

THE G-SPECTRAL ESTIMATOR

by

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Technical Report No. 159
Department of Statistics ONR Contract

May 1982

Research sponsored by the Office of Naval Research
Contract N00014-75-C-0439
Project NR 042-280

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Introduction

A spectral estimator referred to as the G-spectral estimator was introduced by Gray [4] and by Gray and Foster [5]. It was then studied in some detail by Gray, Houston and Morgan [6]. The estimator was based upon an approximate G_n -transform (see Gray, Houston and Morgan) of the sample autocorrelation function and is equivalent to an e_n -transformation [11] of the same function.

Gray, Houston and Morgan (GHM) noted the estimator to have some useful properties but noted some shortcomings as well. In particular it required more autocorrelation values than seemed necessary and included no satisfactory way of selecting the proper value n in the e_n -transform. GHM went on to conclude that an ARMA spectral estimator would probably have better properties than the G-spectral estimator which they had studied.

In this paper, we wish to give a modified definition of the G-spectral estimator which will be seen to avoid the difficulties noted by GHM, and will in fact, be shown to be equivalent to a method of moments ARMA spectral estimator.

Definitions and Theorems

The e_n -transformation has a rich history and has been studied in detail in numerous places (see for instance Shanks [11] and Wynn [12], and for its application to complex sequences McWilliams [9]). We define the e_n -transform formally as follows.

Definition 1. Let $\{f_m\}$ be a sequence of complex numbers defined for

$$m = -k, -k+1, \dots$$

and let

$$F_m = \sum_{j=-k}^m f_j.$$

We then define for $m \geq n-k-1$, $n \geq 1$

$$e_n(F_m) = \begin{vmatrix} F_{m-n+1} & F_{m-n+2} & \cdots & F_{m+1} \\ f_{m-n+1} & f_{m-n+2} & \cdots & f_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_m & f_{m+1} & \cdots & f_{m+n} \end{vmatrix} \div \begin{vmatrix} 1 & 1 & \cdots & 1 \\ f_{m-n+1} & f_{m-n+2} & \cdots & f_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_m & f_{m+1} & \cdots & f_{m+n} \end{vmatrix}$$

whenever the indicated division is defined. Otherwise, $e_n(F_m) = e_{n-1}(F_{m-1})$.

We define $e_0(F_m) = F_m$.

When considering ARMA processes, a natural class of sequences to consider is the class of sequences eventually satisfying a linear homogeneous difference equation with constant coefficients. For ease of reference, we make the following definition.

Definition 2. Let $\{f_j\}$ be a sequence of complex numbers. We say $\{f_j\}$ is an element of $L(n, s, \Delta)$ if n is the smallest whole number for which there exists an integer ℓ and complex numbers $\phi_1, \dots, \phi_n$, such that $f_j - \phi_1 f_{j-1} - \dots - \phi_n f_{j-n} = 0$ for all $j > \ell$. We then take s to be the infimum of all such ℓ . If $f_j = 0$ for $j > s$ and $f_s \neq 0$ then we say $\{f_j\}$ is an element of $L(0, s, \Delta)$.

For k finite, it is well-known that if $\{f_j\}$ is an element of $L(n, s, \Delta)$, then

$$\sum_{j=-k}^{\infty} |f_j| < \infty$$

if and only if all of the solutions to the algebraic equation (called the characteristic equation)

$$\phi(r) = 1 - \phi_1 r - \phi_2 r^2 - \dots - \phi_n r^n = 0$$

are strictly outside the unit circle.

Assuming only that $\sum_{j=-k}^{\infty} |f_j| < \infty$ numerous results are available giving sufficient conditions for $e_n(F_m)$ to speed up the rate of convergence of $\{F_m\}$. In particular the following exactness result is well-known.

Theorem 1. If $\{f_m\}$ is an element of $L(n, s, \Delta)$ and all of the roots of $\phi(r)$ are outside the unit circle, then

$$e_n(F_m) = \sum_{j=-k}^{\infty} f_j$$

for all $m \geq s$, provided k is finite and $e_n(F_m)$ is defined (i.e., provided $m \geq n-k-1$).

Proof. The proof for $\{f_m\}$ a real sequence is given in GHM and the proof for $\{f_m\}$ a complex sequence follows in the same manner.

Noting that for a wide-sense stationary real stochastic process, the spectral density can be expressed as

$$s(\omega) = 1 + 2 \sum_{j=1}^{\infty} \rho(j) \cos(2\pi\omega j),$$

Gray was led to consider applying the e_n -transformation to the sequence $\{f_m\}$ defined by

$$f_m = \begin{cases} 1, m=0 \\ 2\rho(m) \cos(2\pi\omega m), m \geq 1. \end{cases} \quad (1)$$

If $\{X(t)\}$ is an ARMA (p, q) process, $\{\rho(m)\}$ is an element of $L(p, q, \Delta)$ from which it follows that $\{\rho(m) \cos(2\pi\omega m)\}$ is an element of $L(p+n(\omega), q'(\Delta))$ where $0 \leq n(\omega) \leq p$. Thus, if one knows (or has estimated) the order of the process $\{X(t)\}$, it is natural to replace $\rho(m)$ in (1) by $\hat{\rho}(m)$ and then to perform an $e_{p+n(\omega)}$ -transform on the sequence thus obtained. The result of this procedure is the GHM version of the G-spectral estimator.

When the G-spectral estimator was originally proposed, there was no adequate method available for choosing the order (p,q) of the process. That, in conjunction with the fact that the order of the difference equation satisfied by the sequence given in (1) changes with frequency, necessitated that essentially a different model be chosen at every frequency. This made the procedure extremely cumbersome to implement and also required additional smoothing of the resulting estimate. Furthermore since $n(\omega)$ is typically equal to p , a larger number of autocorrelations was required in the estimate than seemed necessary.

Gray, Kelley and McIntire [7] proposed a method for determining (p,q) based upon the so-called R- and S-arrays. This procedure makes the order selection procedure tractable. The procedure, which is unambiguous if the true autocorrelation function is known, has been shown to be robust to stochastic disturbance. In addition, other methods for determining (p,q) have been proposed such as AIC [1] or BIC [2]. However, in this paper we will make use only of the Gray, Kelley, McIntire method.

For those readers unfamiliar with the method, the theoretical S-array is given in Table 1 for an ARMA (p,q) process. The procedure then consists of choosing p and q so that the sample S-array best matches the theoretical pattern.

As an alternative to the sequence given in (1) it now appears more sensible to use the sequence

$$f_m = \exp(2\pi i \omega m) \rho(m), \quad m = -k, -k+1, \dots, \quad (2)$$

where k is chosen to be positive and larger than $p+q-2$. In that case, $\{f_m\}$ will be an element of $L(p,q,\Delta)$ for all frequencies. From Theorem 1, taking $\{f_m\}$ as in (2), it is immediate that if $\rho(m)$ is the autocorrelation function

of an ARMA (p,q) process and $\{f_m\}$ is given by (2), then

$$e_p(F_m) = \sum_{j=-k}^{\infty} f_j \quad \text{for all } m \geq q.$$

Thus the spectral density $s(\omega)$ is given by

$$s(\omega) = 2 \operatorname{Real} [e_p(F_m) - F_0] + 1 \quad \text{for all } m \geq q, \quad (3)$$

where $F_v = \sum_{j=-k}^v f_j.$

From the discussion given above, the following modified definition of the G-spectral estimator should appear natural.

Definition 3. Let $s(\omega)$ be the spectral density of a wide-sense stationary stochastic process and $\hat{\rho}(j)$ an estimate of its autocorrelation function.

The order (n,m) G-spectral estimator is then defined as

$$G_{n,m}(\omega) = 2 \operatorname{Real} (e_n(F_m) - F_0) + 1$$

where

$$F_v = \sum_{j=-k}^v \exp(2\pi i \omega j) \hat{\rho}(j), \quad v = -k, -k+1, \dots$$

and

$$k > \max(0, n-m-2).$$

We will show below that the modified G-spectral estimator given in Definition 3 is equivalent to an ARMA spectral estimator. To facilitate the discussion we make the following definition.

Definition 4. Let $\hat{\rho}(j)$ be the sample autocorrelation estimator for $\rho(j)$ and let $\hat{\phi}_1, \dots, \hat{\phi}_n$ be the solution to the equations

$$\begin{aligned} \hat{\rho}(m+1) &= \hat{\phi}_1 \hat{\rho}(m) + \dots + \hat{\phi}_n \hat{\rho}(m-n+1) \\ &\vdots \\ \hat{\rho}(m+n) &= \hat{\phi}_1 \hat{\rho}(m+n-1) + \dots + \hat{\phi}_n \hat{\rho}(m). \end{aligned} \quad (4)$$

Then let $\hat{\theta}_1, \dots, \hat{\theta}_m$ be the solution to the equations

$$\hat{\rho}(\ell) = \rho(\ell; \hat{\phi}_1, \dots, \hat{\phi}_n, \hat{\theta}_1, \dots, \hat{\theta}_m), \quad \ell = 1, \dots, m \quad (5)$$

where

$$\rho(\ell; \phi_1, \dots, \phi_n, \theta_1, \dots, \theta_m)$$

is the autocorrelation function of the ARMA (n,m) process with parameters $\phi_1, \dots, \phi_n$ and $\theta_1, \dots, \theta_m$. For uniqueness, we require that the solution to (5) be such that the resulting ARMA process is invertible (i.e., the roots of $\theta(r) = 1 - \theta_1 r - \theta_2 r^2 - \dots - \theta_m r^m = 0$ are outside the unit circle). We then call the spectral density of the above estimated ARMA process

$$\begin{aligned} \hat{s}_{n,m}(\omega) &= \sum_{j=-\infty}^{\infty} \exp(2\pi i \omega j) \rho(j; \hat{\phi}_1, \dots, \hat{\phi}_n, \hat{\theta}_1, \dots, \hat{\theta}_m) \\ &= \frac{\hat{\sigma}_a^2 |\hat{\theta}(\exp(2\pi i \omega))|^2}{\hat{\sigma}_x^2 |\hat{\phi}(\exp(2\pi i \omega))|^2} \end{aligned}$$

the order (n,m) method of moments ARMA spectral density estimator.

In the above definition it was tacitly assumed that a solution to (4) and (5) exists which corresponds to a stationary ARMA (n,m) process. However, for $m > 0$, even if $\hat{\rho}(j)$ is positive definite, that need not be the case. The solution to (4) may have a characteristic equation with roots inside the unit circle and there may exist no solution to (5) at all. In that case, we say that the method of moments spectral estimator is undefined.

The relationship between the two spectral estimators is as follows.

Theorem 2. Let $G_{n,m}(\omega)$ and $\hat{s}_{n,m}(\omega)$ be as defined above and suppose that $\hat{s}_{n,m}(\omega)$ exists. Then

$$G_{n,m}(\omega) \equiv \hat{s}_{n,m}(\omega) .$$

That is $G_{n,m}(\omega)$ is an ARMA spectral estimator which avoids the computation of the moving average parameter estimates.

Proof. It is immediate from the definition that

$$\hat{\rho}(\ell) = \rho(\ell; \hat{\phi}_1, \dots, \hat{\phi}_n, \hat{\theta}_1, \dots, \hat{\theta}_m), \ell = 1, \dots, m+n.$$

Further $\{\rho(\ell; \hat{\phi}_1, \dots, \hat{\phi}_n, \hat{\theta}_1, \dots, \hat{\theta}_m)\}$ is an element of $L(n, m, \Delta)$. The result then follows from equation (3).

The above theorem also provides an alternative means of calculating the method of moments parameter estimates utilizing the so-called inverse autocorrelations introduced by Cleveland [3]. In particular we have the following easily proved result.

Theorem 3. Let $\hat{\phi}_1, \dots, \hat{\phi}_n, \hat{\theta}_1, \dots, \hat{\theta}_m$ and $G_{n,m}(\omega)$ be as defined above and suppose that $\hat{s}_{n,m}(\omega)$ exists. Then if

$$ci(k) = \int_0^{.5} \frac{\cos(2\pi i \omega k)}{G_{n,m}(\omega)} d\omega, \quad (6)$$

$\hat{\theta}_1, \dots, \hat{\theta}_m$ are the solutions to

$$\begin{aligned} ci(n+1) &= \theta_1 ci(n) + \dots + \theta_m ci(n-m+1) \\ &\vdots \\ ci(n+m) &= \theta_1 ci(n+m-1) + \dots + \theta_m ci(n). \end{aligned} \quad (7)$$

The above procedure for calculating $\hat{\theta}_1, \dots, \hat{\theta}_m$ provides a better numerical procedure than the usual non-linear scheme (where (6) is approximated by Simpson's Rule for instance). The solution to (7) is unique and ensures that an invertible solution will be obtained. Further details will appear in a later paper by Morton.

Since the spectral estimator in Definition 3 is an ARMA spectral estimator, when the latter exists the former is non-negative. However,

since $G_{n,m}(\omega)$ may exist when $\hat{s}_{n,m}(\omega)$ does not, it need not always be non-negative. It is, in that case, natural to consider shifting the estimator upward or truncating it from below. Two methods for shifting the estimator upward are contained in the following theorem. Their relative merits are unclear, but the second estimator seems more intuitively appealing. The second estimator has the additional virtue of providing initial "estimates" (using Theorem 3 on the "shifted" spectrum) for $\hat{\theta}_1, \dots, \hat{\theta}_m$ when no solution to the equations in (5) exists.

Theorem 4. Let $G_{n,m}(\omega)$ and $\hat{\phi}_1, \dots, \hat{\phi}_n$ be as defined above and let C_1 and C_2 be constants (possibly depending on the data) such that

$$(a) \quad \hat{s}_1(\omega) = k_1(G_{n,m}(\omega) + C_1) \geq 0$$

$$(b) \quad \hat{s}_2(\omega) = k_2(G_{n,m}(\omega) + C_2/|\hat{\phi}(e^{2\pi i\omega})|^2) \geq 0$$

where

$$\hat{\phi}(e^{2\pi i\omega}) = 1 - \hat{\phi}_1 e^{2\pi i\omega} - \dots - \hat{\phi}_n e^{2\pi i\omega n}$$

and k_1 and k_2 are chosen such that

$$\int_{-.5}^{.5} \hat{s}_1(\omega) d\omega = \int_{-.5}^{.5} \hat{s}_2(\omega) d\omega = 1.$$

Then $\hat{s}_1(\omega)$ is the spectral density of an ARMA(n, l_1) process where $l_1 \leq \max(n, m)$ and $\hat{s}_2(\omega)$ is the spectral density of an ARMA(n, l_2) process where $l_2 \leq m$.

Proof. It is straightforward to show that

$$G_{n,m}(\omega) = \frac{\sum_{j=-m}^m a_j e^{2\pi i\omega j}}{|1 - \hat{\phi}_1 e^{2\pi i\omega} - \dots - \hat{\phi}_n e^{2\pi i\omega n}|^2}$$

where the a_j are real and $a_{-j} = a_j$. Thus $\hat{s}_1(\omega)$ is of the form

$$\hat{s}_1(\omega) = \frac{\sum_{j=-\ell_1}^{\ell_1} b_j e^{2\pi i \omega j}}{|1 - \hat{\phi}_1 e^{2\pi i \omega} - \dots - \hat{\phi}_n e^{2\pi i \omega n}|^2} \geq 0$$

and the result follows since $\sum_{j=-\ell_1}^{\ell_1} b_j e^{2\pi i \omega j}$ can then be factored. (See Hannan [8] p. 14). The result for $\hat{s}_2(\omega)$ follows similarly.

As a final point, we make the observation that the autoregressive parameters need not be calculated in order to calculate the G-spectral estimator. The calculation can be made using the so-called ϵ -algorithm introduced by Wynn [12]. However, the calculated coefficients themselves provide meaningful information and more importantly offer a more efficient method for calculating the G-spectral estimator using the following result.

Theorem 5. Let $\hat{\phi}_1, \dots, \hat{\phi}_n$ be as given in Definition 4 and let

$$f_j = \hat{\rho}(j) \exp(2\pi i \omega j), \quad F_j = \sum_{\ell=-k}^j f_\ell,$$

$$\alpha_j = \hat{\phi}_j \exp(2\pi i \omega j).$$

Then

$$G_{n,m}(\omega) = 2 \operatorname{Real} \left[\frac{F_{m+1} - \alpha_1 F_m - \dots - \alpha_n F_{m-n+1}}{1 - \alpha_1 - \dots - \alpha_n} - F_0 \right] + 1. \quad (8)$$

Proof. By elementary column operations, we have that

$$e_n(F_m) = \frac{F_{m+1} - \alpha_1 F_m - \dots - \alpha_n F_{m-n+1}}{1 - \alpha_1 - \dots - \alpha_n}$$

and the result follows.

As a more compact formula, we may let $s_j = F_j - F_0 + 1/2$ and (8)

becomes

$$G_{n,m}(\omega) = 2 \operatorname{Real} \left[\frac{s_{m+1} - \alpha_1 s_m - \dots - \alpha_n s_{m-n+1}}{1 - \alpha_1 - \dots - \alpha_n} \right]$$

If f is the number of frequencies at which the spectrum is calculated, the method using the ϵ -algorithm requires roughly $5fn^2$ operations while the method using (8) requires roughly $n^3/3$ operations. Taking $f = 100$, for instance, the break-even point is just less than $n = 1300$, a much higher order autoregressive operator than is ever used in practice.

Examples

In this section, the performance of the spectral estimator introduced in Definition 3 is investigated on several simulated data sets. For comparison purposes, an autoregressive spectral estimate and a Parzen window spectral estimate are also calculated. The order of the autoregressive operator to be used in the autoregressive estimator is chosen according to the minimum FPE criterion of Akaike [1]. The window spectral estimator will be calculated using 15% of the sample autocorrelations.

Example 1. The first example we consider is a realization of length 200 from the MA(3) process

$$X_t = a_t - .9a_{t-1} + .81a_{t-2} - .729a_{t-3}.$$

The true log spectral density for the above process is given in Figure 1.

The first 10 sample autocorrelations are given in Table 2, and the process is identified rather easily as a third order moving average process. The $G_{0,3}(\omega)$ spectral estimate (which here simply reduces to the finite Fourier transform $1 + 2 \sum_{j=1}^3 \hat{\rho}(j) \cos(2\pi\omega j)$) is given in Figure 2.

The minimum FPE criterion chooses a ninth order autoregressive model for this data set. The estimated AR(9) spectral density is given in Figure 3. The numerous false peaks shown there is typical for an autoregressive fit to a moving average spectral density, especially near the non-invertible

region.

The Parzen window spectral estimate using 30 autocorrelations is given in Figure 4 and is somewhat similar to the autoregressive spectral estimate.

Example 2. We now consider a realization of length 400 from the ARMA(2,4) process

$$X_t - 1.38X_{t-1} + .64X_{t-2} = a_t + .95a_{t-1} + .9a_{t-2} + .86a_{t-3} + .81a_{t-4} .$$

The true spectral density for the above process is given in Figure 5.

The sample S-array for this series is given in Table 3, and the correct order is easily identified. The $G_{2,4}(\omega)$ spectral estimate is shown in Figure 6.

The FPE criterion chooses an AR(9) model. The corresponding autoregressive spectral estimate is shown in Figure 7. We note that the intermediate hump shows through only very poorly.

In Figure 8, the Parzen window spectral estimate using 60 autocorrelations is given. It also does not show the intermediate hump as clearly as the $G_{2,4}(\omega)$ estimate and gives a false peak for ω slightly larger than .4.

Example 3. This example illustrates some of the robustness properties of the spectral estimator being investigated. The example is a realization of length 30 from the signal-plus-noise model

$$X_t = \sqrt{20} \cos(.2\pi + \psi) + a_t$$

where $\{a_t\}$ is a variance 1 white noise random sequence and $\psi \sim U(0, 2\pi)$.

The true spectrum is shown in Figure 9, where the spike indicates a Dirac delta function.

The sample S-array is given in Table 4. A second order autoregressive operator appears to be called for, but it is unclear which order moving average operator is most appropriate. A moving average

order of 1 and 3 both give negative spectral estimates. The $G_{2,2}(\omega)$ estimate is given in Figure 10. The peak is rather clearly shown and the estimate is quite flat off the peak.

FPE was minimized for this example using an AR(5) model. The 5th order autoregressive spectrum is given in Figure 11. This estimate also gives good resolution of the peak, however, it is not as flat off the peak as the $G_{2,2}(\omega)$ estimate.

Finally we show the Parzen window spectral estimate using 5 autocorrelations. The bias of the window estimator for short data sets is apparent as there is no indication of a peak at all.

Example 4. Our last example illustrates what we feel is an important point. Namely, that when filtering is necessary to identify the complete model, the spectrum should be estimated in a "piecewise" manner as we illustrate below.

The example is a realization of length 300 from the ARMA(2,4) process $X_t - 1.65X_{t-1} + .9X_{t-2} = a_t + .9a_{t-1} + .81a_{t-2} + .73a_{t-3} + .66a_{t-4}$. This model is similar to that of Example 2 except that the roots of the characteristic equation are nearer to the unit circle and the moving average operator is further from the non-invertibility region. The true spectral density is shown in Figure 13.

The sample S-array is given in Table 5. The near constancy in the second column of the S-array indicates a second order near non-stationarity (see Gray, Kelley and McIntire [7]). The estimated transformation to stationarity (from the sample Yule-Walker equations adjusted for frequency) is $(1 - 1.68B + .94B^2)$.

The first 10 autocorrelations of the transformed series are given in

Table 6. The residual series is then identified very easily as an MA(4) process and the series is thus modeled as an ARMA(2,4) process.

The $G_{2,4}(\omega)$ spectral estimate is given in Figure 14. The estimate is rather disappointing in that it shows no hump at the intermediate frequencies. The reason for that, recalling Theorem 2, is that the method of moments estimate for the moving average parameters here is quite poor (see Table 7). For comparison purposes, we calculated the maximum likelihood estimate for the parameters (see Table 8). That estimate is a considerable improvement over the method of moments estimate as is the accompanying spectral estimate (see Figure 15).

Actually, the direct calculation of $G_{2,4}(\omega)$ is not what we would have recommended from the start. That is, in view of the Findley-Quinn theorem (see [10]) it is clear that the estimate for the moving average coefficients must be poor if one is sufficiently close to the non-stationary region. The domination of the autoregressive operator near the non-stationary region is clearly demonstrated in GKM [7]. The whole problem can easily be avoided by transforming to stationarity, i.e., prefiltering. The approach we now investigate as an alternative to obtaining the maximum likelihood estimate is similar to the pre-filtering procedure of Tukey. First estimate the second-order autoregressive operator from the Yule-Walker equations associated with an ARMA (2,4) model. The estimated operator is $1 - 1.65B + .91B^2$. Then transform by the estimated operator and estimate the spectrum of the residual series by the $G_{0,4}(\omega)$ estimate. The resulting spectral estimate is then $\hat{s}(\omega)$ given by

$$\hat{s}(\omega) = K |1 - 1.65 \exp(2\pi i \omega) + .91 \exp(4\pi i \omega)|^{-2} G_{0,4}(\omega),$$

where K is chosen so that

$$\int_{-.5}^{.5} \hat{s}(\omega) d\omega = 1.$$

The spectral estimate using the above procedure is given in Figure 16. It compares favorably with the maximum likelihood estimate and is actually slightly better in an L^1 sense for this particular example. It is also far easier to calculate than the maximum likelihood estimate.

FPE is minimized using a 5th order autoregressive operator. That spectral estimate is given in Figure 17. It appears similar to the initial $G_{2,4}(\omega)$ estimate, showing no hump at the intermediate frequency (using the maximum likelihood estimate for this model shows no real improvement).

Finally, the Parzen window spectral estimate using 45 autocorrelations is given in Figure 18. The estimate is similar to the autoregressive estimate, but does not resolve the peak quite as well.

Summary and Discussion

In this paper we have defined a modified version of the so-called G-spectral estimator introduced by Gray [5]. We showed that the modified G-spectral estimator is an extension of a method of moments ARMA spectral estimator. Moreover when this method of moments ARMA spectral estimator exists, the G-spectral estimator is equivalent and does not require calculation of the moving average parameters.

We feel that the primary virtues of this spectral estimation procedure are the inherent flexibility of the ARMA model over the AR model and the spectral estimator's computational simplicity. The G-spectral estimator presented here furnishes a closed form expression for the method of moments ARMA(p,q) a spectral estimator, even though no closed form solution exists for the method of moments estimate of the moving average parameters. This is important from a theoretical and a

practical perspective and is currently being exploited by the authors in a number of areas.

The primary difficulty of the procedure is that the method of moments estimate of the moving average parameters is often not very good. That difficulty can frequently be mitigated however, by the prefiltering procedure described in Example 4.

TABLE 1

	1	. . .	p	p+1	p+2	
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
-q-3			C_2	U	U	. . .
-q-2	.		C_2	U	U	. . .
-q-1	$s_1(f_{-q-1})$	. . .	C_2	$\pm\infty$	$\pm\infty$	. . .
-q	$s_1(f_{-q})$	. . .	$s_p(f_{-q})$	$s_{p+1}(f_{-q})$	$s_{p+2}(f_{-q})$	. . .
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
-1	$s_1(f_{-1})$	. . .	$s_p(f_{-1})$	$s_{p+1}(f_{-1})$	$s_{p+2}(f_{-1})$	. . .
0	$s_1(f_0)$	. . .	$s_p(f_0)$	$s_{p+1}(f_0)$	$s_{p+2}(f_0)$	. . .
1	$s_1(f_1)$	. . .	$s_p(f_1)$	$s_{p+1}(f_1)$	$s_{p+2}(f_1)$	. . .
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
q-1	$s_1(f_{q-1})$	. . .	$s_p(f_{q-1})$	$s_{p+1}(f_{q-1})$	$s_{p+2}(f_{q-1})$	. . .
q	$s_1(f_q)$	. . .	C_1	$-C_1$	C_1	
q+1	.		C_1	U	U	. . .
.	.		.	.	.	.
.	.		.	.	.	.
.	.		.	.	.	.

Theoretical S-Array for ARMA(p,q) process.

TABLE 2

1	2	3	4	5
-.71	.48	-.24	.01	-.04
6	7	8	9	10
.01	-.05	.01	.00	-.02

1st 10 sample autocorrelations for Example 1.1

TABLE 3

	S1	S2	S3	S4	S5
-8	.47	.41	-.29	.02	-.46
-7	.05	.41	-.71	.45	-.59
-6	-.35	.41	-.21	-11.24	4.22
-5	-1.16	<u>.41</u> [↑] *	16.44	-7.86	-5.64
-4	9.79	.35	.12	-.08	.03
-3	.94	.28	.09	.06	.64
-2	.37	.23	.06	-.68	.88
-1	.10	.20	.23	-.46	2.19
0	-.09	.18	-.10	.08	-.08
1	-.27	.19	-.04	.06	-.13
2	-.48	.21	-.06	.03	.46
3	-.91	<u>.25</u> [↓] *	-.07	-.50	.15
4	-7.13	.29	-.28	.28	-.28
5	.53	.29	.31	.67	.78
6	-.04	.29	-.68	1.29	-.06
7	-.32	.29	.74	.02	-.67

Sample S-Array for Example 2

*Constancy from these points on implies ARMA(2,4)

TABLE 4

	S1	S2	S3	S4	S5	S6
-7	1.722	.372	-.296	.838	-2.538	1.293
-6	.299	.399	-1.179	1.094	.030	-.850
-5	-.144	.375	-1.049	5.823	.838	-1.191
-4	-.544	.350	-.755	-.505	-7.376	6.851
-3	-1.720	.426	-1.059	4.628	-7.046	25.049
-2	2.167	.243	-2.025	-2.669	-5.507	-17.620
-1	.448	.933	-1.443	2.892	-3.858	6.868
0	-.309	.462	.680	.890	-1.157	1.391
1	-.684	.240	-.446	.289	-.610	.500
2	-2.389	.393	-.509	.905	-.656	.841
3	1.192	.304	-.698	.225	-.605	-21.266
4	.169	.375	-.562	.646	-1.628	.226
5	-.230	.361	-.530	-12.943	-.067	4.528
6	-.633	.351	1.011	2.485	-4.366	4.265

Sample S-array for Example 3 $f_m = \hat{\rho}(m)$

TABLE 5

	S1	S2	S3	S4
-7	-1.972	3.985	-13.810	29.307
-6	-1.721	3.992	.907	-2.805
-5	-1.321	3.986	2.555	-20.639
-4	.641	3.991	-17.473	20.799
-3	-3.566	3.989	-9.757	16.537
-2	-2.444	3.958	-7.687	-49.474
-1	-2.115	3.912	-20.561	-20.285
0	-1.897	3.684	-4.487	3.675
1	-1.692	3.634	-7.399	5.227
2	-1.390	3.589	-6.039	11.284
3	-.380	3.570	-4.626	8.715
4	-4.119	3.570	-1.393	.754
5	-2.387	3.567	-.661	7.425
6	-2.029	3.572	-5.024	4.873

Sample S-array for Example 6 $f_m = \hat{\rho}(m)$

TABLE 6

.79	.57	.38	.20	.01	.00	.02	.01	.03	.08
-----	-----	-----	-----	-----	-----	-----	-----	-----	-----

1st 10 autocorrelations from the residual series of Example 4

TABLE 7

$$\hat{\phi}_1 = 1.65 \quad \hat{\phi}_2 = -.91$$

$$\hat{\theta}_1 = .036 \quad \hat{\theta}_2 = -.26 \quad \hat{\theta}_3 = -.20 \quad \hat{\theta}_4 = -.14$$

Method of Moments Parameter Estimates for Example 4

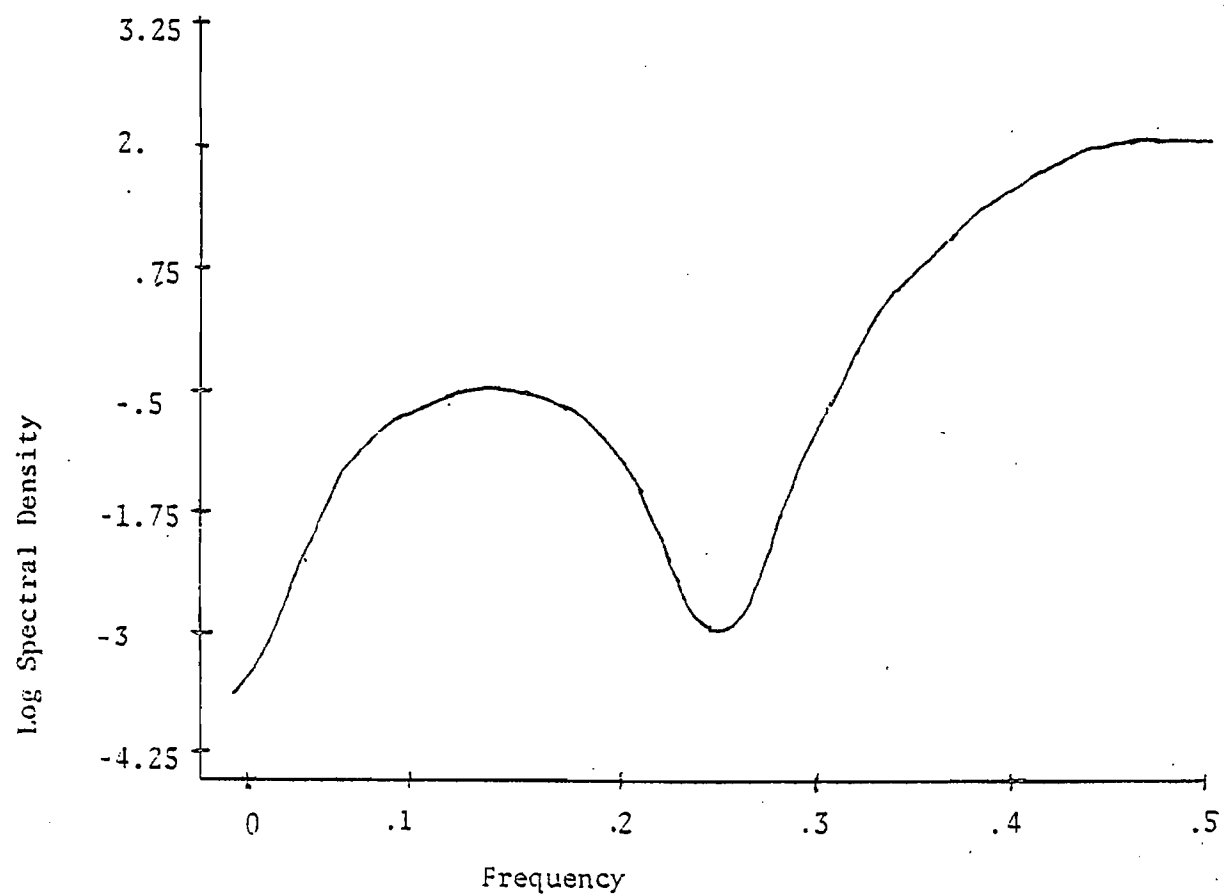
TABLE 8

$$\hat{\phi}_1 = 1.66 \quad \hat{\phi}_2 = -.92$$

$$\hat{\theta}_1 = -.92 \quad \hat{\theta}_2 = -.78 \quad \hat{\theta}_3 = -.66 \quad \hat{\theta}_4 = -.59$$

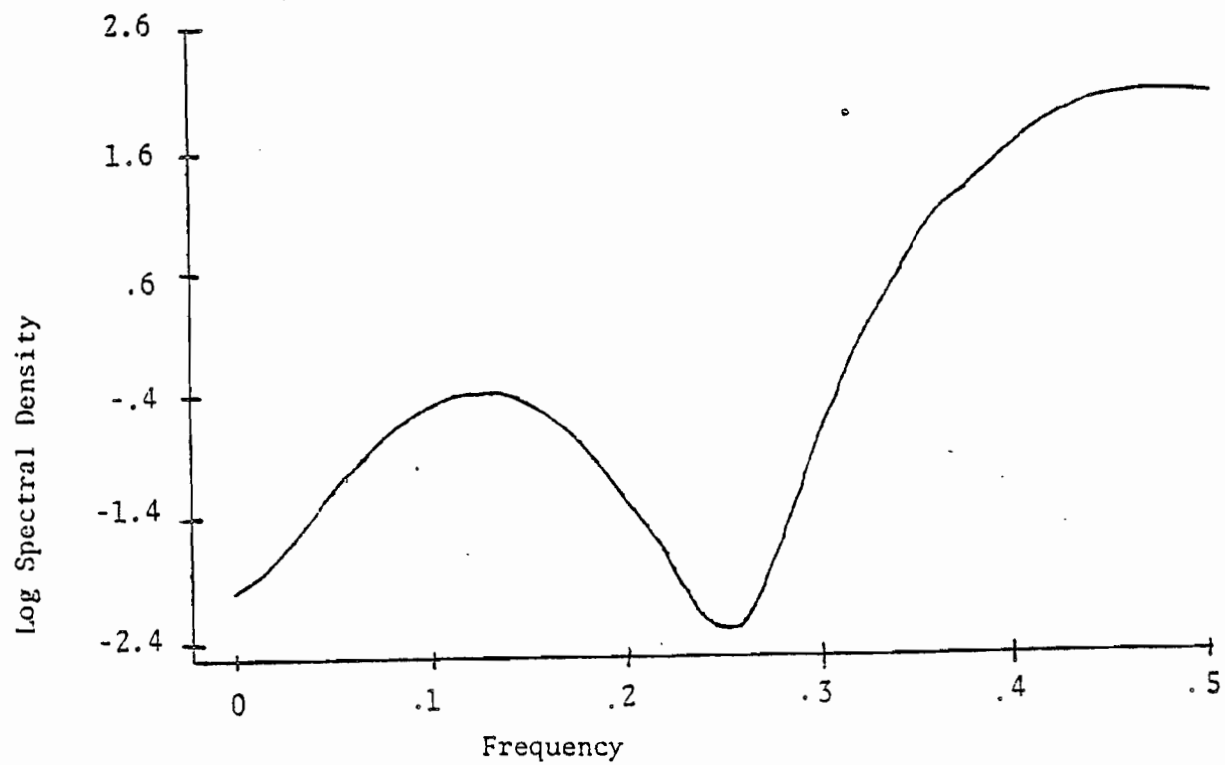
Maximum Likelihood Parameter Estimates for Example 4

FIGURE 1



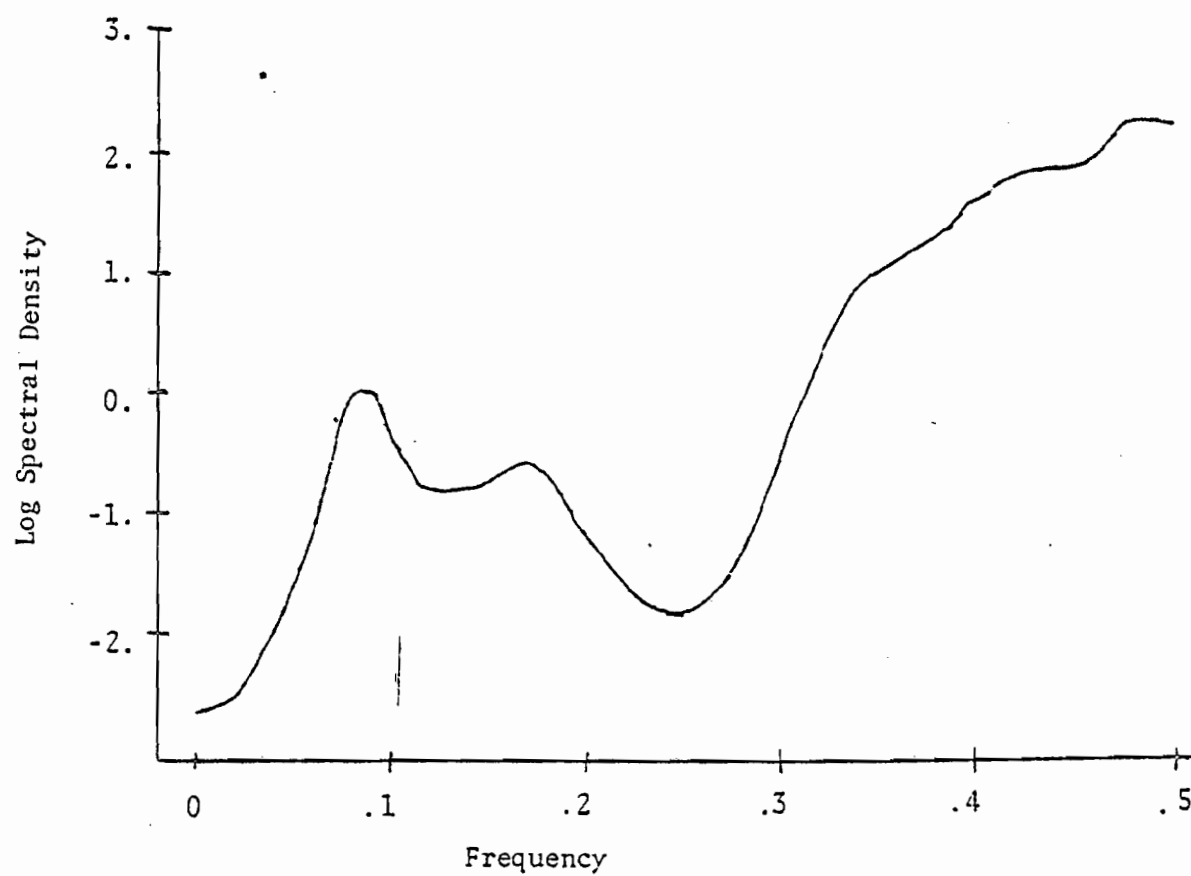
TRUE SPECTRAL DENSITY FOR EXAMPLE 1

FIGURE 2



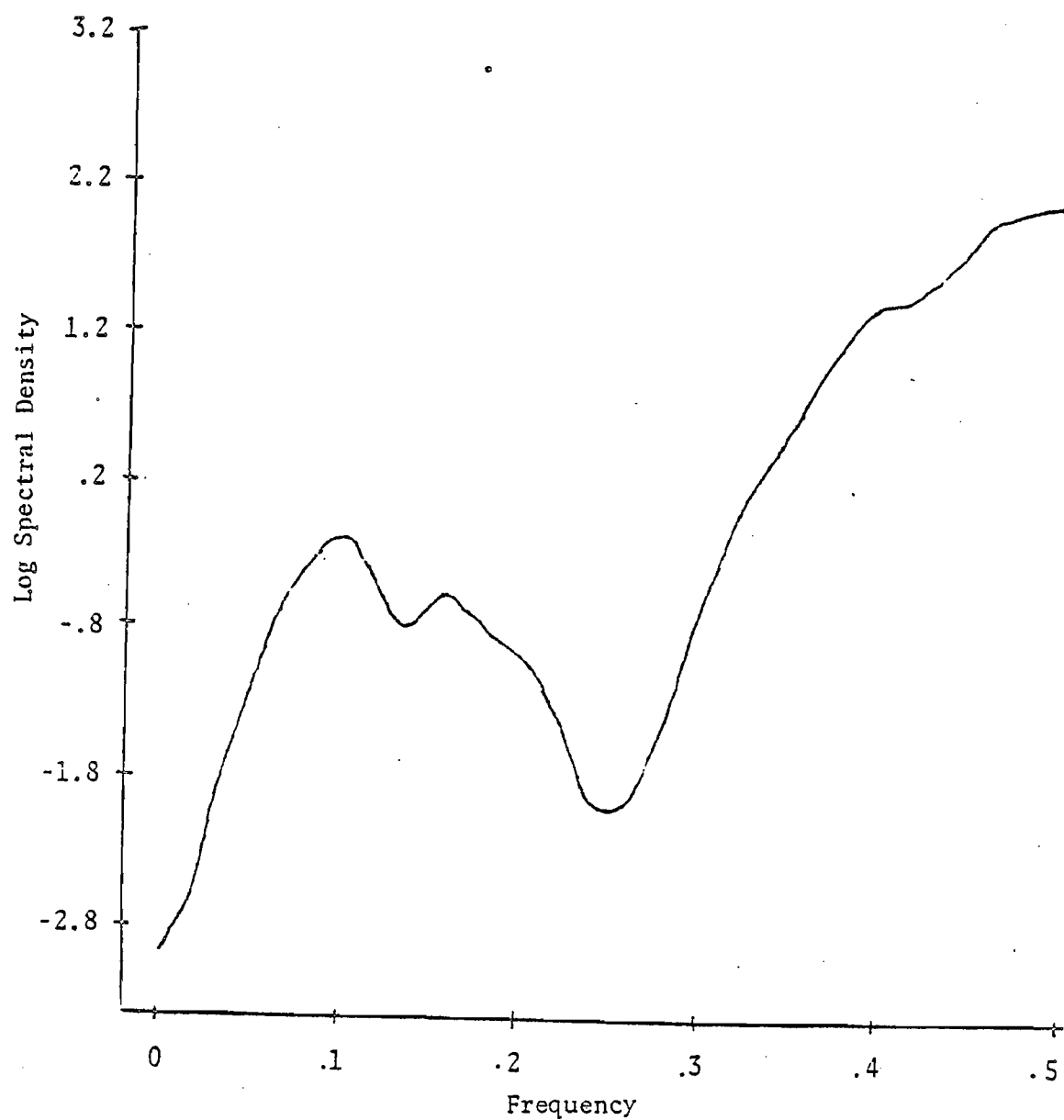
LOGARITHM OF THE $G_{0,3}(\omega)$ SPECTRAL ESTIMATE FOR EXAMPLE 1

FIGURE 3



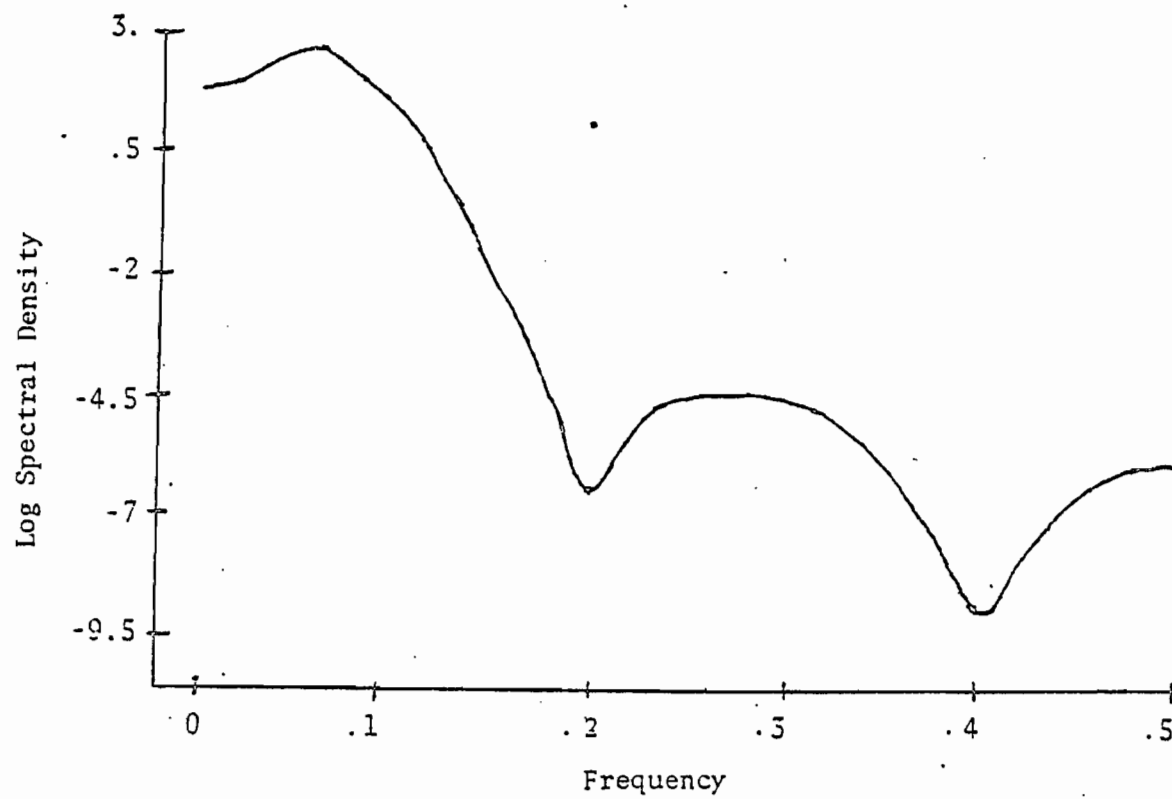
LOGARITHM OF THE AR(9) SPECTRAL ESTIMATE FOR EXAMPLE 1

FIGURE 4



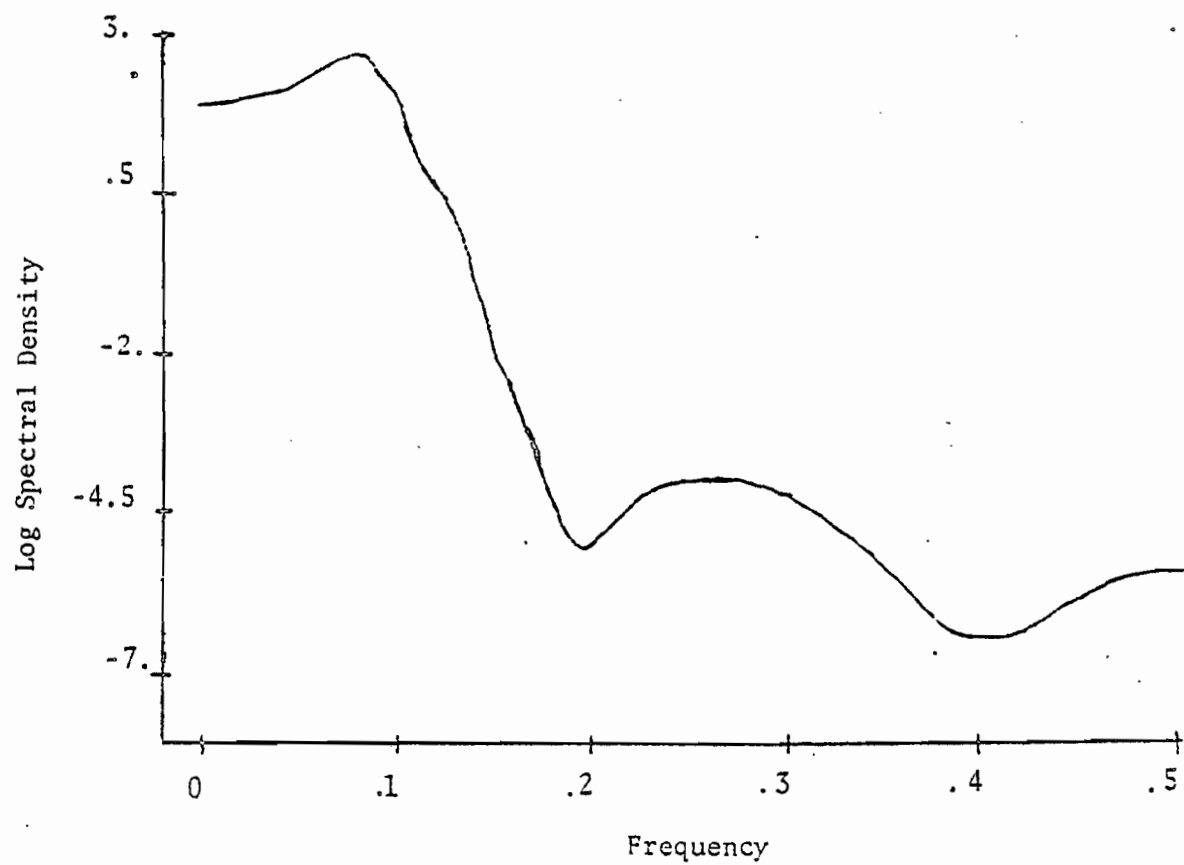
LOGARITHM OF THE PARZEN WINDOW ESTIMATE FOR EXAMPLE 1

FIGURE 5



LOGARITHM OF THE TRUE SPECTRAL
DENSITY FOR EXAMPLE 2

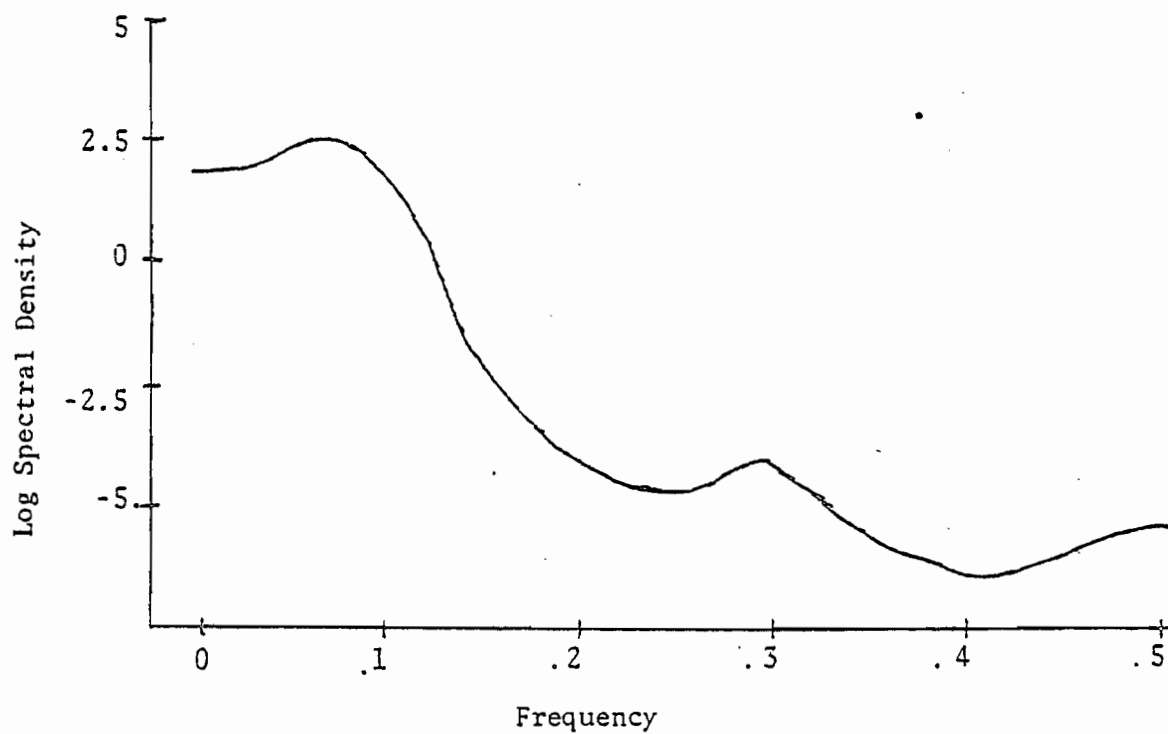
FIGURE 6



LOGARITHM OF THE $G_{2,4}(\omega)$ SPECTRAL

ESTIMATE FOR EXAMPLE 2

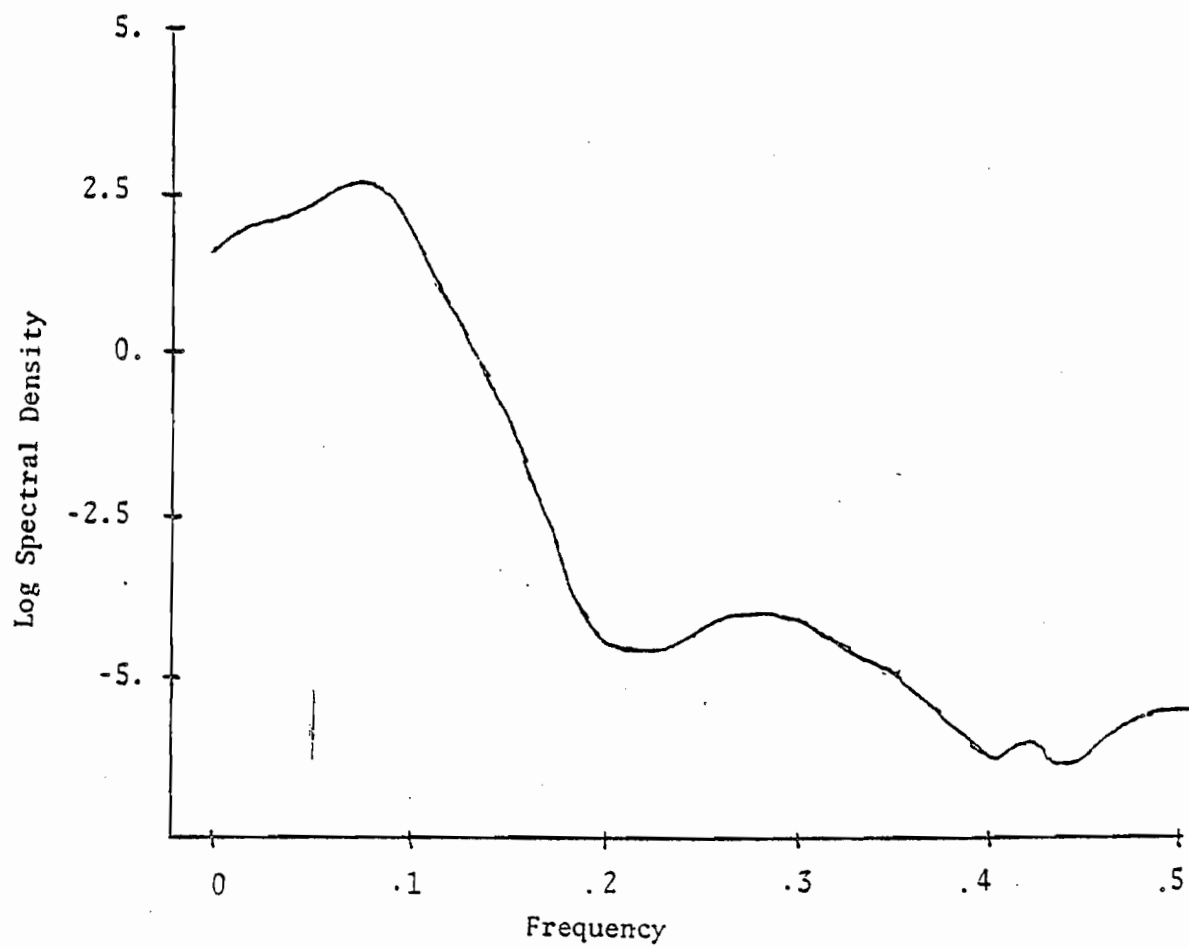
FIGURE 7



LOGARITHM OF THE AR(9) SPECTRAL

ESTIMATE FOR EXAMPLE 2

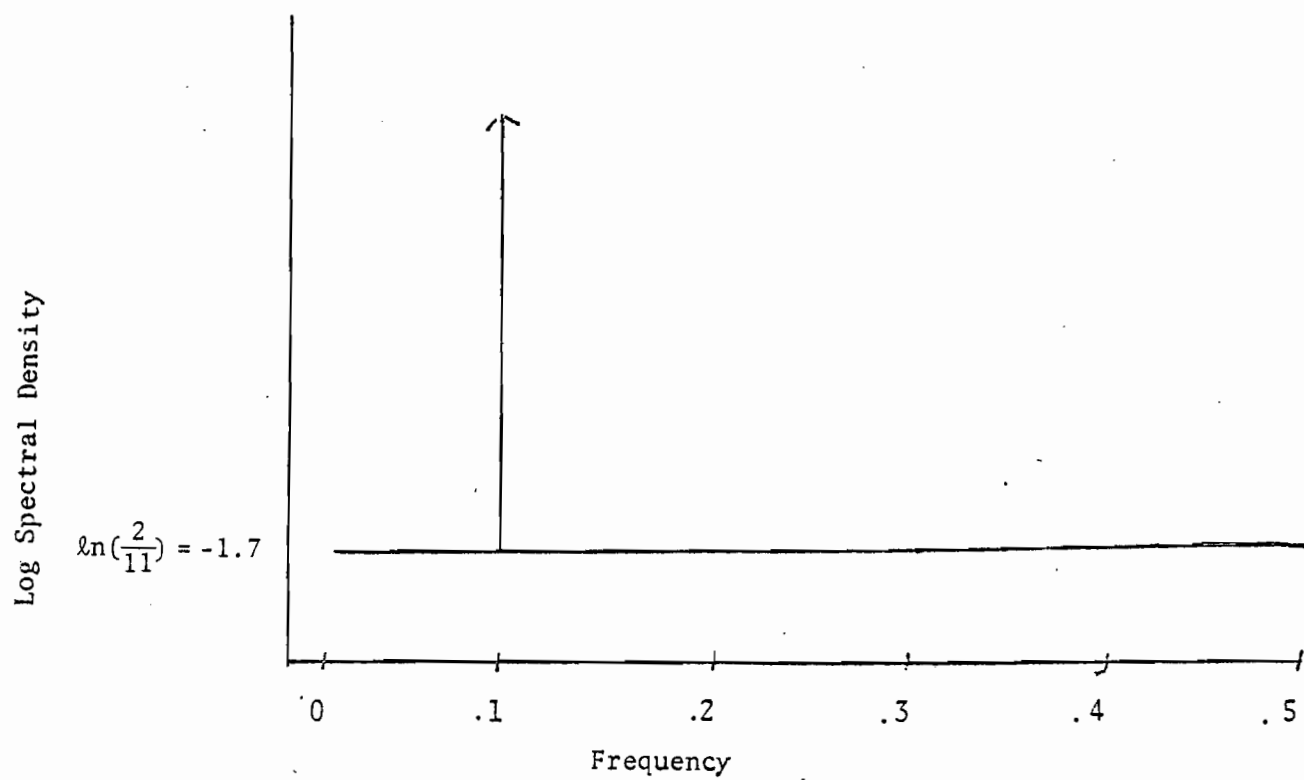
FIGURE 8



LOGARITHM OF THE PARZEN WINDOW SPECTRAL

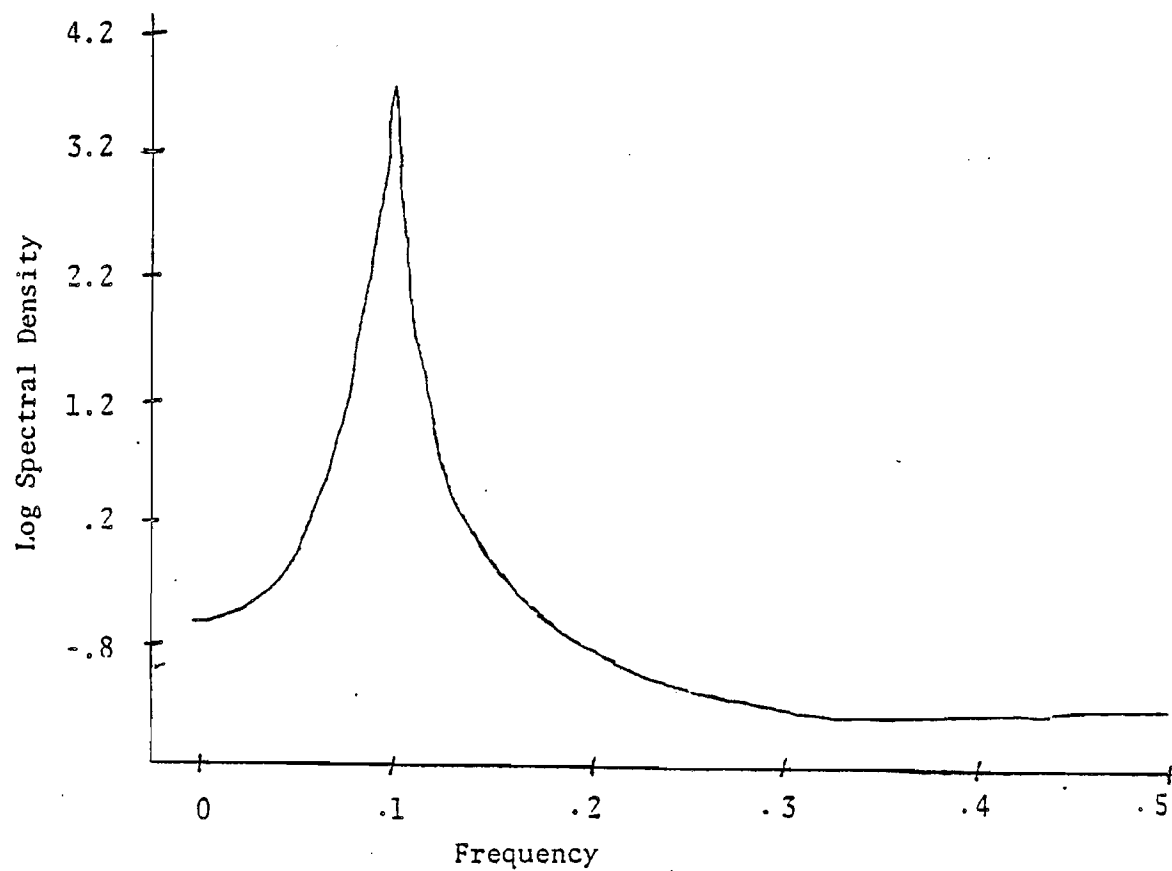
ESTIMATE FOR EXAMPLE 2

FIGURE 9



LOGARITHM OF THE TRUE SPECTRAL
DENSITY FOR EXAMPLE 3

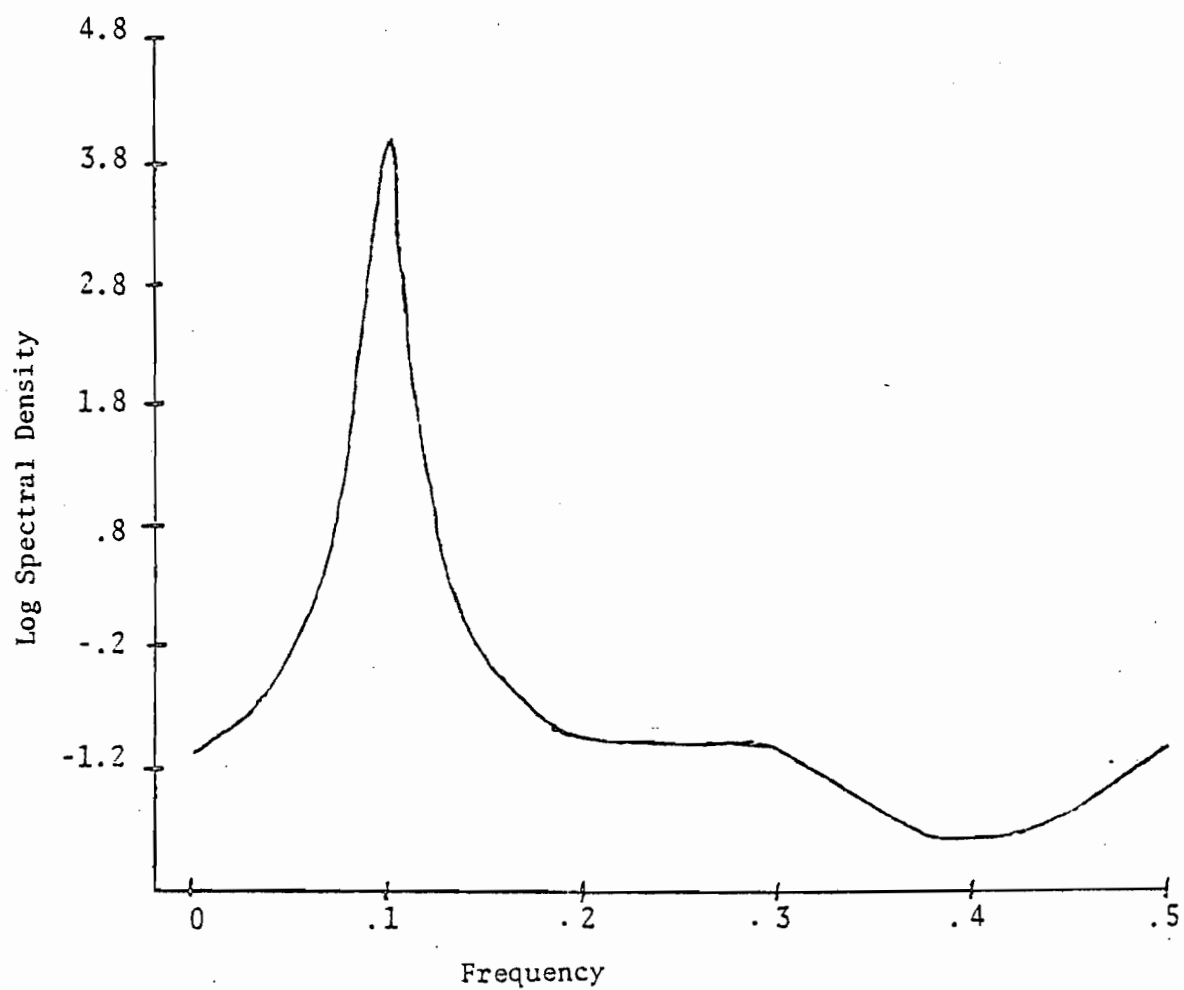
FIGURE 10



LOGARITHM OF THE $G_{2,2}(\omega)$ SPECTRAL

ESTIMATE FOR EXAMPLE 3

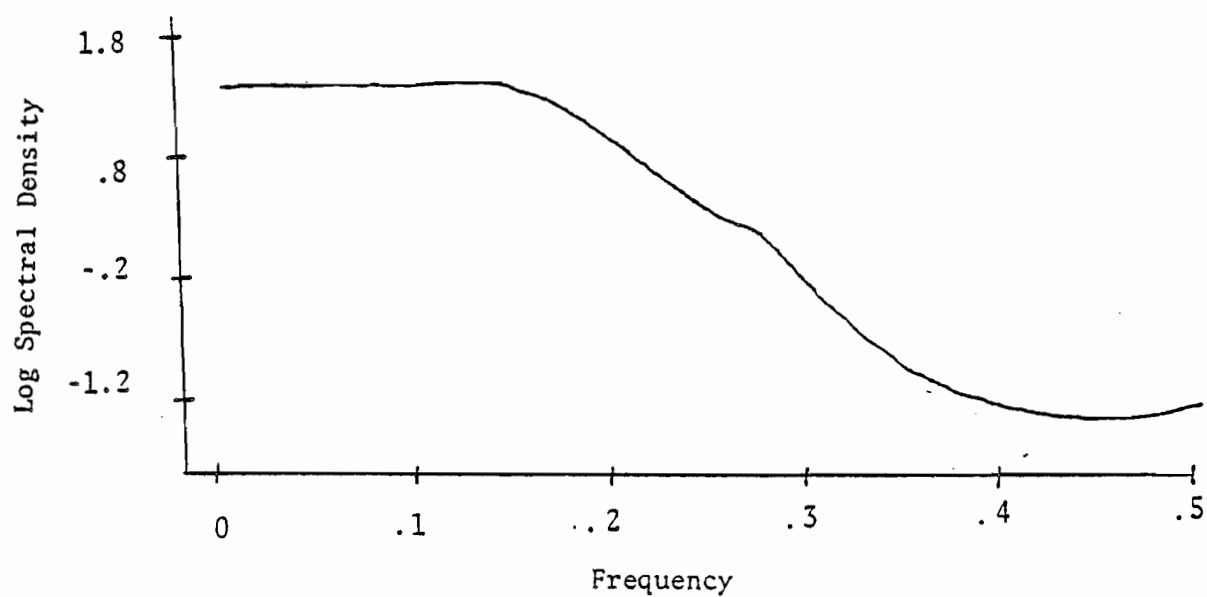
FIGURE 11



LOGARITHM OF THE AR(5) SPECTRAL

ESTIMATE FOR EXAMPLE 3

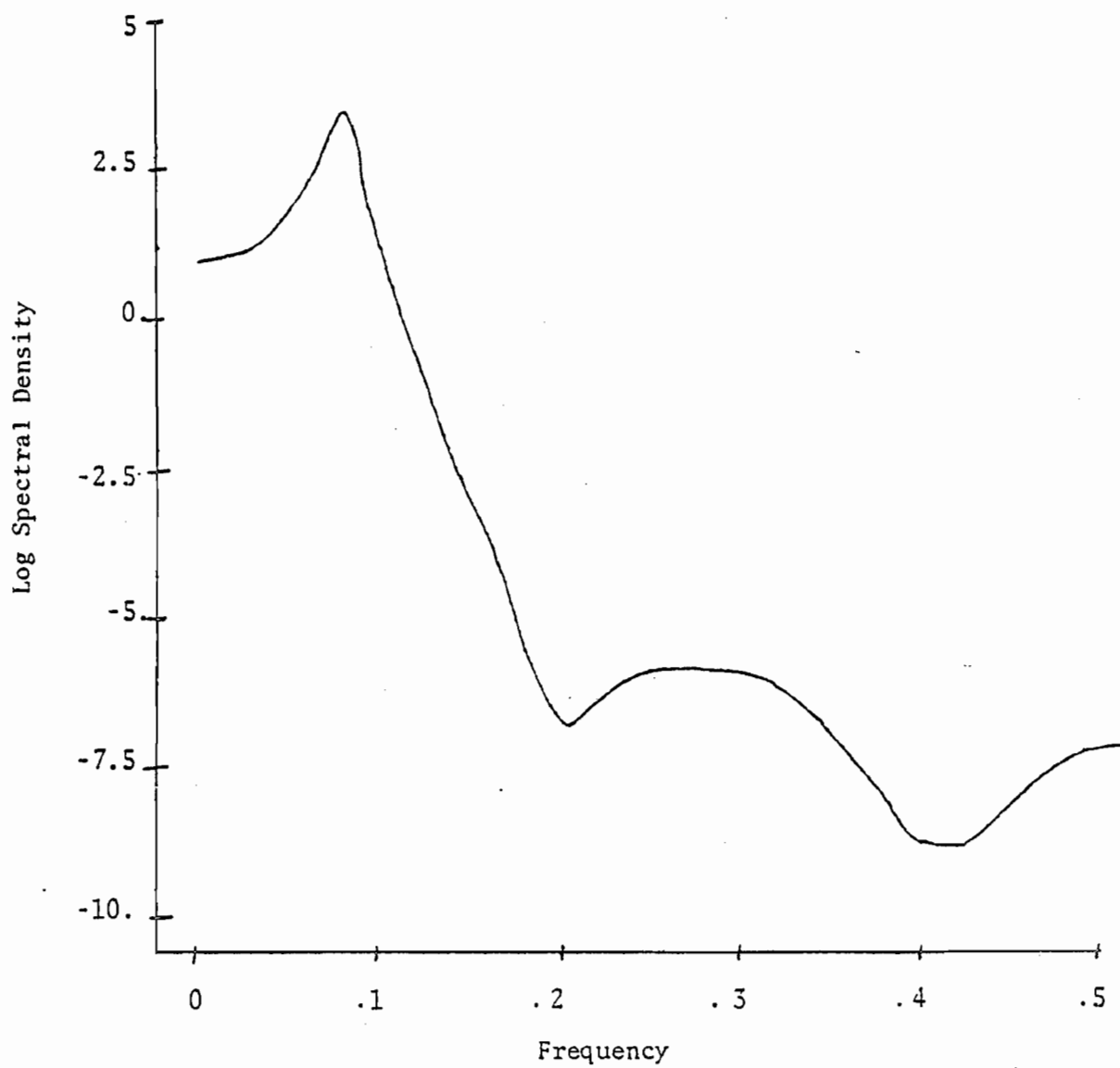
FIGURE 12



LOGARITHM OF THE PARZEN WINDOW SPECTRAL

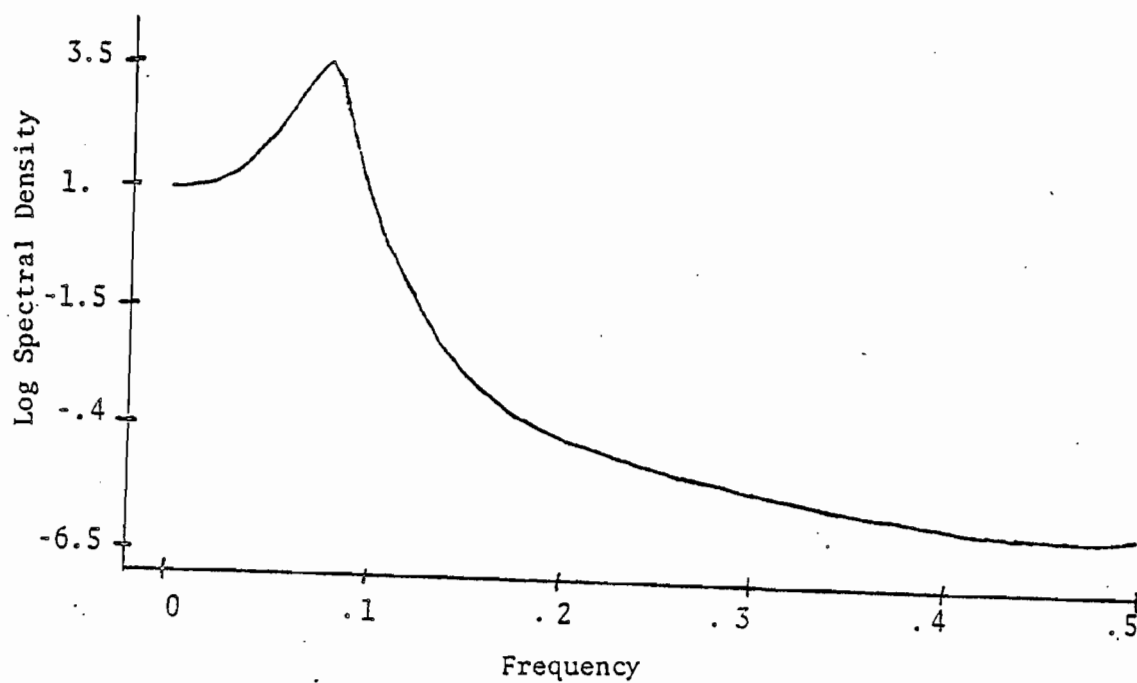
ESTIMATE FOR EXAMPLE 3

FIGURE 13



LOGARITHM OF THE TRUE SPECTRAL
DENSITY FOR EXAMPLE 4

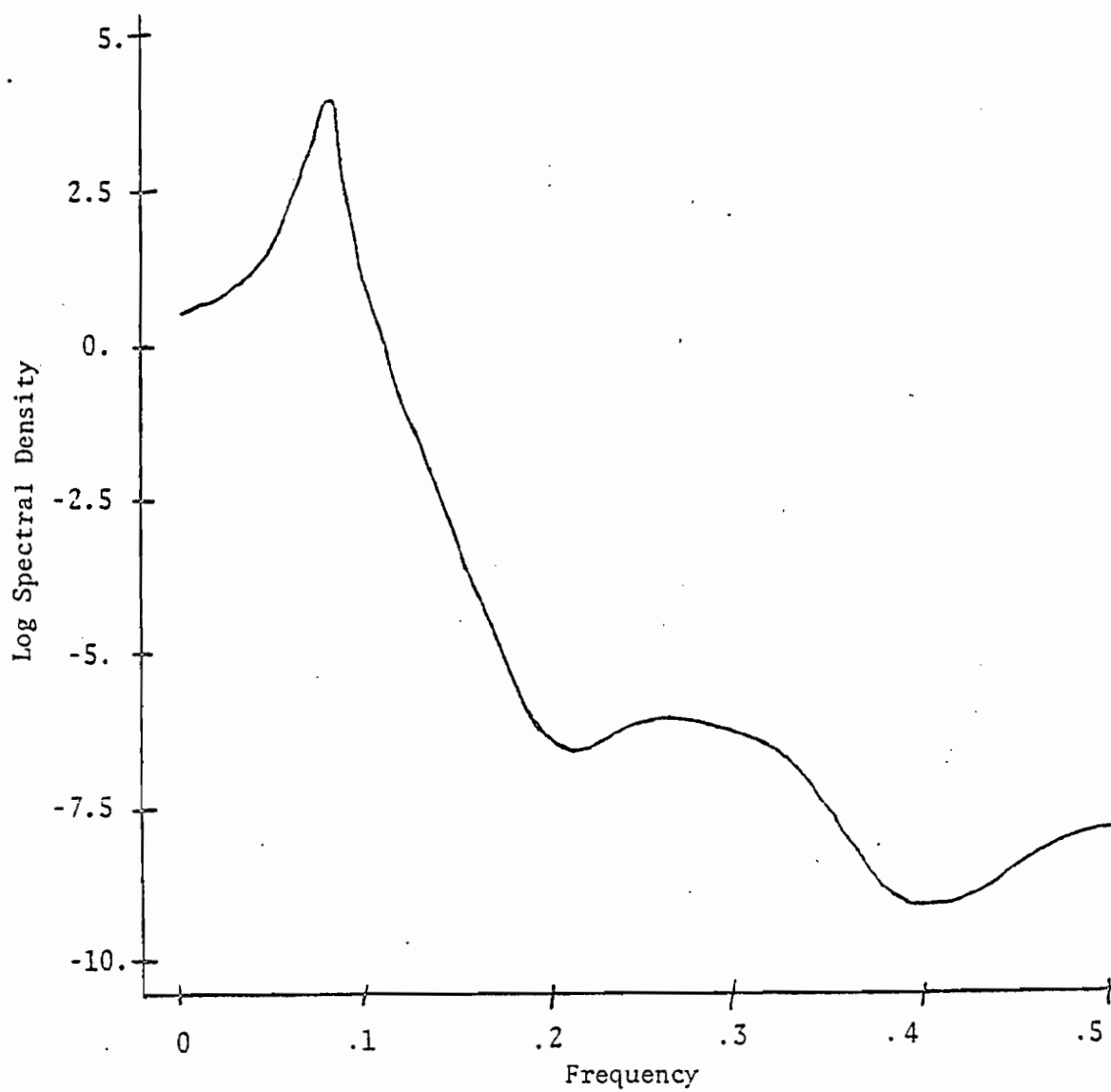
FIGURE 14



LOGARITHM OF THE $G_{2,4}(\omega)$ SPECTRAL

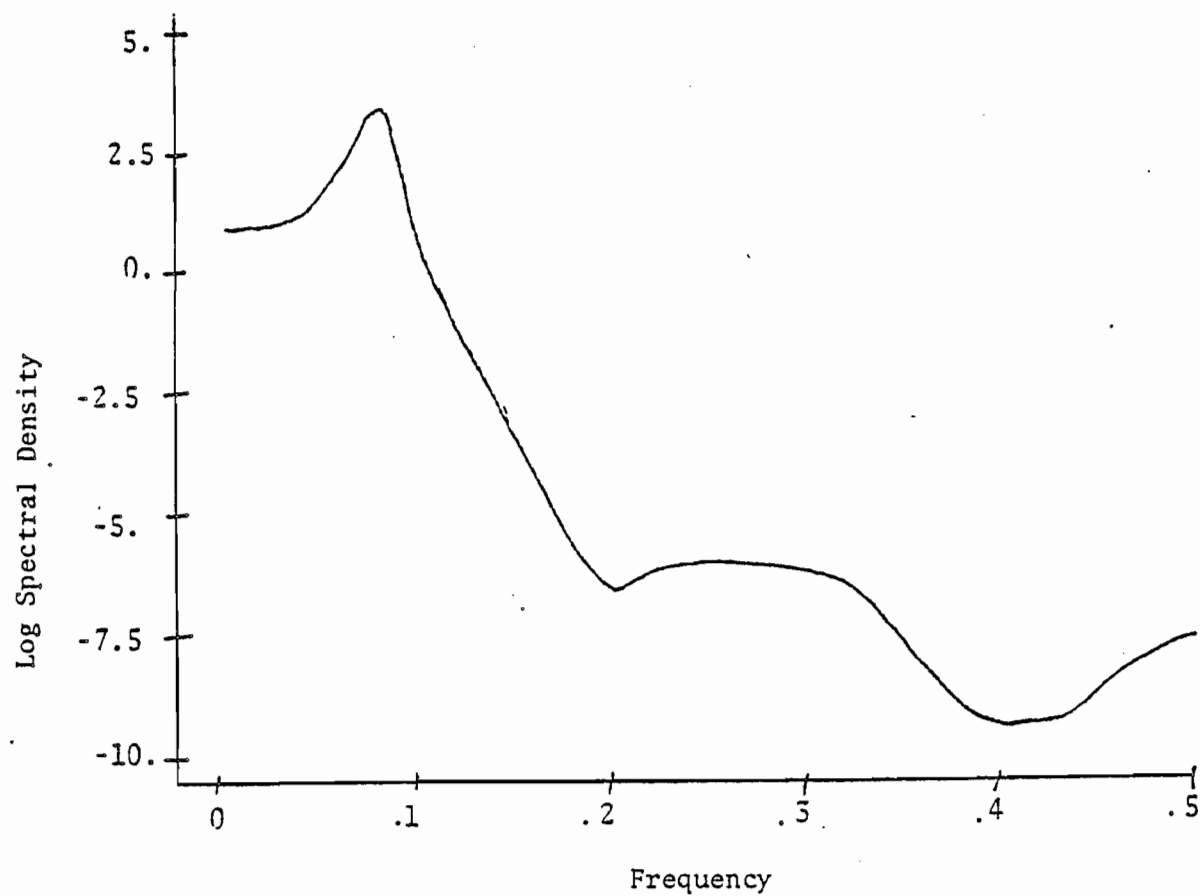
ESTIMATE FOR EXAMPLE 4

FIGURE 15



LOGARITHM OF THE MAXIMUM LIKELIHOOD ESTIMATE OF
THE SPECTRUM FOR EXAMPLE 4

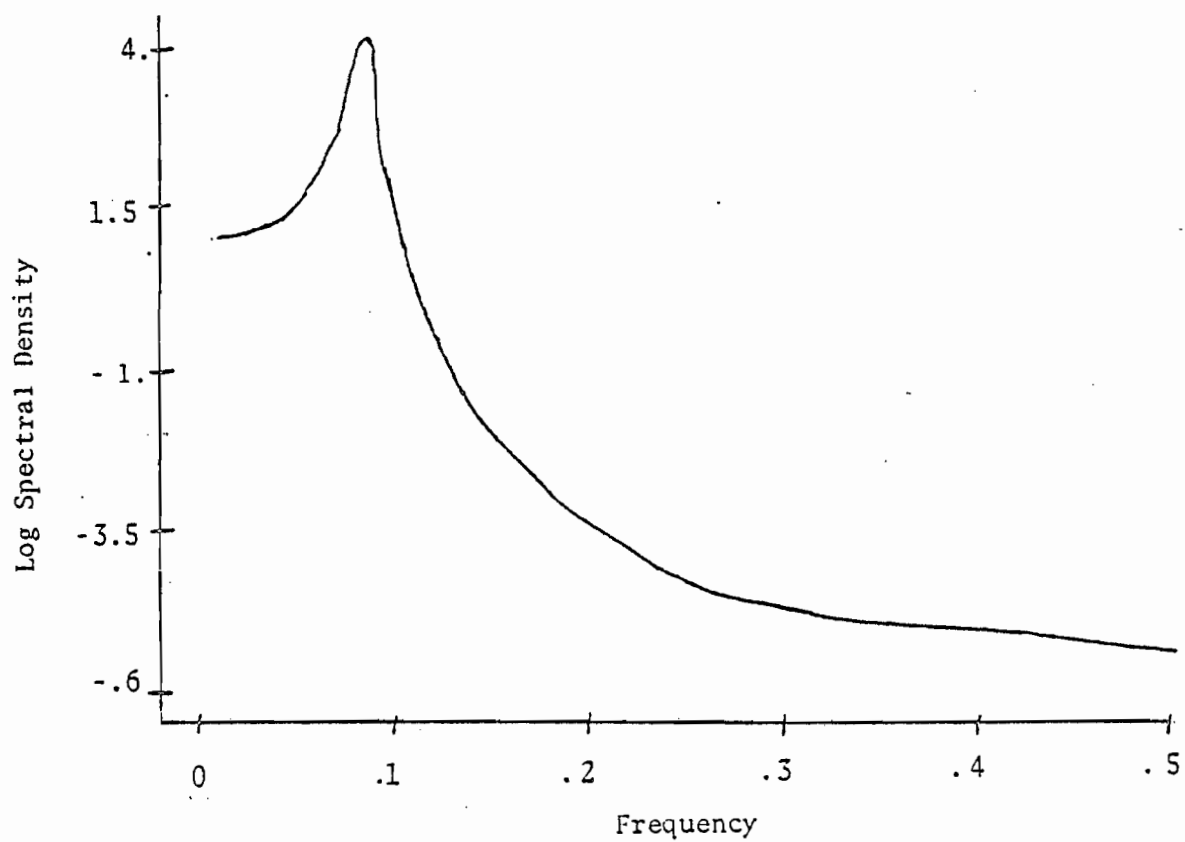
FIGURE 16



LOGARITHM OF $K|1 - \hat{\phi}_1 e^{2\pi i \omega} - \hat{\phi}_2 e^{4\pi i \omega}|^{-2} G_{0,4}(\omega)$. THIS

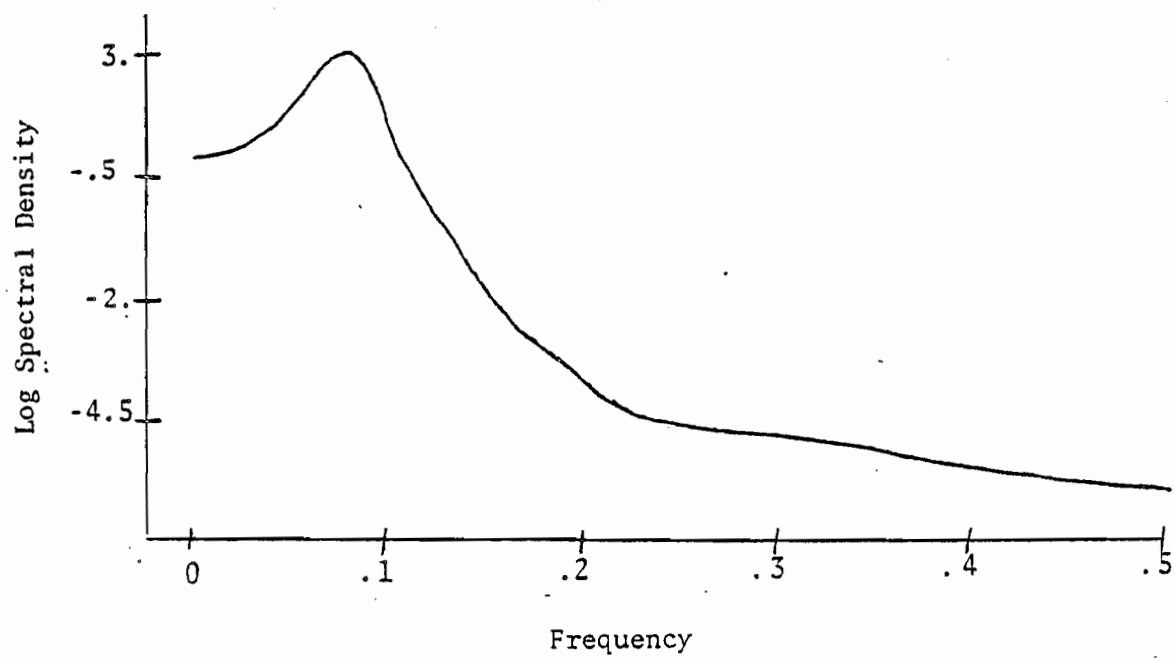
IS THE PIECEWISE SPECTRAL ESTIMATE FOR EXAMPLE 4

FIGURE 17



LOGARITHM OF THE AR(5) SPECTRAL
ESTIMATE FOR EXAMPLE 4.

FIGURE 18



LOGARITHM OF THE PARZEN WINDOW SPECTRAL

ESTIMATE FOR EXAMPLE 4

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 159	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) The G-Spectral Estimator		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER 159
7. AUTHOR(s) M. J. Morton H. L. Gray		8. CONTRACT OR GRANT NUMBER(s) N00014-75-C-0439
9. PERFORMING ORGANIZATION NAME AND ADDRESS		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 042 280
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE May 1982
		13. NUMBER OF PAGES 42
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report)
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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)		
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