A GENERAL METHOD FOR APPROXIMATING TAIL PROBABILITES

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ABSTRACT

A new transformation referred to as the $G_n^{(m)}$-transformation is introduced to establish a general methodology for finding functions which are easy to evaluate and give very good approximations to tail probabilities. The transformation is based only on a general class of differential equations that include the specified density function in the solution set. As a result this new method applies to a broad class of distributions. Not only is the method general but it is also very accurate. Unlike many other approximation methods, the approximation functions produced maintain their high degree of accuracy even in the extreme tails, making them also suitable for extreme tail probability approximation. In this paper the method is applied to the Pearson family members: the distribution of the normal, t, gamma, F and the Pearson type IV and the inverse Gaussian which does not belong to the Pearson family. Tables are included which show the high accuracy of this method even with small values of n (n=1, 2, 3). In most cases satisfactory approximation functions can be obtained which are sufficiently simple so that they can all be calculated on a hand calculator.

KEW WORDS: $G_n^{(m)}$-transformation; Generalized jackknife; Inverse Gaussian; Pearson family.

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1. Introduction

The problem of approximating tail probabilities is one that has been of interest in statistics since its beginning and remains of strong interest today. One of the more popular approaches to the problem is to produce an approximation function which is easy to evaluate. There are however few general methodologies for producing such functions, most being the product of ad-hockery. An exception to this is the B_n-transform method introduced by Gray and Lewis (1971). Actually the B_n-transform method has been shown by Gray (1988) to be a special case of the Generalized Jackknife. In this paper we follow the approach of Gray (1988) to show how the Generalized Jackknife can be employed to establish a very general method for obtaining approximation functions for most tail probabilities. The method has the advantage, over most methods, of producing approximation functions which maintain a high degree of accuracy even in the extreme tails, thus making it suitable even when extreme tail probabilities are needed.

2. Background

Given a collection of estimators $\hat{\theta}_1$, $\hat{\theta}_2$, ..., $\hat{\theta}_{k+1}$ and a set constants c_j such that

$$E\left[\hat{\theta}_j - c_j\theta\right] = \sum_{i=1}^{\infty} a_{ij}b_i(\theta), \quad j=1, 2, \ldots, k+1 , \tag{1}$$

where $c_1 = 1$, the a_{ij} are given constants, and the $b_i(\theta)$ are unknown functions of θ, the k-th order Generalized Jackknife is defined by

$$G(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_{k+1}; a_{ij}) = \frac{H_{k+1}(\hat{\theta}_j; a_{ij})}{H_{k+1}(c_j; a_{ij})} , \tag{2}$$

where

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$$H_{k+1}(z_j; a_{ij}) = \begin{vmatrix} z_1 & z_2 & \cdot \cdot \cdot & z_{k+1} \\ a_{11} & a_{12} & \cdot \cdot \cdot & a_{1,k+1} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{k,1} & \cdot \cdot \cdot & & a_{k,k+1} \end{vmatrix} \qquad \cdot \qquad (3)$$

The Gneeralized Jackknife defined by (2) is a slight extension of the original definition given in Schucany, Gray and Owen (1971). The value of this extension has however been demonstrated in Gray (1988). If $a_{ij} b_i(\theta) \equiv 0$ for $i > k$ in (1), then taking the expectation of both sides in (2) easily shows that $G(\hat{\theta}_1, \ldots, \hat{\theta}_{k+1}; a_{ij})$ is an unbiased estimate for $\hat{\theta}$. If $a_{ij} b_i(\theta) \not\equiv 0$ for $i > k$, then the jackknife is not unbiased but, under general conditions, is lower order bias.

The fact that the Generalized Jackknife is a general method for bias reduction is well known; see Schucany, Gray and Owen (1971). It is however not so well known that the Jackknife also furnishes a general methodology for reducing the error in a numerical approximation. That is, suppose the $\hat{\theta}_j$ are degenerate, i.e., each has its probability mass concentrated at a single point, then they are more commonly referred to as approximations (rather than estimators) and $E[\hat{\theta}_j - \theta] = \hat{\theta}_j - \theta$, so that the bias is in fact the numerical error. Therefore when the Jackknife is applied to a collection of approximations it can be viewed as a general approach to reducing the error in the approximation. This was pointed out by Gray (1988) where it was also noted that such an observation can be used to show that such well known numerical methods as Simpson's Rule, Lagrange interpolation, Romberg integration, Newton-Cotes' method, Newton's Rule, the e_n-transformation of Shanks, Gray's G_n-transformation and many others

can all be viewed as Generalized Jackknives. In the next section we will show how, in addition, this observation leads to a general method for approximating tail probabilites.

3. The Jackknife Method for Approximating Tail Probabilities

Let f be a probability density function and let

$$F(x) = \int_a^x f(t)dt \qquad (4)$$

and suppose $F(x) \to S$ as $x \to \infty$. Then let $\varepsilon(x) = S - F(x)$ and

$$U_k(x) = x^{\ell_k} \sum_{i=0}^{\infty} \frac{\alpha_{k,i}}{x^i} \quad , \qquad (5)$$

where $\alpha_{k,0} \neq 0$ and ℓ_k is an integer with $\ell_k \leq k$. Now suppose m is the smallest possible integer such that $\varepsilon(x)$ satisfies the differential equation

$$U_m(x)y^{(m)} + U_{m-1}(x)y^{(m-1)} + \ldots + U_1(x)y' - y = 0, \qquad (6)$$

for some collection of U_k's. Then if $\alpha_{1,i} = \alpha_{2,i} = \ldots = \alpha_{m,i} \equiv 0$ when $i \geq n_1, n_2, \ldots, n_m$ respectively, we have

$$\sum_{i=0}^{n_m-1} \alpha_{m,i} x^{\ell_m-i} \varepsilon^{(m)}(x) + \ldots + \sum_{i=0}^{n_1-1} \alpha_{1,i} x^{\ell_1-i} \varepsilon'(x) - \varepsilon(x) = 0.$$

Then

$$F(x)-S = \sum_{i=0}^{n_m-1} \alpha_{m,i} x^{\ell_m-i} f^{(m-1)}(x) + \ldots + \sum_{i=0}^{n_1-1} \alpha_{1,i} x^{\ell_1-i} f(x) \quad , \qquad (7)$$

and

$$f^{(k)}(x) = \sum_{i=0}^{n_m-1} \alpha_{m,i} \{x^{\ell_m-i} f^{(m-1)}(x)\}^{(k+1)} + \ldots + \sum_{i=0}^{n_1-1} \alpha_{1,i} \{x^{\ell_1-i} f(x)\}^{(k+1)}, \qquad (8)$$

$$k = 0, 1, \ldots, N-1, \quad N = \sum_{i=1}^{m} n_i \quad .$$

The system of equations defined by equations (7) and (8) is of the same form as (1) with $c_1 = 1$ and $c_j = 0$ for $j = 2, 3, \ldots, N+1$, and with the $\alpha_{j,i}$ corresponding to the $b_i(\theta)$. Equations (7) and (8) therefore define a Generalized Jackknife which is exact, i.e., produces the exact tail probability when applied to the $N+1$ functions $F^{(k)}(x)$, $k = 0, 1, \ldots, N$, if $\epsilon(x)$ satisfies equation (6). That is if $\epsilon(x)$ satisfies (6) for $x \gtrless a$, then for any $x \gtrless a$

$$G\left[F(x),\ F'(x),\ \ldots,\ F^{(N)}(x);\ a_{ij}(x)\right] \equiv S \qquad (9)$$

if the a_{ij} are properly defined by (7) and (8). Since (9) holds for all $x \gtrless a$, we can take $x = a$ and $F(x) = 0$ so that no integration is required at all in (9). To be more specific, suppose $m = 1$, $n_1 = n$ and $x = a$. Then $N = n$ and denoting $G\left[0, F'(x), \ldots, F^{(n)}(x); a_{ij}(x)\right]$ by $G_n^{(m)}[f(x); a_{ij}(x)]$ we have

$$G_n^{(1)}\left[f(x);\ a_{ij}(x)\right] = \cfrac{\begin{vmatrix} 0 & f(x) & \cdots & f^{(n-1)}(x) \\ a_{11}(x) & a_{12}(x) & \cdots & a_{1,n+1}(x) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n,1}(x) & a_{n,2}(x) & \cdots & a_{n,n+1}(x) \end{vmatrix}}{\begin{vmatrix} 1 & 0 & \cdots & 0 \\ a_{11}(x) & a_{12}(x) & \cdots & a_{1,n+1}(x) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n,1}(x) & a_{n,2}(x) & \cdots & a_{n,n+1}(x) \end{vmatrix}}, \qquad (11)$$

where

$$a_{ij}(x) = \{x^{\ell_1 - i + 1} f(x)\}^{(j-1)} . \tag{12}$$

From our earlier remarks it therefore follows that

$$G_n^{(1)}[f(x); a_{ij}(x)] = \int_x^{\infty} f(t)\, dt. \tag{13}$$

Now suppose $f(x)$ satisfies (6) for some set of $U_k(x)$ defined by (5) and $\lim_{x \to \infty} U_k^{(i-1)}(x) f^{(k-i)}(x) = 0$, $k = i$, $i+1$, ..., $i = 1$, 2, ... Then Levin and Sidi (1981) have shown that there exist $\alpha'_{k,i}$ such that

$$U_m^*(x)\, \varepsilon^{(m)}(x) + U_{m-1}^*(x)\, \varepsilon^{(m-1)}(x) + \ldots + U_1^*(x) \varepsilon'(x) - \varepsilon(x) = o(x^{-n}), \tag{14}$$

where

$$U_k^*(x) = \sum_{i=0}^{n-1} \alpha'_{k,i}\, x^{\ell_k - i} . \tag{15}$$

This leads us to define the Generalized Jackknife tail probability approximation function as follows.

Definition 1. Let $f(x)$ be a probability density function with infinite support and suppose $f(x)$ satisfies the differential equation in equation (6) for some m and some set of U_k's. Then we define $G_n^{(m)}$-transformation of $f(x)$ as the Generalized Jackknife approximation of $\int_x^{\infty} f(t)dt$ corresponding to equation (14), i.e.,

$$G_n^{(m)}[f(x); a_{ij}(x)] = G[0, f(x), \ldots, f^{(mn-1)}(x); a_{ij}(x)] , \tag{16}$$

where

$$a_{ij}(x) = \begin{cases} \left\{ x^{\ell_1 - i + 1} f(x) \right\}^{(j-1)}, & i=1,\ldots, n, \ j=1,\ldots, nm+1, \\[2em] \left\{ x^{\ell_2 - i + n + 1}, f'(x) \right\}^{(j-1)}, & i=n+1, \ldots, 2n, \ j=1, \ldots, nm+1, \\[1em] \quad \vdots \\[1em] \left\{ x^{\ell_m - i + (m-1)n + 1} f^{(m-1)}(x) \right\}^{(j-1)}, & i=(m-1)n+1,\ldots, nm, \ j=1,\ldots, mn+1. \end{cases}$$

It is clear that $G_n^{(m)}[f(x); a_{ij}(x)] \to \int_x^\infty f(t)dt$ as n increases for

$x > 1$. One should therefore expect very rapid converge of $G_n^{(m)}[f(x);$

$a_{ij}(x)]$ to $F(\infty)$ for large x and n in view of the error being $o(x^{-n})$. One

important property of the $G_n^{(m)}$-transform is that it does not require

any knowledge about $\alpha_{k,i}$ in (6) or $\alpha'_{k,i}$ in (14).

4. Applications

The $G_n^{(m)}$-transform defined by equation (16) can be used to obtain a

good approximation to most tail probabilities. In all of the examples we

consider, m=1. This is certainly not necessary and is simply a consequence

of the fact that many of the more popular probability density functions

(PDF) satisfy a first order differential equation. It is however easy to

find PDFs for which $m > 1$.

More specifically, we demonstrate the application of the

$G_n^{(m)}$-transform to the Pearson family and to the inverse Gaussian PDFs and

in each of these cases m=1. Some comments are in order however before we

proceed. Although one could apply $G_n^{(m)}$ to any diffentiable PDF by simply

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selecting m (usually in practice we take $m \leqq 2$) and increasing n, the most efficient use of $G_n^{(m)}$ is when m can be ascertained, i.e., if it can be shown that f(x) satisfies an m-th order differential equation of the type of (6). In this event there is the added advantage that the differential equation itself normally will furnish a recursion formula for the required derivatives so that one need only calculate directly the first m-1 derivatives. Making use of the observations and the fact that if $g(x) = x^s f(x)$, then

$$g^{(r)}(x) = \sum_{k=0}^{r} \binom{r}{k} s(s-1) \ldots (s-k+1) \, x^{s-k} f^{(r-k)}(x), \tag{17}$$

the $G_n^{(m)}$-transform can be easily calculated. In the case m=1,

$$f(x) = U_1(x) f'(x), \tag{18}$$

$$U_1(x) = a_0 x^{\ell_1} + a_1 x^{\ell_1 - 1} + a_2 x^{\ell_1 - 2} + \ldots, \text{ as } x \to \infty.$$

For the Pearson family of PDFs

$$f(x) = \frac{b_0 + b_1 x + b_2 x^2}{x - a} \, f'(x), \tag{19}$$

which is clearly of the form in (18). The tail probabilities of all of these distributions may therefore be well approximated by $G_n^{(1)}[f(x); a_{ij}(x)]$. For those distributions with finite support, the method can still be employed simply by transformating the data to a distribution with infinite support.

For larger values of m and n the form of the $G_n^{(m)}$-transform given by equation (16) is a convenient one, and one which is easy to calculate on a micro-computer. However for m=1 and $n \leqq 3$ the transform can be simplified to the extent that it is easily computed on a hand calculator. Specifically we have (shortening our notation)

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$$G_1^{(1)}[f(x)] = \frac{-x\ f^2(x)}{x\ f'(x) + \ell_1 f(x)} \quad , \tag{20}$$

$$G_2^{(1)}[f(x)] = \frac{x\ f^2(x)\ A(x)}{x^2 B(x) - \ell_1(\ell_1-1)f^2(x) - xf'(x)A(x)} \quad , \tag{21}$$

and

$$G_3^{(1)}[f(x)] = \frac{xf^2(x)\ C(x)}{x^3 D(x) + 3(\ell_1-2)x^2 f(x)[B(x)-(f'(x))^2] + (\ell_1-1)(\ell_1-2)f^2(x)E(x)}, \tag{22}$$

where $A(x) = xf'(x) + 2(\ell_1-1)f(x),$

   $B(x) = f(x)\ f''(x) - (f'(x))^2,$

   $C(x) = 3(\ell_1-2)f(x)[xf'(x) + (\ell_1-1)f(x)] - x^2[2B(x)-(f'(x))^2],$

   $D(x) = -f^2(x)f^{(3)}(x) + 6f'(x)\ B(x)$

   $E(x) = -3xf'(x) - \ell_1 f(x).$

As we will see in the following examples, $n \leq 3$ is sufficient in these
applications. Since ℓ_1 is often 0 or 1, $G_n^{(1)}$ in (20), (21) and (22) is
usually even simpler than it appears in these equations. In all the
examples we consider here $G_n^{(1)}[f(x)]$ is a rational function times $f(x)$.

Example 1. The Normal distribution.

Finding a proper functional approximation to the tail of a normal
distribution is a problem which has a rich history in statistics and many
approximations have been proposed. In some applications one requires a
tail probability that is accurate in the extreme tails as well as at the
usual nominal significance levels. This problem has recently been
addressed by Hawkes (1982) who employed an ad hoc approach to obtain a
specialized approximation which is accurate in the very extreme tails. In
the event f is a normal PDF, i.e., $f(x) = (2\pi)^{-1/2} \exp\{-x^2/2\}$, then

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$$f'(x) = -xf(x)$$

and clearly m=1 and $\ell_1 = -1$. Furthermore

$$f^{(k)}(x) = -xf^{(k-1)}(x) + (1-k) f^{(k-2)}(x),$$

so that $G_n^{(1)}[f(x); a_{ij}(x)]$ is easy to calculate. Using (16) and (17) one could calculate the tail probability to any desired degree of accuracy by increasing n. However for most purposes the simple formula in (20), (21) or (22) will suffice. This is in fact the case in all of the examples which follow.

In the case of the standardized normal, it is easily derived that

$$G_1^{(1)}[f(x)] = \frac{x}{x^2+1} f(x),$$

$$G_2^{(1)}[f(x)] = \frac{x(x^2+4)}{(x^2+1)(x^2+4)-2} f(x),$$

and

$$G_3^{(1)}[f(x)] = \frac{x(x^2+2)(x^2+9)}{x^2(x^2+3)(x^2+9)+6} f(x) .$$

Table 1 compares $G_1^{(1)}$, $G_2^{(1)}$ and $G_3^{(1)}$ with the the approximations Q_{L2} and Q_{H2} given in Hawkes (1982); see Hawkes (1982) for details. Note that $G_2^{(1)}[f(x)]$ is essentially as good as the best of Q_{L2} and Q_{H2} and $G_3^{(1)}[f(x)]$ is better. In fact, these new approximations are exact to many more significant digits than those shown in Table 1 in the extreme tails.

Example 2. The t distribution.

Good (1986) gives two asymptotic formulas to approximate the extreme tails. The formulas are easily calculated on a hand calculator with great accuracy in the extreme tails. The $G_n^{(1)}$-transform yields approximations which are equally easy to calculate and more accurate. Moreover as in the previous example $G_n^{(1)}[f(x)]$ is also a satisfactory approximation at standard nominal significance levels.

The density of t_k, where k is the degree of freedom, is

$$f(x) = \{\Gamma((k+1)/2)/[(\pi k)^{1/2}\Gamma(k/2)]\}(1+x^2/k)^{-(k+1)/2},$$

so that

$$(k+x^2)f'(x) = -(k+1)xf(x). \tag{23}$$

Dividing both sides of (23) by x shows that m=1 and ℓ_1=1. Taking (r-1)th derivative on both sides of (23) and after some simple algebra, we have

$$f^{(r)}(x) = [(1-2r-k)xf^{(r-1)}(x) + (r-1)(1-r-k)f^{(r-2)}(x)]/(k+x^2). \tag{24}$$

Using (16), (17) and (24) the tail probability can be calculated to any desired degree of accuracy. However for most purposes $G_2^{(1)}[f(x)]$ and $G_3^{(1)}[f(x)]$ are more than adequate. Simplified expressions for them in this example are as follows:

$$G_1^{(1)}[f(x)] = \frac{x(x^2+k)}{k(x^2-1)} f(x),$$

$$G_2^{(1)}[f(x)] = \frac{x(x^2+k)}{k(x^2+1)} f(x),$$

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and

$$G_3^{(1)}[f(x)] = \frac{x(x^2+k)\{(k+2)x^2+5k\}}{k\{(k+2)x^4+6kx^2+3k\}} f(x) .$$

The comparisons are given in Table 2. The quantities Q_2 and Q_R^* here are the two seemingly best asymptotic formulas given by Good (1986); see Good (1986) for details. From the table we see that $G_2^{(1)}$ is more accurate than Q_2 and Q_R^* and $G_3^{(1)}$ is even better.

Example 3. The gamma distribution.

The density of the gamma distribution with parameters α and β is

$$f(x) = \{\Gamma(\alpha+1)\}^{-1} \beta^{-(\alpha+1)} x^\alpha e^{-x/\beta} , \qquad x \geq 0.$$

Therefore

$$f'(x) = (\alpha/x - \beta^{-1}) f(x) . \qquad (25)$$

Equation (25) implies m=1 and dividing both sides of (25) by $\alpha/x - \beta^{-1}$ shows $\ell_1 = 0$. Rewriting (25) as

$$xf'(x) = (\alpha - x/\beta)f(x)$$

and taking (r-1)th derivative on both sides leads to

$$f^{(r)}(x) = [-\beta^{-1} + (\alpha-r+1)(r-1)/x]f^{(r-1)}(x)-[(r-1)/(\beta x)]f^{(r-2)}(x). \qquad (26)$$

Using (26), (16) and (17), $G_n^{(1)}$ can be calculated for any n. However once again n=2 and n=3 are both quite accurate so that (21) or (22) suffices. It is easily calculated that in this case,

$$G_1^{(1)}[f(x)] = \frac{\beta x}{x-\alpha\beta} f(x),$$

$$G_2^{(1)}[f(x)] = \frac{\beta x\{x+\beta(2-\alpha)\}}{x^2-2\beta(\alpha-1)x+\beta^2\alpha(\alpha-1)} f(x),$$

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and

$$G_3^{(1)}[f(x)] = \frac{x\{\beta x^2 - \beta^2(2\alpha-6)x + \beta^3(\alpha^2-4\alpha+6)\}}{x^3 + \beta(\alpha+6)x^2 + \beta^2(3\alpha^2-9\alpha+6)x - \beta^3\alpha(\alpha-1)(\alpha-2)} f(x).$$

Gray and Lewis (1971) introduced the B_n-transform method of approximating the tail probabilities for a class of distributions where the density approximately satisfies some homogenous differential equation with constant coefficients. Because B_n uses the derivatives up to $f^{(2n-1)}(x)$ while $G_n^{(1)}$ uses those up to $f^{(n)}(x)$ only, it is sensible to compare, say, B_2 and $G_3^{(1)}$. Consider now more closely the special case where $\beta=2$ and $\alpha=k/2-1$, i.e., the chi-squared distribution. Interestingly, from Table 3 we see that $G_2^{(1)}$ and B_2 give exactly the same approximations in this special case. In general $G_3^{(1)}$ is clearly more accurate than B_2, or $G_3^{(1)}$. Note that B_2, $G_2^{(1)}$ and $G_3^{(1)}$ are exact for k=4. But only $G_3^{(1)}$ is exact for k=6.

Example 4. The F distribution.

The density function of an F distribution with degrees of freedom k and s is given by

$$f(x) = (k/s)^{k/2} x^{(k-2)/2} \{1 + (k/s)x\}^{(k+s)/2}/B(k/2, s/2).$$

Simple calculation shows that

$$x(1+(k/s)x)f'(x) = \{k/2-1 + [k(k+s/2-1)/s]x\}\, f(x).$$

Therefore for $r = 2, 3, \ldots,$

$$f^{(r)}(x) = \{[(k/2-r)s + k(k+s/2-2r+1)x]f^{(r-1)}(x) +$$
$$(r-1)k(k+s/2-r+1)f^{(r-2)}(x)\}/\{x(s+kx)\}. \qquad (27)$$

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It is easily seen that $m=1$, $\ell_1=1$. It is straightforward to calculate $G_n^{(1)}[f(x)]$ using (16), (17) and (27). Table 4 compares $G_n^{(1)}$ with the true values for $n = 2, 3, 4$, and demonstrates the great accuracy of $G_n^{(1)}$. Again $n=3$ is sufficient for most practical purposes. The simplified versions of (20), (21) and (22) for $G_n^{(1)}$, $n=1, 2, 3$ can be easily obtained in this case as in Examples 1-3. This is also true for the following examples. We omit these formulas in the remaining portion of this paper.

Example 5. The Pearson type IV distribution.

As Woodward (1976) pointed out, the type IV curve is the only one of Pearson's curves for which the probability integral cannot be reduced to known integrals such as X^2 integrals or incomplete gamma functions and thus it is more difficult to handle in practice. Its density is as follows:

$$f(x) = c(1+x^2/a^2)^{-k}e^{-s \arctan(x/a)}, \quad -\infty < x < \infty, \quad a > 0,$$

where

$$c^{-1} = a \int_{-\pi/2}^{\pi/2} e^{-s\theta} \cos^{2k-2}\theta \, d\theta.$$

The values of the integral above can be found in Pearson (1914) or calculated in a standard program. We may take the scale parameter $a = 1$. Then

$$(1+x^2)f'(x) = (-2kx-s)f(x) . \tag{28}$$

Taking $(r-1)$th derivative on both sides of (28), we have

$$f^{(r)}(x) = \{[2(r+k-1)x-s]f^{(r-1)}(x) + (r-1)(r+2k-2)f^{(r-2)}(x)\}/(1+x^2). \tag{29}$$

Using (29) and (17) we are able to compute $G_n^{(1)}$ with $m=1$ and $\ell_1=1$ for all n. Woodward (1976) applied the B_n-transform to this case and compared B_n with Shenton and Carpenter's (1965) approximation C_n. Table 5 compares

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$G_3^{(1)}$ with B_2 and C_2 for the same reason mentioned in Example 3. It is seen from Table 5 that $G_3^{(1)}$ tends to be more accurate than B_2 when x increases and is uniformly more accurate than C_2. Again we notice that $G_n^{(1)}$ converges to the true values quickly as n increases. Approximations B_2 and C_2 were taken from Woodward (1976).

All examples considered so far belong to the Pearson family. We now consider the last example which does not belong to the family.

Example 6. The Inverse Gaussian.

The density of Inverse Gaussian distribution is

$$f(x) = (\lambda/2\pi x^3)^{1/2} e^{-\lambda(x-\mu)^2/2\mu^2 x} , \quad x > 0, \quad \lambda > 0, \quad \mu > 0.$$

For simplicity, let $\lambda = \mu = 1$. It is easily calculated that

$$x^2 f'(x) = [(-x^2-3x+1)/2]f(x) ,$$

and therefore m=1, $\ell_1=0$. Similarly to the previous examples, for $r \geq 2$,

$$f^{(r)}(x) = \{[(-x^2-(4r-5x+1)/2] f^{(r-1)}(x) - (r-1)(x+r-1/2)f^{(r-2)}(x)\}/x^2. \quad (30)$$

Our approximation is tabulated in Table 6 for n=2, 3 and 4. Again $G_n^{(1)}$ is shown to be accurate in approximating tail probabilities, especially in the extreme tails.

5. Concluding Remarks

In this paper we have developed a general method for approximating the tail probabilities of continuous random variables. The method is three fold. First it is very accurate. It is theoretically and numerically sound that $G_n^{(m)}$ converges rapidly as $n \to \infty$. In fact, n = 2 or 3 is enough

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for most applications, especially in the extreme tails when $G_2^{(m)}$ or $G_3^{(m)}$ is more than adequate. Secondly, the $G_n^{(m)}$-transformation is very simple. When m=1, simple formulas for $G_n^{(m)}$, n=1, 2, 3 have been obtained. These formulas can easily be simplified even further for each given distribution and are suitable for computation on a hand calculator. For higher order m and n, a short FORTRAN program which calls any standard subroutine of calculating a determinant is sufficient to carry out the calculation. Thirdly, the method is general. Unlike most simple approximations designed for some particular distributions the $G_n^{(m)}$-transformation can be applied to a very broad class of distributions. Finally we should point out that in this paper we have determined m and ℓ_i in order to obtain efficiency of calculation. However $G_n^{(m)}$ is very robust in general and in most cases very accurate approximations can be obtained by letting $\ell_i=i$, m=1 or 2 and simply increasing n until the approximation remains essentially constant. Due to its simplicity a macro program could easily be written into statistical packages such as GLIM for the calculation of the $G_n^{(m)}$-transformation.

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Table 1. Approximations to the upper tail of the normal distribution.

x	true tail	Q_{L2}	Q_{H2}	$G_1^{(1)}$	$G_2^{(1)}$	$G_3^{(1)}$
1.2	.11507	.11737	.11402	.09550	.11244	.11504
1.6	.05480	.05589	.05465	.04985	.05452	.05486
2.0	.02275	.02314	.02274	.02160	.02273	.02276
2.4	.008198	.008312	.008197	.007951	.008199	.008200
3.0	.001350	.001364	.001350	.001330	.001350	.001350
4.5	3.398×10^{-6}	3.414×10^{-6}	3.396×10^{-6}	3.385×10^{-6}	3.398×10^{-6}	3.398×10^{-6}
6.0	9.866×10^{-10}	9.881×10^{-10}	9.861×10^{-10}	9.853×10^{-10}	9.866×10^{-10}	9.866×10^{-10}
8.0	6.221×10^{-16}	6.229×10^{-16}	6.218×10^{-16}	6.218×10^{-16}	6.221×10^{-16}	6.221×10^{-16}
10.0	7.620×10^{-24}	7.625×10^{-24}	7.618×10^{-24}	7.618×10^{-24}	7.620×10^{-24}	7.620×10^{-24}
14.0	7.794×10^{-45}	7.796×10^{-45}	7.793×10^{-45}	7.793×10^{-45}	7.794×10^{-45}	7.794×10^{-45}
18.0	9.741×10^{-73}	9.742×10^{-73}	9.741×10^{-73}	9.741×10^{-73}	9.741×10^{-73}	9.741×10^{-73}

Table 2. Approximations to the double-tail of the t distribution.

k	x	true tail	Q_2	Q_R^*	$G_2^{(1)}$	$G_3^{(1)}$
10	1.812	0.1001	0.2426	0.1196	0.0917	0.0986
10	2.228	0.0500	0.0853	0.0569	0.0474	0.0497
10	3.169	0.01000	0.01261	0.01075	0.00978	0.00999
10	4.587	0.001000	0.001124	0.001037	0.000990	0.01000
20	6.927	1.000×10^{-6}	1.008×10^{-6}	1.018×10^{-6}	0.998×10^{-6}	1.000×10^{-6}
60	5.449	1.000×10^{-6}	1.002×10^{-6}	1.030×10^{-6}	0.9970×10^{-6}	1.000×10^{-6}
120	20	5.105×10^{-40}	5.105×10^{-40}	5.118×10^{-40}	5.105×10^{-40}	5.105×10^{-40}

Table 3. Approximations to the upper tail of the X_k^2 distribution.

k	x	true tail	B_2	$G_2^{(1)}$	$G_3^{(1)}$
4	5	.28730	.28730	.28730	.28730
	7	.13589	.13589	.13589	.13589
	9	.06110	.06110	.06110	.06110
	13	.01128	.01128	.01128	.01128
5	6	.30622	.30272	.30272	.30608
	8	.15624	.15561	.15561	.15622
	10	.07524	.07510	.07510	.07523
	14	.01561	.01560	.01560	.01561
6	7	.32085	.31252	.31252	.32085
	9	.17358	.17190	.17190	.17190
	12	.06197	.06178	.06178	.06197
	16	.01375	.01374	.01374	.01375
7	9	.25266	.24616	.24616	.25285
	11	.13862	.13709	.13709	.13865
	14	.05118	.05098	.05098	.05118
	18	.01197	.01195	.01195	.01197

Table 4. Approximations to the upper tail of the F distribution.

k	S	x	true tail	$G_2^{(1)}$	$G_3^{(1)}$	$G_4^{(1)}$
1	1	5	.267720	.271148	.268009	.267751
	1	20	.140049	.140170	.140052	.140049
	1	200	.044941	.044941	.044941	.044941
1	10	3	.113938	.120508	.115110	.114228
	10	6	.034291	.034790	.034343	.034297
	10	11	.007792	.007825	.007794	.007792
1	60	2	.162470	.183623	.167128	.163875
	60	4	.050036	.051645	.050262	.050080
	60	8	.006354	.006395	.006354	.006351
10	1	10	.241668	.241267	.241673	.241668
	1	110	.074077	.074076	.074077	.074077
	1	1000	.024605	.024605	.024605	.024605
10	10	2	.144846	.136565	.145669	.144797
	10	3	.048927	.048094	.048980	.048925
	10	8	.001449	.001447	.001449	.001449
10	60	1.5	.161863	.142537	.164436	.161660
	60	1.8	.080036	.075538	.080491	.080007
	60	3.0	.003919	.003887	.003921	.003919
40	1	10	.246526	.246068	.246533	.246526
	1	30	.143946	.143917	.143946	.143946
	1	125	.070824	.070823	.070824	.070824
40	10	1.9	.138782	.130405	.140300	.138544
	10	3.0	.033326	.032873	.033372	.033322
	10	6.0	.002299	.002295	.002299	.002299
40	60	1.4	.117146	.104819	.120301	.116413
	60	1.7	.030729	.029676	.030907	.030700
	60	2.5	.000641	.000637	.000641	.000641

Table 5. Approximations to the upper tail of the Pearson type IV distribution.

k	s	x	true value	B_2	C_2	$G_3^{(1)}$
7	4	.07	.087553	.078528	.071427	.077052
		.15	.049286	.046772	.043898	.047860
		.61	.001022	.001016	.001010	.001019
5	-4	1.03	.100113	.097962	.097095	.100298
		1.24	.050102	.049055	.049428	.050180
		2.50	.001080	.001070	.001080	.001080
3	-2	1.25	.100993	.094930	.099260	.100662
		1.60	.050398	.041595	.050076	.050382
		4.40	.001004	.000961	.001004	.001005
3	2	.18	.099795	.091947	.077568	.086955
		.36	.050747	.048978	.045573	.047920
		1.38	.001146	.001124	.001142	.001144
3.5	0	.59	.100093	.096507	.093037	.095589
		.79	.050563	.048932	.049020	.049853
		2.10	.001064	.001040	.001063	.001064

Table 6. Approximations to the upper tail of the
inverse Gaussian distribution.

x	true value	$G_2^{(1)}$	$G_3^{(1)}$	$G_4^{(1)}$
1.5	.1892	.1759	.1839	.1868
2.0	.1145	.1091	.1126	.1138
3.0	.04681	.04562	.04649	.04671
4.5	.01430	.01413	.01427	.01429
6.0	.004850	.004818	.004845	.004849
8.0	.001260	.001256	.001260	.001260
10.0	.0003504	.0003498	.003504	.0003504
16.0	9.439×10^{-6}	9.434×10^{-6}	9.439×10^{-6}	9.439×10^{-6}
32.0	4.1206×10^{-10}	4.1204×10^{-10}	4.1206×10^{-10}	4.1206×10^{-10}

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