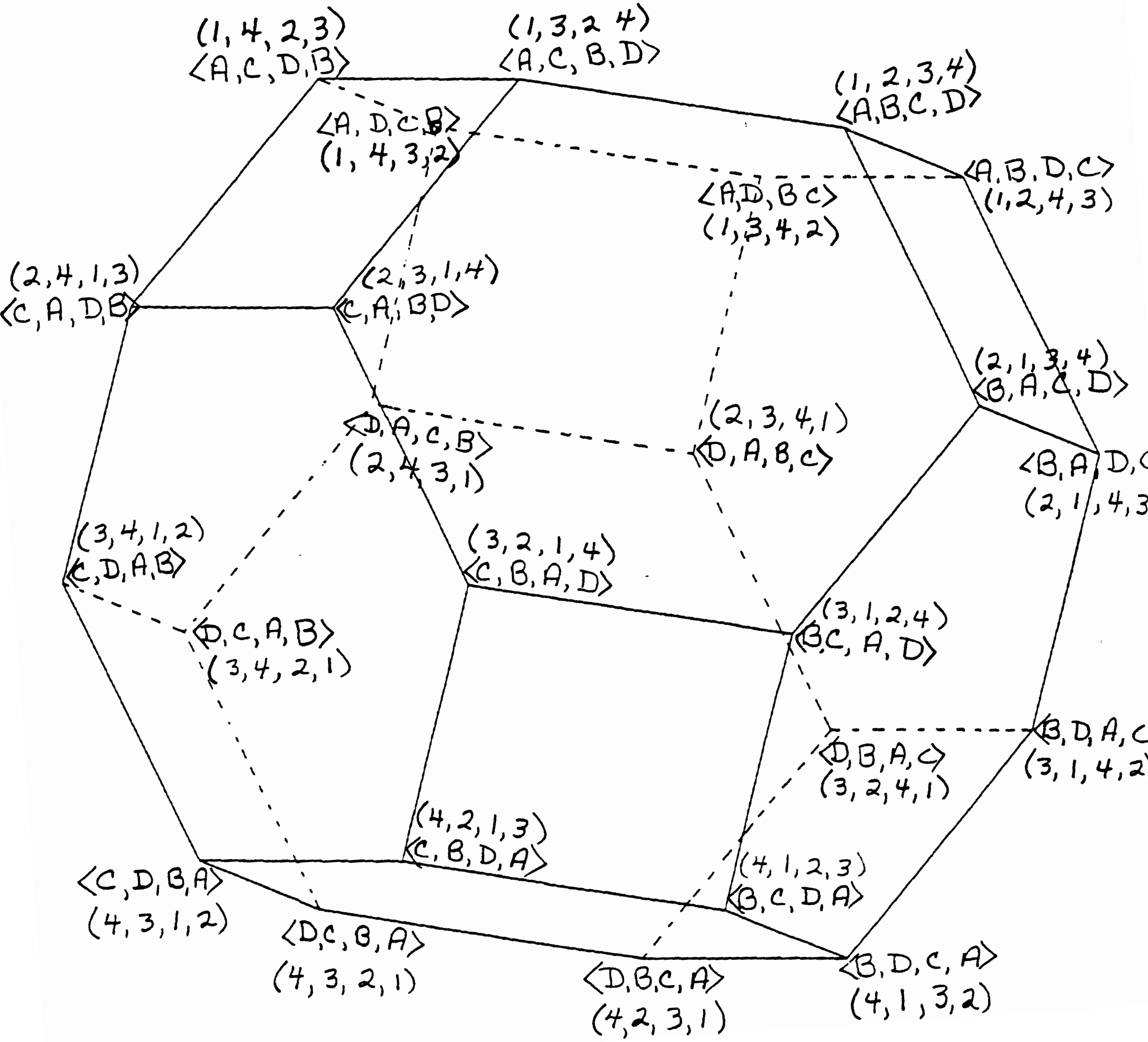

(1, 2, 3)
$\langle a, b, c \rangle$
(2, 1, 3)
$\langle b, a, c \rangle$
(3, 1, 2)
$\langle b, c, a \rangle$
(3, 2, 1)
$\langle c, b, a \rangle$
(2, 3, 1)
$\langle c, a, b \rangle$
(1, 3, 2)
$\langle a, c, b \rangle$

Figure 1

Figure 2


(1, 4, 2, 3)
<A, C, D, B>
(1, 3, 2 4)
<A, C, B, D>
(1, 2, 3, 4)
<A, B, C, D>
<A, D, C, B>
(1, 4, 3, 2)
<A, B, D, C>
(1, 2, 4, 3)
<A, D, B C>
(1, 3, 4, 2)
(2, 4, 1, 3)
<C, A, D, B>
(2, 3, 1, 4)
<C, A, B, D>
(2, 1, 3, 4)
<B, A, C, D>
(2, 3, 4, 1)
<D, A, B, C>
<D, A, C, B>
(2, 4, 3, 1)
<B, A, D, C
(2, 1, 4, 3
(3, 4, 1, 2)
<C, D, A, B>
(3, 2, 1, 4)
<C, B, A, D>
<D, C, A, B>
(3, 4, 2, 1)
(3, 1, 2, 4)
<B, C, A, D>
<B, D, A, C
(3, 1, 4, 2
<D, B, A, C>
(3, 2, 4, 1)
(4, 2, 1, 3)
<C, B, D, A>
(4, 1, 2, 3)
<B, C, D, A>
<C, D, B, A>
(4, 3, 1, 2)
<D, C, B, A>
(4, 3, 2, 1)
<D, B, C, A>
(4, 2, 3, 1)
<B, D, C, A>
(4, 1, 3, 2)

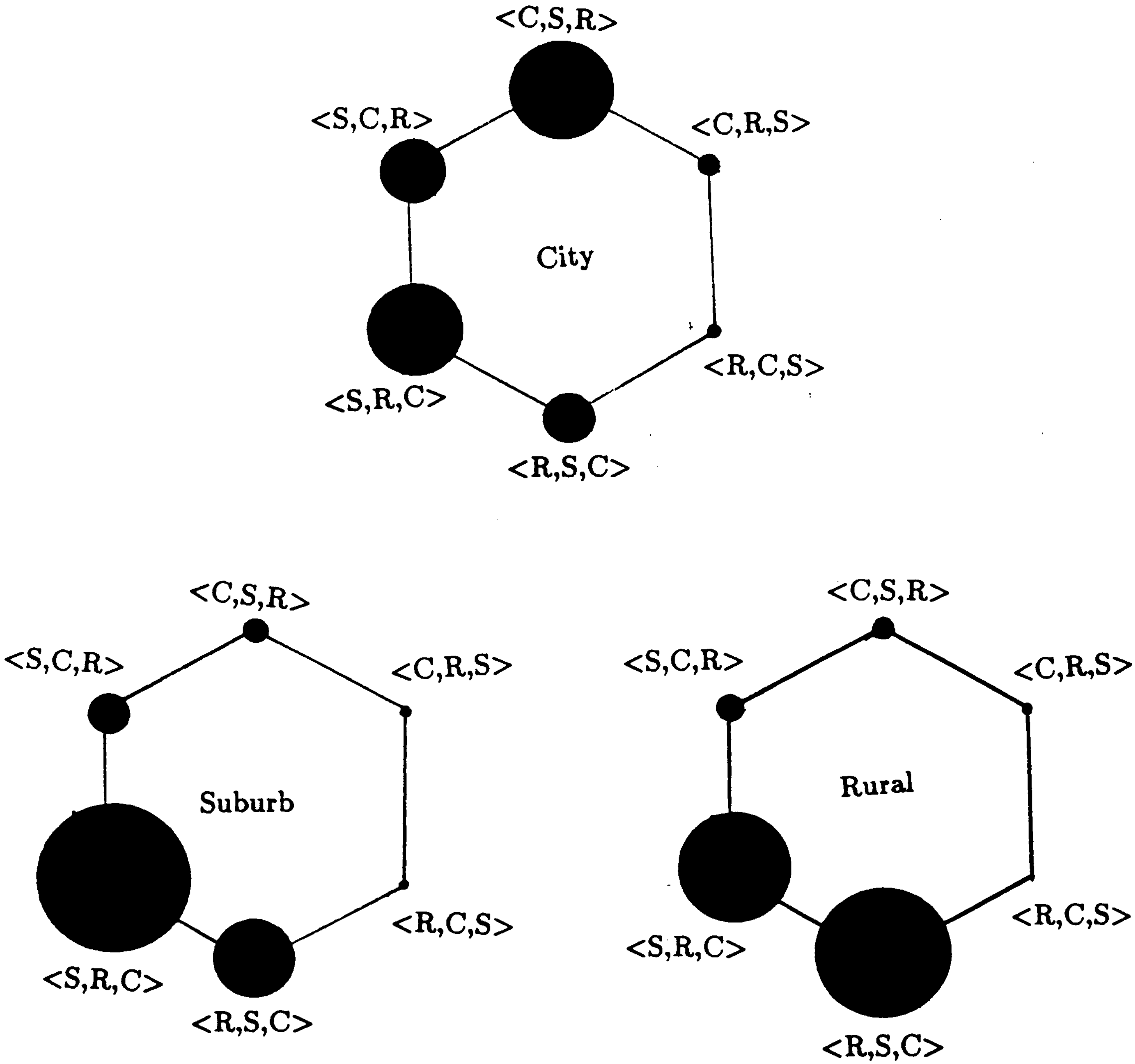

<C,S,R>
<S,C,R>
<C,R,S>
City
<R,C,S>
<S,R,C>
<R,S,C>
<C,S,R>
<S,C,R>
<C,R,S>
Suburb
<R,C,S>
<S,R,C>
<R,S,C>
<C,S,R>
<S,C,R>
<C,R,S>
Rural
<R,C,S>
<S,R,C>
<R,S,C>

Figure 3

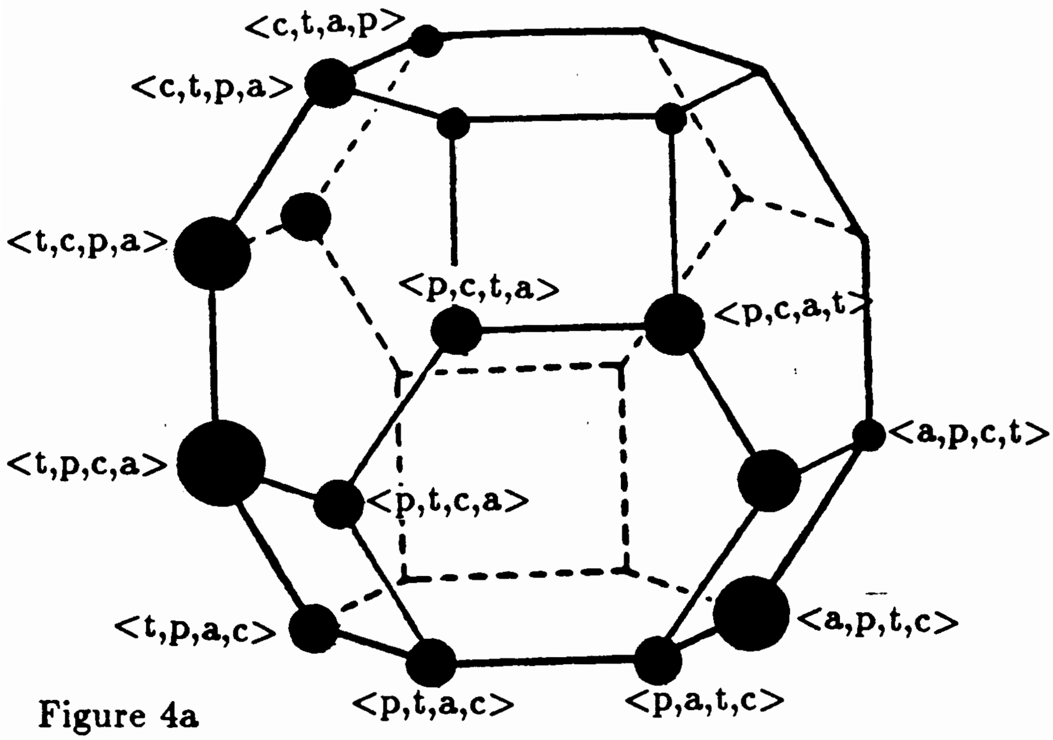



Figure 4a

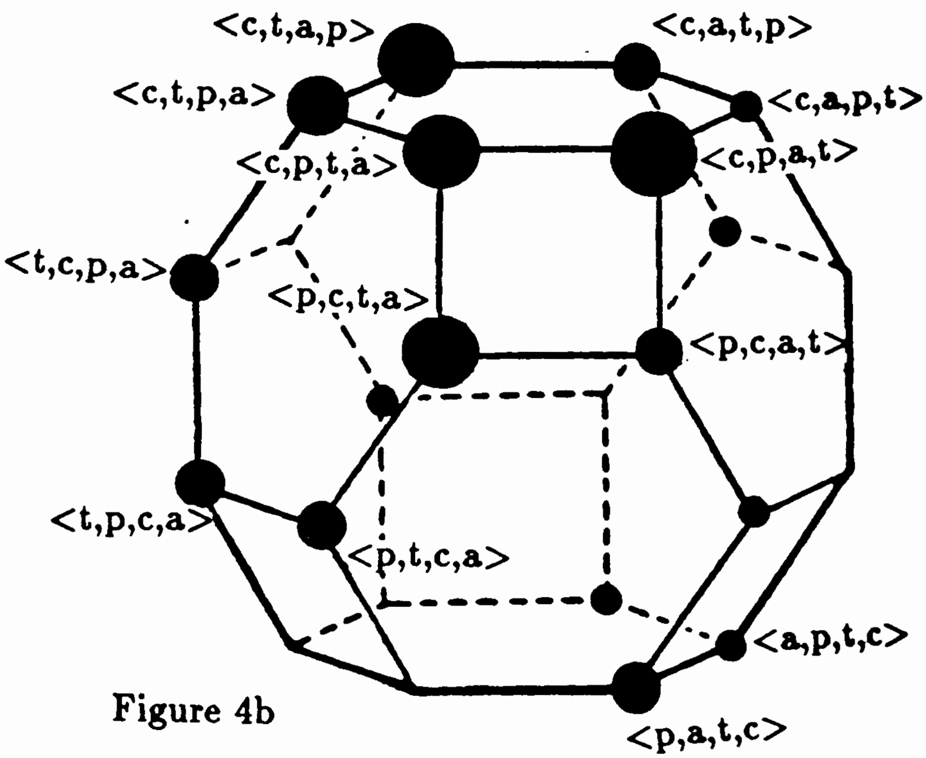
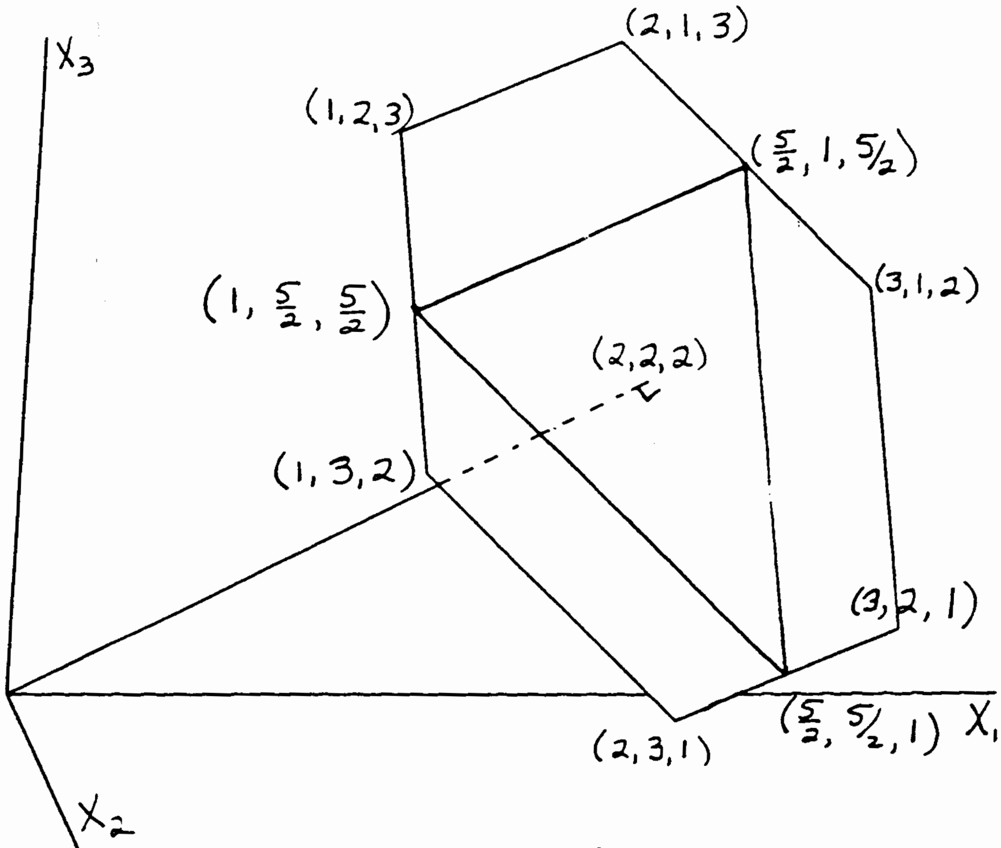



Figure 4b



$X_3$
(2,1,3)
(1,2,3)
$(\frac{5}{2}, 1, 5/2)$
(3,1,2)
$(1, \frac{5}{2}, \frac{5}{2})$
(2,2,2)
(1,3,2)
(3,2,1)
(2,3,1)
$(\frac{5}{2}, 5/2, 1)$
$X_1$
$X_2$


Figure 5

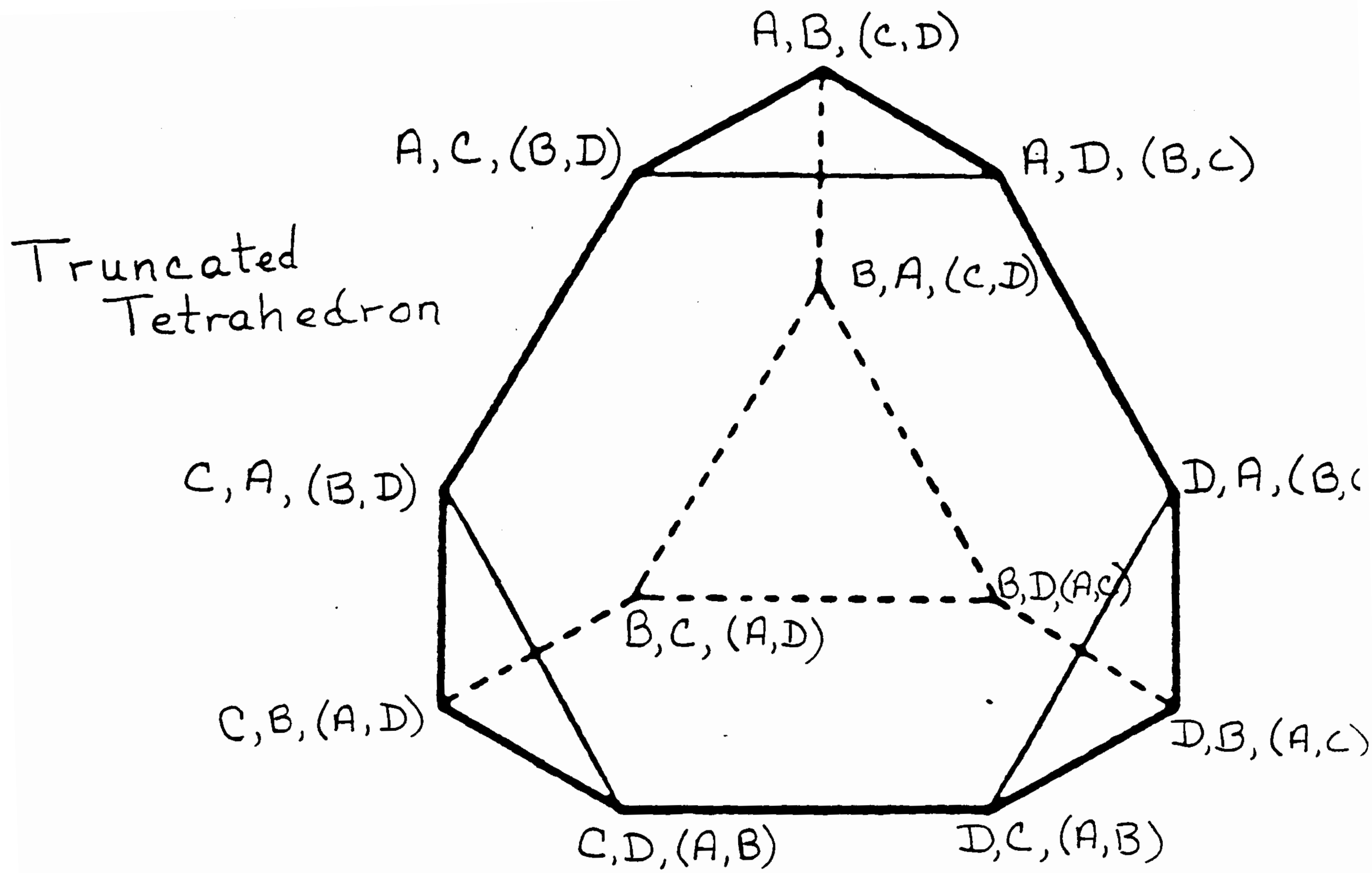

A, B, (C, D)
A, C, (B, D)
A, D, (B, C)
Truncated
Tetrahedron
B, A, (C, D)
C, A, (B, D)
D, A, (B, C
B, D, (A, C)
B, C, (A, D)
C, B, (A, D)
D, B, (A, C)
C, D, (A, B)
D, C, (A, B)

Figure 6

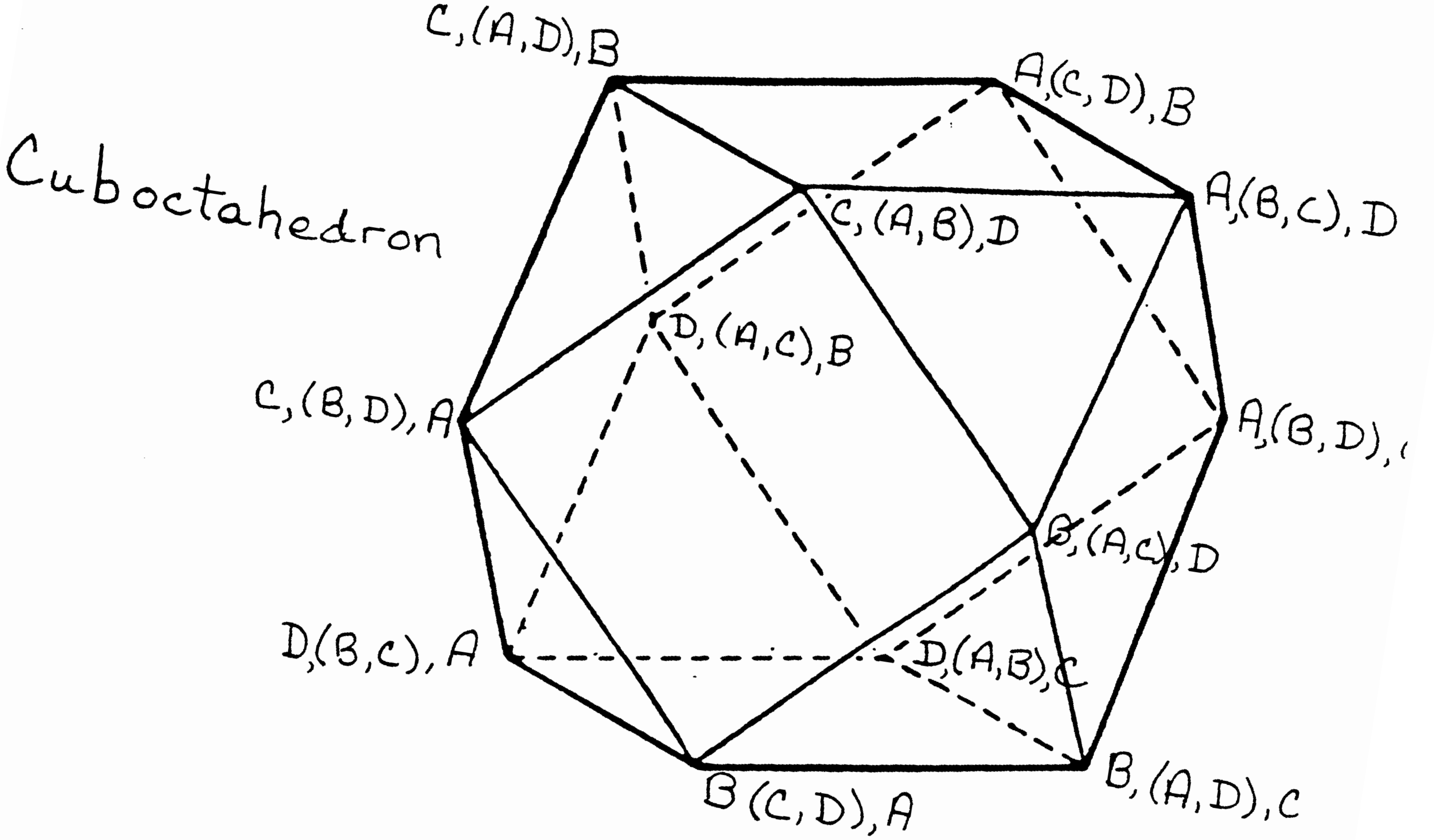

Cuboctahedron
C,(A,D),B
A,(C,D),B
A,(B,C),D
C,(A,B),D
D,(A,C),B
C,(B,D),A
A,(B,D),
B,(A,C),D
D,(B,C),A
D,(A,B),C
B(C,D),A
B,(A,D),C

Figure 7

Figure 8
Tetrahedron

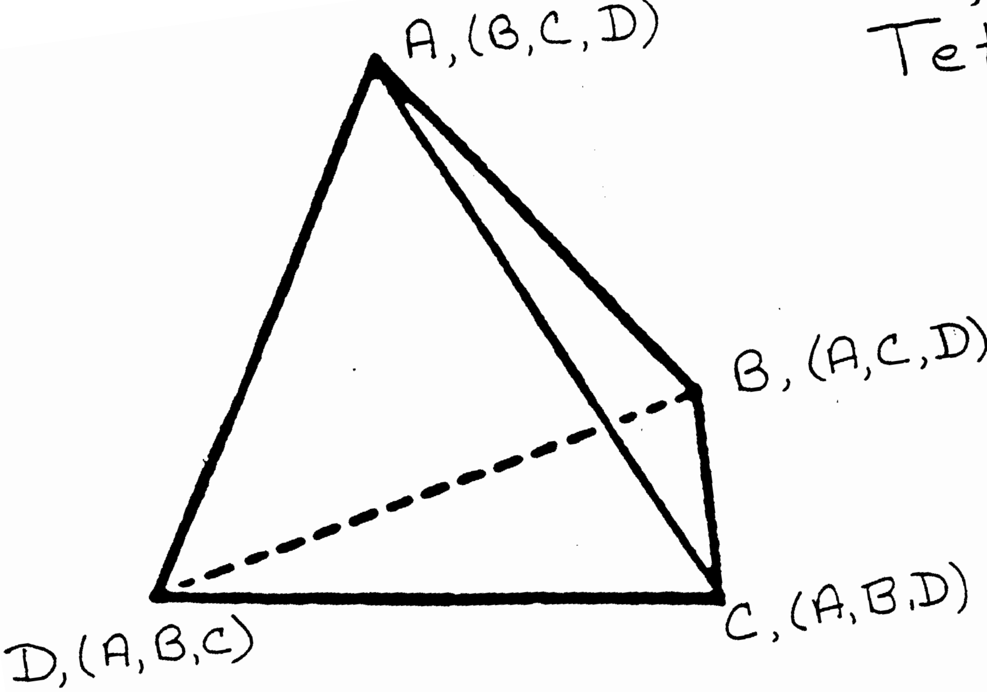
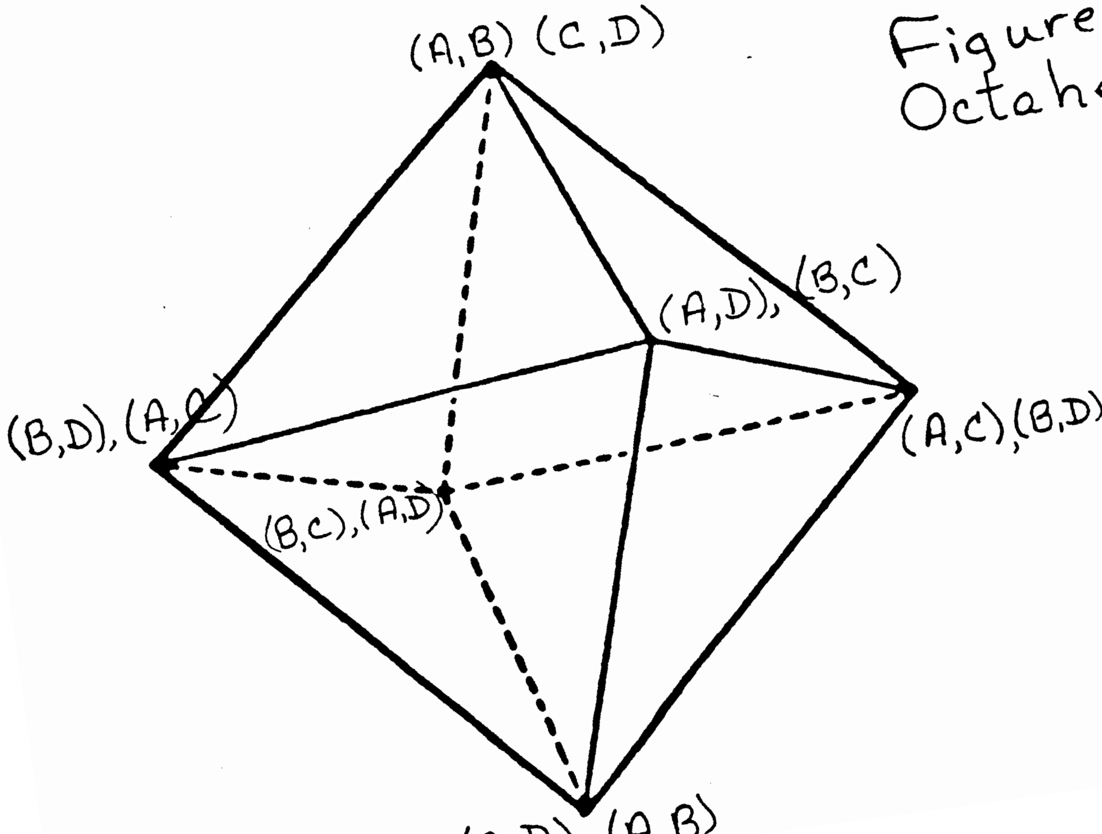
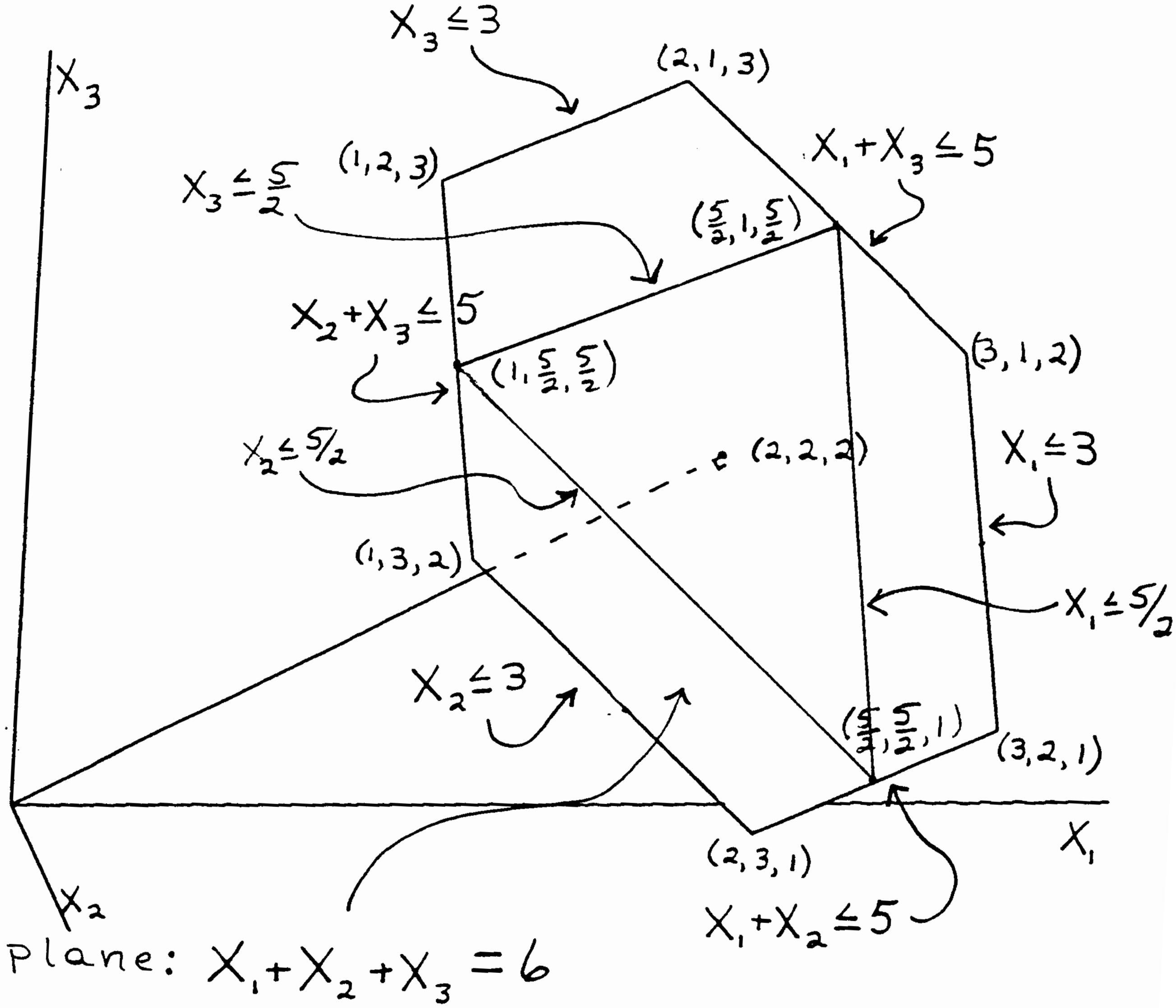



Figure 9
Octahedron

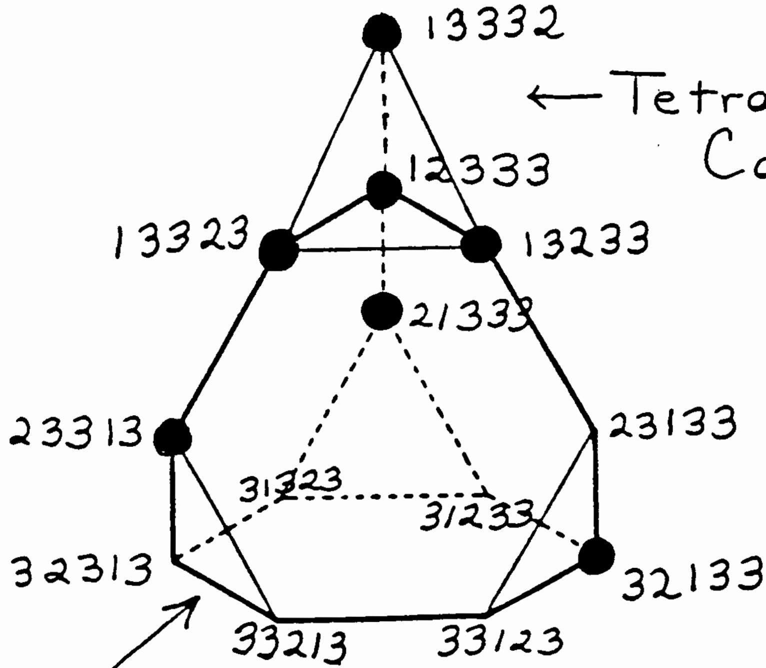




X3
X3 ≤ 3
(2, 1, 3)
(1, 2, 3)
X1 + X3 ≤ 5
X3 ≤ 5/2
(5/2, 1, 5/2)
X2 + X3 ≤ 5
(1, 5/2, 5/2)
(3, 1, 2)
X2 ≤ 5/2
(2, 2, 2)
X1 ≤ 3
(1, 3, 2)
X1 ≤ 5/2
X2 ≤ 3
(5/2, 5/2, 1)
(3, 2, 1)
X1
(2, 3, 1)
X2
X1 + X2 ≤ 5
plane: X1 + X2 + X3 = 6


Figure 10

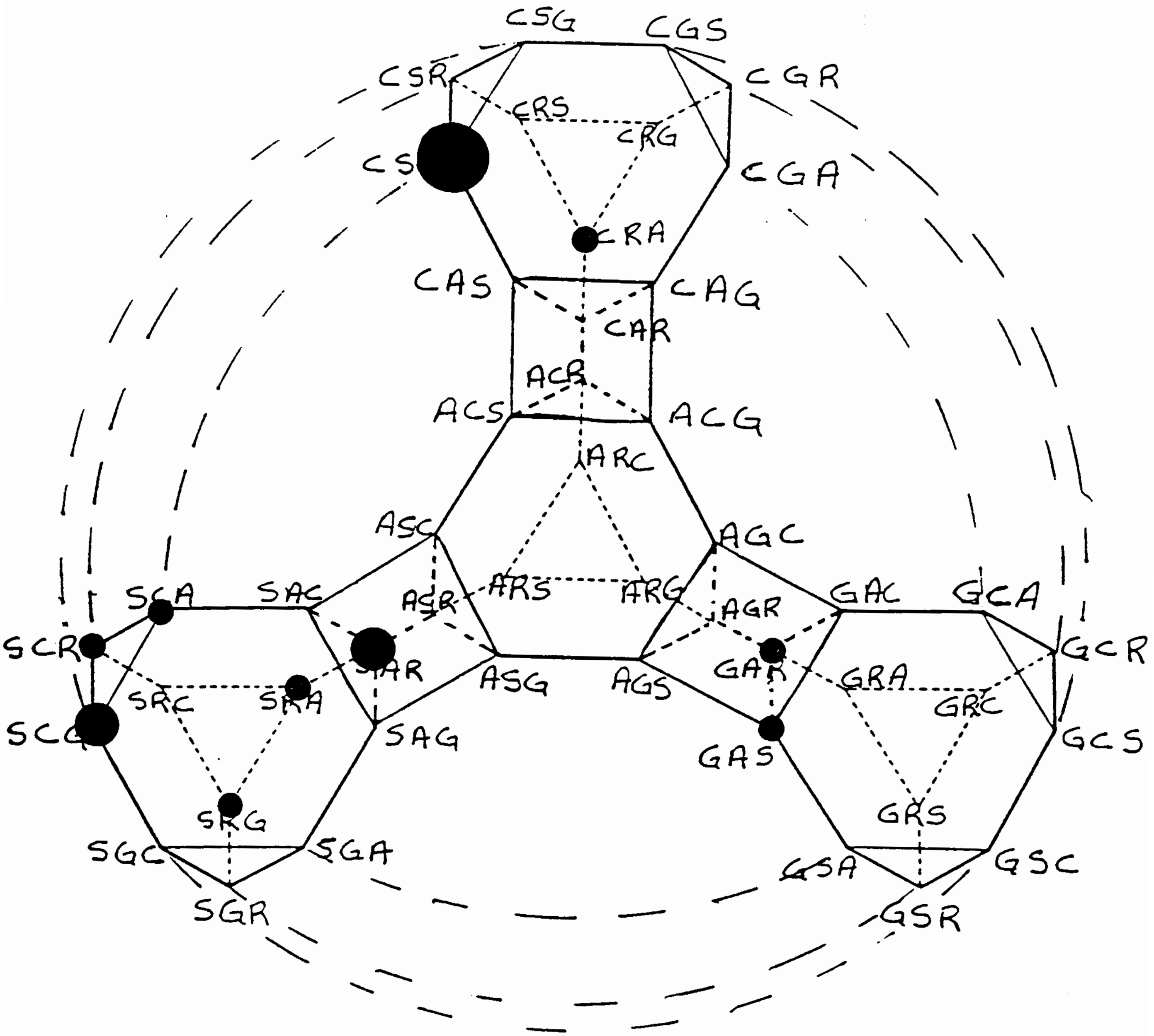
Figure 11




← Tetrahedron = Candidate A is ranked 1st

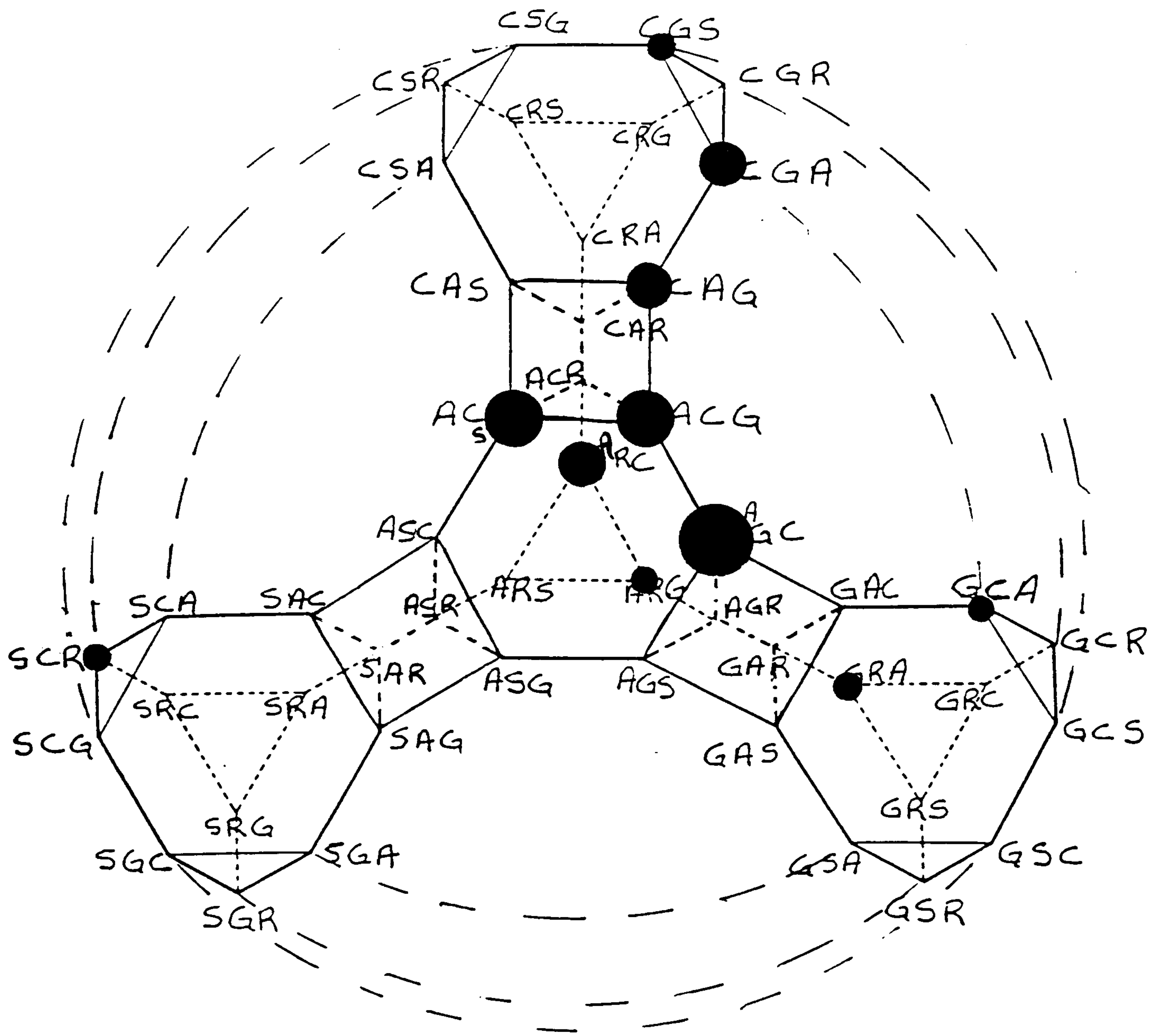
Truncated tetrahedron = Candidate E is ranked among the last 3.

Figure 12a: mothers' rankings




omit: RSC
RAG

Figure 12b: boys' rankings




Omit RAC

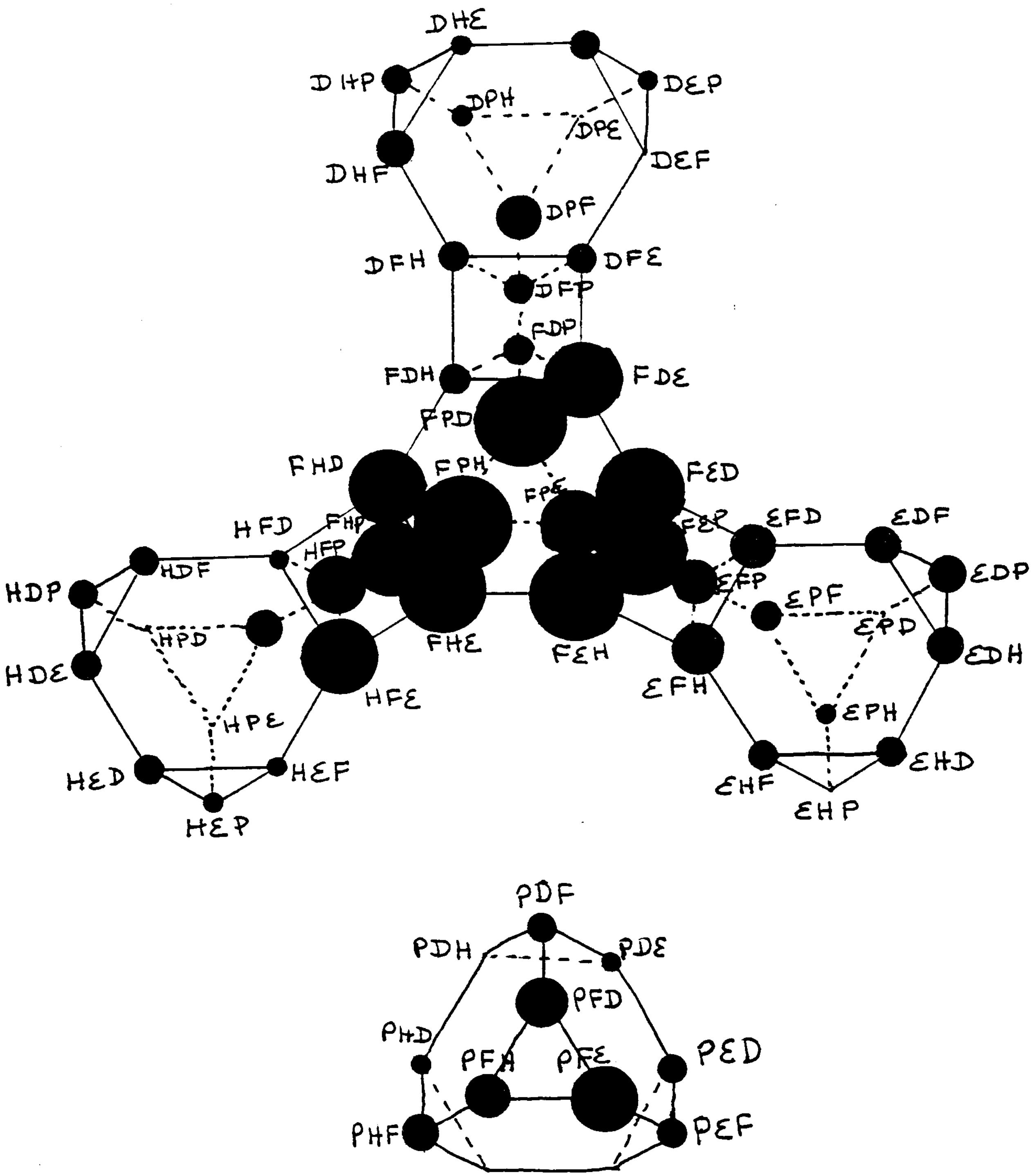

DHE
DHP
DPH
DEP
DPE
DEF
DHF
DPF
DFH
DFE
DFP
FDP
FDH
FDE
FPD
FHD
FPH
FED
FPE
HFD
FHP
FEP
EFD
EDF
HFP
HDF
HDP
EDP
EFP
EPF
HPD
EPD
HDE
FHE
FEH
EDH
HFE
EFH
HPE
EPH
HED
HEF
EHD
EHF
HEP
EHP
PDF
PDH
PDE
PFD
PHD
PFH
PFE
PED
PHF
PEF

Figure 13

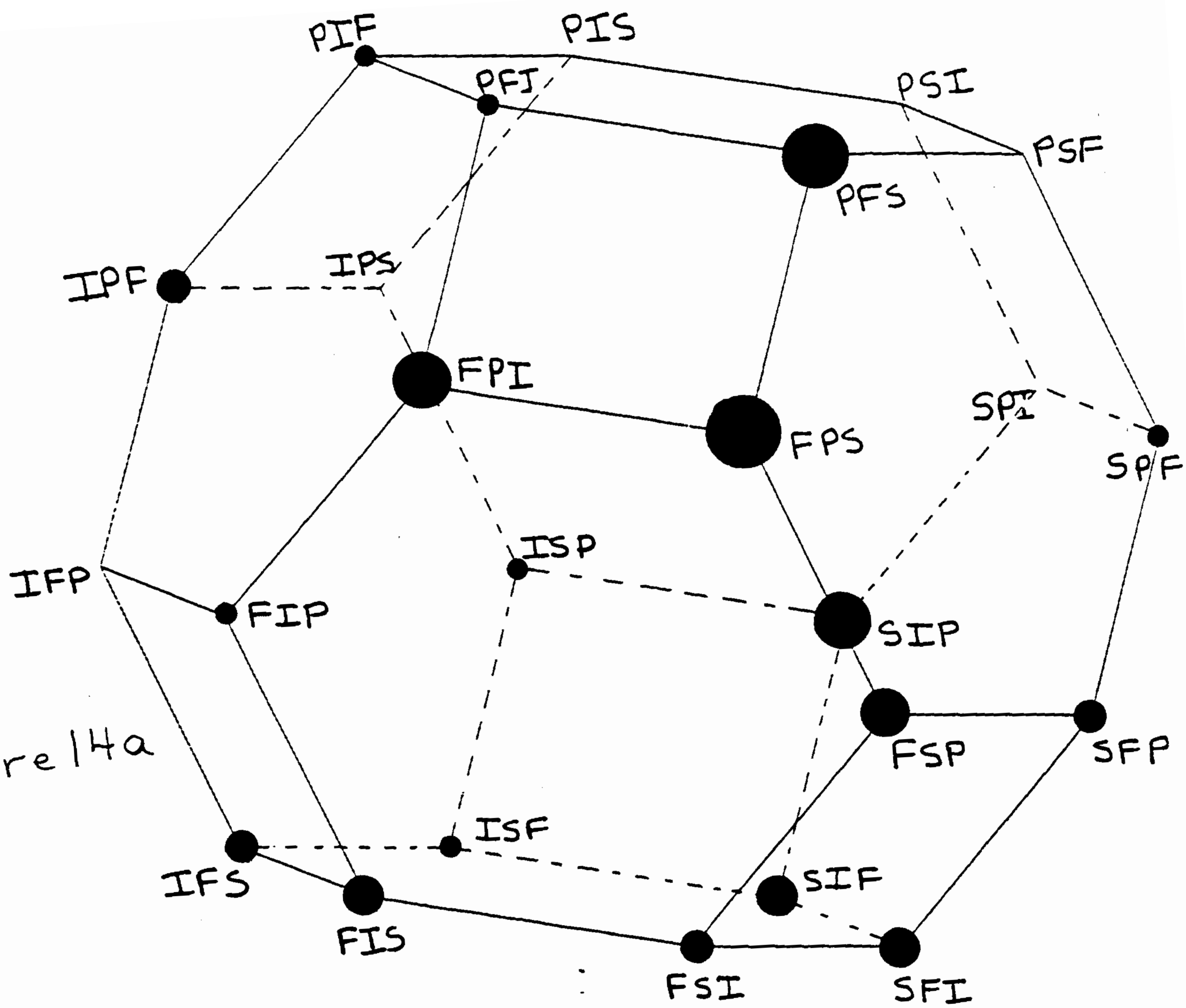



Figure 14a

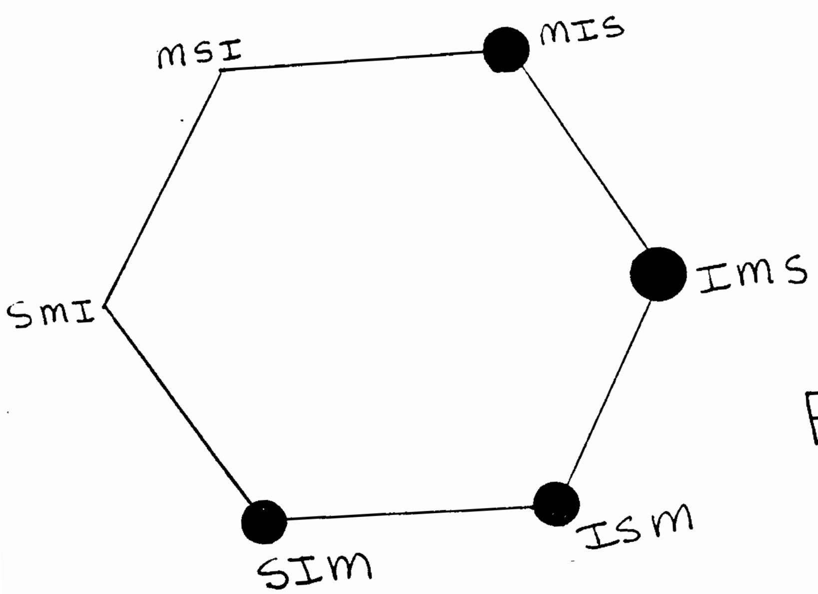



Figure 14b