NUMBER OF SUCCESSES IN MARKOV TRIALS

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ABSTRACT

Markov trials are a sequence of dependent trials with two

outcomes, success and failure, which are the states of a Markov chain.

A new Markov chain is defined by augmenting the state space to include

information on the number of successes. The distribution

characteristics of the number of successes in n trials and the first

passage time for a specified number of successes are obtained using

this augmented Markov chain.

Key words: Markov trials, Markov chain, first passage time.

1. Introduction

Distribution characteristics of a sequence of Bernoulli trials (independent trials with constant probabilities for success and failure) are some of the basic topics covered in any introductory course in probability. Instead of a sequence of independent trials if we consider a sequence of Markov dependent trials (when probabilities of success and failure are governed by a Markov chain structure), their distribution characteristics are not easily determined. Using combinatorial arguments Gabriel (1959) has given the distribution and moments of the number of successes. Here we provide an alternative and simpler method which gives the distribution characteristics of the number of successes as well as the first passage time for a specified number of successes. This method has the added attraction of being much more suitable for numerical calculations.

The key feature of this approach is an augmented Markov chain defined by incorporating the information on the number of successes. The definition is given in section 2. Distribution of the number of successes is derived in section 3 and distribution characteristics of the first passage time to a specified number of successes are presented in section 4. Some remarks on the use of these results are made in the final section.

2. An Augmented Markov Chain

Let $\{Y_n,\ n=0,1,2...\}$ be a two state Markov chain with states 0 (failure) and 1 (success) and transition probabilities

$$p_{ij} = P(Y_{n+1} = j \mid Y_n = i) \qquad n = 0,1,2,\ldots$$

$$i,j = 0,1.$$

We assume that the transition probability matrix is given by

$$
\begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \qquad (0 \leq a, b \leq 1).
\tag{1}
$$

Let

$$
p_{ij}^{(n)} = P(Y_n = j \mid Y_0 = i) \qquad n = 1, 2, \ldots
$$

When $|1-a-b| < 1$, the limiting probabilities

$$
\pi_j = \lim_{n \to \infty} p_{ij}^{(n)} \qquad (j = 0, 1)
$$

exist independent of the initial state, and are given by

$$
(\pi_0, \pi_1) = \left(\frac{b}{a+b}, \frac{a}{a+b} \right) = (1-p, \ p), \text{ say.}
\tag{2}
$$

Define an augmented Markov chain (X_n, Y_n) where

X_n = number of successes in n trials

Y_n = state at the nth trial.

The state space for this Markov chain can be given as

$$
\{00, \ 11, \ 10, \ 21, \ 20, \ 31, \ 30, \ \ldots \ldots\}
$$

with the corresponding transition probabilities

$$
P_{ij,k\ell} = P(X_n = k, \ Y_n = \ell \mid X_0 = i, \ Y_0 = j)
$$

and the matrix $[p_{ij,k\ell}]$

$$
P = \begin{array}{c} \\ 00 \\ 11 \\ 10 \\ 21 \\ 20 \\ 31 \\ \cdot \\ \cdot \\ \cdot \end{array}
\begin{array}{cccccc}
00 & 11 & 10 & 21 & 20 & 31 \quad \cdot \quad \cdot \quad \cdot \\
\left[\begin{array}{cccccc}
1-a & a & & & & \\
 & & b & 1-b & & \\
 & & 1-a & a & & \\
 & & & & b & 1-b \\
 & & & & 1-a & a \\
 & & & & & \ddots \\
 & & & & & & \ddots \\
 & & & & & & & \ddots
\end{array} \right]
\end{array}
\tag{3}
$$

3

The matrix P possesses the necessary information and the structure to determine the distribution characteristics related to the number of successes mentioned earlier.

3. Distribution of the Number of Successes

The first two rows of the matrix P^n (nth power of P) give the distribution of the number of defectives in n trials (after the initial trial). Let $p_{ij,k\ell}^{(n)}$ be the $(ij,k\ell)$ element of P^n. Assuming that the process is in equilibrum we may write

$$P(X_0 = 0, Y_0 = 0) = 1-p$$

$$P(X_0 = 1, Y_0 = 1) = p.$$

Then

$$P(X_n = k) = (1-p)\left[p_{00,k1}^{(n)} + p_{00,k0}^{(n)} \right]$$

$$+ p\left[p_{11,k1}^{(n)} + p_{11,k0}^{(n)} \right]$$

$$k = 0,1,2,\ldots\ldots n. \tag{4}$$

A simple way of obtaining $p_{ij,k\ell}^{(n)}$ is by matrix multiplication after truncating the state space of the countably infinite state Markov chain appropriately. If explicit expressions for $p_{ij,k\ell}^{(n)}$ as well as (4) are needed, they can be determined as follows.

Partition P as

4

$$
\begin{array}{c}
 \\
00 \\
11 \\
\\
10 \\
21 \\
\\
30 \\
31 \\
\end{array}
\begin{array}{c}
\begin{array}{cccccccc}
00 & 11 & & 10 & 21 & & 20 & 31 & \quad \cdot \quad \cdot \quad \cdot \\
\end{array} \\
\left[
\begin{array}{cc|cc|cc|cc}
1-a & a & 0 & 0 & & & & \\
0 & 0 & b & 1-b & & & & \\
\hline
& & 1-a & a & 0 & 0 & & \\
& & 0 & 0 & b & 1-b & & \\
\hline
& & & & 1-a & a & & \\
& & & & 0 & 0 & & \\
\hline
\end{array}
\right]
\end{array}
$$

$$
=
\left[
\begin{array}{cccccc}
A & B & & & & \\
& A & B & & & \\
& & A & B & & \\
& & & & \ddots & \\
& & & & & \ddots
\end{array}
\right]
\tag{5}
$$

where

$$
A = \begin{bmatrix} 1-a & a \\ 0 & 0 \end{bmatrix} \quad , \quad
B = \begin{bmatrix} 0 & 0 \\ b & 1-b \end{bmatrix} \quad .
\tag{6}
$$

Clearly

$$
P^{n} =
\left[
\begin{array}{ccccccc}
D_{0}^{(n)} & D_{1}^{(n)} & D_{2}^{(n)} & D_{3}^{(n)} & \cdot & \cdot & \cdot \\
& D_{0}^{(n)} & D_{1}^{(n)} & D_{2}^{(n)} & \cdot & \cdot & \cdot \\
& & D_{0}^{(n)} & D_{1}^{(n)} & \cdot & \cdot & \cdot \\
& & & D_{0}^{(n)} & \cdot & \cdot & \cdot \\
& & & & \cdot & \cdot & \cdot
\end{array}
\right]
\tag{7}
$$

5

where

$$D_0^{(n)} \equiv A^n$$

$$D_i^{(n+1)} = D_{i-1}^{(n)}B + D_i^{(n)}A, \quad i = 1,2,\ldots \tag{8}$$

By recursive substitutions, from (8) we get

$$D_i^{(n)} = \sum_{j=i-1}^{n-1} D_{i-1}^{(j)}B \, A^{n-1-j}, \quad i = 1,2,\ldots \tag{9}$$

with

$$D_0^{(0)} \equiv I$$

$$D_i^{(j)} \equiv 0 \text{ for } j \leq i-1$$

$$D_i^{(j)} = B^j \text{ for } j = i.$$

In deriving explicit expressions for the elements of $D_i^{(n)}$, we first note the following results given as a lemma.

Lemma 1. For A and B defined in (6) and for $n \geq 0$, we have

$$A^n = \begin{bmatrix} (1-a)^n & \delta_{0n}^*(1-a)^{n-1}a \\ 0 & \delta_{0n} \end{bmatrix} \tag{10}$$

$$B^n = \begin{bmatrix} \delta_{0n} & 0 \\ \delta_{0n}^*(1-b)^{n-1}b & (1-b)^n \end{bmatrix} \tag{11}$$

$$B^m A^n = \begin{bmatrix} \delta_{0m}(1-a)^n & \delta_{0m}\delta_{0n}^*(1-a)^{n-1}a \\ \delta_{0m}^*(1-b)^{m-1}b(1-a)^n & \delta_{0m}^*\delta_{0n}^*(1-b)^{m-1}ba(1-a)^{n-1}+\delta_{0n}(1-b)^m \end{bmatrix}$$

$$\tag{12}$$

6

where

$$\delta_{0n} = 1 \text{ if } n = 0$$

$$= 0 \text{ if } n > 0$$

$$\delta_{0n}^* = 1 - \delta_{0n} .$$

Explicit expressions for the elements of $D_i^{(n)}$ are given in the following theorem.

Theorem 1. For $n \geq i$, let

$$D_i^{(n)} = \begin{bmatrix} d_i^{(n)}(11) & d_i^{(n)}(12) \\ d_i^{(n)}(21) & d_i^{(n)}(22) \end{bmatrix} \tag{13}$$

and define

$$\delta_{gh}^{**} = 1 \quad \text{if } g < h$$

$$= 0 \text{ if } g \geqq h$$

$$\delta_{0in}^{**} = \delta_{0i}^{**}\delta_{0n}^{**}(1-\delta_{ni}^{**}) .$$

We have

$$d_i^{(n)}(11) = \sum_{j=\min(1,i)}^{i} \delta_{i+j-1,n}^{**} \binom{i-1}{j-1}\binom{n-i}{j}(1-a)^{n-i-j}(1-b)^{i-j}a^j b^j$$

$$d_i^{(n)}(12) = \sum_{j=1}^{i+1} \delta_{i+j-1,n}^{**} \binom{i}{j-1}\binom{n-i-1}{j-1}(1-a)^{n-i-j}(1-b)^{i+1-j}a^j b^{j-1}$$

$$d_i^{(n)}(21) = \delta_{0in}^{**}(1-a)^{n-i}(1-b)^{i-1}b +$$

$$\sum_{j=1}^{i-1} \delta_{i+j-1,n}^{**} \binom{i-1}{j}\binom{n-i}{j}(1-a)^{n-i-j}(1-b)^{i-1-j}a^j b^{j+1}$$

7

$$d_i^{(n)}(22) = \delta_{in}(1-b)^i + \sum_{j=1}^{i} \delta_{i+j-1,n}^{**} \binom{i}{j}\binom{n-i-1}{j-1}(1-a)^{n-i-j}(1-b)^{i-j}a^j b^j.$$

(14)

<u>Proof.</u> Results in (14) are proved by induction on i for each element of $D_i^{(n)}$.

For i = 0, from (14) we get

$$D_0^{(n)} = \begin{bmatrix} (1-a)^n & \delta_{0n}^{**}(1-a)^{n-1}a \\ 0 & \delta_{0n} \end{bmatrix}$$

$$= A^n \quad [\text{from (10)}].$$

Thus (14) holds for i=0. Assume that it holds for some i > 0. Now the proof is completed by showing that it holds for i+1 after simplifications of the recursive relation (9) for i = i+1, which can be written as

$$D_{i+1}^{(n)} = \sum_{j=i}^{n-1} D_i^{(j)} B\, A^{n-1-j} \quad .$$

Details of these simplification are given in the appendix.

Using the above expression in (4) the distribution of the number of successes in (n+1) trials (including the initial trial) follows.

$$P(X_n=k) = (1-p) \sum_{j=\min(1,k)}^{k} \binom{k-1}{j-1}(1-a)^{n-k-j}(1-b)^{k-j}a^j b^{j-1}$$

$$(b\,\delta_{k+j-1,n}^{**}\binom{n-k}{j} + (1-a)\delta_{0k}^{**}\,\delta_{k+j-2,n}^{**}\binom{n-k}{j-1}))$$

$$+ p\,\delta_{0k}^{**}(1-b)^{k-1}(\delta_{k-1,n} + \delta_{0kn}^{**}b(1-a)^{n-k})$$

$$+ \ p \sum_{j=1}^{k-1} \binom{k-1}{j}(1-a)^{n-k-j}(1-b)^{k-1-j}a^j b^j (b\overset{**}{\delta}_{k+j-1,n}\binom{n-k}{j}$$

$$+ \ (1-a)\overset{**}{\delta}_{k+j-2,n}\binom{n-k}{j-1})) \tag{15}$$

$$k = 0,1,2,\ldots,n+1$$

4. First Passage Time for More than c Successes

To determine distribution characteristics of first passage time for more than c successes convert state $(c+1,1)$ into an absorbing state. Now the transition probability matrix R partitioned as in section 3 can be given as

$R =$

	00	11	10	21	20	31	...	c0	(c+1,1)
00	1-a	a	0	0					
11	0	0	b	1-b					
10			1-a	a	0	0			
21			0	0	b	1-b			
20					1-a	a			
31					0	0			
.							.		
.							.		
.							.		
c1								b	1-b
c0								1-a	a
(c+1,1)								0	1

9

$$
= \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \cdot \\ \cdot \\ \cdot \\ K-1 \\ \\ K \end{array}
\begin{array}{c} 1 \quad 2 \qquad\qquad\quad K-1 \ \big| \ K \end{array}
\left[
\begin{array}{ccccc|c}
A & B & & & & \\
 & A & B & & & \\
 & & A & B & & \\
 & & & \ddots & & \\
 & & & & A & B \\
\hline
 & & 0 & & & C
\end{array}
\right]
\tag{16}
$$

where A and B are given in (6), and

$$
C = \begin{bmatrix} 1-a & a \\ 0 & 1 \end{bmatrix} , \tag{17}
$$

and $K = c+1$. The dimension of the matrix in (16) is $N = 2K$.

Let T_{ij} be the number of transitions required to visit state $(c+1,1)$. Because of the absorbing state $(c+1,1)$, the elements

$R_{ij,\ell m}^{(n)}$ of the nth power of $R(=R^n)$ can be interpreted as follows.

For $\ell = 0,1,2,\ldots,c$, $m = 0,1$

$$
R_{ij,\ell m}^{(n)} = P\left[X_n=\ell,\ Y_n=m \mid X_0=i,\ Y_0=j,\ T_{ij}>n\right] \tag{18}
$$

and for $\ell = c+1$, $m=1$

$$
R_{ij,(c+1,1)}^{(n)} = P(X_n=c+1,\ Y_n=1 \mid X_0=i,\ Y_0=j,\ T_{ij} \leqq n] \tag{19}
$$

Hence for a process which starts with states having an equilibrium distribution $[(1-p),p]$,

10

$$P(T \leq n) = (1-p)R_{00,(c+1,1)}^{(n)} + p\, R_{11,(c+1,1)}^{(n)} \tag{20}$$

where we have used T to represent the unconditional first passage time.

To derive explicit expressions for elements of R^n we proceed as follows. We have

$$R^n = \left[\begin{array}{ccccc|c} D_0^{(n)} & D_1^{(n)} & D_2^{(n)} & \cdots \cdots & D_{K-2}^{(n)} & E_1^{(n)} \\ & D_0^{(n)} & D_1^{(n)} & \cdots \cdots & D_{K-3}^{(n)} & E_2^{(n)} \\ & & D_0^{(n)} & \cdots \cdots & D_{K-4}^{(n)} & E_3^{(n)} \\ & & & & \vdots & \\ & & & & D_0^{(n)} & E_{K-1}^{(n)} \\ \hline & & 0 & & & C^n \end{array} \right] \tag{21}$$

where $D_i^{(n)}$ are given in equations (8) and (9) and Theorem 1, and $E_i^{(n)}$ can be obtained from recurrence relations

$$E_i^{(n+1)} = D_{K-1-i}^{(n)} B + E_i^{(n)} C \tag{22}$$

$$i = 1,2\ldots,K-1$$

with

$$E_{K-1}^{(1)} \equiv B, \quad E_i^{(1)} \equiv 0, \quad i=1,2,\ldots,K-2.$$

Recursive substitutions in (22) give

$$E_i^{(n)} = \sum_{j=K-1-i}^{n-1} D_{K-1-i}^{(j)} B C^{n-1-j}, \quad i=1,2,\ldots,K-1 \tag{23}$$

where $E_i^{(j)} \equiv ($ for $j \leq K-1-i$

$E_i^{(j)} = B^j$ for $j = K-i$.

11

Explicit expressions for the elements of $E_i^{(n)}$, are presented in a lemma and a theorem below.

Lemma 2: For B defined in (6), C defined in (17), and for $n \geq 0$, we have

$$C^n = \begin{bmatrix} (1-a)^n & 1-(1-a)^n \\ 0 & 1 \end{bmatrix} \tag{24}$$

$$B^m C^n = \begin{bmatrix} \delta_{0m}(1-a)^n & \delta_{0m}[1-(1-a)^n] \\ \delta_{0m}^*(1-a)^n(1-b)^{m-1}b & (1-b)^m + \delta_{0m}^*(1-b)^{m-1}[1-(1-a)^n]b \end{bmatrix}$$

$$\tag{25}$$

where δ functions are defined in Lemma 1.

These results follow directly by matrix multiplication.

Theorem 2. For $n \geq K-i$, $i=1,2,\ldots,K-1$, let

$$E_i^{(n)} = \begin{bmatrix} e_i^{(n)}(11) & e_i^{(n)}(12) \\ e_i^{(n)}(21) & e_i^{(n)}(22) \end{bmatrix} \tag{26}$$

and define δ functions as in Theorem 1. We have

$$e_i^{(n)}(11) = \sum_{j=1}^{K-i} \delta_{K-1-i+j,n}^{**} \binom{n-K+i}{j}\binom{K-1-i}{j-1}(1-a)^{n-K+i-j}(1-b)^{K-i-j}a^j b^j$$

$$e_i^{(n)}(12) = \sum_{j=1}^{K-i} \delta_{K-1-i+j,n}^{**} \left(\sum_{g=0}^{n-K+i-j} \binom{j-1+g}{g}(1-a)^g\right)\binom{K-1-i}{j-1}(1-b)^{K-i-j}a^j b^{j-1}$$

$$- e_i^{(n)}(11)$$

$$e_i^{(n)}(21) = \delta_{K-1-i,n}^{*} (1-a)^{n-K+i}b(1-b)^{K-1-i}$$

12

$$+ \sum_{j=1}^{K-1-i} \delta^{**}_{K-1-i+j,n} \binom{n-K+i}{j}\binom{K-1-i}{j}(1-a)^{n-K+i-j}(1-b)^{K-1-i-j}a^j b^{j+1}$$

$$e_i^{(n)}(22) = \delta^{**}_{K-1-i,n}(1-b)^{K-1-i}$$

$$+ \sum_{j=1}^{K-1-i} \delta^{**}_{K-1-i+j,n}\left(\sum_{g=0}^{n-K+i-j}\binom{j-1+g}{g}(1-a)^g\right)\binom{K-1-i}{j}$$

$$(1-b)^{K-1-i-j}\,a^j b^j - e_i^{(n)}(21)$$

These results follow directly from (23) on simplifications, after using appropriate terms from Theorem 1, Lemma 2, and noting that

$$\sum_{g=K-1-i}^{n-1} \delta^{**}_{K-2-i+j,g}\binom{g-K+i}{j-1}(1-a)^{g-K+1+i-j} = \delta^{**}_{K-1-i+j,n}\sum_{g=0}^{n-K+i-j}\binom{j-1+g}{g}(1-a)^g$$

Let $T_{ij,k\ell}$ be the number of visits of the process to state $(k\ell)$ before it eventually gets absorbed in state $(c+1,1)$, having originally started from state (ij). Mean and variance of $T_{ij,k\ell}$ are obtained in terms of the fundamental matrix M of the transition probability matrix R. [For a definition of the fundamental matrix and expressions for mean and variance see Kemeny and Snell (1960), Ch. 3 or Bhat (1984), Ch. 4]. First we modify the partition in (16) as

$$R = \begin{bmatrix} A & B & & & & & \\ & A & B & & & & \\ & & A & B & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & A & b & 1-b \\ & & & & & & 1-a & a \\ \hline & & & 0 & & & & 1 \end{bmatrix} \qquad (27)$$

$$= \begin{bmatrix} Q & R_1 \\ 0 & 1 \end{bmatrix} \qquad (28)$$

From the theory of finite Markov chains (see references cited above), we have (using $|| \bullet ||$ to denote a matrix)

$$|| E(T_{ij,k\ell}) || = M = (I-Q)^{-1} \qquad (29)$$

$$|| V(T_{ij,k\ell} || = M(2M_D - I) - M_2 \qquad (30)$$

where M_D is a matrix with only the diagonal elements of M and M_2 is a matrix whose elements are the squared elements of M. Clearly row sums of (29) and (30) give mean and variance of T_{ij} which is the first passage time conditional on the initial state (ij). These results are given below as Theorem 3. Proof of the theorem follows from the well-known method of inversion of a triangular matrix (I-Q) (see Faddeev and Faddeeva (1963)).

Theorem 3. The mean and variance of the first passage times $T_{ij,k\ell}$ and T_{ij} are given as follows

14

$$||E(T_{ij,k\ell})|| = \begin{array}{c} \\ 00 \\ 11 \\ 10 \\ 21 \\ \cdot \\ \cdot \\ \cdot \\ (c-1,0) \\ c1 \\ c0 \end{array} \begin{array}{cccccccccc} 00 & 11 & 10 & 21 & 20 & 31 & & (c-1,0) & c1 & c0 \\ 1/a & 1 & b/a & 1 & b/a & 1 & \ldots & b/a & 1 & b/a \\ & 1 & b/a & 1 & b/a & 1 & \ldots & b/a & 1 & b/a \\ & & 1/a & 1 & b/a & 1 & \ldots & b/a & 1 & b/a \\ & & & 1 & b/a & 1 & \ldots & b/a & 1 & b/a \\ & & & & & & \cdot & \cdot & \cdot & \cdot \\ & & & & & & \cdot & \cdot & \cdot & \cdot \\ & & & & & & \cdot & \cdot & \cdot & \cdot \\ & & & & & & & 1/a & 1 & b/a \\ & & & & & & & & 1 & b/a \\ & & & & & & & & & 1/a \end{array}$$

$$\tag{31}$$

Let $\alpha = (1-a)/a^2$ and $\beta = (2-a-b)b/a^2$. Then

$$||V(T_{ij,k\ell})|| = \begin{array}{c} \\ 00 \\ 11 \\ 10 \\ 21 \\ \cdot \\ \cdot \\ \cdot \\ (c-1,0) \\ c1 \\ c0 \end{array} \begin{array}{ccccccccc} 00 & 11 & 10 & 21 & & & (c-1,0) & c1 & c0 \\ \alpha & 0 & \beta & 0 & \ldots & & \beta & 0 & \beta \\ & 0 & \beta & 0 & \ldots & & \beta & 0 & \beta \\ & & \alpha & 0 & \ldots & & \beta & 0 & \beta \\ & & & 0 & \ldots & & \beta & 0 & \beta \\ & & & & & \cdot & \cdot & \cdot & \cdot \\ & & & & & \cdot & \cdot & \cdot & \cdot \\ & & & & & \cdot & \cdot & \cdot & \cdot \\ & & & & & & \alpha & 0 & \beta \\ & & & & & & & 0 & \beta \\ & & & & & & & & \alpha \end{array}$$

$$\tag{32}$$

Furthermore

$$||E(T_{ij})|| = \begin{array}{c} 00 \\ 11 \\ 10 \\ 21 \\ \cdot \\ \cdot \\ \cdot \\ (c-1,0) \\ c1 \\ c0 \end{array} \left[\begin{array}{c} (c(a+b)+1)/a \\ c(a+b)/a \\ ((c-1)(a+b)+1)/a \\ (c-1)(a+b)/a \\ \cdot \\ \cdot \\ \cdot \\ ((a+b)+1)/a \\ (a+b)/a \\ 1/a \end{array} \right] , \tag{33}$$

and

$$
||V(T_{ij})|| =
\begin{matrix}
00 \\
11 \\
10 \\
21 \\
\cdot \\
\cdot \\
\cdot \\
(c-1,0) \\
c1 \\
c0
\end{matrix}
\left[
\begin{matrix}
\alpha+c\beta \\
c\beta \\
\alpha+(c-1)\beta \\
(c-1)\beta \\
\cdot \\
\cdot \\
\cdot \\
\alpha+\beta \\
\beta \\
\alpha
\end{matrix}
\right] \quad . \tag{34}
$$

5. Applications

The results obtained in sections 3 and 4 are useful in developing techniques for the quality control of production processes in which the quality characteristics (whether defective or not) of successive items are Markov dependent. The distribution of the number of successes can be used to obtain control charts, and the first passage probabilities can be used to develop sequential inspection plans. These applications will be reported in a separate paper. Nevertheless, we believe that the results reported in this paper are important on their own merit in providing properties of Markov trials in much the same way as the properties of Bernoulli trials.

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To show that Theorem 1 holds for i+1 when we assume that it holds for i, consider

$$D_{i+1}^{(n)} = \sum_{j=i}^{n-1} D_i^{(j)} B \, A^{n-1-j}$$

$$= \sum_{j=1}^{n-1} \left[\begin{array}{c|l} X & \sum_{h=1}^{i+1} \delta_{i+h-1,j}^{**} \binom{i}{h-1}\binom{j-i-1}{h-1}(1-a)^{j-i-h}(1-b)^{i+1-h}a^h b^{h-1} \\ \hline & \delta_{ij}(1-b)^i + \\ X & \sum_{h=1}^{i} \delta_{i+h-1,j}^{**} \binom{i}{h}\binom{j-i-1}{h-1}(1-a)^{j-i-h}(1-b)^{i-h}a^h b^{h} \end{array} \right]$$

$$x \left[\begin{array}{c|c} 0 & 0 \\ \hline b(1-a)^{n-1-j} & \delta_{0,n-1-j}^{*} ba(1-a)^{n-2-j} + \delta_{0,n-1-j}(1-b) \end{array} \right]$$

$$= \left[\begin{array}{c|c} I & III \\ \hline II & IV \end{array} \right] \quad ,$$

where

$$I = \sum_{j=i}^{n-1} \sum_{h=1}^{i+1} \delta_{i+h-1,j}^{**} \binom{i}{h-1}\binom{j-i-1}{h-1}(1-a)^{n-1-i-h}(1-b)^{i+1-h}a^h b^h \quad ,$$

$$II = \sum_{j=i}^{n-1} \delta_{ij}(1-b)^i b(1-a)^{n-1-j} +$$

$$\sum_{j=i}^{n-1} \sum_{h=1}^{i} \delta_{i+h-1,j}^{**} \binom{i}{h}\binom{j-i-1}{h-1}(1-a)^{n-1-i-h}(1-b)^{i-h}a^h b^{h+1} \quad ,$$

$$\text{III} = \sum_{j=i}^{n-1} \sum_{h=1}^{i+1} \binom{i}{h-1}\binom{j-i-1}{h-1}\delta^{**}_{i+h-1,j}(1-a)^{j-i-h}(1-b)^{i+1-h}a^h b^{h-1}$$

$$(\delta^{*}_{0,n-1-j}ba(1-a)^{n-2-j} + \delta_{0,n-1-j}(1-b)) \ ,$$

and

$$\text{IV} = \sum_{j=i}^{n-1} \delta_{ij}\delta^{*}_{0,n-1-j}(1-a)^{n-2-j}(1-b)^i ab + \sum_{j=i}^{n-1} \delta_{ij}\delta_{0,n-1-j}(1-b)^{i+1}$$

$$+ \sum_{j=i}^{n-1} \sum_{h=1}^{i} \binom{i}{h}\binom{j-i-1}{h-1}\delta^{**}_{i+h-1,j}(1-a)^{j-i-h}(1-b)^{i-h}a^h b^h$$

$$(\delta^{*}_{0,n-1-j}ab(1-a)^{n-2-j} + \delta_{0,n-1-j}(1-b)).$$

In order to show that expression for $D^{(n)}_{i+1}$ are consistent with (14), we simplify I, II, III, and IV as follows.

Consider

$$\text{I} = \sum_{h=1}^{i+1}\left(\sum_{j=i}^{n-1} \delta^{**}_{i+h-1,j}\binom{j-i-1}{h-1}\right)\binom{i}{h-1}(1-a)^{n-1-i-h}(1-b)^{i+1-h}a^h b^h \ .$$

Note that

$$\sum_{j=i}^{n-1} \delta^{**}_{i+h-1,j}\binom{j-i-1}{h-1} = \delta^{**}_{i+h-1,n-1}\sum_{j=i+h}^{n-1}\binom{j-i-1}{h-1} = \delta^{**}_{i+h,n}\binom{n-1-i}{h}. \quad \text{(a)}$$

On substituting the result in (a) in the expression for I, we get

$$\text{I} = \sum_{h=1}^{i+1} \delta^{**}_{i+h,n}\binom{n-1-i}{h}\binom{i}{h-1}(1-a)^{n-1-i-h}(1-b)^{i+1-h}a^h b^h \ . \quad \text{(b)}$$

Consider

$$\text{II} = (1-b)^i b\left(\sum_{j=i}^{n-1}\delta_{ij}(1-a)^{n-1-j}\right) +$$

$$\sum_{h=1}^{i}\left(\sum_{j=i}^{n-1}\delta^{**}_{i+h-1,j}\binom{j-i-1}{h-1}\right)\binom{i}{h}(1-a)^{n-1-i-h}(1-b)^{i-h}a^h b^{h+1}$$

18

$$= \delta_{0,i+1,n}^{**}(1-b)^i b(1-a)^{n-1-i}$$

$$+ \sum_{h=1}^{i} \delta_{i+h,n}^{**}\binom{n-1-i}{h}\binom{i}{h}(1-a)^{n-1-i-h}(1-b)^{i-h}a^h b^{h+1} \ . \tag{c}$$

Consider

$$III = \sum_{h=1}^{i+1}(\sum_{j=i}^{n-1}\delta_{i+h-1,j}^{**}\delta_{0,n-1-j}^{*}\binom{j-i-1}{h-1}\binom{i}{h-1}(1-a)^{n-2-i-h}$$

$$(1-b)^{i+1-h}a^{h+1}b^h$$

$$+ \sum_{h=1}^{i+1}(\sum_{j=i}^{n-1}\delta_{i+h-1,j}^{**}\delta_{0,n-1-j}\binom{j-i-1}{h-1}(1-a)^{j-i-h})\binom{i}{h-1}(1-b)^{i+2-h}a^h b^{h-1}$$

$$= \sum_{h=1}^{i+1}\delta_{i+h+1,n}^{**}\binom{n-2-i}{h}\binom{i}{h-1}(1-a)^{n-2-i-h}(1-b)^{i+1-h}a^{h+1}b^h$$

$$+ \sum_{h=1}^{i+1}\delta_{i+h,n}^{**}\binom{n-2-i}{h-1}\binom{i}{h-1}(1-a)^{n-1-i-h}(1-b)^{i+2-h}a^h b^{h-1}$$

$$= \delta_{i+i+2,n}^{**}\binom{n-2-i}{i+1}\binom{i}{i}(1-a)^{n-2-i-i-1}(1-b)^{i+1-i-1}a^{i+1+1}b^{i+1}$$

$$+ \sum_{h=2}^{i+1}\delta_{i+h,n}^{**}\binom{n-2-i}{h-1}(\binom{i}{h-2} + \binom{i}{h-1})(1-a)^{n-1-i-h}(1-b)^{i+2-h}a^h_n b^{h-1}$$

$$+ \delta_{i+1,n}^{**}\binom{n-2-i}{0}\binom{i}{0}(1-a)^{n-1-i-1}(1-b)^{i+2-1}a^1 b^{1-1}$$

$$= \sum_{h=1}^{i+2}\delta_{i+h,n}^{**}\binom{n-2-i}{h-1}\binom{i+1}{h-1}(1-a)^{n-1-i-h}(1-b)^{i+2-h}a^h b^{h-1} \ . \tag{d}$$

19

Consider

$$IV = (1-b)^i ab \sum_{j=i}^{n-1} \delta_{ij}\delta^*_{0,n-1-j}(1-a)^{n-2-j} + (1-b)^{i+1}\sum_{j=i}^{n-1} \delta_{ij}\delta_{0,n-1-j}$$

$$+ \sum_{h=1}^{i}(\sum_{j=i}^{n-1}\delta^{**}_{i+h-1,j}\delta_{0,n-1-j}\binom{j-i-1}{h-1}(1-a)^{j-i-h})\binom{i}{h}(1-b)^{i+1-h}a^h b^h$$

$$+ \sum_{h=1}^{i}(\sum_{j=i}^{n-1}\delta^{**}_{i+h-1,j}\delta^*_{0,n-1-j}\binom{j-i-1}{h-1})\binom{i}{h}(1-a)^{n-2-i-h}(1-b)^{i-h}a^{h+1}b^{h+1}$$

$$= \delta^{**}_{i+1,n}(1-a)^{n-2-i}(1-b)^i ab + \delta_{i+1,n}(1-b)^{i+1}$$

$$+ \sum_{h=1}^{i}\delta^{**}_{i+h,n}\binom{n-2-i}{h-1}\binom{i}{h}(1-a)^{n-1-i-h}(1-b)^{i+1-h}a^h b^h$$

$$+ \sum_{h=1}^{i}\delta^{**}_{i+h+1,n}\binom{n-2-i}{h}\binom{i}{h}(1-a)^{n-2-i-h}(1-b)^{i-h}a^{h+1}b^{h+1}$$

$$= \delta_{i+1,n}(1-b)^{i+1} + \delta^{**}_{i+1,n}\binom{n-2-i}{1-1}(1 + \binom{i}{1})(1-a)^{n-2-i}(1-b)^{i+1-1}a^1 b^1$$

$$+ \sum_{h=2}^{i}\delta^{**}_{i+h,n}\binom{n-2-i}{h-1}(\binom{i}{h} + \binom{i}{h-1})(1-a)^{n-1-i-h}(1-b)^{i+1-h}a^h b^h$$

$$+ \delta^{**}_{i+i+1,n}\binom{n-2-i}{i}\binom{i}{i}(1-a)^{n-2-i-i}(1-b)^{i-i}a^{i+1}b^{i+1}$$

$$= \delta_{i+1,n}(1-b)^{i+1}$$

$$+ \sum_{h=1}^{i+1}\delta^{**}_{i+h,n}\binom{n-2-i}{h-1}\binom{i+1}{h}(1-a)^{n-1-i-h}(1-b)^{i+1-h}a^h b^h \ . \tag{e}$$

From (b), (c), (d) and (e), one can see that (14) holds for i+1.

20