A CORRELATED RANDOM WALK MODEL FOR A QUEUE WITH MARKOVIAN ARRIVALS AND DEPARTURES

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ABSTRACT

Using a random walk model for single server queues with Bernoulli arrivals and geometric service times is an analysis technique well known in queueing theory. When the arrivals and departures are Markov dependent, the number of customers in the queue can be modeled as a correlated random walk (CRW). Previous investigations of CRW have used mostly transform techniques. In this paper, using a CRW model for the queue, representing it as a bivariate Markov chain, and exploiting the structural properties of its transition probability matrix, explicit results for the equilibruim solution and some first passage characteristics, including the mean busy period, have been obtained in the following cases: (i) with no restrictions on the waiting space and (ii) with limited waiting space.

Key words: Queues with correlated arrivals and departures, correlated random walk, steady state solution, busy period.

1. <u>Introduction</u>

Consider a single server queueing system in discrete time at epochs 0, t, 2t, 3t, ... Let the arrivals of customers at time epochs nt and (n+1)t (n = 0, 1, 2, . . .) follow a two state Markov chain with states A (arrival) and $\bar{A}$ (no arrival), and transition probability matrix

$$\begin{array}{c c} & \begin{array}{cc} A & \bar{A} \end{array} \\ \begin{array}{c} A \\ \bar{A} \end{array} & \left[\begin{array}{cc} 1\text{-}a & a \\ b & 1\text{-}b \end{array} \right] \end{array} \tag{1.1}$$

$$0 \le a, b \le 1.$$

When there are customers in the system, let their departures at time epochs nt and (n+1)t (n=0, 1, 2, ...) also follow a two state Markov chain with states D (departure) and $\bar{D}$ (no departure), and transition probability matrix

$$\begin{array}{c c} & \begin{array}{cc} D & \bar{D} \end{array} \\ \begin{array}{c} A \\ \bar{D} \end{array} & \left[\begin{array}{cc} 1\text{-}c & c \\ d & 1\text{-}d \end{array} \right] \end{array} \tag{1.2}$$

$$0 \le c, d \le 1.$$

With these assumptions, the correlated arrival process has the following properties:

(i) The probability distribution of an arrival at nt, as $n \to \infty$ is given by $\left(\frac{b}{a+b}, \frac{a}{a+b} \right)$.

(ii) Assuming that the arrival process starts with the limiting distribution in (i), its serial correlation (of lag j) is given by

$$\rho_j = (1 - a - b)^j \tag{1.3}$$

(iii) The interarrival time X_A between sucessive arrivals has the distribution

$$
\begin{aligned}
&P(X_A = 1) = 1\text{-}a \\
&P(X_A = k) = ab(1\text{-}b)^{k-2} \qquad\qquad k = 2, 3, \ldots
\end{aligned}
\tag{1.4}
$$

Similar properties hold for the departure process as well, when appropriate.

Queueing systems with the Markovian arrival process similar to the one described above (but with a = b) have been considered by Chaudhry (1966, 1967) and Sharda (1981). Both these researchers use transform techniques to derive equilibrium properties of various systems with the standard departure process. Other types of dependence incorporated into the arrival or the service process have been through making the interarrival times or service times sequentially dependent. In this context we may cite Runnenberg (1961), Cinlar (1967a, b), Pestalozzi (1968), Tin (1985) and Langaris (1986). Conolly (1968) and Conolly and Hadidi (1969) assume that the inter-arrival time and the service time of the same customer are correlated.

For the discrete time single server queueing system with Markovian dependence in both the arrival and service processes, we determine the limiting distribution and the mean busy period in the following sections. In section 2 we discribe the correlated random walk (CRW) model for the number of customers in the system. In section 3, the equilibrium solution and some first passage characteristics, including the mean busy period, are given for a queue with unlimited waiting space and in section 4 similar results are obtained for a queue with limited waiting space.

2. A CRW Model for the Queue

When the arrival and departure processes are Markov dependent with transition probabilities given by (1.1) and (1.2), the number of customers in the system, if greater than 0, changes by +1, 0 or -1 between time epochs nt and (n+1)t, also as a Markov chain. Corresponding to the four different combinations of arrival and departure states of (1.1) and (1.2), consider a process $\{U_n, n = 0, 1, 2, \ldots\}$ with the following states representing the change: $(A\bar{D}, AD, \bar{A}\bar{D}, \bar{A}D) = (+1, +0, -0, -1)$. Note that the state 0 of change results from two arrival-departure combinations AD and $\bar{A}\bar{D}$. The transition probability matrix of the Markov chain $\{U_n\}$ is given below, along with a notational change, made for convenience.

$$P_U = \begin{array}{c} \\ +1 \\ +0 \\ -0 \\ -1 \end{array} \begin{array}{c} \begin{array}{cccc} +1 & +0 & -0 & -1 \end{array} \\ \begin{bmatrix} (1-a)(1-d) & (1-a)d & a(1-d) & ad \\ (1-a)c & (1-a)(1-c) & ac & a(1-c) \\ b(1-d) & bd & (1-b)(1-d) & (1-b)d \\ bc & b(1-c) & (1-b)c & (1-b)(1-c) \end{bmatrix} \end{array} \tag{2.1}$$

$$= \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix} \quad \text{, say,} \tag{2.2}$$

where $a_4 = 1 - a_1 - a_2 - a_3$, $b_4 = 1 - b_1 - b_2 - b_3$, $c_4 = 1 - c_1 - c_2 - c_3$ and $d_4 = 1 - d_1 - d_2 - d_3$.

Let Q_n be the number of customers in the system at time epoch nt. For consistency of definition we assume that the system is observed just after the transition. Define the bivariate process $\{(Q_n, U_n), n = 0, 1, 2, \ldots\}$; clearly it is a Markov chain and may be identified as a correlated random walk with the transition probability matrix P given below. When $Q_n = 0$, the state (0, -1) should be interpreted as simply the combination of events $Q_n = 0$ and no arrival.

P =

$$
\begin{array}{c}
 \\
(0,-1) \\
 \\
(1,+1) \\
(1,+0) \\
(1,-0) \\
P = (1,-1) \\
 \\
(2,+1) \\
(2,+0) \\
(2,-0) \\
(2,-1) \\
 \\
\vdots
\end{array}
\begin{array}{c}
\begin{array}{c|cccc|cccc|c}
(0,-1) & (1,+1) & (1,+0) & (1,-0) & (1,-1) & (2,+1) & (2,+0) & (2,-0) & (2,-1) & \ldots
\end{array} \\
\left[\begin{array}{c|cccc|cccc|c}
1\text{-}b & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\hline
a_4 & 0 & a_2 & a_3 & 0 & a_1 & 0 & 0 & 0 & \\
b_4 & 0 & b_2 & b_3 & 0 & b_1 & 0 & 0 & 0 & \ldots \\
c_4 & 0 & c_2 & c_3 & 0 & c_1 & 0 & 0 & 0 & \\
d_4 & 0 & d_2 & d_3 & 0 & d_1 & 0 & 0 & 0 & \\
\hline
0 & 0 & 0 & 0 & a_4 & 0 & a_2 & a_3 & 0 & \\
0 & 0 & 0 & 0 & b_4 & 0 & b_2 & b_3 & 0 & \ldots \\
0 & 0 & 0 & 0 & c_4 & 0 & c_2 & c_3 & 0 & \\
0 & 0 & 0 & 0 & d_4 & 0 & d_2 & d_3 & 0 & \\
\hline
 & & & \vdots & & & & \vdots & &
\end{array}\right]
\end{array}
\qquad (2.3)
$$

$$
= \begin{bmatrix}
A & B & 0 & 0 & 0 & . & . & . \\
C & Z & Y & 0 & 0 & . & . & . \\
0 & X & Z & Y & 0 & . & . & . \\
0 & 0 & X & Z & Y & . & . & . \\
\vdots & \vdots & \vdots & \vdots & \vdots & & &
\end{bmatrix} \qquad (2.4)
$$

where the submatrices A, B, C, X, Y and Z have the corresponding definitions.

The transition probability matrix (2.4) is similar to the transition probability matrix of the CRW with stay (i.e., when the walk is allowed to stay in the same location in consecutive transitions) considered by Lal and Bhat (1988b), but with major differences in the structure of submatrices. Previous other work on CRW with stay is by Nain and Sen (1979) who consider an unrestricted CRW and study various first passage related characteristics using probability generating functions. For previous work on CRW without stay references are provided in Lal and Bhat (1988b). The common approach taken by all previous researchers is the use of difference equations and generating functions to obtain primarily first passage characteristics of the CRW. In this paper we extend the direct method used in Lal and Bhat (1988b) and obtain explicit solutions which are convenient for numerical computations, for equilibrium and busy period results for the process $\{Q_n\}$. The method exploits the structure of the submatrices A through Z and the transition probability matrix P.

3. The Queue with Unlimited Waiting Space

The CRW model for this queue is a Markov chain with the transition probability matrix given by (2.3). First we shall consider its equilibrium behavior. The condition for the existence of the equilibrium solution is given by the following theorem.

Theorem 3.1

The necessary and sufficient condition for the queue with unlimited waiting space to have an equilibrium solution is given by

$$\rho = \frac{a_1\eta + b_1g_1 + c_1g_2}{(1-d_1)\eta - b_1g_3 - c_1g_4} < 1 \tag{3.1}$$

where

$$\begin{aligned} \eta &= (1 - b_2)(1 - c_3) - b_3c_2 \\ g_1 &= a_3c_2 + a_2(1 - c_3) \\ g_2 &= a_2b_3 + a_3(1 - b_2) \\ g_3 &= c_2d_3 + (1 - c_3)d_2 \\ g_4 &= b_3d_2 + (1 - b_2)d_3 \end{aligned} \tag{3.2}$$

Proof: Following Neuts (1981, p. 32), the necessary and sufficient condition for the existence of the equilibrium solution may be stated as

$$\underline{\pi}_U Y \underline{e} = \underline{\pi}_U X \, \underline{e} \tag{3.3}$$

where $\underline{\pi}_U = (\pi_{+1}, \pi_{+0}, \pi_{-0}, \pi_{-1})$ is the limiting distribution of the Markov chain $\{U_n\}$, whose transition probability matrix is given by (2.2), and $\underline{e}$ is a unit column vector. From the first and the last equations of $\underline{\pi}_U P_U = \underline{\pi}_U$, we find

$$\underline{\pi}_U Y \underline{e} = \pi_{+1} \tag{3.4}$$

$$\underline{\pi}_U X \underline{e} = \pi_{-1}$$

Thus condition (3.3) reduces to

$$\pi_{+1} < \pi_{-1} \; . \tag{3.5}$$

Eliminating π_{+0} and π_{-0} from first equation of $\underline{\pi}_U P_U = \underline{\pi}_U$ with the help of the second and third equations, we get

$$\pi_{+1} = \frac{d_1 \eta + b_1 g_3 + c_1 g_4}{(1-a_1)\eta - b_1 g_1 - c_1 g_2} \; \pi_{-1} \tag{3.6}$$

The theorem now follows from (3.5) after re-arranging terms in the inequality. □

Let

$$\underline{x} = (x_0, \underline{x}_1, \underline{x}_2, \; . \; . \; . \; .) \tag{3.7}$$

where

$$x_0 = x_{0,-1}$$

$$\underline{x}_i = (x_{i,+1}, \; x_{i,+0} \, , \; x_{i,-0}, \; x_{i,-1})$$

$$i = 1, 2, \ldots$$

be the limiting distribution of the Markov chain $\{Q_n, U_n\}$. It is determined by solving the system of equations

$$x_0 A + \underline{x}_1 C = x_0 \tag{3.8}$$

$$x_0 B + \underline{x}_1 Z + \underline{x}_2 X = \underline{x}_1 \tag{3.9}$$

$$\underline{x}_{i-1} Y + \underline{x}_i Z + \underline{x}_{i+1} X = \underline{x}_i, \qquad i \geq 2 \tag{3.10}$$

$$x_0 + \sum_{i=1}^{\infty} \underline{x}_i \, \underline{e}_i = 1 \tag{3.11}$$

In (3.11) we use the unit vectors $\underline{e}_i$ with appropriate dimensions.

From Neuts (1981, p. 25) we also have

$$\underline{x}_i = \underline{x}_1 R^{i-1} \qquad i = 1, 2, \ldots \tag{3.12}$$

where R is the rate matrix of the Markov chain P. It is obtained in the following lemma.

<u>Lemma</u> <u>3.1</u>

$$R = \begin{bmatrix} a_1 & a_1\alpha & a_1\beta & a_1\rho \\ b_1 & b_1\alpha & b_1\beta & b_1\rho \\ c_1 & c_1\alpha & c_1\beta & c_1\rho \\ d_1 & d_1\alpha & d_1\beta & d_1\rho \end{bmatrix} \tag{3.13}$$

and

$$R^j = \rho^{j-1} R \qquad j = 1, 2, 3, \ldots . \tag{3.14}$$

where

$$\alpha = \frac{c_1(a_3d_2 - a_2d_3) + c_2(a_1d_3 + a_3(1-d_1)) + (1-c_3)(a_1d_2 + a_2\,(1-d_1))}{(1-d_1)[(1-b_2)(1-c_3) - b_3c_2 - d_2(c_1b_3 + b_1(1-c_3)) - d_3(b_1c_2 + c_1(1-b_2))]}$$

$$\beta = \frac{b_1(a_2d_3 - a_3d_2) + b_3(a_1d_2 + a_2(1-d_1)) + (1-b_2)(a_1d_3 + a_3(1-d_1))}{(1-d_1)[(1-b_2)(1-c_3) - b_3c_2 - d_2(c_1b_3 + b_2(1-c_3)) - d_3(b_1c_2 + c_1(1-b_2))]} \tag{3.15}$$

and ρ is given by (3.1)

Proof: The rate matrix R satisfies the matrix equation

$$R = Y + RZ + R^2X$$

which can be written as

$$R = Y(I - Z)^{-1} + R^2X\,(I - Z)^{-1} \tag{3.16}$$

In order to solve this equation iteratively, let R_n be the nth iterate of R such that

$$R_n = Y\,(I - Z)^{-1} + R_{n-1}^2\, X(I - Z)^{-1} \tag{3.17}$$

We have

$$(I - Z)^{-1} = \begin{bmatrix} 1 & g_1/\eta & g_2/\eta & 0 \\ 0 & (1-c_3)/\eta & b_3/\eta & 0 \\ 0 & c_2/\eta & (1-b_2)/\eta & 0 \\ 0 & g_3/\eta & g_4/\eta & 1 \end{bmatrix}$$

Substituting this back in (3.17) we get

$$R_n = \begin{bmatrix} a_1 & a_1g_1/\eta & a_1g_2/\eta & 0 \\ b_1 & b_1g_1/\eta & b_1g_2/\eta & 0 \\ c_1 & c_1g_1/\eta & c_1g_2/\eta & 0 \\ d_1 & d_1g_1/\eta & d_2g_2/\eta & 0 \end{bmatrix}$$

$$+ R_{n-1}^2 \begin{bmatrix} 0 & a_4g_3/\eta & a_4g_4/\eta & a_4 \\ 0 & b_4g_3/\eta & b_4g_4/\eta & b_4 \\ 0 & c_4g_3/\eta & c_4g_4/\eta & c_4 \\ 0 & d_4g_3/\eta & d_4g_4/\eta & d_4 \end{bmatrix} \tag{3.18}$$

Substituting iteratively, with $R_0 = 0$ and by induction, for $n \geq 1$, we get

$$R_n = \begin{bmatrix} a_1 & a_1(g_1+g_3K_n)/\eta & a_1(g_2+g_4K_n)/\eta & a_1K_n \\ b_1 & b_1(g_1+g_3K_n)/\eta & b_1(g_2+g_4K_n)/\eta & b_1K_n \\ c_1 & c_1(g_1+g_3K_n)/\eta & c_1(g_2+g_4K_n)/\eta & c_1K_n \\ d_1 & d_1(g_1+g_3K_n)/\eta & d_1(g_2+g_4K_n)/\eta & d_1K_n \end{bmatrix} \tag{3.19}$$

where K_n satisfies the recurrence relation

$$K_n = [a_1 + (b_1g_1 + c_1g_2)/\eta + (d_1 + (b_1g_3 + c_1g_4)/\eta)\, K_{n-1}]$$
$$\bullet\, [1 - a_1 - (b_1g_1 + c_1g_2)/\eta + (1 - d_1 - (b_1g_3 + c_1g_4)/\eta)\, K_{n-1}] \tag{3.20}$$
$$K_1 = 0\ .$$

As $n \to \infty$ $R^n \to R$; hence $K_n \to K$. Thus K can be determined by solving the quadratic equation in K

$$\begin{aligned} &(d_1 + (b_1g_3 + c_1g_4)/\eta)\,(1 - d_1 - (b_1g_3 + c_1g_4)/\eta)\, K^2 \\ &+ [(a_1 + (b_1g_1 + c_1g_2)/\eta)\,(1 - d_1 - (b_1g_3 + c_1g_4)/\eta) \\ &+ \ (1 - a_1 - (b_1g_1 + c_1g_2)/\eta)\,(d_1 + (b_1g_3 + c_1g_4)/\eta) - 1]\, K \\ &+ (a_1 + (b_1g_1 + c_1g_2)/\eta)\,(1 - a_1 - (b_1g_1 + c_1g_2)/\eta) = 0 \end{aligned} \tag{3.21}$$

After simplification, the discriminant of this equation is obtained as

$$[a_1 + (b_1g_1 + c_1g_2)/\eta + \ d_1 + (b_1g_3 + c_1g_4)/\eta - 1]^2 \tag{3.22}$$

The two possible roots of the equation are

$$K = \frac{a_1 + (b_1g_1 + c_1g_2)/\eta}{1 - d_1 - (b_1g_3 + c_1g_4)/\eta} \quad ; \quad \frac{1 - a_1 - (b_1g_1 + c_1g_2)/\eta}{d_1 + (b_1g_3 + c_1g_4)/\eta} \tag{3.23}$$

From (3.12) note that $R^j \to 0$ as $j \to \infty$. However, the second root in (3.23) results in $R^j = R$ for $j \geq 1$; therefore it is inadmissible. The lemma now follows using the first solution for K in (3.19) and simplifying. In deriving (3.14) we use the identity

$$a_1 + b_1\alpha + c_1\beta + d_1\rho = \rho \tag{3.24}$$

in simplifications. □

Note that the left hand side of the identity (3.24) is the sum of the diagonal elements of R. Another identity related to the elements of R, which will be used later is given as

$$(1 - a_1 - a_2 - a_3) + (1 - b_1 - b_2 - b_3)\alpha + (1 - c_1 - c_2 - c_3)\beta + (1 - d_1 - d_2 - d_3)\rho = 1. \tag{3.25}$$

The limiting distributions of the CRW and the correlated queue are given in the following theorem.

Theorem 3.2

(a) When the equilibrium condition (3.1) holds the limiting distribution of the CRW is given by

$$x_0 = x_{0,-1}$$

$$= \frac{1 - \rho}{1 - \rho + (1+\alpha+\beta+\rho)b} \quad , \tag{3.26}$$

$$\underline{x}_j = (x_{j,+1} \, , x_{j,+0}, x_{j,-0}, x_{j,-1})$$

$$= \frac{b(1-\rho)\rho^{j-1}}{1-\rho + (1+\alpha+\beta+\rho)\, b} \; [1, \ \alpha, \ \beta, \ \rho\,] \, , \quad j \geq 1 \tag{3.27}$$

(b) Let $\underline{\pi} = (\pi_0, \pi_1, \pi_2, \ldots)$ be the limiting distribution of the number of customers in the queueing system. When the equilibrium condition (3.1) holds, the limiting distribution is given by

$$\pi_0 = \frac{1 - \rho}{1 - \rho + (1+\alpha+\beta+\rho)\, b}$$

$$\pi_j = \frac{b\,(1+\alpha+\beta+\rho)\,(1-\rho)\,\rho^{j-1}}{1 - \rho + (1+\alpha+\beta+\rho)\, b} \quad , \quad j \geq 1\,. \tag{3.28}$$

Proof: From (3.9) and (3.12) we get

$$\underline{x}_1 = x_0 B\,(I - Z - RX)^{-1} \tag{3.29}$$

and

$$\underline{x}_j = x_0\; B\,(I - Z - RX)^{-1}\, R^{j-1} \quad , \quad j \geq 1$$

$$= x_0 \rho^{j-2} \; B(I - Z - RX)^{-1} R \;, \quad j \geq 1 \tag{3.30}$$

where x_0 is obtained form the normalizing condition

$$x_0 \, [1 + B \, (I - Z - RX)^{-1} \, (I - R)^{-1} \, \underline{e} \,] = 1 \tag{3.31}$$

The following results are derived by direct substitution and simplification using the identity (3.25).

$$RX = \begin{bmatrix} 0 & 0 & 0 & a_1 \\ 0 & 0 & 0 & b_1 \\ 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & d_1 \end{bmatrix} \tag{3.32}$$

$(I - Z - RX)^{-1} =$

$$\begin{bmatrix} 1 & \alpha & \beta & \rho \\ 0 & ((1\text{-}c_3)(1\text{-}d_1) - c_1 d_3)\gamma & (b_1 d_3 + b_3(1\text{-}d_1))\gamma & (b_3 c_1 + b_1(1\text{-}c_3))\gamma \\ 0 & (c_1 d_2 + c_2(1\text{-}d_1))\gamma & ((1\text{-}b_2)(1\text{-}d_1) - b_1 d_2)\gamma & (b_1 c_2 + (1\text{-}b_2)c_1)\gamma \\ 0 & (c_2 d_3 + (1\text{-}c_3)\, d_2)\gamma & (b_3 d_2 + (1\text{-}b_2)\, d_3)\gamma & ((1\text{-}b_2)(1\text{-}c_3) - c_2 b_3)\gamma \end{bmatrix} \tag{3.33}$$

where

$\gamma = [(1 - d_1)((1 - b_2)(1 - c_3) - b_3 c_2) - d_2(b_1(1 - c_3) + b_3 c_1) - d_3 \, ((1 - b_2) \, c_1 + b_1 c_2)]^{-1}$.

Therefore

$$B(I - Z - RX)^{-1} = (b, \, b\alpha, \, b\beta, \, b\rho) \; . \tag{3.34}$$

We also get

$$B(I - Z - RX)^{-1} \, (I - R)^{-1}$$

$$\Longrightarrow B\,(I - Z - RX)^{-1}\,(I + \sum_{j=1}^{\infty} R^j)$$

$$= B\,(I - Z - RX)^{-1}\,(I + \frac{R}{1-\rho}) \tag{3.35}$$

But using identity (3.24) it can be easily shown that

$$B\,(I - Z - RX)^{-1}\,R = \rho\,B\,(I - Z - RX)^{-1} \tag{3.36}$$

Then, (3.35) simplifies to

$$B\,(I - Z - RX)^{-1}\,(I - R)^{-1} = \frac{b}{1-\rho}\,(\,1,\,\alpha,\,\beta,\,\rho) \tag{3.37}$$

giving

$$B\,(I - Z - RX)^{-1}\,(I - R)^{-1}\underline{e} = \frac{b(1+\alpha+\beta+\rho)}{1-\rho} \tag{3.38}$$

Substituting in (3.31), we get

$$x_0 = [\,1 + \frac{(1+\alpha+\beta+\rho)\,b}{1-\rho}\,]^{-1}$$

which gives (3.26). The result (3.27) follows directly from (3.26), (3.30) and (3.36).

The limiting distribution of the number of customers in the system is obtained by summing over the elements of $\underline{x}_j$, $j = 1, 2, 3, \ldots$. □.

The limiting distribution of the queue length process derived in Theorem 3.2 can be used to determine the mean busy period E(B) of the system in a simple manner as follows.

Theorem 3.3

$$E(B) = \frac{1+\alpha+\beta+\rho}{1-\rho} \tag{3.39}$$

Proof: Mean busy period E(B) is the expected amount of time the queue length process takes for first passage from state 1 to state 0. In terms of E(B), the mean recurrence time π_0^{-1} can be expressed as

$$\frac{1}{\pi_0} = (1 - b) + b\,[E(B) + 1]$$
$$= 1 + b\,E(B)\,. \tag{3.40}$$

From (3.28), we then get

$$1 + b\,E(B) = \frac{1 - \rho + (1+\alpha+\beta+\rho)\,b}{1 - \rho}$$

which proves the theorem. □

Incidentally we may note that the method used above in the determination of E(B) works only if the busy period starts every time with the arrival of a single customer, as in this system. Otherwise a method of using the fundamental matrix (Kemeny and Snell, 1960) can be employed. By this method, one can also determine the expected number of visits to any state during a busy period. Consider the following partition of the transition probability matrix P.

$$P = \begin{array}{c c} & \begin{array}{cc} y_0 & y_1 \end{array} \\ \begin{array}{c} y_0 \\ y_1 \end{array} & \left[\begin{array}{c|c} 1-b & B^* \\ \hline C^* & D^* \end{array}\right] \end{array} \tag{3.41}$$

where y_0 is state (0, -1) and y_1 is the set of all remaining states. Let x_0^* and $\underline{x}_1^*$ be the limiting probabilities corresponding to the sets y_0 and y_1 respectively. Using the reduced system arguments (see Lal and Bhat, 1987, 1988a) we have the relation

$$\underline{x}_1^* = x_0^*\,B^*\,(I - D)^{-1} \tag{3.42}$$

From (3.29) and (3.30), noting the correspondence of x_0^* with x_0 and $\underline{x}_1^*$ with $(\underline{x}_1, \underline{x}_2, \ldots)$, we have

$$\underline{x}_1^* = x_0^*\;B\,(I - Z - RX)^{-1}\,(I, R, R^2, \ldots) \tag{3.43}$$

Comparing (3.42) and (3.43) we get

$$B^*(I - D^*)^{-1} = B\,(I - Z - RX)^{-1}\,(I, R, R^2, \ldots) \tag{3.44}$$

It is well known that in (3.41), the expected number of visits to a state in y_1 before first passage to state y_0, having initially started from state 1 in y_1, is given by the elements of the first row of

$(I - D^*)^{-1}$. Since $B^* = (b, 0, 0, \ldots)$, $B^* \ (I - D^*)^{-1}$ gives the elements of the first row multiplied by b. From the right hand side of (3.44), we have

$$B \ (I - Z - RX)^{-1} \ R^{j-1} = b\rho^{j-1} \ (1, \alpha, \beta, \rho)$$
$$j = 1, 2, 3, \ldots \tag{3.45}$$

Consequently, expected number of visits to state $\underline{j}$ is given by $\rho^{j-1} \ (1, \alpha, \beta, \rho)$.

4. <u>The Queue with Limited Waiting Space</u>

Consider the queue discribed earlier, but now with a limited waiting space for only N customers in the system. The corresponding CRW has the transition probability matrix

$$\begin{array}{cc}
 & \begin{array}{ccccccc} 0 & \underline{1} & \underline{2} & \underline{3} & \cdots & \underline{N-1} & \underline{N} \end{array} \\
\begin{array}{c} 0 \\ \underline{1} \\ \underline{2} \\ \underline{3} \\ \vdots \\ \underline{N-1} \\ \underline{N} \end{array} &
\left[\begin{array}{ccccccc} A & B & 0 & 0 & \cdots & 0 & 0 \\ C & Z & Y & 0 & \cdots & 0 & 0 \\ 0 & X & Z & Y & \cdots & 0 & 0 \\ 0 & 0 & X & Z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & Z & F \\ 0 & 0 & 0 & 0 & & E & D \end{array}\right]
\end{array}$$

where $0 = (0, -1)$

$\underline{i} = \{(i, +1), (i, +0), (i, -0), (i, -1)\} \quad i = 1, 2, \ldots, N\text{-}1$

$\underline{N} = \{(N, +1), (N, +0), (N, -0)\}$

and

$A = 1 - b \, , \quad B = [\, b, 0, 0, 0]$

$$C = \begin{bmatrix} 1 - a_1 - a_2 - a_3 \\ 1 - b_1 - b_2 - b_3 \\ 1 - c_1 - c_2 - c_3 \\ 1 - d_1 - d_2 - d_3 \end{bmatrix} \qquad D = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 & 0 & 1-a_1-a_2-a_3 \\ 0 & 0 & 0 & 1-b_1-b_2-b_3 \\ 0 & 0 & 0 & 1-c_1-c_2-c_3 \end{bmatrix} \quad F = \begin{bmatrix} a_1 & 0 & 0 \\ b_1 & 0 & 0 \\ c_1 & 0 & 0 \\ d_1 & 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & 0 & 0 & 1-a_1-a_2-a_3 \\ 0 & 0 & 0 & 1-b_1-b_2-b_3 \\ 0 & 0 & 0 & 1-c_1-c_2-c_3 \\ 0 & 0 & 0 & 1-d_1-d_2-d_3 \end{bmatrix}$$

$$Y = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 \\ c_1 & 0 & 0 & 0 \\ d_1 & 0 & 0 & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 0 & a_1 & a_2 & 0 \\ 0 & b_1 & b_2 & 0 \\ 0 & c_1 & c_2 & 0 \\ 0 & d_1 & d_2 & 0 \end{bmatrix} \tag{4.2}$$

The limiting distributions $\underline{x} = (x_0, \underline{x}_1, \underline{x}_2, \ldots, \underline{x}_N)$ of the CRW and $\underline{\pi} = (\pi_0, \pi_1, \pi_2, \ldots, \pi_N)$ of the queueing system can be determined using the backward recursion (see, Lal and Bhat, 1987, 1988a) as in the following theorem. Note that

$$x_0 = x_{0,-1}$$

$$\underline{x}_i = (x_{i,+1},\ x_{i,+0},\ x_{i,-0},\ x_{i,-1}) \qquad i = 1, 2, \ldots, N\text{-}1$$

$$\underline{x}_N = (x_{N,+1},\ x_{N,+0},\ x_{N,-0}) \ .$$

<u>Theorem</u> <u>4.1</u>

(a)

$$x_{0,-1} = \left[\ 1 + b\left[\left(\frac{\eta+g_1+g_2}{(1-a_1)\eta - b_1g_1 - c_1g_2}\right)\rho^{N-1} + (1+\alpha+\beta+\rho)\,\frac{1-\rho^{N-1}}{1-\rho}\right]\right]^{-1} \tag{4.3}$$

$$\left.\begin{aligned} x_{i,+1} &= b\ \rho^{i-1}\ x_{0,-1} \\ x_{i,+0} &= b\alpha\ \rho^{i-1}\ x_{0,-1} \\ x_{i,-0} &= b\beta\ \rho^{i-1}\ x_{0,-1} \\ x_{i,-1} &= b\ \rho^{i}\ x_{0,-1} \end{aligned}\right\}\quad i = 1, 2, \ldots, N\text{-}1 \tag{4.4}$$

$$\begin{aligned} x_{N,+1} &= \frac{b\eta\ \rho^{N-1}}{(1-a_1)\ \eta - b_1g_1 - c_1g_2}\ x_{0,-i} \\ x_{N,+0} &= \frac{bg_1\ \rho^{N-1}}{(1-a_1)\ \eta - b_1g_1 - c_1g_2}\ x_{0,-1} \\ x_{N,-0} &= \frac{bg_2\ \rho^{N-1}}{(1-a_1)\ \eta - b_1g_1 - c_1g_2}\ x_{0,-1} \end{aligned} \tag{4.5}$$

where ρ, η, g_1, g_2, α and β are defined as (3.1), (3.2), and (3.15).

(b)

$$\pi_0 = \left[\ 1 + b\left[\left(\frac{\eta + g_1 + g_2}{(1-a_1)\ \eta - b_1g_1 - c_1g_2}\right)\rho^{N-1} + (1+\alpha+\beta+\rho)\,\frac{1-\rho^{N-1}}{1-\rho}\right]\right]^{-1}$$

$$\pi_i = b\ (1+\alpha+\beta+\rho)\ \rho^{i-1}\ \pi_0\ , \qquad i = 1, 2, \ldots N\text{-}1$$

$$\pi_N = \frac{b\,(\eta + g_1 + g_2)\,\rho^{N-1}}{(1 - a_1)\,\eta - b_1 g_1 - c_1 g_2}\,\pi_0 \quad . \tag{4.6}$$

Proof: consider

$$\underline{x}\,P = \underline{x} \tag{4.7}$$

$$\underline{x}\,\underline{e} = 1 \tag{4.8}$$

To use the backward recursion, we start with the last 4 equations in (4.7).

$$x_{N,-0} = c_3 x_{N,-0} + b_3 x_{N,+0} + a_3 x_{N,+1} \tag{4.9}$$

$$x_{N,+0} = c_2 x_{N,-0} + b_2 x_{N,+0} + a_2 x_{N,+1} \tag{4.10}$$

$$x_{N,+1} = c_1 x_{N,-0} + b_1 x_{N,+0} + a_1 x_{N,+1} + d_1 x_{N-1,-1} + c_1 x_{N-1,-0} + b_1 x_{N-1,+0} + a_1 x_{N-1,+1} \tag{4.11}$$

$$x_{N-1,-1} = (1-c_1-c_2-c_3)\,x_{N,-0} + (1-b_1-b_2-b_3)\,x_{N,+0} + (1-a_1-a_2-a_3)\,x_{N,+1} \tag{4.12}$$

From (4.9),

$$x_{N,-0} = \frac{b_3}{1-c_3}\,x_{N,+0} + \frac{a_3}{1-c_3}\,x_{N,+1}$$

From (4.10)

$$x_{N,+0} = \frac{a_2\,(1 - c_3) + a_3 c_2}{(1-b_2)\,(1-c_3) - b_3 c_2}\,x_{N,+1} = \frac{g_1}{\eta}\,x_{N,+1}$$

$$x_{N,-0} = \frac{a_3\,(1 - b_2) + a_2 b_3}{(1-b_2)\,(1-c_3) - b_3 c_2}\,x_{N,+1} = \frac{g_2}{\eta}\,x_{N,+1} \tag{4.13}$$

From (4.12) and (4.13), we get after simplification

$$x_{N,+1} = \frac{\eta}{(1\text{-}a_1)\ \eta - b_1 g_1 - c_1 g_2}\ x_{N-1,-1}$$

$$x_{N,+0} = \frac{g_1}{(1\text{-}a_1)\ \eta - b_1 g_1 - c_1 g_2}\ x_{N-1,-1} \tag{4.14}$$

$$x_{N,-0} = \frac{g_2}{(1\text{-}a_1)\ \eta - b_1 g_1 - c_1 g_2}\ x_{N-1,-1}$$

From (4.11) and (4.14) we get

$$x_{N-1,-1} = \frac{a_1}{1\text{-}d_1}\, x_{N-1,+1} + \frac{b_1}{1\text{-}d_1}\, x_{N-1,+0} + \frac{c_1}{1\text{-}d_1}\, x_{N-1,-0} \tag{4.15}$$

Now, 4 equations next to the least 4 are

$$\begin{aligned}
x_{N-1,-0} &= d_3\, x_{N-1,-1} + c_3 x_{N-1,-0} + b_3\, x_{N-1,+0} + a_3\, x_{N-1,+1} \\
x_{N-1,+0} &= d_2\, x_{N-1,-1} + c_2 x_{N-1,-0} + b_2\, x_{N-1,+0} + a_2\, x_{N-1,+1} \\
x_{N-1,+1} &= d_1\, x_{N-2,-1} + c_1 x_{N-2,-0} + b_1\, x_{N-2,+0} + a_1\, x_{N-2,+1} \\
x_{N-2,-1} &= (1 - d_1 - d_2 - d_3)\, x_{N-1,-1} + (1 - c_1 - c_2 - c_3)\, x_{N-1,-0} \\
&\quad + (1 - b_1 - b_2 - b_3)\, x_{N-1,+0} + (1 - a_1 - a_2 - a_3)\, x_{N-1,+1}
\end{aligned} \tag{4.16}$$

From (4.15) and (4.16), after simplifications we get

$$\begin{aligned}
x_{N-1,+0} &= \alpha\, x_{N-1,+1} \\
x_{N-1,-0} &= \beta\, x_{N-1,+1} \\
x_{N-1,-1} &= \rho\, x_{N-1,+1}
\end{aligned} \tag{4.17}$$

From the last equation of (4.16), (4.17) and identity (3.25) we get

$$x_{N-2,-1} = x_{N-1,+1} \tag{4.18}$$

This leads to the set of relations

$$x_{N,-0} = \frac{g_2 \rho}{(1\text{-}a_1)\ \eta - b_1 g_1 - c_1 g_2}\ x_{N-2,-1}$$

$$x_{N,+0} = \frac{g_1\rho}{(1-a_1)\,\eta - b_1g_1 - c_1g_2}\, x_{N-2,-1}$$

$$x_{N,+1} = \frac{\eta\,\rho}{(1-a_1)\,\eta - b_1g_1 - c_1g_2}\, x_{N-2,-1} \tag{4.19}$$

$$x_{N-1,-1} = \rho\, x_{N-2,-1}$$

$$x_{N-1,-0} = \beta\, x_{N-2,-1}$$

$$x_{N-1,+0} = \alpha\, x_{N-2,-1}$$

On repeatedly using the above recursive procedure and noting that $x_{1,+1} = b\, x_{0,-1}$, we get

$$x_{N,-0} = \frac{b\, g_2\rho^{N-1}}{(1-a_1)\,\eta - b_1g_1 - c_1g_2}\; x_{0,-1}$$

$$x_{N,+0} = \frac{b\, g_1\rho^{N-1}}{(1-a_1)\,\eta - b_1g_1 - c_1g_2}\; x_{0,-1}$$

$$x_{N,+1} = \frac{b\,\eta\,\rho^{N-1}}{(1-a_1)\,\eta - b_1g_1 - c_1g_2}\; x_{0,-1}$$

$$\tag{4.20}$$

$$x_{i,-1} = b\,\rho^{i}\, x_{0,-1}$$

$$x_{i,-0} = b\beta\,\rho^{i-1}\, x_{0,-1}$$

$$x_{i,+0} = b\alpha\,\rho^{i-1}\, x_{0,-1}$$

$$x_{i,+1} = b\,\rho^{i-1}\, x_{0,-1}$$

$$i = 1, 2, \ldots, N-1 .$$

Part (a) of the theorem now follows when we use the normalizing condition. Part (b) is obtained by noting that $\pi_0 = x_{0,-1}$, $\pi_i = x_{i,+1} + x_{i,+0} + x_{i,-0} + x_{i,-1}$ $(i = 1, 2, \ldots, N-1)$ and $\pi_N = x_{N,+1} + x_{N,+0} + x_{N,-0}$. □

The mean busy period E(B) of the queueing system can be obtained directly from the mean recurrence time π_0^{-1}, as in the case of the unlimited waiting space system. We have

Theorem 4.2

$$E(B) = \frac{\eta + g_1 + g_2}{(1 - a_1)\,\eta - b_1 g_1 - c_1 g_2}\,\rho^{N-1} + (1+\alpha+\beta+\rho)\,\frac{1-\rho^{N-1}}{1-\rho}\ . \tag{4.21}$$

Proof: Arguing as in Theorem 3.3, we get

$$\frac{1}{\pi_0} = 1 + b\,E(B)\ .$$

The theorem follows, if we substitute the value of π_0 from Theorem 4.1. □

To determine the mean and variance of first passage times in Markov chains (busy period is one such characteristic) fundamental matrix (Kemeny and Snell, 1960) is a convenient tool. For instance, in the system discussed in this section, suppose, we wish to obtain the first passage characteristics of the queue length process from (to) the set of states $S_r = (r+1, r+2, \ldots, N)$ to (from) the set of states $S_r^c = (0, 1, 2, \ldots, r)$. Mean and variance of such first passage can be obtained in terms of elements of the fundamental matrix $(I - H_2)^{-1}$ $[(I - H_1)^{-1}]$ where H_1 and H_2 are defined in the partitioned matrix P as follows.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ r \\ r+1 \\ \vdots \\ N-1 \\ N \end{array}\!\!\begin{array}{c} \begin{array}{ccccc|cccc} 0 & 1 & 2 & \cdots & r & r+1 & \cdots & N-1 & N \end{array} \\ \left[\begin{array}{ccccc|cccc} A & B & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ C & Z & Y & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & X & Z & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Z & Y & \cdots & 0 & 0 \\ \hline 0 & 0 & 0 & \cdots & X & Z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & Z & F \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & E & D \end{array}\right] \end{array}$$

$$= \left[\begin{array}{c|c} H_1 & T \\ \hline R & H_2 \end{array}\right] \tag{4.22}$$

Except for the row and column on outer edges, H_1 and H_2 have identical component matrix H given by

$$H = \begin{bmatrix} Z & Y & 0 & 0 & \cdots & 0 & 0 \\ X & Z & Y & 0 & \cdots & 0 & 0 \\ 0 & X & Z & Y & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & Z & Y \\ 0 & 0 & 0 & 0 & \cdots & X & Z \end{bmatrix} \tag{4.23}$$

Given below is the outline of the procedure for the determination of $(I - H)^{-1}$ in explicit terms. For convenience we use N as the dimension of H.

<u>Lemma 4.1</u>

Let $M = (I - H)^{-1} = [M_{ij}]$. Then

$$M_{ij} = \sum_{l=0}^{\min(i-1,\, j-1)} \left(\prod_{q=l+1}^{i-1} X_{N-q} \right) R_{N-l} \left(\prod_{s=l+1}^{j-1} Y_{N-s} \right) \tag{4.24}$$

where R_l satisfies the relation

$$\begin{aligned} R_l &= (I - Z - YR_{l-1}X)^{-1} \qquad l = 2, 3, \ldots, N \\ R_1 &= (I - Z)^{-1}, \quad R_0 = 0 \end{aligned} \tag{4.25}$$

and

$$Y_l = YR_l \text{ and } X_l = R_lX, \quad l = 1, 2, \ldots, N \tag{4.26}$$

In (4.24) $\prod_{q=l+1}^{i-1} X_{N-q}$ is taken in pre-order as q increases, while $\prod_{s=l+1}^{j-1} Y_{N-s}$ is taken in post-order as s increases.

Proof: The result follows from the recursive algorithm following Theorem 2.2 of Lal and Bhat (1988a). □

In what follows, in addition to the notations introduced in (3.2), we shall use the following notations.

$$\left.\begin{aligned}
g_5 &= b_3c_1 + b_1(1-c_3) \\
g_6 &= b_1c_2 + (1-b_2)c_1 \\
g_7 &= a_3c_1 + a_1(1-c_3) \\
g_8 &= a_2b_1 + a_1(1-b_2) \\
g_9 &= c_1d_3 + (1-c_3)d_1 \\
g_{10} &= b_1d_2 + (1-b_2)d_1
\end{aligned}\right\} \tag{4.27}$$

$$\left.\begin{aligned}
\phi_{1l} &= (1-d_1K_l)\,\eta - d_2g_5K_l - d_3g_6K_l \\
\phi_{2l} &= (1-d_1K_l)\,g_1 + d_2g_7K_7 + d_3(a_1c_2 - a_2c_1)K_l \\
\phi_{3l} &= (1-d_1K_l)\,g_2 + d_2(a_1b_3 - b_1a_3)K_l + d_3g_8K_l \\
\phi_{4l} &= a_1\eta K_l + a_3g_6K_l + a_2g_5K_l \\
\phi &= a_1\eta + a_2g_5 + a_3g_6 \\
\psi &= (1-d_1)\eta - d_2g_5 - d_3g_6
\end{aligned}\right\} \tag{4.28}$$

where K_l is defined by the recursion

$$\begin{aligned}
K_0 &= 0 \\
K_l &= 1 + \frac{\phi(K_{l-1}-1)}{\phi_{1,l-1}}
\end{aligned} \tag{4.29}$$

Theorem 4.3

Let $M_{ij} = [M_{ij,kl}]$, $k, l = 1, 2, 3, 4$. (4.30)

We have

$$M_{ij,11} = S_{ij}^0 + \phi S_{ij}^6 + (\psi - \eta)\, K_{N-i+1} S_{ij}^8$$

$$M_{ij,12} = S_{ij}^3 + \left[\phi S_{ij}^6 + (\psi - \eta)\, K_{N-i+1} S_{ij}^8\right] \frac{\phi_{2,N-j}}{\phi_{1,N-j}}$$

$$M_{ij,13} = S_{ij}^4 + \left[\phi S_{ij}^6 + (\psi - \eta)\, K_{N-i+1} S_{ij}^8\right] \frac{\phi_{3,N-j}}{\phi_{1,N-j}}$$

$$M_{ij,14} = S_{ij}^5 + \left[\phi S_{ij}^6 + (\psi - \eta)\, K_{N-i+1} S_{ij}^8\right] \frac{\phi K_{N-j}}{\phi_{1,N-j}}$$

$$M_{ij,21} = g_5 S_{ij}^6 + (\psi - \eta)\, (\phi_{1,N-j} + g_5\, (K_{N-i}-1)\, \frac{S_{ij}^8}{\phi_{1,N-j}}$$

$$M_{ij,22} = (1-c_3)\, S_{ij}^1 - g_9 S_{ij}^2 + \frac{\phi_{2,N-j}}{\phi_{1,N-j}}\, M_{ij,21} + g_3\, (\phi_{1,N-j} + g_5(K_{N-i} - 1))\, \frac{S_{ij}^7}{\phi_{1,N-j}}$$

$$M_{ij,23} = b_3 S_{ij}^1 + (b_1 d_3 - b_3 d_1)\, S_{ij}^2 + \frac{\phi_{3,N-j}}{\phi_{1,N-j}} + g_4(\phi_{1,N-j} + g_5(K_{N-i} - 1))\, \frac{S_{ij}^7}{\phi_{1,N-j}}$$

$$M_{ij,24} = g_5 S_{ij}^2 + \frac{\phi K_{N-j}}{\phi_{1,N-j}} + \eta\, (\phi_{1,N-j} + g_5(K_{N-i} - 1))\, \frac{S_{ij}^7}{\phi_{1,N-j}}$$

$$M_{ij,31} = g_6 S_{ij}^6 + (\psi - \eta)\, (\phi_{1,N-j} + g_6\, (K_{N-i} -1)\, \frac{S_{ij}^8}{\phi_{1,N-j}}$$

$$M_{ij,32} = c_2 S_{ij}^1 + (c_1 d_2 - c_2 d_1)\, S_{ij}^3 + \frac{\phi_{2,N-j}}{\phi_{1,N-j}}\, M_{ij,31} + (\phi_{1,N-j} + g_6(K_{N-i} - 1)\, \frac{g_3 S_{ij}^7}{\phi_{1,N-j}}$$

$$M_{ij,33} = (1-b_2)\, S_{ij}^1 - g_{10} S_{ij}^3 + \frac{\phi_{3,N-j}}{\phi_{1,N-j}}\, M_{ij,31} + (\phi_{1,N-j} + g_6(K_{N-i} - 1))\, \frac{g_4 S_{ij}^7}{\phi_{1,N-j}}$$

$$M_{ij,34} = g_6 S_{ij}^2 + \frac{\phi K_{N-j}}{\phi_{1,N-j}}\, M_{ij,31} + (\phi_{1,N-j} + g_6(K_{N-i} - 1))\, \frac{\eta\, S_{ij}^7}{\phi_{1,N-j}}$$

$$M_{ij,41} = (\psi - \eta)\left[S_{ij}^{6} + \frac{\psi\, S_{ij}^{8}}{\phi_{1,N-j}}\right]$$

$$M_{ij,42} = g_3\left[S_{ij}^{1} + \frac{\psi\, S_{ij}^{7}}{\phi_{1,N-j}}\right] + \frac{\phi_{2,N-j}}{\phi_{1,N-j}}\, M_{ij,41}$$

$$M_{ij,43} = g_4\left[S_{ij}^{1} + \frac{\psi\, S_{ij}^{7}}{\phi_{1,N-j}}\right] + \frac{\phi_{3,N-j}}{\phi_{1,N-j}}\, M_{ij,41}$$

$$M_{ij,44} = \eta\left[S_{ij}^{1} + \frac{\psi\, S_{ij}^{7}}{\phi_{1,N-j}}\right] + \frac{\phi K_{N-j}}{\phi_{1,N-j}} \tag{4.30}$$

where

$$S_{ij}^{0} = \sum_{l=0}^{IJ} \delta_{i,l+2}^{**}\, \delta_{j,l+2}^{**}$$

$$S_{ij}^{1} = \sum_{l=0}^{IJ} \delta_{i,l+2}^{**}\, \delta_{j,l+2}^{**}\, \phi_{1,N-l-1}^{-1}$$

$$S_{ij}^{2} = \sum_{l=0}^{IJ} \delta_{i,l+2}^{**}\, \delta_{j,l+2}^{**}\left(\frac{K_{N-l-1}}{\phi_{1,N-l-1}}\right)$$

$$S_{ij}^{3} = \sum_{l=0}^{IJ} \delta_{i,l+2}^{**}\, \delta_{j,l+2}^{**}\left(\frac{\phi_{2,N-l-1}}{\phi_{1,N-l-1}}\right)$$

$$S_{ij}^{4} = \sum_{l=0}^{IJ} \delta_{i,l+2}^{**}\, \delta_{j,l+2}^{**}\left(\frac{\phi_{3,N-l-1}}{\phi_{1,N-l-1}}\right)$$

$$S_{ij}^{5} = \sum_{l=0}^{IJ} \delta_{i,l+2}^{**}\, \delta_{j,l+2}^{**}\left(\frac{\phi_{4,N-l-1}}{\phi_{1,N-l-1}}\right)$$

$$S_{ij}^{6} = \sum_{l=0}^{IJ} \delta_{i,l+2}^{**}\, \delta_{j,l-1}^{**}\left(\frac{\kappa_{i-l-2}^{2}}{\phi_{1,N-l-1}}\right)$$

$$S_{ij}^{7} = \sum_{l=0}^{IJ} \delta_{l,i-1}^{**}\, \delta_{j,l+2}^{**} \left(\frac{\kappa_{i-l-2}^{1}}{\phi_{1,N-l-1}} \right)$$

$$S_{ij}^{8} = \sum_{l=0}^{IJ} \delta_{l,i-1}^{**}\, \delta_{l,j-1}^{**} \left[\frac{\kappa_{i-l-2}^{1}\, \kappa_{i-l-2}^{2}}{\phi_{1,N-l-1}} \right] \tag{4.31}$$

with, IJ = min (i - 1, j - 1), $\delta_{jk}^{**} = 1$ if $j < k$, and $=0$ if $j \geq k$ and

$$\kappa_{r}^{1} = \prod_{q=1}^{r} \left(\frac{\psi}{\phi_{1,N-i+q}} \right)$$

$$\kappa_{r}^{2} = \prod_{s=1}^{r} \left(\frac{\phi}{\phi_{1,N-j+s}} \right)$$

Proof: Let $R_l = [r_{ij}^{(l)}]$, i, j = 1, 2, 3, 4. From (4.25) we get after simiplification,

$$R_l = \begin{bmatrix} 1 & -a_2 & -a_3 & a_1 K_{l-1} \\ 0 & 1-b_2 & -b_3 & b_1 K_{l-1} \\ 0 & -c_2 & 1-c_3 & c_1 K_{l-1} \\ 0 & -d_2 & -d_3 & 1-d_1 K_{l-1} \end{bmatrix} \tag{4.32}$$

where we have written

$K_l = (1 - a_1 - a_2 - a_3)\, r_{11}^{(l)} + (1 - b_1 - b_2 - b_3)\, r_{12}^{(l)} + (1 - c_1 - c_2 - c_3)\, r_{13}^{(l)} + (1 - d_1 - d_2 - d_3)\, r_{14}^{(l)}$.

Upon inversion from (4.32) we get

$$R_l = \begin{bmatrix} 1 & \frac{\phi_{2,l-1}}{\phi_{1,l-1}} & \frac{\phi_{3,l-1}}{\phi_{1,l-1}} & \frac{\phi_{4,l-1}}{\phi_{1,l-1}} \\ 0 & \frac{(1-d_1K_{l-1})\,(1-c_3) - c_1d_3K_{l-1}}{\phi_{1,l-1}} & \frac{(1-d_1K_{l-1})b_3+b_1d_3K_{l-1}}{\phi_{1,l-1}} & \frac{g_5K_{l-1}}{\phi_{1,l-1}} \\ 0 & \frac{(1-d_1K_{l-1})c_2+c_1d_2K_{l-1}}{\phi_{1,l-1}} & \frac{(1-d_1K_{l-1})\,(1-b_2) - b_1d_2K_{l-1}}{\phi_{1,l-1}} & \frac{g_6K_{l-1}}{\phi_{1,l-1}} \\ 0 & \frac{g_3}{\phi_{1,l-1}} & \frac{g_4}{\phi_{1,l-1}} & \frac{\eta}{\phi_{1,l-1}} \end{bmatrix} \tag{4.33}$$

We also find that K_l satisfies the recursion given in (4.29).

Substituting from (4.33), we get

$$Y_l = \begin{bmatrix} a_1 & a_1\frac{\phi_{2,l-1}}{\phi_{1,l-1}} & a_1\frac{\phi_{3,l-1}}{\phi_{1,l-1}} & a_1\frac{\phi_{4,l-1}}{\phi_{1,l-1}} \\ b_1 & b_1\frac{\phi_{2,l-1}}{\phi_{1,l-1}} & b_1\frac{\phi_{3,l-1}}{\phi_{1,l-1}} & b_1\frac{\phi_{4,l-1}}{\phi_{1,l-1}} \\ c_1 & c_1\frac{\phi_{2,l-1}}{\phi_{1,l-1}} & c_1\frac{\phi_{3,l-1}}{\phi_{1,l-1}} & c_1\frac{\phi_{4,l-1}}{\phi_{1,l-1}} \\ d_1 & d_1\frac{\phi_{2,l-1}}{\phi_{1,l-1}} & d_1\frac{\phi_{3,l-1}}{\phi_{1,l-1}} & d_1\frac{\phi_{4,l-1}}{\phi_{1,l-1}} \end{bmatrix} \tag{4.34}$$

$$
X_l = \begin{bmatrix}
0 & 0 & 0 & K_l \\[2ex]
0 & 0 & 0 & 1 + \dfrac{g_5(K_{l-1}-1)}{\phi_{1,l-1}} \\[2ex]
0 & 0 & 0 & 1 + \dfrac{g_6(K_{l-1}-1)}{\phi_{1,l-1}} \\[2ex]
0 & 0 & 0 & \dfrac{\psi}{\phi_{1,l-1}}
\end{bmatrix} \tag{4.35}
$$

$$l = 1, 2, \ldots .$$

The theorem now follows by substituting from (4.33) - (4.35) in (4.24) and simplifying. □

As discussed earlier, if we are interested in the properties of first passage from the set of states $S_r = (r+1, r+2, \ldots, N)$ to the set of sets $S_r^c = (0, 1, 2, \ldots, r)$, we need to determine the elements of the fundamental matrix $(I - H_2)^{-1} = M^{(2)}$ where

$$
H_2 = \left[\begin{array}{c|c}
 & 0 \\
H & \vdots \\
 & F \\
\hline
0 \;\ldots\; E & D
\end{array}\right] \tag{4.36}
$$

If, on the other hand, we are interested in the properties of first passage from the set S_r^c to the set S_r, we need to determine the elements of the fundamental matrix $(I - H_1)^{-1} = M^{(1)}$ where

$$H_1 = \left[\begin{array}{c|c} A & B \; . \; . \; . \; 0 \\ \hline C & \\ . & H \\ 0 & \\ 0 & \end{array}\right] \tag{4.37}$$

The elements $M_{ij}^{(2)}$ of $M^{(2)}$ and $M_{ij}^{(1)}$ of $M^{(1)}$ can be expressed in terms of the elements M_{ij} of Theorem 4.3 using methods of Theorem 2.2 of Lal and Bhat (1988a). We shall not detail them due to their cumbersome nature.

It should be noted that elements $M_{ij}^{(2)}$ of $M^{(2)}$ give the expected number of visits of the process to state j before eventually entering a state in S_r^c, having originally started from $i \in S_r$. For the formula for variance of first passage times, see Kemeny and Snell (1960).

5. Remark.

Some of the explicit results given in this paper may look involed and complicated at first glance. Nevertheless, as anyone well-versed in scientific computations can realize, their explicit nature makes them convenient for numerical computations.

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