# SOME EXPLICIT RESULTS FOR CORRELATED RANDOM WALKS

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ABSTRACT

In a correlated random walk (CRW) the probabilities of movement to the positive and negative direction are given by the transition probabilities of a Markov chain. The walk can be represented as a Markov chain if we use a bivariate state space, with the location of the particle and the direction of movement as the two variables. In this paper we derive explicit results for the following characteristics of the walk directly from its transition probability matrix: (i) n-step transition probabilities for the unrestricted CRW, (ii) equilibrium distribution and first passage probabilities for the CRW restricted on one side, and (iii) equilibrium distribution and first passage characteristics for the CRW restricted on both sides (i.e., with finite state space).

Key Words: Correlated random walk, Markov chain, equilibrium solution, first passage problem.

## 1. Introduction

Consider a particle moving a unit distance along a straight line in a unit interval of time. In the classical random walk problem, the probabilities of moving to the right and to the left from any position are independent of the previous move. In a correlated random walk (CRW), we assume that the probabilities of movement to the positive and negative direction are given by the following transition probability matrix of a Markov chain:

$$\begin{array}{cc} & \begin{array}{cc} -1 & +1 \end{array} \\ \begin{array}{c} -1 \\ +1 \end{array} & \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \end{array} \tag{1.1}$$

$$0 \leq a,b \leq 1.$$

Starting with Gillis (1955) and Mohan (1955) several authors have investigated problems related to this CRW, and its special cases such as the symmetric CRW with b=a in (1), using primarily the classical approach of difference equations and generating functions. See, Gupta (1958), Seth (1963), Jain (1971, 1973), Darroch and Whitford (1972), Proudfoot and Lampard (1972), Nain and Sen (1979a, b, 1980), Renshaw and Henderson (1981), Henderson et al. (1983, 1984), Bender and Richmond (1984) and Roerdink (1985). In most of these investigations first passage problems are of primary interest. Other significant problems discussed include, the characteristic function and the exact expression for the dispersion matrix of the n-step transition probabilities by Henderson et al. (1984), the generating function of higher order transition probabilities as a discrete domain Green's function by Proudfoot and Lampard (1972), the exact expression for the n-step transition probability obtained through combinatorial arguments in a symmetric CRW by Renshaw and Henderson (1981) and the characterization of the transition probability matrix for CRW by Nain and Sen (1980). Significant omissions are explicit expressions for the n-step transition probabilities in the unrestricted CRW and the equilibrium distribution of the state of the process in the restricted CRW. These are the focus of the present investigation.

The general characterization of the state space used in this study is similar to that employed by Nain and Sen (1980). The state is represented by a bivariate process $\{(W_n, U_n)\ n = 0,1,2,\ldots\}$ where $W_n$ is the location of the particle after n steps and $U_n$ is the nature of that step. Thus in the unrestricted CRW the state space is the product space with factors:

$$W = (\ \ldots, -2, -1, 0, +1, +2, \ldots)$$

$$U = (-1, +1)$$

The two-state Markov chain $\{U_n\}$ with transition probability matrix (1.1) has an equilibrium distribution (when $|\ 1-a-b\ | < 1$),

$$(\pi_{-1}, \pi_{+1}) = \left[\frac{b}{a+b}, \frac{a}{a+b}\right] . \tag{1.2}$$

If we assume that the distribution of the initial state $U_0$ is given by (1.2), for the process $\{U_n\}$ we have

$$E[U_n] = \frac{a-b}{a+b}, \quad Var[U_n] = \frac{4ab}{(a+b)^2}, \quad n=1,2,\ldots$$

If $\rho_j$ is the serial correlation (with lag j) for the process $\{U_n\}$, we get

$$\rho_j = (1-a-b)^j \ , \ j=1,2,3,\ldots \ .$$

Define the transition probability

$$p_{ij,k\ell} = P(W_{n+1} = k,\ U_{n+1} = \ell \mid W_n = i,\ U_n = j)$$
$$i,k = \ldots -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$
$$j,\ell = -1,\ +1$$

Using the transition probabilities for the $\{U_n\}$ process given by (1.1), we get

$$\begin{aligned} &p_{i,-1,i-1,-1} = 1-a, \quad p_{i,\ -1,\ i+1,\ +1} = a \\ &p_{i,+1,i-1,-1} = b, \quad p_{i,\ +1,\ i+1,\ +1} = 1-b \end{aligned} \tag{1.3}$$

Exhibiting these probabilities in matrix form, for the transition probability matrix, we have

$$
P = \begin{array}{r|c|cc|cc|cc|c|}
 & & (-1,+1) & (-1,-1) & (0,+1) & (0,-1) & (1,+1, & 1,-1) & \\
\hline
 & \bullet & \multicolumn{2}{c|}{\bullet} & \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & 0 \\
\hline
(-1,+1) & \bullet & 0 & 0 & 1-b & 0 & \multicolumn{2}{c|}{0} & 0 \\
(-1,-1) & & 0 & 0 & a & 0 & & & \\
\hline
(0,+1) & 0 & 0 & b & 0 & 0 & 1-b & 0 & 0 \\
(0,-1) & & 0 & 1-a & 0 & 0 & a & 0 & \\
\hline
(1,+1) & 0 & \multicolumn{2}{c|}{0} & 0 & b & 0 & 0 & \bullet \\
(1,-1) & & & & 0 & 1-a & 0 & 0 & \\
\hline
(2,+1) & 0 & \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & 0 & b & \bullet \\
(2,-1) & & & & & & 0 & 1-a & \\
\hline
 & 0 & \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & \bullet
\end{array} \qquad (1.4)
$$

Writing

$$
X = \begin{bmatrix} 0 & b \\ 0 & 1-a \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 1-b & 0 \\ a & 0 \end{bmatrix}, \qquad (1.5)
$$

we may display P as

$$P = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & Y & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & Y & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & X & 0 & Y & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & X & 0 & Y & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 0 & X & 0 & Y & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 0 & 0 & X & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & X & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{1.6}$$

We shall exploit the structure of the submatrices X and Y and the transition probability matrix P to obtain explicit expressions for the distribution characteristics of the CRW.

In section 2 we derive n-step transition probabilites of the unrestricted CRW with the transition probability matrix given by (1.6). In section 3 we consider CRW with the restriction on one-side and obtain its equilibrium distribution and probabilities related to first passage transitions. Finally in section 4 we consider a CRW with restrictions on both sides (i.e., CRW with finite state space) and obtain its equilibrium distribution and first passage characteristics.

## 2. Unrestricted CRW

Using matrix algebra on (1.6), for $n \geq 0$ we can show that

$$
P^n = \begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & P_{-2}^{(n)} & P_{-1}^{(n)} & P_{0}^{(n)} & P_{1}^{(n)} & P_{2}^{(n)} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & P_{-3}^{(n)} & P_{-2}^{(n)} & P_{-1}^{(n)} & P_{0}^{(n)} & P_{1}^{(n)} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & P_{-4}^{(n)} & P_{-3}^{(n)} & P_{-2}^{(n)} & P_{-1}^{(n)} & P_{0}^{(n)} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix} \tag{2.1}
$$

where $P_0^{(0)} \equiv I, \quad P_i^{(0)} \equiv 0 \qquad i \neq 0$

$P_{-1}^{(1)} \equiv X, \quad P_1^{(1)} \equiv Y, \quad P_i^{(1)} = 0 \qquad |i| > 1$

and other elements can be obtained using the recursion

$$
P_i^{(h)} = XP_{i+1}^{(h-1)} + YP_{i-1}^{(h-1)} \tag{2.2}
$$

Using (2.2) repeatedly and by induction, we can establish the following results.

$$
\begin{gathered}
P_i^{(n)} = 0 \quad \text{and} \quad P_{-i}^{(n)} = 0, \quad i > n \\
P_i^{(i+2k+1)} = 0 \quad \text{and} \quad P_{-1}^{(i+2k+1)} = 0, \quad i,k \geq 0 \\
P_i^{(i)} = Y^i \quad \text{and} \quad P_{-i}^{(i)} = X^i, \quad i > 0
\end{gathered} \tag{2.3}
$$

and

$$
P_i^{(i+2k)} = \sum_{g_1=0}^{i+k} \sum_{g_2=0}^{i+k-G_1} \cdots \sum_{g_k=0}^{i+k-G_{k-1}} \left[ \prod_{j=1}^{k} Y^{g_j} X^{\delta^{**}_{j,k+1}} \right] Y^{i+k-G_k} \tag{2.4}
$$

$$P_{-i}^{(i+2k)} = \sum_{g_1=0}^{i+k} \sum_{g_2=0}^{i+k-G_1} \cdots \sum_{g_k=0}^{i+k-G_{k-1}} \left[ \prod_{j=1}^{k} X^{g_j} Y^{\delta^{**}_{j,k+1}} \right] X^{i+k-G_k} \tag{2.5}$$

$$G_h = g_1 + g_2 + \dots + g_h$$

$$\delta^{**}_{jk} = 1 \quad \text{if} \quad j < k$$

$$= 0 \quad \text{if} \quad j \geq k$$

and the matrix product in (2.4) and (2.5) is taken in post order as subscript j increases.

The results (2.4) and (2.5) are not easy to work with. In order to simplify them we state the following lemma without proof.

<u>Lemma 2.1</u> For $h > 0$, we have

$$X^h = \begin{bmatrix} \delta_{0h} & \delta^{**}_{0h} b(1-a)^{h-1} \\ 0 & (1-a)^h \end{bmatrix} \tag{2.6}$$

$$Y^h = \begin{bmatrix} (1-b)^h & 0 \\ \delta^{**}_{0h} a(1-b)^{h-1} & \delta_{0h} \end{bmatrix} \tag{2.7}$$

$$P_{i,22}^{(i+2k)} = \delta_{0,i+k}(1-a)^k +$$

$$\sum_{j=1}^{k} \delta^{**}_{0,i+k+1-j} \binom{i+k-1}{j-1}\binom{k}{j}(ab)^j(1-a)^{k-j}(1-b)^{i+k-j}$$

where $\delta_{0h} = 1$ if $h = 0$, and $= 0$ if $h \neq 0$

$\delta^{**}_{jk} = 1$ if $j < k$, and $= 0$ if $j \geq k$.

Proof: Let $i \geq 0$. We shall prove (2.10) by induction on k.

When $k = 0$

$$P_i^{(i+2(0))} = \left[\begin{array}{c|c} (1-b)^i & 0 \\ \hline \delta^{**}_{0i}a(1-b)^{i-1} & \delta_{0i} \end{array}\right] = Y^i \tag{2.11}$$

Since (2.10) holds for k=0, we can assume that (2.10) holds for nonnegative integers less than or equal to some k. To complete the induction proof we need to show that (2.10) holds for $k = k + 1$. Consider,

$$P_i^{(i+2(k+1))} = \sum_{g_1=0}^{i+k+1} Y^{g_1} X^{\delta^{**}_{1,k+2}} \left( \sum_{g_2=0}^{i+k+1-g_1} \cdots \sum_{g_{k+1}}^{i+k+1-G_k} \left( \prod_{j=2}^{k+1} Y^{g_j} X^{\delta^{**}_{j,k+2}} \right) \right) Y^{i+k+1-G_{k+1}}$$

(2.12)

From (2.4) and (2.10), we can write (2.12) as

$$P_i^{(i+2(k+1))} = \sum_{g_1=0}^{i+k+1} \left[\begin{array}{c|c} 0 & b(1-b)^{g_1} \\ \hline 0 & (1-a)\delta_{0g_1} + \delta^{**}_{0g_1} ab(1-b)^{g_1-1} \end{array}\right] \cdot$$

$$\left[\begin{array}{c|c} X & X \\ \hline \sum_{j=0}^{k} \delta^{**}_{0,i+k+1-g_1-j} \binom{i+k+1-g_1-1}{j}\binom{k}{j} \cdot & \delta_{0,i+1-g_1+k}\,(1-a)^k + \\ & \sum_{j=1}^{k} \delta^{**}_{0,i+1-g_1+k-j+1} \binom{i+1-g_1+k-1}{j-1}\binom{k}{j}(ab)^j \cdot \\ a(ab)^j(1-a)^{k-j}(1-b)^{i+k+1-g_1-j-1} & (1-a)^{k-j}(1-b)^{i+1-g_1+k-j} \end{array}\right] \tag{2.13}$$

$$= \left[\begin{array}{c|c} P^{(i+2(k+1))}_{i,11} & P^{(i+2(k+1))}_{i,12} \\ \hline P^{(i+2(k+1))}_{i,21} & P^{(i+2(k+1))}_{i,22} \end{array}\right] \tag{2.14}$$

where

$$P^{(i+2(k+1))}_{i,11} = \sum_{g_1=0}^{i+k+1} b(1-b)^{g_1} \sum_{j=0}^{k} \delta^{**}_{0,i+k+1-g_1-j} \binom{i+k+1-g_1-1}{j}\binom{k}{j} a(ab)^j(1-a)^{k-j} \cdot (1-b)^{i+k-g_1-j} \tag{2.15}$$

$$P^{(i+2(k+1))}_{i,12} = \sum_{g_1=0}^{i+k+1} b(1-b)^{g_1} \delta_{0,i+1-g_1+k}(1-a)^k +$$

$$\sum_{g_1=0}^{i+k+1} b(1-b)^{g_1} \sum_{j=1}^{k} \delta^{**}_{0,i+2+k-g_1-j} \binom{i+k-g_1}{j-1} \binom{k}{j} (ab)^j (1-a)^{k-j} \cdot (1-b)^{i+1-g_1+k-j} \qquad (2.16)$$

$$P^{(i+2(k+1))}_{i,21} = \sum_{g_1=0}^{i+k+1} (1-a)\delta_{0g_1} \sum_{j=0}^{k} \delta^{**}_{0,i+k+1-g_1-j} \binom{i+k+1-g_1-1}{j} \binom{k}{j} a(ab)^j (1-a)^{k-j} \cdot (1-b)^{i+k-g_1-j}$$

$$+ \sum_{g_1=0}^{i+k+1} \delta^{**}_{0,g_1} ab(1-b)^{g_1-1} \sum_{j=0}^{k} \delta^{**}_{0,i+k+1-g_1-j} \binom{i+k+1-g_1-1}{j} \binom{k}{j} a(ab)^j (1-a)^{k-j} \cdot (1-b)^{i+k-g_1-j-1}$$

$$(2.17)$$

$$P^{(i+2(k+1))}_{i,22} = \sum_{g_1=0}^{i+k+1} \left((1-a)\delta_{0g_1} + \delta^{**}_{0g_1} ab(1-b)^{g_1-1}\right) \delta_{0,i+1-g_1+k} (1-a)^k +$$

$$+ \sum_{g_1=0}^{i+k+1} \left((1-a)\delta_{0g_1} + \delta^{**}_{0g_1} ab(1-b)^{g_1-1}\right) \sum_{j=1}^{k} \delta^{**}_{0,i+1-g_1+k-j+1} \binom{i+1-g_1+k-1}{j-1} \binom{k}{j} (ab)^j \cdot (1-a)^{k-j} (1-b)^{i+1-g_1+k-j}$$

$$(2.18)$$

Now, we would like to show that (2.14) (including (2.15)-(2.18)) is the same as (2.10) when k=k+1.

Consider

$$P^{(i+2(k+1))}_{i,11} = \sum_{g_1=0}^{i+k+1} \sum_{j=0}^{k} \delta^{**}_{0,i+k+1-g_1-j} \binom{i+k-g_1}{j} \binom{k}{j} (ab)^{j+1} (1-a)^{k-j} (1-b)^{i+k-j}$$

$$= \sum_{j=0}^{k} \left( \sum_{g_1=0}^{i+k+1} \delta^{**}_{0,i+k+1-g_1-j} \binom{i+k-g_1}{j} \right) \binom{k}{j} (ab)^{j+1} (1-a)^{k-j} (1-b)^{i+k-j}$$

$$= \sum_{j=1}^{k} \left( \sum_{g_1=0}^{i+k+1} \delta^{**}_{0,i+k+2-g_1-j} \binom{i+k-g_1}{j-1} \right) \binom{k}{j-1} (ab)^{j} (1-a)^{k+1-j} (1-b)^{i+1+k-j}$$

$$= \sum_{j=1}^{k+1} \delta^{**}_{0,i+k+2-j} \binom{i+k+1}{j} \binom{k}{j-1} (ab)^{j} (1-a)^{k+1-j} (1-b)^{i+1+k-j} \tag{2.19}$$

$$P^{(i+2(k+1))}_{i,12} = \sum_{g_1=0}^{i+k+1} \delta_{0,i+1-g_1+k} (1-a)^{k} b (1-b)^{g_1}$$

$$+ \sum_{g_1=0}^{i+k+1} \sum_{j=1}^{k} \delta^{**}_{0,i+k+2-g_1-j} \binom{i+k-g_1}{j-1} \binom{k}{j} b (ab)^{j} (1-a)^{k-j} (1-b)^{i+k+1-j}$$

$$= \delta^{**}_{0,k+1} (1-a)^{k} b (1-b)^{i+k+1} +$$

$$\sum_{j=1}^{k} \left( \sum_{g_1=0}^{i+k+1} \delta^{**}_{0,i+k+2-g_1-j} \binom{i+k=g_1}{j-1} \right) \binom{k}{j} b (ab)^{j} (1-a)^{k-j} (1-b)^{i+k+1-j}$$

$$= \delta^{**}_{0,k+1} (1-a)^{k} b (1-b)^{i+k+1} +$$

$$\sum_{j=1}^{k} \delta^{**}_{0,i+k+2-j} \binom{i+k+1}{j-1} \binom{k}{j} b (ab)^{j} (1-a)^{k-j} (1-b)^{i+k+1-j} \tag{2.20}$$

$$P^{(i+2(k+1))}_{i,21} = \sum_{g_1=0}^{i+k+1} \sum_{j=0}^{k} \delta_{0g_1} \delta^{**}_{0,i+k+1-g_1-j} \binom{i+k-g_1}{j} \binom{k}{j} a (ab)^{j} (1-a)^{k+1-j} (1-b)^{i+k-j-g_1}$$

$$+ \sum_{g_1=0}^{i+k+1} \sum_{j=0}^{k} \delta^{**}_{0g_1} \delta^{**}_{0,i+k+1-g_1-j} \binom{i+k-g_1}{j} \binom{k}{j} a (ab)^{j+1} (1-a)^{k-j} (1-b)^{i+k-j-1}$$

$$= \sum_{j=0}^{k} \delta^{**}_{0,i+k+1-j} \binom{i+k}{j}\binom{k}{j} a(ab)^{j}(1-a)^{k+1-j}(1-b)^{i+k-j}$$

$$+ \sum_{j=0}^{k} \left( \sum_{g_1=0}^{i+k+1} \delta^{**}_{0g_1} \delta^{**}_{0,i+k+1-j-g_1} \binom{i+k-g_1}{j} \right) \binom{k}{j} a(ab)^{j+1}(1-a)^{k-j}(1-b)^{i+k-j-1}$$

$$= \sum_{j=0}^{k} \delta^{**}_{0,i+k+1-j} \binom{i+k}{j}\binom{k}{j} a(ab)^{j}(1-a)^{k+1-j}(1-b)^{i+k-j}$$

$$+ \sum_{j=0}^{k} \delta^{**}_{0,i+k-j} \binom{i+k}{j+1}\binom{k}{j} a(ab)^{j+1}(1-a)^{k-j}(1-b)^{i+k-j-1}$$

$$+ \delta^{**}_{0,i+k+1} \binom{i+k}{0}\binom{k}{0} a(ab)^{0}(1-a)^{k+1-0}(1-b)^{i+k-0}$$

$$+ \sum_{j=1}^{k} \delta^{**}_{0,i+k+1-j} \binom{i+k}{j} \left( \binom{k}{j} + \binom{k}{j-1} \right) a(ab)^{j}(1-a)^{k+1-j}(1-b)^{i+k-j}$$

$$+ \delta^{**}_{0,i} \binom{i+k}{k+1}\binom{k}{k} a(ab)^{k+1}(1-a)^{0}(1-b)^{i-1}$$

$$= \sum_{j=0}^{k+1} \delta^{**}_{0,i+k+1-j} \binom{i+k}{j}\binom{k+1}{j} a(ab)^{j}(1-a)^{k+1-j}(1-b)^{i+k-j} \qquad (2.21)$$

$$P^{(i+2(k+1))}_{i,22} = \sum_{g_1=0}^{i+k+1} \delta_{0g_1} \delta_{0,i+1-g_1+k} (1-a)^{k+1} + \sum_{g_1=0}^{i+k+1} \delta^{**}_{0g_1} \delta_{0,i+k+1-g_1} (1-a)^{k}(1-b)^{g_1-1}$$

$$+ \sum_{g_1=0}^{i+k+1} \sum_{j=1}^{k} \delta_{0g_1} \delta^{**}_{0,i+k+2-g_1-j} \binom{i+k-g_1}{j-1}\binom{k}{j} (ab)^{j}(1-a)^{k+1-j}(1-b)^{i+k+1-g_1-j}$$

$$+ \sum_{g_1=0}^{i+k+1} \sum_{j=1}^{k} \delta^{**}_{0g_1} \delta^{**}_{0,i+k+2-g_1-j} \binom{i+k-g_1}{j-1}\binom{k}{j} (ab)^{j+1}(1-a)^{k-j}(1-b)^{i+k-j}$$

$$= \delta_{0,i+k+1}(1-a)^{k+1} + \delta^{**}_{0,i+k+1}(1-a)^{k}ab(1-b)^{i+k}$$

$$+ \sum_{j=1}^{k} \delta^{**}_{0,i+k+2-j}\binom{i+k}{j-1}\binom{k}{j}(ab)^{j}(1-a)^{k+1-j}(1-b)^{i+k+1-j}$$

$$+ \sum_{j=1}^{k} \left( \sum_{g_1=0}^{i+k+1} \delta^{**}_{0,i+k+2-g_1-j}\binom{i+k-g_1}{j-1}\right)\binom{k}{j}(ab)^{j+1}(1-a)^{k-j}(1-b)^{i+k-j}$$

$$= \delta_{0,i+k+1}(1-a)^{k+1}+\delta^{**}_{0,i+k+1}(1-a)^{k}ab(1-b)^{i+k}+\delta^{**}_{0,i+k+1}k(ab)(1-a)^{k}(1-b)^{i+k}$$

$$+ \sum_{j=2}^{k} \delta^{**}_{i+k+2-j}\binom{i+k}{j-1}\binom{k}{j}(ab)^{j}(1-a)^{k+1-j}(1-b)^{i+k+1-j}$$

$$+ \sum_{j=1}^{k} \delta^{**}_{0,i+k+1-j}\binom{i+k}{j}\binom{k}{j}(ab)^{j+1}(1-a)^{k-j}(1-b)^{i+k-j}$$

$$= \delta_{0,i+k+1}(1-a)^{k+1}+\delta^{**}_{0,i+k+1}\binom{k+1}{1}(1-a)^{k}ab(1-b)^{i+k}$$

$$+ \sum_{j=2}^{k} \delta^{**}_{0,i+k+2-j}\binom{i+k}{j-1}\left(\binom{k}{j} + \binom{k}{j-1}\right)(ab)^{j}(1-a)^{k+1-j}(1-b)^{i+k+1-j}$$

$$+ \delta^{**}_{0,i+1}\binom{i+k}{k}\binom{k}{k}(ab)^{k+1}(1-a)^{0}(1-b)^{i}$$

$$= \delta_{0,i+k+1}(1-a)^{k+1} +$$

$$\sum_{j=1}^{k+1} \delta^{**}_{0,i+k+2-j}\binom{i+k}{j-1}\binom{k+1}{j}(ab)^{j}(1-a)^{k+1-j}(1-b)^{i+k+1-j} \qquad (2.22)$$

From (2.19)-(2.22), we see that (2.10) holds for k=k+1. This completes the proof of the theorem. □

We state without proof, which is quite similar to the proof of Theorem 2.1, the following theorem for $P_{-1}^{(i+2k)}$.

Theorem 2.2 For $i,k \geq 0$, $P_{-1}^{(i+2k)}$ of (2.5) can be simplified to give

$$P_{-1}^{(i+2k)} = \left[\begin{array}{c|c} P_{-i,11}^{(i+2k)} & P_{-i,12}^{(i+2k)} \\ \hline P_{-i,21}^{(i+2k)} & P_{-i,22}^{(i+2k)} \end{array}\right] \tag{2.23}$$

where

$$P_{-i,11}^{(i+2k)} = \delta_{0,i+k}(1-b)^k +$$
$$\sum_{j=1}^{k} \delta^{**}_{0,i+k+1-j} \binom{i+k-1}{j-1}\binom{k}{j}(ab)^j(1-a)^{i+k-j}(1-b)^{k-j}$$

$$P_{-i,12}^{(i+2k)} = \sum_{j=0}^{k} \delta^{**}_{0,i+k-j} \binom{i+k-1}{j}\binom{k}{j} b(ab)^j(1-a)^{i+k-j-1}(1-b)^{k-j}$$

$$P_{-i,21}^{(i+2k)} = \delta^{**}_{0,k}(1-a)^{i+k}a(1-b)^{k-1} +$$
$$\sum_{j=1}^{k-1} \delta^{**}_{0,i+k+1-j} \binom{i+k}{j}\binom{k-1}{j} a(ab)^j(1-a)^{i+k-j}(1-b)^{k-1-j}$$

$$P_{-i,22}^{(i+2k)} = \sum_{j=\min(1,k)}^{k} \delta^{**}_{0,i+k+1-j} \binom{i+k}{j}\binom{k-1}{j-1}(ab)^j(1-a)^{i+k-j}(1-b)^{k-j}$$

## 3. CRW Restricted on One Side.

We shall restrict the state space of the CRW to nonnegative integers. Consequently the transition probability matrix takes the form

$$P = \begin{bmatrix} 0 & B & 0 & 0 & 0 & \cdots \\ C & 0 & Y & 0 & 0 & \cdots \\ 0 & X & 0 & Y & 0 & \cdots \\ 0 & 0 & X & 0 & Y & \cdots \\ 0 & 0 & 0 & X & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix} \tag{3.1}$$

where X and Y are given by (1.5) and

$$B = [1,\ 0] \quad \text{and} \quad C = \begin{bmatrix} b \\ 1-a \end{bmatrix} \tag{3.2}$$

The condition for the existence of an equilibrium solution is given by the following theorem.

Theorem 3.1

The necessary and sufficient condition for the Markov chain with transition probability matrix (3.1) to have an equilibrium distribution is given by $b > a$.

Proof: Following Neuts (1981, p. 32), the necessary and sufficient condition for the existence of an equilibrium solution may be stated as

$$\underline{\pi} X \underline{e} > \underline{\pi} Y \underline{e} \tag{3.3}$$

where $\underline{\pi}$ is the limiting distribution $(\pi_1, \pi_2)$ of the Markov chain with transition probability matrix X+Y, and $\underline{e}$ is the unit vector. Solving

$$[\pi_1, \pi_2][X+Y] = \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}, \quad \pi_1 + \pi_2 = 1$$

we get

$$[\pi_1, \pi_2] = \left[\frac{a}{a+b}, \frac{b}{a+b}\right] \tag{3.4}$$

The stated condition now follows from (3.3) by direct substitution and simplification. □

Let $\underline{x} = (x_0, \underline{x}_1, \underline{x}_2, \ldots)$, where $x_0$ is a scalar and $\underline{x}_i$, $(i=1,2,\ldots)$ are two element row vectors, be the limiting distribution of the Markov chain. Following Neuts (1981, p. 25), we have

$$\underline{x}_i = \underline{x}_1 R^{i-1} \qquad i \geq 1 \tag{3.5}$$

where R is the rate matrix of the Markov chain P. Also, $x_0$ and $\underline{x}_1$ can be obtained from

$$\begin{aligned} x_0 B + \underline{x}_2 X &= \underline{x}_1 \\ \underline{x}_1 C &= x_0 \\ x_0 + \sum_{i=1}^{\infty} \underline{x}_i \underline{e}_i &= 1 \end{aligned} \tag{3.6}$$

The rate matrix R is found iteratively in the following lemma.

Lemma 3.1

$$R = \begin{bmatrix} 1-b & (1-b)^2/(1-a) \\ a & a(1-b)/(1-a) \end{bmatrix} \qquad (3.7)$$

$$R^j = \left(\frac{1-b}{1-a}\right)^{j-1} R \qquad j \geq 1 \qquad (3.8)$$

Proof: The rate matrix R satisfies the matrix equation (Neuts, 1981, p.19)

$$R = Y + R^2 X \qquad (3.9)$$

Let $R_n$ be the $n^{th}$ iterate and set $R_0 \equiv 0$. Using (3.9) iteratively, we get

$$R_n = Y + R_{n-1}^2 X \qquad (3.10)$$

Using induction, for $n \geq 1$ we can show that

$$R_n = \begin{bmatrix} 1-b & b(1-b)^2 g_n \\ a & ab(1-b)g_n \end{bmatrix} \qquad (3.11)$$

where

$$g_n = (1 + abg_{n-1})(1 + (1-a)(1-b)g_{n-1})$$

with $g_1 = 0$.

Since $R_n \to R$ as $n \to \infty$, $g_n \to g$ as well. Thus g can be obtained by solving

$$g = (1 + abg)(1 + (1-a)(1-b)g) \qquad (3.12)$$

This quadratic equation has two solutions

$$\frac{1}{b(1-a)} \;, \quad \frac{1}{a(1-b)} \qquad (3.13)$$

From (3.5) we may note that $R^j \to 0$ as $j \to \infty$. Using this condition we can easily show that the second solution is inadmissible. The lemma now follows by simple substitution of the value of $g = [b(1-a)]^{-1}$ in (3.11). □

We may also note that the condition $b > a$ is necessary for $R^j \to 0$ as $j \to \infty$.

Using R, the limiting distribution $\underline{x}$ is obtained in the following theorem.

<u>Theorem 3.2</u>

When $b > a$, the limiting distribution of CRW is given by

$$\underline{x} = (x_0, \underline{x}_1, \underline{x}_2, \ldots)$$

where

$$x_0 = \frac{1}{2}\left(\frac{b-a}{1-a}\right) \qquad (3.14)$$

$$\underline{x}_j = \left[\left(\frac{1-b}{1-a}\right)^{j-1}, \left(\frac{1-b}{1-a}\right)^{j}\right] x_0 \qquad j = 1,2,\ldots \qquad (3.15)$$

Clearly, the fact that the probability that the process eventually returns to state (0,-1) is 1, results from the positive recurrence of the Markov chain. Also, the mean recurrence time for this state is

$$x_0^{-1} = \frac{2(1-a)}{b-a} \tag{3.20}$$

Both these results can also be established by relating the rate matrix to the reduced system defined by Lal and Bhat (1987, 1988), which is a convenient technique in deriving first passage characteristics.

Consider two complementary sets of states

$$S_i = \{(0,-1),\ (1,+1),\ (1,-1),\ldots.(i,+1),(i,-1)\}$$

$$S_i^c = \{i+1,+1),\ (i+1,-1),\ldots.\ \}$$

Let ${}_iP_{ii}$ be the first return probability to set $S_i$ (i.e., the first passage probability of the process to states in the set $S_i$, after visiting states in $S_i^c$, while avoiding states $S_i$, having left set $S_i$ initially). We have

Theorem 3.3

$$[{}_iP_{ii}] = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ i-1 \\ i \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & i \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & RX \end{bmatrix} \end{array} \tag{3.21}$$

where

$$RX = \begin{bmatrix} 0 & 1-b \\ 0 & a \end{bmatrix}$$

Proof: When a transition probability matrix P is partitioned as

$$P = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \end{array}$$

it is well known that the elements of the matrix $B(I-D)^{-1}C$ give the first return probability (in the sense described above) into states $y_1$. Relating the reduced system method of determining the equilibrium solution outlined in Lal and Bhat (1987, 1988) to the rate matrix method used earlier, we can show that

$$B(I-D)^{-1} = \begin{bmatrix} 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & & \\ R & R^2 & \cdots \end{bmatrix}$$

Noting that the corresponding C matrix is

$$C = \begin{bmatrix} 0 & 0 & \cdots & X \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

we get (3.21) as $B(I-D)^{-1}C$.

It may be noted that 1-b is the probability of the first passage from (i,+1) to (i,-1) and a is the probability of first passage from (i,-1) to (i,-1), through higher states. The elements of X, viz. b and 1-a, provide first passage probabilities through lower states, so that the recurrence probability for each state is 1.

## 4. CRW Restricted on Both Sides

Let the state space of the $W_n$ process (location of the particle after n steps) be restricted to {0, 1, 2, ..., N}. Now the transition probability matrix takes the form

$$
P = \begin{array}{c} \\ (0,-1) \\ (1,+1) \\ (1,-1) \\ \vdots \\ (N-1,+1) \\ (N-1,-1) \\ (N,+1) \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} (0,-1) & (1,+1) & (1,-1) & (2,+1) & (2,-1) & \cdots & (N-1,+1) & (N-1,-1) & (N,+1) \end{array} \\
\left[\begin{array}{c|cc|cc|c|cc|c}
0 & 1 & 0 & & & \cdots & & 0 & 0 \\
\hline
b & 0 & 0 & 1-b & 0 & \cdots & & 0 & 0 \\
1-a & 0 & 0 & a & 0 & \cdots & & 0 & 0 \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots \\
\hline
 & & & & & \cdots & & & 1-b \\
0 & & 0 & & 0 & & & 0 & \\
 & & & & & \cdots & & & a \\
\hline
0 & & 0 & & 0 & \cdots & 0 & 1 & 0
\end{array}\right]
\end{array}
$$

$$
= \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 0 & 1 & 2 & 3 & \cdots & N-1 & N \end{array} \\
\left[\begin{array}{ccccccc}
0 & B & 0 & & & 0 & 0 \\
C & 0 & Y & 0 & \cdots & 0 & 0 \\
0 & X & 0 & Y & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & F \\
0 & 0 & 0 & 0 & \cdots & E & 0
\end{array}\right]
\end{array}
\qquad (4.1)
$$

where B and C are as defined in (3.2) and

$$
E = [0,\ 1] \qquad \text{and} \qquad F = \begin{bmatrix} 1-b \\ a \end{bmatrix} \qquad (4.2)
$$

Limiting probabilites $\underline{x} = (x_0, \underline{x}_1, \ldots, x_N)$ can be determined using a backward recursion (see, Lal and Bhat, 1987, 1988) as in the following theorem.

Theorem 4.1

$$x_{0,-1} = \frac{(b-a)(1-a)^{N-1}}{2[(1-a)^N - (1-b)^N]} \tag{4.3}$$

$$x_{i,+1} = \left(\frac{1-b}{1-a}\right)^{i-1} x_{0,-1} \qquad i=1,2,\ldots, N \tag{4.4}$$

$$x_{i,-1} = \left(\frac{1-b}{1-a}\right)^{i} x_{0,-1} \qquad i=1,2,\ldots, N. \tag{4.5}$$

Using W to represent the location of the particle as $n \to \infty$, we have

$$E[W] = \frac{1}{2}\left[\frac{2-a-b}{b-a} - \frac{2N(1-b)^N}{(1-a)^N-(1-b)^N}\right]$$

$$\mathrm{Var}[W] = \frac{1}{4}\left(\frac{2-a-b}{b-a}\right)\left(\frac{3b-a-2}{b-a}\right) - N^2\frac{(1-a)^N(1-b)^N}{[(1-a)^N-(1-b)^N]^2}$$

Proof

From the last two equations of

$$\underline{x}P = \underline{x} \tag{4.6}$$

we have

$$\left.\begin{aligned} x_{N,+1} &= (1-b)x_{N-1,+1} + a\, x_{N-1,-1} \\ x_{N-1,-1} &= x_{N,+1} \end{aligned}\right\} \tag{4.7}$$

giving

$$x_{N,+1} = x_{N-1,-1} = \left(\frac{1-b}{1-a}\right) x_{N-1,+1} \tag{4.8}$$

From the next two equations from the last two of (4.6), we have

$$x_{N-1,+1} = (1-b)x_{N-2,+1} + a\, x_{N-2,-1} \tag{4.9}$$

$$x_{N-2,-1} = b\, x_{N-1,+1} + (1-a)x_{N-1,-1} \tag{4.10}$$

Using (4.8) in (4.9) and (4.10) we get

$$x_{N-1,+1} = x_{N-2,-1} = \left(\frac{1-b}{1-a}\right) x_{N-2,+1} \tag{4.11}$$

Going back to (4.7), we find

$$x_{N,+1} = x_{N-1,-1} = \left(\frac{1-b}{1-a}\right)^2 x_{N-2,+1} \tag{4.12}$$

Extending this backward recursion, we get

$$x_{i,+1} = x_{i-1,-1} = \left(\frac{1-b}{1-a}\right)^{i-1} x_{1,+1} \qquad i = 2,3,\ldots, N \tag{4.13}$$

$$x_{0,-1} = x_{1,+1} \tag{4.14}$$

The theorem now follows using the normalizing condition $\underline{x}\,\underline{e} = 1$. □

In a CRW restricted on both sides, besides the equilibrium distribution, there are significant first passage problems for consideration. For instance, in the context of a gambler's ruin problem, it is of interest to determine the probabilities of ruin and win as well as expected duration of the game. A computational procedure to obtain these characteristics is to determine what has come to be known as the fundamental matrix of the appropriate transition probability matrix. See Kemeny and Snell (1960) for the definition of the matrix and the interpretation of results. In the finite state CRW considered in this section, the basic fundamental matrix is given by

$$M = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & & & N \end{array} \\ \left[\begin{array}{ccccccc} I & -Y & & & & & \\ -X & I & -Y & & & & \\ & -X & I & -Y & & & \\ & & & \ddots & & & \\ & & & & -X & I & -Y \\ & & & & & -X & I \end{array}\right]^{-1} \end{array} \qquad (4.15)$$

where X and Y are submatrices defined in (1.5). Let $M_{ij}$ be the (i,j)th element of M. We have

<u>Lemma 4.1</u>

$$M_{ij} = \sum_{\ell=0}^{\min(i-1,j-1)} \left[\prod_{g=\ell+1}^{i-1} X_{N-g}\right] R_{N-\ell} \left[\prod_{h=\ell+1}^{j-1} Y_{N-h}\right] \qquad (4.16)$$

where $R_\ell$ satisfies the relation

$$R_\ell = (I - YR_{\ell-1}X)^{-1}\ , \qquad \ell = 2,3,\ldots, N \tag{4.17}$$

$$R_1 = I$$

and

$$Y_\ell = YR_\ell \quad \text{and} \quad X_\ell = R_\ell X \tag{4.18}$$

$$\ell = 1,2 \ldots N.$$

In (4.6) $\prod_{g=\ell+1}^{i-1} X_{N-g}$ is taken in preorder as g increases, while $\prod_{h=\ell+1}^{j-1} Y_{N-h}$ is taken in postorder as h increases.

Proof: The result follows from the recursive algorithm following Theorem 2.2 of Lal and Bhat (1988). □

Theorem 4.2

When X and Y submatrices are given as

$$X = \begin{bmatrix} 0 & b \\ 0 & 1-a \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 1-b & 0 \\ a & 0 \end{bmatrix}$$

for the fundamental matrix M of equation (4.15), we have

$$M_{ij} = \begin{bmatrix} \mu_{ij}^{(11)} & \mu_{ij}^{(12)} \\ \mu_{ij}^{(21)} & \mu_{ij}^{(22)} \end{bmatrix} \tag{4.19}$$

where

$$\mu_{ij}^{(11)} = S_{ij}^{0} + (1-b)S_{ij}^{3} + aK_{N-i+1}S_{ij}^{5}$$

$$S_{ij}^{5} = \sum_{\ell=0}^{\min(i-1,j-1)} \frac{\delta^{**}_{\ell,i-1}\ \delta^{**}_{\ell,j-1}}{1-aK_{N-\ell-1}} \left(\prod_{g=1}^{i-\ell-2} \frac{1-a}{(1-aK_{N-i+g})}\right)\left(\prod_{h=1}^{j-\ell-2} \frac{1-b}{(1-aK_{N-j+h})}\right)$$

Note that $\delta^{**}_{jk} = 1$ if $j < k$, and $= 0$ if $j \geq k$. (4.22)

Proof: Starting with equation (4.17) and defining

$$R_\ell = \begin{bmatrix} r_1^{(\ell)} & r_2^{(\ell)} \\ r_3^{(\ell)} & r_4^{(\ell)} \end{bmatrix} \tag{4.23}$$

we get

$$R_\ell = \begin{bmatrix} 1 & -(1-b)(b\ r_1^{(\ell-1)} + (1-a)r_2^{(\ell-1)}) \\ 0 & 1-a(b\ r_1^{(\ell-1)} + (1-a)r_2^{(\ell-1)}) \end{bmatrix}^{-1} \tag{4.24}$$

Writing

$$K_\ell = b\ r_1^{(\ell)} + (1-a)r_2^{(\ell)}$$

from (4.24) we get

$$R_\ell = \begin{bmatrix} 1 & \dfrac{(1-b)K_{\ell-1}}{1-aK_{\ell-1}} \\ 0 & \dfrac{1}{1-aK_{\ell-1}} \end{bmatrix} \tag{4.25}$$

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