Regression Type Tests for Parameteric Hypotheses Based on Sums of Squared L-Statistics

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Regression Type Tests for Parameteric Hypotheses

Based on Sums of Squared L-Statistics

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Abstract

A procedure is developed for testing the hypothesis $H_0 : \underline{\theta} = \underline{\theta}_0$ against the alternative $H_A: \underline{\theta} \neq \underline{\theta}_0$ for a continuous, univariate distribution depending on a parameter vector $\underline{\theta}$. The statistic used for the test is a sum of squared L-statistics that is asymptotically equivalent in distribution, under both the null hypothesis and local alternatives, to the generalized likelihood ratio statistic for testing H_0.

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1. **Introduction.** Let $X_1,\ldots,X_n$ be independent identically distributed random variables with common absolutely continuous distribution function (d.f.) $F(x;\underline{\theta})$, for $\underline{\theta}$ a p×1 vector of parameters. In this paper a simple test procedure is developed for testing H_0: $\underline{\theta} = \underline{\theta}_0$ against the composite alternative H_A: $\underline{\theta} \neq \underline{\theta}_0$.

One method of testing H_0 against H_A, which is applicable to general F, is based on the generalized likelihood ratio statistic (G.L.S), Λ_n. This statistic is the ratio of the sample likelihood under H_0 to the likelihood function with $\underline{\theta}$ replaced by its maximum likelihood estimate. It is well known that, subject to regularity conditions, $S_n = -2 \ln \Lambda_n$ converges in distribution to a central chi-squared random variable with p degrees of freedom when H_0 holds. Thus, S_n is frequently used for testing H_0.

In the present paper a regression framework is utilized to derive an alternative test statistic, applicable to continuous random variables, that is asymptotically equivalent in distribution to S_n under both the null hypothesis and sequences of local alternatives. The proposed statistic is a sum of squared L-statistics that, unlike the G.L.S., does not require parameter estimation. It, therefore, has a computational advantage over the G.L.S. in many cases.

An intuitive development of the test statistic, from the viewpoint of (continuous time) regression analysis of the quantile process, is presented in the next section along with our principle asymptotic results. Section 3 contains a discussion of the case when $\underline{\theta}' = (\mu,\sigma)$, a vector of location and scale parameters.

2. Test Derivation and Main Results. Let $f(x;\underline{\theta})$, for $\underline{\theta} \in \Theta$ an open subset of $\mathbb{R}^p$, denote the density function for $F(x;\underline{\theta})$. The quantile function associated with F is defined by $Q(u;\underline{\theta}) = \inf\{x: F(x;\underline{\theta}) \geq u\}$, $0 < u < 1$, and the density-quantile function is $fQ(u;\underline{\theta}) = f(Q(u;\underline{\theta});\underline{\theta})$ (see Parzen 1979a,b for discussions of these functions). When $\underline{\theta} = \underline{\theta}_0$ the notational conventions $Q_0(u) = Q(u;\underline{\theta}_0)$ and $f_0Q_0(u) = fQ(u;\underline{\theta}_0)$ will also be employed.

Let $X_{1,n} \leq \ldots \leq X_{n,n}$ denote the sample order statistics and define the sample quantile function by

$$Q_n(u) = X_{j,n}, \qquad \frac{j-1}{n} < u \leq \frac{j}{n}, \quad j = 1,\ldots,n. \tag{1}$$

It follows from results in Csörgő (1983) that, under appropriate restrictions,

$$\sup_{0<u<1} \left| \sqrt{n}\, fQ(u;\underline{\theta})(Q_n(u) - Q(u;\underline{\theta})) - B(u) \right| \overset{p}{\to} 0 , \tag{2}$$

where "$\overset{p}{\to}$" denotes "converges in probability" and $\{B(u): 0 \leq u \leq 1\}$ is a Brownian bridge process, i.e., a zero mean normal process with covariance kernel $K(s,t) = \min(s,t) - st$.

Suppose H_0: $\underline{\theta} = \underline{\theta}_0$ holds. Then, (2) may be used to justify the approximate model

$$\sqrt{n}\, f_0Q_0(u)Q_n(u) = \sqrt{n}\, f_0Q_0(u)Q_0(u) + B(u). \tag{3}$$

The test statistic will be derived, using (3), from a regression analysis perspective.

To detect departures from H_0 (i.e., departures from model (3)) we fit, in a figurative sense, the model

$$\sqrt{n}\, f_0Q_0(u)(Q_n(u) - Q_0(u)) = \sum_{i=1}^{p} \delta_i f_0Q_0(u)D_i(u) + B(u) \ , \qquad (4)$$

where $D_i(u) = \left.\frac{\partial Q(u;\underline{\theta})}{\partial \theta_i}\right|_{\underline{\theta}=\underline{\theta}_0}$ and δ_i is the i^{th} element of $\underline{\delta} = \sqrt{n}(\underline{\theta} - \underline{\theta}_0)$. This approach closely parallels the goodness-of-fit method for testing the specification of a nonlinear regression model pioneered by Hartley (1964) and others (see Gallant 1975).

Model (4) can be viewed as a continuous time regression model in the stochastic process $Y(u) = \sqrt{n}\, f_0Q_0(u)(Q_n(u) - Q_0(u))$ with regression functions

$$g_i(u) = f_0Q_0(u)D_i(u) \ , \qquad i = 1,\ldots,p \ , \qquad (5)$$

regression coefficients $\delta_1, \ldots, \delta_p$ and error process $B(u)$. Our objective is to test H_0: $\delta_i = 0$, $i = 1,\ldots,p$.

Parzen (1961, Section 8) addresses the problem of hypothesis testing in continuous time linear models and derives a test statistic which can be used for this purpose. Since (4) is only an approximate model his results are not directly applicable. Nonetheless, we use them to suggest the form of our test statistic and then give a rigorous justification (c.f. Theorem 1 below) for its use.

By application of results in the proof of Theorem 8A of Parzen (1961) to the case of a Brownian bridge error process a discretized version of Parzen's test statistic for model (4) is found to be

$$T_n = \underline{Z}'I^{-1}\underline{Z} \ , \tag{6}$$

where $\underline{Z}$ has typical element

$$Z_i = -n^{-\frac{1}{2}} \sum_{j=1}^{n} g_i''\left(\frac{j}{n+1}\right) f_0Q_0\left(\frac{j}{n+1}\right)\left[Q_n\left(\frac{j}{n+1}\right) - Q_0\left(\frac{j}{n+1}\right)\right],$$

$$i = 1, \ldots, p, \tag{7}$$

and I is the Fisher information matrix, i.e.,

$$I = \{ \textstyle\int_0^1 g_i'(u)g_j'(u)du\}_{i,j=1}^{p} \ .$$

H_0 is to be rejected at significance level α if T_n exceeds its upper α level critical value.

Initial inspection may leave the impression that the computation of T_n may be somewhat involved. For most cases of practical interest this is typically not true, however. It is easy to see that I is the usual Fisher information matrix evaluated at $\underline{\theta} = \underline{\theta}_0$. Consequently, the elements of I can be found in the literature for many important problems. Closed form expressions for the quantile and density-quantile functions required for computation of $\underline{Z}$ can be found in Parzen (1979a, 1982) for a variety of standard distribution types. In cases where Q_0 does not have a closed form the necessary values can be obtained using, e.g., IMSL subroutine MDFI.

The statistic T_n has a representation as a sum of squared L-statistics. To see this, let $I^{\frac{1}{2}}$ denote the symmetric square root of I (assumed positive definite) and define $\underline{v}(t)' = (g_1''(t)f_0Q_0(t), \ldots, g_p''(t)f_0Q_0(t))$. By considering the vector of weight functions $\underline{w}(t) = I^{-\frac{1}{2}}\underline{v}(t)$, it is readily verified that

$$T_n = \sum_{i=1}^{p} L_{in}^2 \ , \tag{8}$$

where

$$L_{in} = -n^{-\frac{1}{2}} \sum_{j=1}^{n} w_i\left(\frac{j}{n+1}\right)\left[Q_n\left(\frac{j}{n+1}\right) - Q_0\left(\frac{j}{n+1}\right)\right]. \qquad (9)$$

Equations (8)-(9) make it possible to recognize that T_n is closely related to the sums of squared L-statistics studied in a paper by LaRiccia and Mason (1985) which we hereafter refer to as LM. In LM such statistics were derived for the purpose of goodness-of-fit tests for location/scale families. While our objectives are somewhat different, the similarity between the form of T_n and the statistics used in LM has the consequence that many of their results carry over to the present setting. We will therefore rely heavily on their work and refer the reader to LM for a detailed exposition of results and conditions not specifically described here.

The asymptotic distribution theory for T_n will be derived under H_0 as well as a sequence of local alternatives. In this regard we make the following definition.

Definition. Let $\underline{\beta}$ be a fixed, but arbitrary, element of $\mathbb{R}^p - \{\underline{0}\}$ which satisfies $\underline{\theta}_0 + \underline{\beta}n^{-\frac{1}{2}} \in \Theta$ for all $n \geq 1$. Then, any sequence $\{X_i^{(n)}\}_{i=1}^n$, $n \geq 1$, where the $X_i^{(n)}$, $i = 1, \ldots, n$, are independent random variables with common distribution function $F(\cdot; \underline{\theta}_0 + \underline{\beta}n^{-\frac{1}{2}})$, is termed a sequence of local alternatives.

Our principal asymptotic results for T_n are summarized in Theorem 1. For Theorem 1 to hold two sets of technical restrictions are required. The first set consists of Conditions B-H of LM which include smoothness and boundary restrictions for the g_i. The second set of restrictions is provided by conditions (IV) - (V) of LM which assures the correct limiting behavior for $\underline{Z}$. The reader is referred to LM for a detailed development.

Theorem 1. Under Conditions B-H and (IV) - (V) of LM, and the assumption that $\underline{Z} \xrightarrow{d} N_p(\underline{0},I)$ under H_0,

$$T_n \xrightarrow{d} \chi_p^2(0), \quad \text{under } H_0 ,$$

where $\overset{\text{"d"}}{\rightarrow}$ denotes "converges in distribution" and $\chi_p^2(\Delta)$ indicates the noncentral chi-squared distribution with noncentrality parameter Δ. For any sequence of local alternatives

$$T_n \xrightarrow{d} \chi^2(\underline{\beta}'I\ \underline{\beta}) .$$

A proof of Theorem 1 can be obtained by modifying the proof of Lemma 1 in LM to make it applicable to the L_{in} defined in (9). The key difference is that, in our case, the L_{in} depend on $Q_n\left(\frac{j}{n+1}\right) - Q_0\left(\frac{j}{n+1}\right)$ rather than Q_n alone as in the LM paper. Further details of the proof are left to the reader.

An important difference between our results and those in LM is that Theorem 1 is applicable to tests of hypotheses about the values of location and scale parameters. We illustrate this point in the next section.

Our next result indicates the relationship between T_n and the GLS. Its proof is an immediate consequence of results in Wald (1943) and Theorem 1.

Corollary. Subject to the conditions of Theorem 1 and the regularity conditions in Wald (1943), T_n has the same asymptotic distribution, under both H_0 and any sequence of local alternatives, as $S_n = -2 \ln \Lambda_n$.

As a consequence of the Corollary we see that T_n has the same asymptotic power against local alternatives as S_n. It should be pointed out, however, that unlike the GLS T_n does not require parameter estimation. In addition, although it is beyond the scope of this paper, by imposing further restrictions, using (8) and the strong representation for linear combinations of order statistics given in Govindarajulu and Mason (1983), it can be shown that T_n is asymptotically equivalent in probability to the Wald statistic, S_n.

3. Application to Location/Scale Models. In this section we briefly discuss the use of T_n under a location and scale parameter model. The test statistic has a particularly simple form in this case and provides a natural complement to the estimators of μ and σ proposed by Parzen (1979a,b).

Let $p = 2$ and assume that $F(x;\underline{\theta}) = F_0\left(\frac{x-\mu}{\sigma}\right)$ with $-\infty < \mu < \infty$ and $\sigma > 0$. Making a slight change of notation, let f_0 and Q_0 denote the density and quantile functions corresponding to F_0. Thus, $Q(u;\underline{\theta}) = \mu + \sigma Q_0(u)$, $fQ(u;\underline{\theta}) = \sigma^{-1} f_0Q_0(u)$ and the required functions are $g_1(u) = f_0Q_0(u)/\sigma_0$ and $g_2(u) = f_0Q_0(u)Q_0(u)/\sigma_0$.

If we introduce the score function $J_0(u) = f_0'Q_0(u)/f_0Q_0(u) = (f_0Q_0)'(u)$, the information matrix, I, is seen to have elements $I_{11} = \sigma_0^{-1} \int_0^1 J_0(u)^2 du$, $I_{22} = \sigma_0^{-2} \int_0^1 (1 + J_0(u)Q_0(u))^2 du$, and $I_{12} = I_{21} = \sigma_0^{-2} \int_0^1 (J_0(u) + J_0(u)^2 Q_0(u))du$. Thus, for testing H_0: $(\mu,\sigma) = (\mu_0,\sigma_0)$ versus H_A: $(\mu,\sigma) \neq (\mu_0,\sigma_0)$ we find that

$$T_n = \sigma_0^{-2}(I_{22}Z_1^2 - 2I_{12}Z_1Z_2 + I_{11}Z_2^2)/(I_{11}I_{22} - I_{12}^2) ,$$

where

$$Z_1 = -n^{-\frac{1}{2}} \sum_{i=1}^{n} J'\left(\frac{i}{n+1}\right) f_0Q_0\left(\frac{i}{n+1}\right)\left[Q_n\left(\frac{i}{n+1}\right) - \mu_0 - \sigma_0Q_0\left(\frac{i}{n+1}\right)\right]$$

and

$$Z_2 = -n^{-\frac{1}{2}} \sum_{i=1}^{n} \left\{ \left[J'\left(\frac{i}{n+1}\right) Q_0\left(\frac{i}{n+1}\right) f_0Q_0\left(\frac{i}{n+1}\right) + J\left(\frac{i}{n+1}\right)\right] \times \left[Q_n\left(\frac{i}{n+1}\right) - \mu_0 - \sigma_0Q_0\left(\frac{i}{n+1}\right)\right] \right\} .$$

Let $\underline{E} = (E_1, \ldots, E_n)'$ and $V = \{v_{ij}\}_{i,j=1}^{n}$ denote, respectively, the vector of expected values and variance-covariance matrix for the order statistics in a random sample of size n from F_0. The vector $\underline{Q} = \left(Q_0\left(\frac{1}{n+1}\right), \ldots, Q_0\left(\frac{n}{n+1}\right)\right)'$ and matrix I are actually asymptotic approximations to $\underline{E}$ and $\sigma_0^{-2}X'V^{-1}X$, where $X = [\underline{1}, \underline{E}]$ for $\underline{1}$ a $n\times 1$ vector of all unit elements. Thus, for smaller samples, it may be useful to replace $\underline{Q}$ and I by their finite sample analogs when computing T_n for tests about μ and σ. Tables are available in the literature from which $\underline{E}$ and V can be obtained for several standard distributions.

Finally, it follows from the discussion in Section 2 that for location and scale models $I^{-1}\underline{Z}$ is a discrete approximation to $(\tilde{\mu} - \mu_0, \tilde{\sigma} - \sigma_0)'$, where $\tilde{\mu}$ and $\tilde{\sigma}$ are the optimal estimators of μ and σ proposed by Parzen (1979a,b). Thus, for location/scale models, the results of Section 2 provide a hypothesis testing procedure that can be used to complement Parzen's estimators.

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