THEMIS SIGNAL ANALYSIS STATISTICS RESEARCH PROGRAM

STRESS-WAVE PROPAGATION

IN A THREE-REGION

CYLINDRICAL COMPOSITE MEDIUM

by

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Technical Report No. 22
Department of Statistics THEMIS Contract

November 21, 1968

Research sponsored by the Office of Naval Research
Contract N00014-68-A-0515
Project NR 042-260

DEPARTMENT OF PHYSICS
and
DEPARTMENT OF STATISTICS
Southern Methodist University

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Abstract

A semi-analytical solution is developed for the steady-state pressure in a three-region cylindrical composite medium containing a point sinusoidal source. The geometry is such that conventional analytical methods are not applicable. The scalar wave equation for a viscous homogeneous fluid is solved by separation of variables in each region of the composite medium. Infinite series are set up from these solutions. A finite number of terms in the series are retained for each region, and the interface boundary conditions are applied at a selected finite number of interface boundary points, in order to produce a set of algebraic equations which are linear in the coefficients of the series. The solution of this set then leads to an analytical approximation to the solution of the boundary value problem.

A central problem in this method is the specification of the eigenvalues in each region. There exist no general physically-based procedures for this purpose. In this paper an arbitrary Sturm-Liouville interface boundary condition is applied which enables a set of eigenvalues to be determined. The practical consequences of this step, in terms of numerical calculations, remain to be determined. These calculations are planned in subsequent work.

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Boundary value problems in geometries such as that of figure 1, in which the geometry is of such complexity that conventional analytical attacks will not work, it is the general practice to solve the differential equation, subject to appropriate boundary conditions, by numerical iterative means or finite difference techniques using a digital computer. It is a purpose of the present work to study and develop what might be called semi-analytical procedures for handling problems of this type. Galerkin's method (Collatz, 1960, p. 31 and p. 413) and the method of Garabedian and Thomas (1962) are examples of such methods. In these cases, solutions to the boundary value problem are built up from functions which are not necessarily solutions of the differential equation itself. The method to be applied here is an extension of one proposed by Bobone (1967), in which the separated solutions of the differential equation, expressed as series in each region of the composite medium, are tied together at the interfaces by applying the interface boundary conditions to the truncated series.

The specific purpose of the present paper is to develop the mathematical model for the semi-analytical calculation of the steady-state pressure in a three-region composite cylindrical medium under the influence of a point sinusoidal source. The geometrical arrangement of

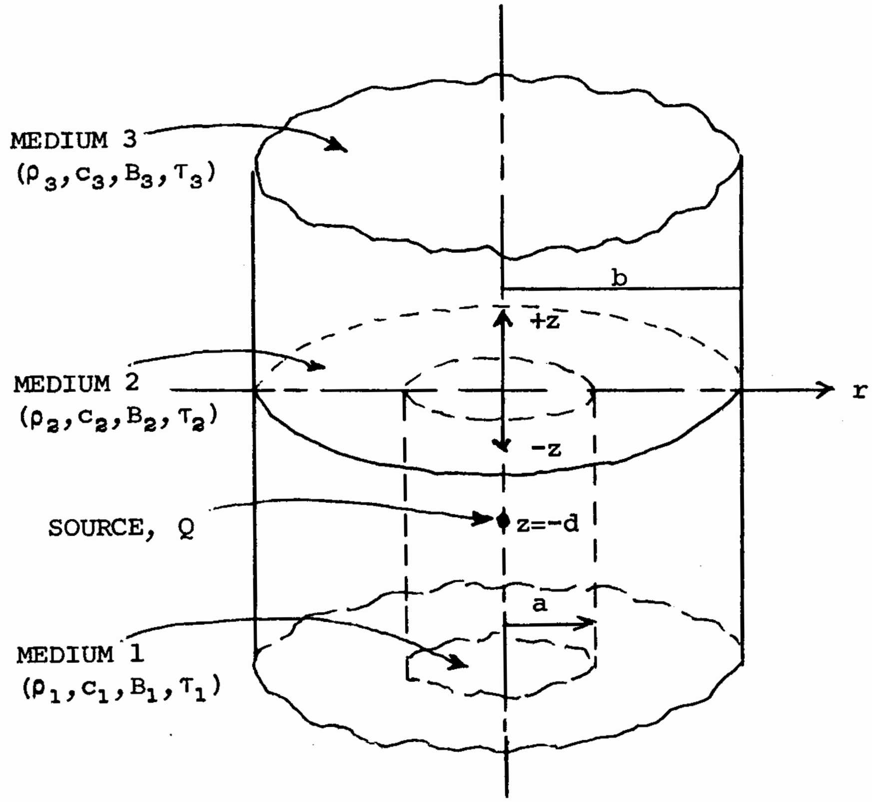

Fig. 1. Geometrical arrangement of source and media.

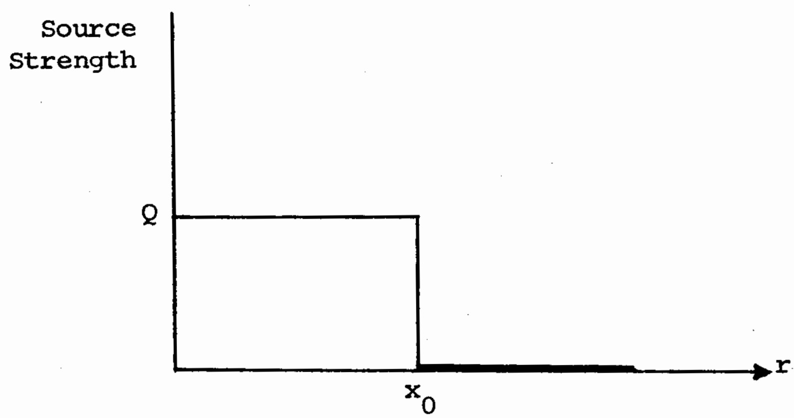

Fig. 2. Source strength as a function of radial distance.

of the source and media are shown in figure 1. The media are each viscous, homogeneous fluids, characterized by their density, ρ; velocity of propagation, c; bulk modulous, B; and relaxation time, τ. Regions 1 and 2 are concentric cylinders; a plane interface perpendicular to the z-axis separates region 3 from regions 1 and 2 at z = 0. The radius of region 1 is a; the radius of regions 2 and 3 is b. The cylinder is infinitely long.

The pressure source is located at z = -d. The variation of source strength with radial distance is shown in figure 2.

The appropriate equation of motion for the scalar displacement potential, Φ, for a viscous, homogeneous fluid is*

$$\nabla^2 \Phi = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \tau \frac{\partial}{\partial t} \nabla^2 \Phi \ . \tag{1}$$

Let

$$\Phi(r,z,t) = \phi(r,z)\ e^{i\omega t}, \tag{2}$$

for harmonic time dependence. Substituting equation (2) into equation (1) yields

$$\nabla^2 \phi\ e^{i\omega t} = -\frac{\omega^2}{c^2} e^{i\omega t}\ \phi - i\omega\tau\ e^{i\omega t}\ \nabla^2 \phi\ .$$

Dividing through by $e^{i\omega t}$ and collecting terms in $\nabla^2 \phi$ gives

$$(1 + i\omega\tau)\ \nabla^2 \phi + \frac{\omega^2}{c^2}\ \phi = 0.$$

Defining

$$k^2 = \frac{\omega^2 / c^2}{1 + i\omega\tau}, \tag{3}$$

*c is the speed of propagation and τ the relaxation time of the medium.

we have

$$\nabla^2\phi + k^2\phi = 0 \ . \tag{4}$$

We seek a solution to equation (4) with standard separation of variables techniques. Assume a solution of the form

$$\phi(r,z) = R(r)Z(z). \tag{5}$$

Substituting equation (5) into equation (4) yields

$$Z(z)\left[\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr}\right] + R(r)\frac{d^2Z(z)}{dz^2} + k^2R(r)Z(z) = 0,$$

since we are employing a cylindrical coordinate system. Dividing by $R(r)Z(z)$ and rearranging terms gives

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} = -\frac{1}{Z}\frac{d^2Z}{dz^2} - k^2 \tag{6}$$

Since the variable r appears only on the left side of equation (6) and the variable z appears only on the right, each side of equation (6) must be constant. Letting that constant be $-\alpha^2$, we have

$$\frac{d^2R}{dr^2} + \frac{1}{R}\frac{dR}{dr} + \alpha^2R = 0 \tag{7}$$

and

$$\frac{d^2Z}{dz^2} + \beta^2Z = 0, \tag{8}$$

where $\beta^2 = k^2 - \alpha^2$.

The solution to equation (8) is given by

$$Z(z) = B\ e^{-i\beta z} + C\ e^{i\beta z}\ , \tag{9}$$

where B and C are arbitrary constants. The solution to equation (7), a form of Bessel's equation, is given by

$$R(r) = J_0(\alpha r) + A\, Y_0(\alpha r) \ , \tag{10}$$

where J_0 and Y_0 are zeroth-order Bessel functions of the first and second kind, respectively, and A is an arbitrary constant. Hence, the solution of equation (1) is

$$\Phi(r,z,t) = \left[J_0(\alpha r) + A\, Y_0(\alpha r)\right] \left[B\, e^{-i\beta z} + C\, e^{i\beta z}\right] e^{i\omega t} \tag{11}$$

Since A, B, C, and β can be complex, write

$$A = A_1 + iA_2,\ B = B_1 + iB_2,\ C = C_1 + iC_2,\ \beta = \beta_1 + i\beta_2$$

Then

$$\begin{aligned}\Phi(r,z,t) = {} & \left[J_0(\alpha r) + (A_1 + iA_2)\, Y_0(\alpha r)\right] \left[\cos \omega t + i\sin \omega t\right] \\ & \cdot\Big\{(B_1 + iB_2)\left[\cos(\beta_1 + i\beta_2)z - i\sin(\beta_1 + i\beta_2)z\right] \\ & + (C_1 + iC_2)\left[\cos(\beta_1 + i\beta_2)z + i\sin(\beta_1 + i\beta_2)z\right]\Big\} \ .\end{aligned} \tag{12}$$

If we define

$$E = B_1 \cos(\beta_1 z - \omega t) + B_2 \sin(\beta_1 z - \omega t),$$

$$F = C_1 \cos(\beta_1 z + \omega t) - C_2 \sin(\beta_1 z + \omega t),$$

$$G = B_1 \sin(\beta_1 z + \omega t) - B_2 \cos(\beta_1 z + \omega t),$$

$$H = C_1 \sin(\beta_1 z - \omega t) + C_2 \cos(\beta_1 z - \omega t),$$

then

$$\begin{aligned}\mathrm{Re}\left[\Phi(r,z,t)\right] = {} & J_0(\alpha r)\left[e^{\beta_2 z}E + e^{-\beta_2 z}F\right] + Y_0(\alpha r)\Big[e^{\beta_2 z}(A_1 E + A_2 G) \\ & + e^{-\beta_2 z}(A_1 F - A_2 H)\Big]\end{aligned} \tag{13}$$

and

$$\begin{aligned}\mathrm{Im}\left[\Phi(r,z,t)\right] = {} & J_0(\alpha r)\left[-e^{\beta_2 z}G + e^{-\beta_2 z}H\right] + Y_0(\alpha r)\Big[e^{\beta_2 z}(-A_1 G + A_2 E) \\ & + e^{-\beta_2 z}(A_1 H + A_2 F)\Big]\end{aligned} \tag{14}$$

BOUNDARY CONDITIONS

In regions 1 and 3 the coefficient of $Y_0(\alpha r)$ must vanish, since the function must be finite at r = 0. Hence, for the real and imaginary parts of Φ, respectively, we have

$$e^{\beta_2 z}(A_1 E + A_2 G) + e^{-\beta_2 z}(A_1 F - A_2 H) = 0$$

and

$$e^{\beta_2 z}(-A_1 G + A_2 E) + e^{-\beta_2 z}(A_1 H + A_2 F) = 0.$$

Solving for A_1 we get

$$A_1 \left[1 + \left(\frac{Ge^{\beta_2 z} - He^{-\beta_2 z}}{Ee^{\beta_2 z} + Fe^{-\beta_2 z}} \right)^2 \right] = 0, \text{ where, } A_2 = A_1 \left(\frac{Ge^{\beta_2 z} - He^{-\beta_2 z}}{Ee^{\beta_2 z} + Fe^{-\beta_2 z}} \right).$$

Hence, either the term inside the parentheses is equal to $\sqrt{-1}$, which it cannot, since all quantities involved are real; or $A_1 = A_2 = 0$, in regions 1 and 3.

Also, with respect to region 3, we require that Φ be finite at $z = \infty$. Since β_2 turns out to be negative, this means that F = H = 0, or

$$C_1 \cos(\beta_1 z + \omega t) - C_2 \sin(\beta_1 z + \omega t) = 0$$

and

$$C_1 \sin(\beta_1 z - \omega t) + C_2 \cos(\beta_1 z - \omega t) = 0.$$

This set of equations can be satisfied only for $C_1 = C_2 = 0$, in region 3.

Hence

$$\text{Re}\left[\Phi_1\right] = J_0(\alpha_1 r)\left[E_1 e^{\beta_{21} z} + F_1 e^{-\beta_{21} z}\right]$$

$$\text{Im}\left[\Phi_1\right] = J_0(\alpha_1 r)\left[-G_1 e^{\beta_{21} z} + H_1 e^{-\beta_{21} z}\right]$$

$$\text{Re}\left[\Phi_2\right] = \left[J_0(\alpha_2 r) + A_1 Y_0(\alpha_2 r)\right]\left[E_2 e^{\beta_{22} z} + F_2 e^{-\beta_{22} z}\right] + A_2 Y_0(\alpha_2 r)\left[G_2 e^{\beta_{22} z} - H_2 e^{-\beta_{22} z}\right]$$

$$\text{Im}\left[\Phi_2\right] = \left[J_0(\alpha_2 r) + A_1 Y_0(\alpha_2 r)\right]\left[-G_2 e^{\beta_{22} z} + H_2 e^{-\beta_{22} z}\right] + A_2 Y_0(\alpha_2 r)\left[E_2 e^{\beta_{22} z} - F_2 e^{-\beta_{22} z}\right]$$

$$\mathrm{Re}\left[\Phi_3\right] = E_3 J_0(\alpha_3 r)\ e^{\beta_{22} z}$$

$$\mathrm{Im}\left[\Phi_3\right] = -G_3 J_0(\alpha_3 r)\ e^{\beta_{22} z}$$

Rather than deal with the scalar displacement potential directly, it is more realistic to work with some functions which are calculated from Φ. Since the particle displacement vector, $\vec{D}$, is defined by

$$\vec{D} = \nabla\Phi = u\vec{a}_r + w\vec{a}_z = \frac{\partial\Phi}{\partial r}\vec{a}_r + \frac{\partial\Phi}{\partial z}\vec{a}_z ,$$

we can readily calculate the radial particle displacement, u, and the axial particle displacement, w. Furthermore, the stress in a fluid is given by $p = \rho\omega^2$. The equations for u, w and p for each of the three regions are given on the following pages.

We have the following interface and outerface boundary conditions:

(a) radial particle displacement vanishes at r = b in region 3;

(b) radial particle displacement vanishes at r = b in region 2;

(c) pressure is continuous across interface at r = a;

(d) radial particle displacement is continuous across interface at r = a;

(e) pressure is continuous across interface at z = 0;

(f) axial particle displacement is continuous across interface at z = 0;

(g) source normalization at z = -d.

Condition (a) implies that

$$J_1(a_{3\nu} b) = 0. \tag{33}$$

This allows us to evaluate the eigenvalues in region 3.

Condition (b) implies (for real and imaginary parts)

Region 1:

$$\mathrm{Re}\left[u_1\right] = -\alpha_1 J_1(\alpha_1 r)\left[E_1 e^{\beta_{21} z} + F_1 e^{-\beta_{21} z}\right] \tag{15}$$

$$\mathrm{Im}\left[u_1\right] = -\alpha_1 J_1(\alpha_1 r)\left[-G_1 e^{\beta_{21} z} + H_1 e^{-\beta_{21} z}\right] \tag{16}$$

$$\begin{aligned}\mathrm{Re}\left[w_1\right] = J_0(\alpha_1 r)\Big\{&\left[(\beta_{21}B_{11} + \beta_{11}B_{21})\cos(\beta_{11}z - \omega t) + (\beta_{21}B_{21} - \beta_{11}B_{11})\sin(\beta_{11}z - \omega t)\right]e^{\beta_{21}z}\\ &+\left[-(\beta_{21}C_{11} + \beta_{11}C_{21})\cos(\beta_{11}z + \omega t) + (\beta_{21}C_{21} - \beta_{11}C_{11})\sin(\beta_{11}z + \omega t)\right]e^{-\beta_{21}z}\Big\}\end{aligned} \tag{17}$$

$$\begin{aligned}\mathrm{Im}\left[w_1\right] = J_0(\alpha_1 r)&\left[(\beta_{21}B_{21} + \beta_{11}B_{11})\cos(\beta_{11}z + \omega t) - (\beta_{21}B_{11} + \beta_{11}B_{21})\sin(\beta_{11}z + \omega t)\right]e^{\beta_{21}z}\\ &-\left[(\beta_{21}C_{21} - \beta_{11}C_{11})\cos(\beta_{11}z - \omega t) + (\beta_{21}C_{11} + \beta_{11}C_{21})\sin(\beta_{11}z - \omega t)\right]e^{-\beta_{21}z}\Big\}\end{aligned} \tag{18}$$

$$\mathrm{Re}\left[p_1\right] = \rho_1\omega^2 J_0(\alpha_1 r)\left[E_1 e^{\beta_{21} z} + F_1 e^{-\beta_{21} z}\right] \tag{19}$$

$$\mathrm{Im}\left[p_1\right] = \rho_1\omega^2 J_0(\alpha_1 r)\left[-G_1 e^{\beta_{21} z} + H_1 e^{-\beta_{21} z}\right] \tag{20}$$

Region 3:

$$\mathrm{Re}\left[u_3\right] = -\alpha_3 E_3 J_1(\alpha_3 r)e^{\beta_{23}z} \tag{21}$$

$$\mathrm{Im}\left[u_3\right] = \alpha_3 G_3 J_1(\alpha_3 r)e^{\beta_{23}z} \tag{22}$$

$$\mathrm{Re}\left[w_3\right] = J_0(\alpha_3 r)\left[(\beta_{23}B_{13} + \beta_{13}B_{23})\cos(\beta_{13}z - \omega t) + (\beta_{23}B_{23} - \beta_{13}B_{13})\sin(\beta_{13}z - \omega t)\right]e^{\beta_{23}z} \tag{23}$$

$$\mathrm{Im}\left[w_3\right] = J_0(\alpha_3 r)\left[(\beta_{23}B_{23} - \beta_{13}B_{13})\cos(\beta_{13}z + \omega t) - (\beta_{23}B_{13} + \beta_{13}B_{23})\sin(\beta_{13}z + \omega t)\right]e^{\beta_{23}z} \tag{24}$$

$$\mathrm{Re}\left[p_3\right] = \rho_3\omega^2 E_3 J_0(\alpha_3 r)\, e^{\beta_{23}z} \tag{25}$$

$$\mathrm{Im}\left[p_3\right] = -\rho_3\omega^2 G_3 J_0(\alpha_3 r)\, e^{\beta_{23}z} \tag{26}$$

Region 2:

$$\mathrm{Re}\left[u_2\right] = -\alpha_2\left\{\left[J_1(\alpha_2 r) + A_1 Y_1(\alpha_2 r)\right]\left[E_2 e^{\beta_{22}z} + F_2 e^{-\beta_{22}z}\right] + A_2 Y_1(\alpha_2 r)\left[G_2 e^{\beta_{22}z} - H_2 e^{-\beta_{22}z}\right]\right\}$$

$$\mathrm{Im}\left[u_2\right] = -\alpha_2\left\{\left[J_1(\alpha_2 r) + A_1 Y_1(\alpha_2 r)\right]\left[-G_2 e^{\beta_{22}z} + H_2 e^{-\beta_{22}z}\right] + A_2 Y_1(\alpha_2 r)\left[E_2 e^{\beta_{22}z} + F_2 e^{-\beta_{22}z}\right]\right\} \tag{27}$$

$$\begin{aligned}\mathrm{Re}\left[w_2\right] = \left[J_0(\alpha_2 r)\ A_1 Y_0(\alpha_2 r)\right]&\left\{\left[(\beta_{22}B_{12} + \beta_{12}B_{22})\cos(\beta_{12}z - \omega t) + (\beta_{22}B_{22} - \beta_{12}B_{12})\sin(\beta_{12}z - \omega t)\right]e^{\beta_{22}z}\right. \\ &- \left[(\beta_{22}C_{12} + \beta_{12}C_{22})\cos(\beta_{12}z + \omega t) - (\beta_{22}C_{22} - \beta_{12}C_{12})\sin(\beta_{12}z + \omega t)\right]e^{-\beta_{22}z}\Big\} \\ &+ A_2 Y_0(\alpha_2 r)\left\{\left[(\beta_{22}B_{12} - \beta_{12}B_{22})\sin(\beta_{12}z + \omega t) - (\beta_{22}B_{22} - \beta_{12}B_{12})\cos(\beta_{12}z + \omega t)\right]e^{\beta_{22}z}\right. \\ &+ \left[(\beta_{22}C_{12} + \beta_{12}C_{22})\sin(\beta_{12}z - \omega t) + (\beta_{22}C_{22} - \beta_{12}C_{12})\cos(\beta_{12}z - \omega t)\right]e^{-\beta_{22}z}\Big\}\end{aligned} \tag{28}$$

$$\begin{aligned}\mathrm{Im}\left[w_2\right] = \left[J_0(\alpha_2 r) + A_1 Y_0(\alpha_2 r)\right]&\left\{\left[(\beta_{22}B_{22} - \beta_{12}B_{12})\cos(\beta_{12}z + \omega t) - (\beta_{22}B_{12} + \beta_{12}B_{22})\sin(\beta_{12}z + \omega t)\right]e^{\beta_{22}z}\right. \\ &- \left[(\beta_{22}C_{22} - \beta_{12}C_{12})\cos(\beta_{12}z - \omega t) + (\beta_{22}C_{12} + \beta_{12}C_{22})\sin(\beta_{12}z - \omega t)\right]e^{-\beta_{22}z}\Big\} \\ &+ A_2 Y_0(\alpha_2 r)\left\{\left[(\beta_{22}B_{12} + \beta_{12}B_{22})\cos(\beta_{12}z - \omega t) + (\beta_{22}B_{22} - \beta_{12}B_{12})\sin(\beta_{12}z - \omega t)\right]e^{\beta_{22}z}\right. \\ &- \left[(\beta_{22}C_{12} + \beta_{12}C_{22})\cos(\beta_{12}z + \omega t) - (\beta_{22}C_{22} - \beta_{12}C_{12})\sin(\beta_{12}z + \omega t)\right]e^{\beta_{22}z}\Big\}\end{aligned} \tag{29}$$

$$\mathrm{Re}\left[p_2\right] = \rho_2\omega^2\left\{\left[J_0(\alpha_2 r) + A_1 Y_0(\alpha_2 r)\right]\left[E_2 e^{\beta_{22}z} + F_2 e^{-\beta_{22}z}\right] + A_2 Y_0(\alpha_2 r)\left[G_2 e^{\beta_{22}z} - H_2 e^{-\beta_{22}z}\right]\right\} \tag{30}$$

$$\mathrm{Im}\left[p_2\right] = \rho_2\omega^2\left\{\left[J_0(\alpha_2 r) + A_1 Y_0(\alpha_2 r)\right]\left[-G_2 e^{\beta_{22}z} + H_2 e^{-\beta_{22}z}\right] + A_2 Y_0(\alpha_2 r)\left[E_2 e^{\beta_{22}z} + F_2 e^{-\beta_{22}z}\right]\right\} \tag{31}$$

(32)

$$\left[J_1(\alpha_2 b) + A_1 Y_1(\alpha_2 b)\right](E_2 e^{\beta_{22} z} + F_2 e^{-\beta_{22} z}) + A_2 Y_1(\alpha_2 b)(G_2 e^{\beta_{22} z} - H_2 e^{-\beta_{22} z}) = 0$$

and

$$\left[J_1(\alpha_2 b) + A_1 Y_1(\alpha_2 b)\right](-G_2 e^{\beta_{22} z} + H_2 e^{-\beta_{22} z}) + A_2 Y_1(\alpha_2 b)(E_2 e^{\beta_{22} z} + F_2 e^{-\beta_{22} z}) = 0.$$

Hence, either

$$D = \begin{vmatrix} E_2 e^{\beta_{22} z} + F_2 e^{-\beta_{22} z} & G_2 e^{\beta_{22} z} - H_2 e^{-\beta_{22} z} \\ -(G_2 e^{\beta_{22} z} - H_2 e^{-\beta_{22} z}) & E_2 e^{\beta_{22} z} + F_2 e^{-\beta_{22} z} \end{vmatrix} = 0$$

or

$$J_1(\alpha_2 b) + A_1 Y_1(\alpha_2 b) = A_2 Y_1(\alpha_2 b) = 0.$$

Evaluating D, we obtain

$$D = (E_2 e^{\beta_{22} z} + F_2 e^{-\beta_{22} z})^2 + (G_2 e^{\beta_{22} z} - H_2 e^{-\beta_{22} z})^2 ,$$

and since all the terms in D are real, D must be a positive number (unless we have the trivial case in which E_2, F_2, G_2 and H_2 are zero). Hence,

$$J_1(\alpha_2 b) + A_1 Y_1(\alpha_2 b) = 0$$

and

$$A_2 Y_1(\alpha_2 b) = 0.$$

Consider the second condition. It implies that either A_2 or $Y_1(\alpha_2 b)$ is zero. If $Y_1(\alpha_2 b) = 0$, we have a set of eigenvalues $\alpha_{2n} = x_n/b$, where x_n is the n^{th} zero of $Y_1(x_n) = 0$. Then the first equation becomes $J_1(\alpha_2 b) = 0$. But for the set of eigenvalues α_{2n}, this equation cannot be satisfied. Hence, we must have

$$A_2 = 0$$

and

$$A_1 = -\frac{J_1(\alpha_2 b)}{Y_2(\alpha_2 b)} \quad .$$

If we now apply condition (c) term-by-term, we find that this requires equality of the z-functions in regions 1 and 2. This means that

$\beta_{11} = \beta_{12}$ and $\beta_{21} = \beta_{22}$.

This then implies that

$$\alpha_1^2 = \alpha_2^2 + k_1^2 - k_2^2,$$

and since k_1 and k_2 are complex, α_1 would be complex. This is certainly mathematically reasonable, but Bessel functions with complex-valued arguments are not readily evaluated numerically. Therefore, we shall impose an arbitrary boundary condition designed to generate real eigenvalues for regions 1 and 2; that is, that the sum of the pressure and a constant times its derivative in region 2, evaluated at r = a, be equal to zero:

$$p_2(r=a) + Kp_2'(r=a) = 0.$$

Using eq. (30) we have

$$J_0(\alpha_{2n}a) + A_{1n}Y_0(\alpha_{2n}a) - K\alpha_{2n}\left[J_1(\alpha_{2n}a) + A_{1n}Y_1(\alpha_{2n}a)\right] = 0, \qquad (34)$$

where K is completely arbitrary and the subscript n has been added to indicate that there is a discrete set of values $y_n = \alpha_{2n}/a$ which satisfy the transcendental equation (34).

Now if we apply condition (c), we get for the real parts

$$\sum_m \rho_1 J_0(\alpha_1 a)\left[E_1 e^{\beta_{21}z} + F_1 e^{-\beta_{21}z}\right] = \sum_n \rho_2\left[J_0(\alpha_2 a) + A_1 Y_0(\alpha_2 a)\right]\left[E_2 e^{\beta_{22}z} + F_2 e^{-\beta_{22}z}\right]. \qquad (35)$$

Substituting eq. (34) into (35) yields

$$\sum_m \rho_1 J_0(\alpha_1 a)\left[E_1 e^{\beta_{21} z} + F_1 e^{-\beta_{21} z}\right] = \sum_n K\rho_2\alpha_2\left[J_1(\alpha_2 a) + A_1 Y_1(\alpha_2 a)\right]\left[E_2 e^{\beta_{22} z} + F_2 e^{-\beta_{22} z}\right]. \tag{36}$$

Condition (d) gives, for real parts,

$$\sum_m \alpha_1 J_1(\alpha_1 a)\left[E_1 e^{\beta_{21} z} + F_1 e^{-\beta_{21} z}\right] = \sum_n \alpha_2\left[J_1(\alpha_2 a) + A_1 Y_1(\alpha_2 a)\right]\left[E_2 e^{\beta_{22} z} + F_2 e^{-\beta_{22} z}\right]. \tag{37}$$

Eqs. (36) and (37) can be combined to yield

$$\sum_m \left[\rho_1 J_0(\alpha_1 a) - K\rho_2\alpha_1 J_1(\alpha_1 a)\right]\left[E_1 e^{\beta_{21} z} + F_1 e^{-\beta_{21} z}\right] = 0. \tag{38}$$

Since eq. (38) must be true for all z, we must have

$$\rho_1 J_0(\alpha_{1m} a) - K\rho_2\alpha_{1m} J_1(\alpha_{1m} a) = 0, \tag{39}$$

where we have added the subscript m to indicate that there is a discrete set of eigenvalues that satisfy eq. (39).

For convenience, redefine the following:

$$\alpha_{1m} = \alpha_m \qquad \beta_{11m} = \lambda_m \qquad \beta_{21m} = \xi_m$$

$$\alpha_{2n} = \beta_n \qquad \beta_{12n} = \mu_n \qquad \beta_{22n} = \zeta_n \qquad A_{1n} = A_n$$

$$\alpha_{3\nu} = \gamma_\nu \qquad \beta_{13\nu} = \eta_\nu \qquad \beta_{23\nu} = \epsilon_\nu \qquad Z_p(y_n) = J_p(y_n) + A_n Y_p(y_n)$$

Then for the determination of the eigenvalues we have

$$\rho_1 J_0(\alpha_m a) - K\rho_2\alpha_m J_1(\alpha_m a) = 0,$$

$$Z_0(\beta_n a) - K\beta_n Z_1(\beta_n a) = 0,$$

and

$$J_1(\gamma_\nu b) = 0.$$

In addition, we have the following relationships:

$$\lambda_m + i\xi_m = (k_1^2 - \alpha_m^2)^{\frac{1}{2}}, \qquad \mu_n + i\zeta_n = (k_2^2 - \beta_n^2)^{\frac{1}{2}}, \qquad \eta_\nu + i\epsilon_\nu = (k_3^2 - \gamma_\nu^2)^{\frac{1}{2}}.$$

Again, for convenience, define the following:

$$B_{11m} = A_m \qquad B_{21m} = B_m \qquad C_{11m} = C_m \qquad C_{21m} = D_m$$

$$B_{12n} = E_n \qquad B_{22n} = F_n \qquad C_{12n} = G_n \qquad C_{22n} = H_n$$

$$B_{13\nu} = P_\nu \qquad B_{23\nu} = T_\nu$$

Now, we can write out the boundary conditions (c) through (g) explicitly, resulting in two equations for each condition, since the conditions must hold for both real and imaginary parts. Note that we must write different relationships for various regions when applying conditions (e) through (g). The equations were given on the following pages.

If we retain M, N, and L terms in the summations over m, n and ν, respectively, then we have 10[2(M + N) + L] coefficients to evaluate with 10 equations. Hence, we need to specify 2(M + N) + L points along the various boundaries in order to evaluate those coefficients. Having done that (via computer), the solution for the pressure at any point is calculated by summing over the range of eigenfunctions for the region of interest.

$$z \le 0: \sum_m \rho_1 J_0(\alpha_m a)\left\{\left[A_m\cos(\lambda_m z-\omega t) + B_m\sin(\lambda_m z-\omega t)\right]e^{\xi_m z} + \left[C_m\cos(\lambda_m z+\omega t) - D_m\sin(\lambda_m z+\omega t)\right]e^{-\xi_m z}\right\}$$

$$-\sum_n \rho_2 Z_0(\beta_n a)\left\{\left[E_n\cos(\mu_n z-\omega t) + F_n\sin(\mu_n z-\omega t)\right]e^{\zeta_n z} + \left[G_n\cos(\mu_n z+\omega t) - H_n\sin(\mu_n z+\omega t)\right]e^{-\zeta_n z}\right\} = 0.$$

$$\sum_m \rho_1 J_0(\alpha_m a)\left\{\left[-A_m\sin(\lambda_m z+\omega t) + B_m\cos(\lambda_m z+\omega t)\right]e^{\xi_m z} + \left[C_m\sin(\lambda_m z-\omega t) + D_m\cos(\lambda_m z-\omega t)\right]e^{-\xi_m z}\right\}$$

$$-\sum_n \rho_2 Z_0(\beta_n a)\left\{\left[-E_n\sin(\mu_n Z+\omega t) + F_n\cos(\mu_n z+\omega t)\right]e^{\zeta_n z} + \left[G_n\sin(\mu_n z-\omega t) + H_n\cos(\mu_n z-\omega t)\right]e^{-\zeta_n z}\right\} = 0.$$

$$\sum_m \alpha_m J_1(\alpha_m a)\left\{\left[A_m\cos(\lambda_m z-\omega t) + B_m\sin(\lambda_m z-\omega t)\right]e^{\xi_m z} + \left[C_m\cos(\lambda_m z+\omega t) - D_m\sin(\lambda_m z+\omega t)\right]e^{-\xi_m z}\right\}$$

$$-\sum_n \beta_n Z_1(\beta_n a)\left\{\left[E_n\cos(\mu_n z-\omega t) + F_n\sin(\mu_n z-\omega t)\right]e^{\zeta_n z} + \left[G_n\cos(\mu_n z+\omega t) - H_n\sin(\mu_n z+\omega t)\right]e^{-\zeta_n z}\right\} = 0.$$

$$\sum_m \alpha_m J_1(\alpha_m)\left\{\left[-A_m\sin(\lambda_m z+\omega t) + B_m\cos(\lambda_m z+\omega t)\right]e^{\xi_m z} + \left[C_m\sin(\lambda_m z-\omega t) + D_m\cos(\lambda_m z-\omega t)\right]e^{-\xi_m z}\right\}$$

$$-\sum_n \beta_n Z_1(\beta_n a)\left\{\left[-E_n\sin(\mu_n z+\omega t) + F_n\cos(\mu_n z+\omega t)\right]e^{\zeta_n z} + \left[G_n\sin(\mu_n z-\omega t) + H_n\cos(\mu_n z-\omega t)\right]e^{-\zeta_n z}\right\} = 0.$$

$$r \le a: \sum_m \rho_1 J_0(\alpha_m r)\left[(A_m + C_m)\cos\omega t - (B_m + D_m)\sin\omega t\right] - \sum_\nu \rho_3 J_0(\gamma_\nu r)\left[P_\nu\cos\omega t - T_\nu\sin\omega t\right] = 0.$$

$$\sum_m \rho_1 J_0(\alpha_m r)\left[-(A_m + C_m)\sin\omega t + (B_m + D_m)\cos\omega t\right] + \sum_\nu \rho_3 J_0(\gamma_\nu r)\left[P_\nu\sin\omega t - T_\nu\cos\omega t\right] = 0.$$

$$\sum_m J_0(\alpha_m r)\left\{\left[\xi_m(A_m - C_m) + \lambda_m(B_m - D_m)\right]\cos\omega t - \left[\xi_m(B_m - D_m) - \lambda_m(A_m - C_m)\right]\sin\omega t\right\}$$

$$-\sum_\nu J_0(\gamma_\nu r)\left[(\epsilon_\nu P_\nu + \eta_\nu T_\nu)\cos\omega t + (\epsilon_\nu T_\nu - \eta_\nu P_\nu)\sin\omega t\right] = 0.$$

$$\sum_m J_0(\alpha_m r)\left\{\left[\xi_m(B_m - D_m) - \lambda_m(A_m + C_m)\right]\cos\omega t - \left[\xi_m(A_m + C_m) + \lambda_m(B_m + D_m)\right]\sin\omega t\right\}$$

$$-\sum_\nu J_0(\lambda_\nu r)\left[(\epsilon_\nu T_\nu - \eta_\nu P_\nu)\cos\omega t - (\epsilon_\nu P_\nu + \eta_\nu T_\nu)\sin\omega t\right] = 0.$$

$$a < r \le b: \quad \sum_n \rho_2 Z_0(\beta_n r)\Big[(E_n + G_n)\cos\omega t - (F_n + H_n)\sin\omega t\Big] - \sum_\nu \rho_3 J_0(\gamma_\nu r)\Big[P_\nu \cos\omega t - T_\nu \sin\omega t\Big] = 0.$$

$$\sum_n \rho_2 Z_0(\beta_n r)\Big[-(E_n + G_n)\sin\omega t + (F_n + H_n)\cos\omega t\Big] - \sum_\nu \rho_3 J_0(\gamma_\nu r)\Big[P_\nu \cos\omega t - T_\nu \sin\omega t\Big] = 0.$$

$$\sum_n Z_0(\beta_n r)\Big\{\Big[\zeta_n(E_n - G_n) + \mu_n(F_n - H_n)\Big]\cos\omega t - \Big[\zeta_n(F_n - G_n) - \mu_n(E_n - G_n)\Big]\sin\omega t\Big\}$$

$$-\sum_\nu J_0(\gamma_\nu)\Big[(\epsilon_\nu P_\nu + \eta_\nu T_\nu)\cos\omega t + (\epsilon_\nu T_\nu - \eta_\nu P_\nu)\sin\omega t\Big] = 0.$$

$$\sum_n Z_0(\beta_n r)\Big\{\Big[\zeta_n(F_n - H_n) - \mu_n(E_n - G_n)\Big]\cos\omega t - \Big[\zeta_n(E_n - G_n) + \mu_n(F_n - H_n)\Big]\sin\omega t\Big\}$$

$$-\sum_\nu J_0(\gamma_\nu r)\Big[(\epsilon_\nu T_\nu - \eta_\nu P_\nu)\cos\omega t - (\epsilon_\nu P_\nu + \eta_\nu T_\nu)\sin\omega t\Big] = 0.$$

$$r \le x_0: \quad \sum_m J_0(\alpha_m r)\Big\{\Big[A_m\cos(\lambda_m d+\omega t) - B_m\sin(\lambda_m d+\omega t)\Big]e^{-\xi_m d} + \Big[C_m\cos(\lambda_m d-\omega t) + D_m\sin(\lambda_m d-\omega t)\Big]e^{\xi_m d}\Big\} = \frac{\pi x_0^2 Q}{\rho_1 \omega^2}$$

$$\sum_m J_0(\alpha_m r)\Big\{\Big[A_m\sin(\lambda_m d-\omega t) + B\cos(\lambda_m d-\omega t)\Big]e^{-\xi_m d} - \Big[C_m\sin(\lambda_m d+\omega t) - D_m\cos(\lambda_m d+\omega t)\Big]e^{\xi_m d}\Big\} = 0.$$

$$x_0 < r \le a: \quad \sum_m J_0(\alpha_m r)\Big\{\Big[A_m\cos(\lambda_m d+\omega t) - B_m\sin(\lambda_m d+\omega t)\Big]e^{-\xi_m d} + \Big[C_m\cos(\lambda_m d-\omega t) + D_m\sin(\lambda_m d-\omega t)\Big]e^{\xi_m d}\Big\} = 0.$$

$$\sum_m J_0(\alpha_m r)\Big\{\Big[A_m\sin(\lambda_m d-\omega t) + B_m\cos(\lambda_m d-\omega t)\Big]e^{-\xi_m d} - \Big[C_m\sin(\lambda_m d+\omega t) - D_m\cos(\lambda_m d+\omega t)\Big]e^{\xi_m d}\Big\} = 0.$$

$$a < r \le b: \quad \sum_n Z_0(\beta_n r)\Big\{\Big[E_n\cos(\mu_n d+\omega t) - F_n\sin(\mu_n d+\omega t)\Big]e^{-\zeta_n d} + \Big[G_n\cos(\mu_n d-\omega t) + H_n\sin(\mu_n d-\omega t)\Big]e^{\zeta_n d}\Big\} = 0.$$

$$\sum_n Z_0(\beta_n r)\Big\{\Big[E_n\sin(\mu_n d-\omega t) + F_n\cos(\mu_n d-\omega t)\Big]e^{-\zeta_n d} - \Big[G_n\sin(\mu_n d+\omega t) - H_n\cos(\mu_n d+\omega t)\Big]e^{\zeta_n d}\Big\} = 0.$$

CONCLUSIONS

The practical validity of this mathematical treatment of the stress-wave propagation in a three-region composite medium can be judged by the accuracy of the numbers it generates, at least with respect to comparisons with either experimental results or the results of purely numerical calculations. Such comparisons have not yet been made.

It may be observed that imposition of the arbitrary Sturm-Liouville interface boundary condition is not justified on physical grounds. While it does allow determination of a set of eigenvalues and the ultimate mathematical solution of the problem, the numerical consequences of this step remain to be studied.

Although use is never made of the fact (it is difficult to see its utility), it turns out that the eigenfunctions in region 1 also form an orthogonal set over region 1, a somewhat surprising result. Proof of this is given in Appendix A.

It might be possible to avoid the complex arguments of the Bessel functions which arose in one treatment of the problem by use of a more wisely chosen r-function in the separation of variables procedure. It may be observed also that the complex arguments would not arise if the absorption term, containing τ, in the differential equation were not present.

APPENDIX A

Orthogonality of the Eigenfunctions in Region 1

In order to prove that the eigenfunctions Φ_m are orthogonal over their interval of validity (0,a), it is only necessary to show that

$$\int_0^a \Phi_m \Phi_n r\, dr = 0.$$

Thus, we have

$$\int_0^a J_0(\alpha_m r)\; J_0(\alpha_n r)\; f_m(z)\; f_n(z)\; r\, dr = f_m(z)\; f_n(z) \int_0^a J_0(\alpha_m r)\; J_0(\alpha_n r)\; r\, dr$$

$$= f_m(z)\; f_n(z)\; \frac{a}{\alpha_m^2 - \alpha_n^2}\left[\alpha_m J_1(\alpha_m a)\; J_0(\alpha_n a) - \alpha_n J_1(\alpha_n a)\; J_0(\alpha_m a)\right]$$

(Jahnke and Emde, 1945).

By the use of eq. (39), which states that $\alpha_m J_1(\alpha_m a) = \frac{1}{K}\frac{\rho_1}{\rho_2} J_0(\alpha_m a)$,

we have that

$$\int_0^a \Phi_m \Phi_n\; r\, dr = f_m(z)\; f_n(z)\; \frac{a}{\alpha_m^2 - \alpha_n^2}\left[\frac{1}{K}\frac{\rho_1}{\rho_2} J_0(\alpha_m a)\; J_0(\alpha_n a) - \frac{1}{K}\frac{\rho_1}{\rho_2} J_0(\alpha_n a)\; J_0(\alpha_m a)\right]$$

$$= 0, \text{ Q.E.D.}$$

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