SADDLEPOINT APPROXIMATIONS FOR BIVARIATE DISTRIBUTIONS

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Abstract

A saddlepoint approximation is derived for the cumulative distribution function of the sample mean of n independent bivariate random vectors. The derivations use Lugannani and Rice's saddlepoint formula and the standard bivariate normal distribution function. The separate versions of the approximation for the discrete cases are also given. A Monte Carlo study shows that the new approximation is very accurate.

Some key words: Asymptotic expansion; Bivariate normal distribution; Bivariate gamma distribution; Confidence region.

1. Introduction

Since Daniels' (1954) pioneering paper introduced the saddlepoint method into the statistical literature, these accurate approximations have gradually become popular among statisticians. More attention has been paid to these approximations in the past decade. Barndorff-Nielsen and Cox (1979) and Reid (1988) demonstrate the importance and usefulness of the saddlepoint approximations in statistics. Lugannani and Rice (1980) derive a very useful saddlepoint approximation to the cumulative distribution of the sample mean of an i.i.d. sample. Recently Davison and Hinkley (1988) apply the saddlepoint method to resampling problems.

Saddlepoint expansions for more general statistics have been considered by Wang (1988) and Srivastava and Yau (1989). Most of the development of saddlepoint theory and applications has been restricted to the univariate problems because more technical difficulties arise in the multivariate case.

The purpose of the present paper is to derive saddlepoint approximations for the cumulative distribution function of the sample mean of n independent bivariate random variables. Such approximations are useful in statistical inference, e.g., constructing confidence regions, when the exact distribution is intractable. The derivations make use of some basic properties of analytic asymptotic expansions including Lugannani and Rice's saddlepoint formula and the standard bivariate normal distribution function. The computational aspect of the bivariate normal distribution function has been well developed; see, for example, Owen (1962).

In Section 2, we give a representation of the inversion formula for the bivariate normal distribution which will be used in the subsequent sections. Section 3 derives the new saddlepoint approximation for continuous bivariate random variables in detail. Section 4 extends the new formula to the discrete case. As an example, in Section 5 we use a bivariate gamma distribution together with a simulation study to illustrate the great accuracy of the new approximation.

2. The Normal Case

Assume that $\underset{\sim}{Z}_1 = (X_1, Y_1), \ldots, \underset{\sim}{Z}_n = (X_n, Y_n)$ are independently and identically distributed bivariate random variables. Without loss of generality, suppose $E(X_1) = E(Y_1) = 0$. Let $\bar{\underset{\sim}{Z}} = (\bar{X}, \bar{Y}) = (n^{-1}\Sigma_1^n X_i, n^{-1}\Sigma_1^n Y_i)$.

The density of $\bar{\underset{\sim}{Z}}$ can be expressed by the inversion formula as

$$f_n(x,y) = \left(\frac{n}{2\pi i}\right)^2 \int_{c_2-i\infty}^{c_2+i\infty} \int_{c_1-i\infty}^{c_1+i\infty} \exp\{n(K(t,u)-tx-uy)\}\, dt\,du \ , \tag{1}$$

where $K(t,u) = \log E(e^{tX+uY})$ is the cumulant generating function (CGF) and $(c_1,\ c_2)$ is within the convergent domain U of $K(t,u)$ in R^2 (c_1 and c_2 are real). As in the univariate case, t and u in (1) are extended to be complex variables.

We wish to accurately approximate the distribution $F_n(x,y)$ of $\bar{\underset{\sim}{Z}}$. Analogously to the univariate case, we will use the standard bivariate normal distribution as a useful tool in the approximations. In this section, we develop a representation of the inversion formula for the bivariate normal distribution function which will play a fundamental role in the subsequent sections. We say that a random variable is standard bivariate normal if its components have zero mean and unit variance. Denote its density and cumulative distribution by $\phi(\bullet,\ \bullet,\ \rho)$ and $\Phi(\bullet,\ \bullet,\ \rho)$ respectively, where ρ is the correlation between the two components. The corresponding CGF is

$$K(t,u) = t^2/2 + \rho tu + u^2/2 \tag{2}$$

and its domain is $U = R^2$. We also denote the univariate standard normal density and distribution by $\phi(\bullet)$ and $\Phi(\bullet)$ and let

$$F_n(x,y) = \Pr(\bar{X} \leq x,\ \bar{Y} \leq y)\ .$$

Lemma 1. Assume that $\underset{\sim}{Z}$ has the distribution $\Phi(\bullet, \bullet, \rho)$. Then

$$F_n(x,y) = \Phi(\sqrt{n}x, \sqrt{n}y, \rho)$$

$$= \left(\frac{1}{2\pi i}\right)^2 \int_{b_2-i\infty}^{b_2+i\infty} \int_{b_1-i\infty}^{b_1+i\infty} \exp\{n(z_1^2+ z_2^2)/2\} \frac{dz_1}{z_1+x-\rho u_0-\rho(1-\rho^2)^{-1/2}z_2}$$

$$\bullet \frac{dz_2}{z_2 + (1-\rho^2)^{1/2}u_0} \{2\pi(1-\rho^2)^{1/2} \phi(\sqrt{n}x, \sqrt{n}y, \rho)\}, \tag{3}$$

where $t_0 = (x-\rho y)/(1-\rho^2)$, $u_0 = (y-\rho x)/(1-\rho^2)$, $b_1 < -t_0 + \rho(1-\rho^2)^{-1/2}b_2$ and $b_2 < -(1-\rho^2)^{1/2}u_0$.

Proof. The first equation is trivial from the normal theory. For the second equation, letting $c_1 < 0$, $c_2 < 0$ in (1) and using (2), we have

$$F_n(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_n(\zeta_1, \zeta_2)\, d\zeta_1\, d\zeta_2$$

$$= \left(\frac{1}{2\pi i}\right)^2 \int_{c_2-i\infty}^{c_2+i\infty} \int_{b_1-i\infty}^{b_1+i\infty} \exp\{nz_1^2/2\} \frac{dz_1}{z_1+\tilde{t}} \exp\{n(u^2-2uy-\tilde{t}^2)/2\}\frac{du}{u} ,$$

where $\tilde{t} = x - \rho u$, $z_1 = t - \tilde{t}$ and $b_1 < -\mathrm{Re}(\tilde{t}) = -x+\rho c_2$. Let $v = (1-\rho^2)^{1/2}u$. Then

$$F_n(x,y) = \left(\frac{1}{2\pi i}\right)^2 \int_{c_2-i\infty}^{c_2+i\infty} \int_{b_1-i\infty}^{b_1+i\infty} \exp\{nz_1^2/2\} \frac{dz_1}{z_1+ x - \rho(1-\rho^2)^{-1/2}v}$$

$$\bullet \exp\{n(v^2-2(1-\rho^2)^{-1/2}v(y-\rho x))/2\} \frac{dv}{v} \exp\{-x^2/2\}$$

$$= \left(\frac{1}{2\pi i}\right)^2 \int_{b_2-i\infty}^{b_2+i\infty} \int_{b_1-i\infty}^{b_1+i\infty} \exp\{nz_1^2/2\} \frac{dz_1}{z_1+x-\rho(1-\rho^2)^{-1/2}[z_2+(1-\rho^2)^{-1/2}(y-\rho x)]}$$

$$\bullet \exp\{nz_2^2/2\} \frac{dz_2}{z_2+(1-\rho^2)^{-1/2}(y-\rho x)} \exp\{n(-x^2+2\rho xy-y^2)/[2(1-\rho^2)]\} , \tag{4}$$

where $z_2 = v - (1-\rho^2)^{-1/2}(y-\rho x)$ and $b_2 < -(1-\rho^2)^{-1/2}(y-\rho x)$. Lemma 1 then follows.

Subroutine MDBNOR in the IMSL library can be easily used to calculate $\Phi(x,y,\rho)$. Owen (1962) also supplies the details of computational aspects as well as tables for various selected values.

3. The General Continuous Case

We now consider the case where the distribution of $\underset{\sim}{Z}$ is continuous but not necessarily normal. Assume that the convergent domain U of the CGF $K(t,u)$ contains an open neighborhood of the origin. Moreover, for fixed (x,y) suppose that there exists unique $(t_0, u_0) \in U$ such that

$$\begin{cases} K_t(t_0, u_0) = x \\ K_u(t_0, u_0) = y \end{cases} \tag{5}$$

and that for each fixed u there exists $\tilde{t} = t(u)$ such that

$$K_t(\tilde{t},u) = x, \tag{6}$$

where $K_t = \partial K/\partial t$, $K_u = \partial K/\partial u$. Note that $d\tilde{t}/du = -K_{tu}/K_{uu}$. Let $K_1(t)$ and $K_2(u)$ be the CGFs of X and Y respectively. These general conditions are implied by the assumptions in Skovgaard (1987). They will be assumed throughout the paper.

The deviations of the main result given in Theorem 1 at the end of this section are lengthy. We divide the derivations into four lemmas. Briefly, Lemma 2 is the saddlepoint expansion of the inner integral of the inverse formula which follows. Lemmas 3 and 4 provide the expansion of the outer integral of the term of the integrand which involves Φ, while Lemma 5 shows this expansion for the term involving ϕ.

We choose $c_1 < 0$, $c_2 < 0$, so that

$$F_n(x,y) = \left(\frac{1}{2\pi i}\right)^2 \int_{c_2-i\infty}^{c_2+i\infty} \int_{c_1-i\infty}^{c_1+i\infty} \exp\{n[K(t,u) - tx - uy]\} \frac{dt}{t} \frac{du}{u} . \tag{7}$$

Notice that the real functions $\phi(\cdot)$ and $\Phi(\cdot)$ may be extended to complex variables in a natural way. Using "~" to indicate the error of approximation as $O(n^{-1})$ relative to the main term as $n \to \infty$, we have

Lemma 2. Under general conditions given above, for fixed u the inner integral (in t) in (7) may be approximated by Lugannani and Rice's saddlepoint formula so that

$$F_n(x,y) \sim \frac{1}{2\pi i} \int_{c_2-i\infty}^{c_2+i\infty} -[\Phi(\sqrt{n}\, w_u) + \phi(\sqrt{n}\, w_u)(w_u^{-1}-\tilde{t}^{-1}\{K_{tt}(\tilde{t},u)\}^{-1/2})/\sqrt{n}]$$

$$\cdot \exp\{n[K_2(u) - uy]\} \frac{du}{u} \quad , \tag{8}$$

where $\tilde{t}$ is defined as in (6),

$$w_u = \{2[h(0)-h(\tilde{t})]\}^{1/2} \operatorname{sgn}(\operatorname{Re}(\tilde{t})) \tag{9}$$

and $h(t) = K(t,u) - tx$ for each fixed u.

Lemma 2 can easily be proved by using a standard saddlepoint approximation (see Daniels, 1987) and the following transformation of variables:

$$(w-w_u)^2/2 = h(t) - h(\tilde{t}) = K_{tt}(\tilde{t},u)(t-\tilde{t})^2/2 + \ldots, \tag{10}$$

where $\operatorname{sgn}(\operatorname{Im}(w-w_u)) = \operatorname{sgn}(\operatorname{Im}(t-\tilde{t}))$ so that $w=0$ and w_u corresopnd to $t=0$ and $\tilde{t}$ respectively. From (9) we obtain that

$$dt/dw\,\Big|_{w=w_u} = \{K_{tt}(\tilde{t},u)\}^{-1/2}$$

which will be used in the proof. The details are omitted here.

In what follows, we will expand the integral in (8). The expansion for the first term is given in Lemmas 3 and 4. The proof uses Lemma 1.

For each u we select a real value b_1 such that $b_1 + \mathrm{Re}(w_u) < 0$. It is readily seen that

$$\begin{aligned}\Phi(\sqrt{n}w_u) &= \int_{-\infty}^{w_u} \frac{n}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} \exp\{n(w^2 - 2wy)/2\}\, dwdy \\ &= -\frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} \exp\{n(w^2 - 2ww_u)/2\} \frac{dw}{w} \\ &= -\frac{1}{2\pi i} \exp\{-nw_u^2/2\} \int_{b_1-i\infty}^{b_1+i\infty} \exp\{nz_1^2/2\} \frac{dz_1}{z_1+w_u} \,. \end{aligned} \tag{11}$$

We let

$$\begin{aligned} v_0 &= \{2[\ell(0) - \ell(u_0)]\}^{1/2}\, \mathrm{sgn}(u_0) \\ &= \{-2[K(t_0,\, u_0) - K_1(\hat{t}_0) - (t_0 - \hat{t}_0)x - u_0 y]\}^{1/2}\, \mathrm{sgn}(u_0), \end{aligned} \tag{12}$$

where $\hat{t}_0$ is the solution to $K_1'(t) = x$ and $\ell(u) = K(\tilde{t},u) - \tilde{t}x - uy$, and introduce a new variable v as

$$(v-v_0)^2/2 = \ell(u) - \ell(u_0) \tag{13}$$

and $\mathrm{Im}(v-v_0) = \mathrm{Im}(u-u_0)$. It is easily seen that v=0 and v_0 if and only if u=0 and u_0 respectively. Furthermore,

$$du/dv\,\Big|_{v=v_0} = (K_{uu} - (K_{tu})^2/K_{tt})^{-1/2}\,\Big|_{v=v_0}.$$

We now derive the expansion for the first term in (8). From (11), (9) and (13),

$$\frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty} -\Phi(\sqrt{n}\,w_u)\exp\{n[K_2(u)-uy]\}\frac{du}{u}$$

$$=\left(\frac{1}{2\pi i}\right)^2\int_{c_2-i\infty}^{c_2+i\infty}\int_{b_1-i\infty}^{b_1+i\infty}\exp\{nz_1^2/2\}\frac{dz_1}{z_1+w_u}\exp\{-n[w_u^2/2-K_2(u)+uy]\}\frac{du}{u}$$

$$=\left(\frac{1}{2\pi i}\right)^2\int_{c_2-i\infty}^{c_2+i\infty}\int_{b_1-i\infty}^{b_1+i\infty}\exp\{nz_1^2/2\}\frac{dz_1}{z_1+w_u}\exp\{n(v^2-2vv_0)/2+n(K_1(t_0)-t_0x)\}\frac{1}{u}\frac{du}{dv}\,dv$$

$$=\left(\frac{1}{2\pi i}\right)^2\int_{b_2-i\infty}^{b_2+i\infty}\int_{b_1-i\infty}^{b_1+i\infty}\exp\{n(z_1^2+z_2^2)/2\}\frac{dz_1}{z_1+w(z_2)}\frac{1}{u}\frac{du}{dv}\,dz_2$$

$$\cdot\exp\{n[K(t_0,u_0)-t_0x-u_0y]\},\tag{14}$$

where $z_2 = v - v_0$, $b_2 < -v_0$ and $w(z_2) = w_u$ as a function of z_2. We have then proved the following intermediate result.

Lemma 3. Assuming the same conditions as in Lemma 2,

$$I_1 = -\frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty}\Phi(\sqrt{n}w_u)\exp\{n[K_2(u)-uy]\}\frac{du}{u}\sim I_{11}+I_{12},\tag{15}$$

where

$$I_{1k}=\left(\frac{1}{2\pi i}\right)^2\int_{b_2-i\infty}^{b_2+i\infty}\int_{b_1-i\infty}^{b_1+i\infty}\exp\{n(z_1^2+z_2^2)/2\}\frac{\psi_k(z_2)}{z_1+w(z_2)}\,dz_1dz_2\;q_n(t_0,u_0),\tag{16}$$

$$\psi_1(z_2)=(z_2+v_0)^{-1},\quad \psi_2(z_2)=u^{-1}\,du/dv-\psi_1(z_2)\text{ and}$$

$$q_n(t_0,u_0)=\exp\{n[K(t_0,u_0)-t_0x-u_0y]\}.$$

Continuing from Lemma 3, we now prove the following important lemma.

Lemma 4. Under the conditions given in Lemma 2,

$$I_{11} \sim \Phi(\surd n x_1, \surd n y_1, \rho_1) \ , \tag{17}$$

$$I_{12} \sim \Phi(\surd n w_{u_0})\phi(\surd n \hat{v}_0)\{v_0^{-1} - (u_0 G)^{-1}\}/\surd n \ , \tag{18}$$

where $x_1 = w(-v_0) = w_u \mid_{u=0}$, $y_1 = (v_0 - bx_1)/(1+b^2)^{1/2}$, $\rho_1 = -b/(1+b^2)^{1/2}$,

$b = (w(0)-w(-v_0))/v_0$, $G = (K_{uu} - (K_{tu})^2/K_{tt})^{1/2} \mid_{(t_0,u_0)}$ and $v_0^2 = -2(K_2(u_0) - u_0 y)$.

Proof. We first notice that the integrands have two critically important points in z_2 ($z_2=0$ and $z_2 = -v_0$). Let $a = w(0) = w_{u_0}$ and

$$\theta_1(z_1,z_2) = \{(z_1 + w(z_2))^{-1} - (z_1 + a + bz_2)^{-1}\}/(z_2+v_0) \ .$$

Then $\theta_1(z_1,0) = 0$ and $\lim_{z_2 \to -v_0} \theta_1(z_1,z_2) = c(z_1)$ for each fixed z_1 with $Re(z_1) = b_1^0$ and some analytic $c(z_1)$, where b_1^0 is small enough to avoid any possible singularities. Therefore $\theta_1(z_1,z_2)$ has a removable removable singularity at $z_2 = -v_0$. For each such fixed z_1 defining $\theta_1(z_1,-v_0) = c(z_1)$ guarantees $\theta_1(z_1,z_2)$ to be analytic in a neighborhood of $z_2=0$ that may contain $z_2=-v_0$. Furthermore

$$\theta_1(z_1,z_2) = z_2 \ g_1(z_1,z_2) \tag{19}$$

for some analytic $g_1(z_1,z_2)$ in z_2 in a neighborhood of $z_2 = 0$. From (16),

$$I_{11}=(\frac{1}{2\pi i})^2 \int_{b_2-i\infty}^{b_2+i\infty}\int_{b_1-i\infty}^{b_1+i\infty} \exp\{n(z_1^2+z_2^2)/2\}\{[(z_1+a+bz_2)(z_2+v_0)]^{-1}+\theta_1(z_1,z_2)\}dz_1dz_2$$

$$\cdot\ q_n(t_0,u_0)$$

$$\sim (\frac{1}{2\pi i})^2 \int_{b_2-i\infty}^{b_2+i\infty}\int_{b_0-i\infty}^{b_1+i\infty} \exp\{n(z_1^2+z_2^2)/2\}\{[(z_1+a+bz_2)(z_2+v_0)]^{-1}\}dz_1dz_2\ q_n(t_0,u_0)$$

$$= \Phi(\sqrt{n}x_1,\sqrt{n}y_1,\rho_1)\ .$$

The last equation follows from Lemma 1 with $a = x_1-\rho_1(y_1-\rho_1x_1)/(1-\rho_1^2)$ $b = -\rho_1(1-\rho_1^2)^{-1/2}$ and $v_0 = (1-\rho_1^2)^{-1/2}(y_1-\rho_1x_1)$ and the identity

$$K(t_0,u_0) - t_0x - u_0y = -(x_1^2 - 2\rho_1x_1y_1 + y_1^2)/\{2(1-\rho_1^2)\}.$$

The above asymptotic expression holds by using the property (19) of θ_1 and applying Watson's lemma (see Daniels, 1954) to the integral in z_2 after interchanging the integrals together with some continuity arguments. Therefore expansion (17) holds.

We now employ the above technique to prove (18). Let

$$\theta_2(z_1,z_2) = \{(z_1+w(z_2))^{-1} - (z_1+w(0))^{-1}\}\ \psi_2(z_2),$$

where $\psi_2(z_2)$ is defined as in (16). It is easily seen that $\lim_{z_2\to -v_0} \psi_2(z_2) =$ constant. Therefore $\theta_2(z_1,z_2)$ has the same analytic property as $\theta_1(z_1,z_2)$, so that for fixed z_1,

$$\theta_2(z_1,z_2) = z_2\ g_2(z_1,z_2) \tag{21}$$

for some analytic $g_2(z_1,z_2)$ in z_2 in a neighborhood of $z_2 = 0$ that may include $z_2 = -v_0$. It follows from (16) and the same arguments as in (20) that

$$I_{12} \sim \left(\frac{1}{2\pi i}\right)^2 \int_{b_2-i\infty}^{b_2+i\infty} \int_{b_1-i\infty}^{b_1+i\infty} \exp\{n(z_1^2+z_2^2)/2\}\psi_2(z_2)/(z_1+w_{u_0})dz_1dz_2\ q_n(t_0,u_0). \quad (22)$$

By Watson's lemma and (11),

$$I_{12} \sim -\Phi(\sqrt{n}\ w_{u_0})\ \exp\{nw_{u_0}^2/2\}\ \frac{\psi_2(0)}{(2\pi n)^{1/2}}\ q_n(t_0,u_0)$$

$$= \Phi(\sqrt{n}w_{u_0})\ \phi(\sqrt{n}v_0)\{v_0^{-1} - (u_0G)^{-1}\}/\sqrt{n} \quad . \quad (23)$$

This completes the proof of Lemma 4. □

Notice that the technique used in the above proof is similar to that of using a linear function to approximate a general analytic function used by Bleistein (1966) and Skovgaard (1987). Also see Wang (1988, Chapter 6).

To approximate $F_n(x,y)$ we also need to perform the expansion for the second integral in (8). The next lemma is devoted to this need.

Lemma 5. Assume the same conditions as in Lemma 2. Then

$$I_2 = \frac{-1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty} \phi(\sqrt{n}w_u)(w_u^{-1} - \tilde{t}^{-1}\{K_{tt}(\tilde{t},u)\}^{-1/2})\ \exp\{n(K_2(u)-uy)\}\frac{du}{\sqrt{nu}}$$

$$\sim I_{21} + I_{22}, \quad (24)$$

where

$$I_{21} = \Phi(\sqrt{n}v_0)\ \phi(\sqrt{n}x_1)(w_{u_0}^{-1} - t_0^{-1}\{K_{tt}(t_0,u_0)\}^{-1/2})/\sqrt{n}, \quad (25)$$

$$I_{22}=\exp\{n[K(t_0,u_0)-t_0x-u_0y]\}(w_{u_0}^{-1}-t_0^{-1}\{K_{tt}(t_0,u_0)\}^{-1/2})[v_0^{-1}-(u_0G)^{-1}]/2\pi n, \quad (26)$$

and $x_1 = \{-2[K_1(\hat{t}_0)-\hat{t}_0x]\}^{1/2}\mathrm{sgn}(\hat{t}_0)=w_u\big|_{u=0}$ as in Lemma 4.

Proof. We first assume $u_0 < 0$ and let $c_2 = u_0$. Using transformation (13) we have

$$I_2 = \frac{-1}{(2\pi)^{3/2} i} \int_{v_0 - i\infty}^{v_0 + i\infty} \exp\{n(v^2 - 2vv_0)/2 + n(K_1(t_0) - t_0 x)\}(w_{u_0}^{-1} - t_0^{-1} K_{tt}(t_0, u_0)^{-1/2}) \frac{dv}{\sqrt{nv}}$$

$$- \frac{1}{(2\pi)^{3/2} i} \int_{v_0 - i\infty}^{v_0 + i\infty} \exp\{n(v^2 - 2vv_0)/2 + n(K_1(t_0) - t_0 x)\}$$

$$\cdot \{[w_u^{-1} - \tilde{t}^{-1}\{K_{tt}(\tilde{t}, u)\}^{-1/2}]u^{-1} du/dv - [w_{u_0}^{-1} - t_0^{-1}\{K_{tt}(t_0, u_0)\}^{-1/2}]v^{-1}\} \frac{dv}{\sqrt{n}}$$

$$\sim I_{21} + I_{22},$$

by (12) and Watson's lemma. Expansion (24) is easily verified for $u_0 > 0$ by proper handling of the pole at $v = 0$ in the usual way. □

Summarizing Lemmas 2-5, we finally reach the following main result.

Theorem 1. Under the general regularity conditions, the distribution function $F_n(x,y)$ of $(\bar{X}, \bar{Y})$ can be approximated by the bivariate saddlepoint formula as follows:

$$F_n(x,y) \sim I_{11} + I_{12} + I_{21} + I_{22}, \tag{27}$$

where I_{ij}, j=1,2, i=1,2, are given in (17), (18), (25) and (26).

Examples and discussions will be given in Section 5.

4. Discrete Variables

So far we have dealt with continuous variables. It is often useful to consider the case where at least one of X and Y is discrete. The proof in

Section 3 can simply be modified for the current situation in a standard way used by Daniels (1987) and Skovgaard (1987).

First assume that one variable is discrete and another is continuous. Without loss of generality assume that X is discrete and that the integers are a minimal lattice for X. Formula (7) then is replaced by

$$F_n(x,y) = \left(\frac{1}{2\pi i}\right)^2 \int_{c_2-i\infty}^{c_2+i\infty} \int_{c_1-i\pi}^{c_1+i\pi} \exp\{n[K(t,u)-tx-uy]\}\frac{dt}{1-e^{-t}}\frac{du}{u} . \tag{28}$$

Applying the discrete version of the Lugannani and Rice formula (Daniels, 1987) to the inner integral in (28) and following the rest of the proof in Section 3, we obtain the following theorem.

Theorem 2. Assume the same conditions as in Theorem 1 except that X is discrete. Then

$$F_n(x,y) \sim I_{11} + I_{12} + I_{21}' + I_{22}' , \tag{29}$$

where I_{11} and I_{12} are given in (17) and (18), I_{21}' and I_{22}' are obtained from (25) and (26) respectively by replacing the factor $w_{u_0}^{-1} - t_0^{-1}\{K_{tt}(t_0,u_0)\}^{-1/2}$ by $w_{u_0}^{-1} - (1-e^{-t_0})^{-1}\{K_{tt}(t_0,u_0)\}^{-1/2}$.

It is straightfoward to extend to the case where both X and Y are discrete. The proof is analogous to that of theorem 2.

Theorem 3. Assume the same conditions as in Theorem 1 except that both X and Y are discrete. Then

$$F_n(x,y) \sim I_{11} + I'_{12} + I'_{21} + I''_{22}\ , \tag{30}$$

where I_{11} is given in (17), I'_{12}, I'_{21} and I''_{22} are obtained from (18), (25) and (26) respectively by replacing the factors $w^{-1}_{u_0} - t_0^{-1}\{K_{tt}(t_0,u_0)\}^{-1/2}$ and $v_0^{-1} - (u_0 G)^{-1}$ by $w^{-1}_{u_0} - (1-e^{-t_0})^{-1}\{K_{tt}(t_0,u_0)\}^{-1/2}$ and $v_0^{-1}\{(1-e^{-u_0})G\}^{-1}$ respectively.

Analogously to the univariate case or conditional distribution case (see Daniels, 1987 and Skovgaard, 1987), we can introduce a continuity correction to each discrete variable. The details are left to the reader.

5. Examples

It is seen that the bivariate saddlepoint formulas developed in the previous sections are easily computed once the CGF K(s,u) is available. We now demonstrate the usefulness and accuracy of the bivariate saddlepoint approximations in some special situations.

We start with the case where X_i and Y_i are independent so that $K(s,u) = K_1(s)\ K_2(u)$. Then $w(z_2) \equiv w_{u_0}$, $G = \{K''_2(u_0)\}^{1/2}$ and $v_0 = \hat{v}_0$. Therefore $y_1 = v_0$, $\rho_1 \equiv 0$. The bivariate saddlepoint formula is hence the product of Lugannani and Rice's first order saddlepoint approximations for $\Pr(\bar{X} \leq x)$ and $\Pr(\bar{Y} \leq y)$ in both continuous and discrete cases.

Another special case is that (X_i,Y_i) has bivariate normal distribution with correlation ρ and variances σ_x^2 and σ_y^2. It is easily seen that I_{12},

I_{21} and I_{22} in Formula (27) are all zero. Therefore the saddlepoint approximation (27) to $F_n(x,y)$ becomes

$$I_{11} = \Phi(\sqrt{n}\, x_1, \sqrt{n}\, y_1, \rho_1) = \Phi(\sqrt{n}\, x/\sigma_x, \sqrt{n}\, y/\sigma_y, \rho) ,$$

which is exact.

In the rest of this section we consider a nontrivial example in some detail. Let U_0, U_1 and U_2 be independent and exponentially distributed random variables with mean 1. Put $X' = U_0 + U_1$ and $Y' = U_0 + U_1$. Then (X',Y') has a bivariate gamma distribution; see Johnson and Kotz (1972, p.217). The random variable $(X,Y) = (X'-2, Y'-2)$ satisfies the conditions in Theorem 1. It is easily calculated that the CGF is

$$K(t,u) = -\log\{1 - (t+u)\} - \log(1-t) - \log(1-u) - 2t - 2u.$$

The unique solution $(t_0,u_0) \in U = (t,u) \mid t < 1,\ u < 1,\ t+u < 1\}$ satisfying (5) for fixed x and y is obtained by solving the cubic equation

$$-(x+2)(x-y)\alpha^3 + [(x+4)(x-y) + 2x + 4]\alpha^2 + (y-2x-5)\alpha + 1 = 0$$

for α and using the relation $1/\alpha - 1/\beta = x-y$, where $\alpha = 1 - t$ and $\beta = 1 - u$. It is straightforward to compute the rest of the quantities used in formula (27).

Notice that when $t_0 = 0$, the factor $w_{u_0}^{-1} - t_0^{-1}\{K_{tt}(t_0,u_0)\}^{-1/2}$ should be replaced by its limit, as $t_0 \to 0$, $K_{ttt}(0,u_0)/\{K_{tt}(0,u_0)\}^{3/2}/6$. The same argument applies to the quantities $v_0^{-1} - (u_0G)^{-1}$, and b when $u_0 = 0$. The limits may well be approximated by the corresponding values evaluated at $u = u^0$ very close to zero.

Subroutine MDBNOR in the IMSL library was used to compute $\Phi(\sqrt{n}x_1, \sqrt{n}\,y_1, \rho_1)$. Table 1 provides the numerical results of the saddlepoint approximations (27) and the normal approximations to $F_n(x,y)$ in the bivariate gamma case with n=10. The "exact" values were also given which were obtained from 10^6 simulated samples. The estimated standard error of the simulated estimate $\hat{p}$ is $\sqrt{\hat{p}(1-\hat{p})}/10^3$ so that the simulation results in Table 1 are reliable to the digits given. Because $F_n(x,y)$ is symmetric, i.e. $F_n(x,y) = F_n(y,x)$, only the upper triangle of Table 1 is needed. Table 1 indicates that the saddlepoint approximations are very accurate, far more accurate than the normal approximations. The values of $F_n(x,y)$ in the four corners of Table 1 are of particular importance in statistical inference such as constructing confidence regions. The saddlepoint approximations are extremely accurate in the corners.

Numerical examples show that when w_{u_0} becomes big in the positive direction, the ignored term in (20) involving θ_1 could be significant although it still has the same error rate. This potential drawback can be remedied by taking the following strategy. Let $(X_i^*, Y_i^*) = (Y_i, X_i)$ and $(x^*,y^*) = (y,x)$. Then $F_n(x,y) = F_n^*(x^*,y^*)$, where F_n^* is the cumulative distribution function of $(\bar{X}^*, \bar{Y}^*)$. Using superscript * to denote quantities for F_n^*, we compute $w^*_{u_0^*}$ in the same way as w_{u_0}. We recommend that if $w_{u_0} \leq w^*_{u_0^*}$, one uses (27) to approximate $F_n(x,y)$; otherwise one should use the corresponding formula for $F_n^*(x^*,y^*)$ with the CGF $K^*(t^*,u^*) = K(u,t)$, where $t^* = u$, $u^* = t$. This treatment is the same as interchanging the integrals in (7) and then applying the Lugannani and Rice formula to the integral in u instead of t. The results are usually satisfactory. In the present example, $w_{u_0} \leqq w^*_{u_0^*}$ for $x \leqq y$.

6. Concluding Remarks

In this paper we have derived accurate bivariate saddlepoint formulas for the cumulative distribution function of the sample mean of a bivariate sample. Although the derivations require some analytic considerations, the resulting formulas are easy to use in practice. Therefore, these approximations are useful in statistical inference involving more than one parameter, especially when the sample size is small to medium. Some care is needed in applying these formulas, as is pointed out at the end of Section 5. In principle the theory developed here can be extended to the higher dimensional case. The results developed here can be used to approximate the bootstrap distributions. The validity of such approximations in the bivariate bootstrap setting can be shown using techniques employed by Wang (1989) in the univariate case.

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References

Barndorff-Nielsen, O.E. and Cox, D.R. (1979). Edgeworth and saddlepoint approximations with statistical applications (with discussion). J.R. Statist. Soc. B, 41, 279-312.

Bleinstein, N. (1966). Uniform asymptotic expansions of integrals with stationary points near algebraic singularity. Comm. Pure Appl. Math., 19, 353-370.

Daniels, H.E. (1954). Saddlepoint approximations in statistics. Ann. Math. Statist., 25, 631-650.

Daniels, H.E. (1987). Tail probability approximations. Intern. Statist. Rev., 55, 37-48.

Davison, A.C. and Hinkley, D.V. (1988). Saddlepoint approximations in resampling methods. Biometrika, 75, 417-431.

Johnson, N.L. and Kotz, S. (1972). Distributions in Statistics: Continuous Multivariate Distributions, Wiley, New York.

Lugannani, R. and Rice, S.O. (1980). Saddlepoint approximation for the distribution of the sum of independent random variables. Adv. Appl. Prob., 12, 475-490.

Owen, D.B. (1962). Handbook of Statistical Tables. Addison-Wesley Publishing Company, Inc., Reading, Massachusetts.

Reid, N. (1988). Saddlepoint methods and statistical inference (with discussion). Statistical Science, 3, 213-238.

Skovgaard, I.M. (1987). Saddlepoint expansions for conditional distributions. J. Appl. Prob., 24, 875-887.

Srivastava, M.S. and Yau, W.K. (1989). Tail probability approximations of a general statistic. Unpublished manuscript.

Wang, S. (1988). Saddlepoint approximations in statistics, including resampling. Ph.D. dissertation, The University of Texas at Austin.

Wang, S. (1989). Saddlepoint approximations in resampling analysis. Ann. Inst. Statist. Math. 41, to appear.

Table 1. Comparisons of the bivariate saddlepoint approximations and normal approximations to the cumulative distribution of the sample mean of bivariate gamma random variables with n=10. For each x and y, the first value is "exact" which was obtained from 10^6 simulated samples. The second and the third are the saddlepoint approximation and normal approximation respectively.

y \ x	-1.2	-1.0	-0.8	-0.5	0.0	0.5	1.0	1.5	2.0
2.0	.00026	.0035	.021	.125	.530	.866	.978	.9975	.99962
	.00027	.0036	.022	.129	.539	.871	.979	.9976	.99955
	.00365	.0127	.037	.132	.500	.868	.987	.9996	.99999
1.5	.00026	.0035	.021	.125	.530	.865	.976	.9955	
	.00027	.0036	.022	.128	.537	.869	.977	.9949	
	.00365	.0127	.037	.132	.500	.868	.987	.9992	
1.0	.00026	.0035	.021	.125	.527	.857	.961		
	.00026	.0036	.022	.127	.532	.857	.958		
	.00365	.0127	.037	.132	.499	.863	.976		
0.5	.00026	.0035	.021	.123	.503	.783			
	.00026	.0035	.022	.124	.504	.778			
	.00364	.0126	.037	.130	.476	.784			
0.0	.00025	.0033	.019	.104	.364				
	.00025	.0033	.019	.104	.360				
	.00349	.0118	.033	.108	.333				
-0.5	.00018	.0021	.010	.042					
	.00017	.0020	.010	.042					
	.00243	.0073	.018	.048					
-0.8	.00008	.0008	.0033						
	.00008	.0008	.0032						
	.00134	.0036	.0079						
-1.0	.00003	.0002							
	.00003	.0002							
	.00073	.0018							
-1.2	.00000								
	.00000								
	.00032								