ON CORNISH-FISHER EXPANSIONS WITH UNKNOWN CUMULANTS

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ABSTRACT

In this paper a new method of approximating the quantiles of one distribution by the quantiles of another is introduced. The method is essentially a modification of the Cornish-Fisher technique which eliminates the necessity of knowing the cumulants of the distribution involved.

1. INTRODUCTION

Let $F(\ ;\lambda)$ and Ψ be probability distribution functions with cumulants k_i and α_i, respectively, such that

$$\lim_{\lambda\to\infty} F(x;\lambda) = \Psi(x) \qquad (1)$$

for all x in the support of $F(\ ;\lambda)$ and let

$$\beta_i = k_i - \alpha_i \qquad (2)$$

where we assume for convenience that $\beta_1 = \beta_2 = 0$. In addition, we assume that

$$\beta_i = O(\lambda^{1-(i/2)}), \quad i = 3,4,\ldots \quad . \tag{3}$$

Let x and u be corresponding quantiles of $F(\ ;\lambda)$ and Ψ respectively, such that

$$F(x;\lambda) = \Psi(u) \ . \tag{4}$$

By inverting the Edgeworth expansion of $F(\ ;\lambda)$ in terms of Ψ [See Cornish & Fisher (1937), Draper & Tierney (1973), Fisher & Cornish (1960), Hill & Davis (1968) or Riordan (1949).], one can solve (4), obtaining the Cornish-Fisher expansion of u in terms of x given by

$$u = u_{cn} + O(\lambda^{-(n+1)/2}) \tag{5}$$

where

$$u_{cn} = x + \sum_{\ell=1}^{s} \delta_\ell(\lambda) W_\ell(x) \tag{6}$$

and the inverse Cornish-Fisher expansion of x in terms of u given by

$$x = x_{cn} + O(\lambda^{-(n+1)/2}) \tag{7}$$

where

$$x_{cn} = u + \sum_{\ell=1}^{s} \delta_\ell(\lambda) Y_\ell(u) \ . \tag{8}$$

In (6) and (8) the δ_ℓ is a function of the cumulants of $F(\ ;\lambda)$ and Ψ and hence a function of λ whereas W_ℓ and Y_ℓ are independent of λ. In this paper, expansions similar to (6) and (8) will be obtained which do not require evaluation of cumulants and therefore do not require evaluation of the δ_ℓ.

2. PRELIMINARIES

In many instances we can write [See Coberly (1972) or Hill & Davis (1968).]

$$F(x;\lambda) - F_n(x;\lambda) = O(\lambda^{-(n+1)/2}) \tag{9}$$

where

$$F_n(x;\lambda) = \Psi(x) + \sum_{i=1}^{k} g_i(\lambda) \Psi^{(m_i)}(x) \tag{10}$$

is the Edgeworth approximation of $F(x;\lambda)$ with the terms arranged such that the m_i are distinct, $g_i(\lambda) \neq 0$ are the resulting coefficients, and k is the number of distinct m_i. We now consider the following approximation of $F(x;\lambda)$:

$$\hat{F}_n(x;\lambda) = H_k[\Psi(x), V_i(x;\lambda); \Psi^{(m_i)}(x)] / H_k[1,0;\Psi^{(m_i)}(x)] \qquad (11)$$

where

$$H_k[A,B_i;\Psi^{(m_i)}(x)] = \begin{vmatrix} A & B_1 & \cdots & B_k \\ \Psi^{(m_1)}(x) & \Psi^{(m_1+1)}(x) & \cdots & \Psi^{(m_1+k)}(x) \\ \vdots & \vdots & & \vdots \\ \Psi^{(m_k)}(x) & \Psi^{(m_k+1)}(x) & \cdots & \Psi^{(m_k+k)}(x) \end{vmatrix}$$

for all x such that the denominator is nonzero, where

$$V_i(x;\lambda) = \Psi^{(i)}(x) - F^{(i)}(x;\lambda), \quad i = 1,\ldots,k , \qquad (12)$$

and k and n are defined by (10).

The approximation $\hat{F}_n$ was introduced in a paper by Gray, Coberly and Lewis (1975) and was shown under certain conditions to have the asymptotic property

$$\hat{F}_n(x;\lambda) - F(x;\lambda) = O(\lambda^{-(n+1)/2}). \qquad (13)$$

For convenience we let $\hat{D}_n(x;\lambda) = \hat{F}_n(x;\lambda) - \Psi(x)$ which by properties of determinants reduces to

$$\hat{D}_n(x;\lambda) = H_k[0, V_i(x;\lambda); \Psi^{(m_i)}(x)] / H_k[1,0;\Psi^{(m_i)}(x)] \qquad (14)$$

where $V_i(x;\lambda)$ is defined in (12).

3. THE NEW APPROXIMATION

In (4) let us consider the problem of approximating u with an expansion in terms of x. Under certain conditions [See Bol'shev (1959), (1963); Draper & Tierney (1973); Hill & Davis (1968), (1973).] we can use Taylor's formula to obtain

$$u = u_L + O(\lambda^{-(L+1)/2}) \qquad (15)$$

where

$$u_L = x + \sum_{r=1}^{L} C_r(x)\{[F(x;\lambda) - \Psi(x)]/\Psi^{(1)}(x)\}^r / r! \qquad (16)$$

and where $C_1(x) = 1$ and

$$C_{r+1}(x) = \{[-r(\Psi^{(2)}(x))/\Psi^{(1)}(x)] + D_x\}C_r(x), \quad r = 1,2,\ldots \ .$$

Equation (16) motivates the following definition.

<u>Definition 1</u>. Let $L \in$ Natural Numbers, $L \geq n$. Then

$$\hat{u}_{n,L} = x + \sum_{r=1}^{L} C_r(x)[\hat{D}_n(x;\lambda)/\Psi^{(1)}(x)]^r/r! \ . \tag{17}$$

The following theorem establishes the asymptotic equivalence of u_{cn} and $\hat{u}_{n,L}$.

<u>Theorem 1</u>. If (9), (13) and (15) are valid then

$$u - \hat{u}_{n,L} = O(\lambda^{-(n+1)/2}) \tag{18}$$

as $\lambda \to \infty$.

<u>Proof</u>: Consider

$$\sum_{r=1}^{L} C_r(x)\{[F(x;\lambda) - \Psi(x)]/\Psi^{(1)}(x)\}^r/r!$$

$$= \sum_{r=1}^{L} C_r(x)\{r![\Psi^{(1)}(x)]^r\}^{-1}[F(x;\lambda) - \hat{F}_n(x;\lambda) + \hat{F}_n(x;\lambda) - \Psi(x)]^r$$

$$= \sum_{r=1}^{L} C_r(x)\{[\hat{F}_n(x;\lambda) - \Psi(x)]/\Psi^{(1)}(x)\}^r/r! + T_L \tag{19}$$

where

$$T_L = \sum_{r=1}^{L} \sum_{k=1}^{r} C_r(x)\{r![\Psi^{(1)}(x)]^r\}^{-1}\binom{r}{k}[F(x;\lambda) - \hat{F}_n(x;\lambda)]^k[\hat{F}_n(x;\lambda) - \Psi(x)]^{r-k} \tag{20}$$

and where $r \geq k \geq 1$. It follows from (9) and (13) that

$$[F(x;\lambda) - \hat{F}_n(x;\lambda)]^k[\hat{F}_n(x;\lambda) - \Psi(x)]^{r-k} = O(\lambda^{-(nk+r)/2}) \ . \tag{21}$$

Now observing that the maximum of $-(nk+r)/2$ occurs when $r = k = 1$, it follows from (20) and (21) that

$$T_L = O(\lambda^{-(n+1)/2}) \ . \tag{22}$$

Now from (15) and (19) we can write $u - \hat{u}_{n,L} = T_L + O(\lambda^{-(L+1)/2})$ and and since $L \geq n$ by definition of $\hat{u}_{n,L}$, we have $u - \hat{u}_{n,L} = O(\lambda^{-(n+1)/2})$.

Q.E.D.

At this point a few comments are in order concerning (18). From (18) we see that the order of $u - \hat{u}_{n,L}$ depends only explicitly on n. However, for finite λ, the value of $\hat{u}_{n,L}$ will depend on L and some values of L may be better than others. No attempt to establish an optimal value of L will be made here and only those values of L; i.e., n and n + 1, given by the following definition, will be considered.

Definition 2. The approximation $\hat{u}_n$, which is the subject of this paper, we now define by

$$\hat{u}_n = x + \sum_{r=1}^{2W(n)} C_r(x)[\hat{D}_n(x;\lambda)/\Psi^{(1)}(x)]^r/r! \tag{23}$$

where W(n) = greatest integer ≤ (n+1)/2.

Note that the approximation of u given by (23) does not depend on the cumulants but instead makes use of the derivatives of $F(\ ;\lambda)$ and Ψ. This is in contrast to the Cornish-Fisher expansion which utilizes the derivatives of Ψ and the cumulants of $F(\ ;\lambda)$. Thus we have traded the problem of integration for one of differentiation which is in general easier. In fact if the Taylor expansion of $F(\ ;\lambda)$ about x is known the derivatives required in (23) can be obtained by inspection of that series. Of course if $F(\ ;\lambda)$ is unknown there is no advantage to $\hat{u}_n$ and we are not advocating it for that situation.

We have immediately from Definition 1, Theorem 1 and Definition 2 that

$$u - \hat{u}_n = O(\lambda^{-(n+1)/2}) \tag{24}$$

as $\lambda \to \infty$.

Since u_{cn} and $\hat{u}_n$ are approximations of u in (4), an obvious application is to use them to approximate $F(x;\lambda)$ when the limiting distribution is easy to evaluate. We are therefore interested in the comparison of $|F(x;\lambda) - \Psi(u_{cn})|$ with $|F(x;\lambda) - \Psi(\hat{u}_n)|$. For the following examples $\Psi(x)$ is N(0,1) and for n = 4, the expansion in (6) is therefore

$$
\begin{aligned}
u_{c4} = x &- \beta_3(x^2-1)/6 - \beta_4(x^3-3x)/24 + \beta_3^2(4x^3-7x)/36 \\
&- \beta_5(x^4-6x^2+3)/120 + \beta_3\beta_4(11x^4-42x^2+15)/144 \\
&- \beta_3^3(69x^4-187x^2+52)/648 - \beta_6(x^5-10x^3+15x)/720 \\
&+ \beta_4^2(5x^5-32x^3+35x)/384 - \beta_3\beta_5(7x^5-48x^3+51x)/360 \\
&- \beta_3^2\beta_4(111x^5-547x^3+456x)/864 \\
&+ \beta_3^4(948x^5-3628x^3+2473x)/7776 \; .
\end{aligned} \tag{25}
$$

Example 1. Let

$$F(x;\lambda) = \int_{-\infty}^{x} \lambda^{(1/2)} g(t\lambda^{(1/2)}+\lambda)\,dt \tag{26}$$

where

$$g(z) = \begin{cases} (\Gamma(z))^{-1} z^{\lambda-1} e^{-z}, & z > 0 \\ 0 , & z \le 0 \end{cases} \tag{27}$$

and

$$\beta_i = \begin{cases} 0 , & i = 1,2 \\ \lambda^{1-(i/2)}(i-1)!, & i = 3,4,\ldots \; . \end{cases} \tag{28}$$

Hence F is the standardized Gamma c.d.f. Now

$$
\begin{aligned}
F^{(m)}(x;\lambda) = {} & [\lambda^{m/2}/\Gamma(\lambda)]e^{-(x\lambda^{1/2}+\lambda)} \\
& \times \sum_{i=0}^{m-1} (-1)^{m-1-i} \binom{m-1}{i} D_u^i u^{\lambda-1} \Big|_{u=x\lambda^{1/2}+\lambda}
\end{aligned} \tag{29}
$$

where D_u denotes differentiation with respect to u. Thus $\hat{u}_n$ and u_{cn} can easily be calculated. The results are compared in Table I.

Example 2. Let

$$F(x;\lambda) = G(x\sigma;\lambda) \tag{30}$$

where

$$\sigma = [\lambda/(\lambda-2)]^{1/2} \tag{31}$$

and

$$G(t;\lambda) = \int_{-\infty}^{t} (\lambda\pi)^{-1/2}\{\Gamma[(\lambda+1)/2]/\Gamma(\lambda/2)\}[1+(w^2/\lambda)]^{-(\lambda+1)/2}dw. \tag{32}$$

Hence $F(\;;\lambda)$ is the standardized Student-t c.d.f. with $\beta_1 = \beta_3 = \beta_5 = \ldots = 0$ and $\beta_2 = 0$, $\beta_4 = 6/(\lambda-4)$, $\beta_6 = 240/(\lambda-4)(\lambda-6)$, ... Also $F^{(m)}(x;\lambda)$ is easily calculated by utilizing the relationship

TABLE I

F=F(x;λ)

x	F	$\|F-\Psi(\hat{u}_1)\|$	$\|F-\Psi(u_{c1})\|$	$\|F-\Psi(\hat{u}_2)\|$	$\|F-\Psi(u_{c2})\|$	$\|F-\Psi(\hat{u}_3)\|$	$\|F-\Psi(u_{c3})\|$	$\|F-\Psi(\hat{u}_4)\|$	$\|F-\Psi(u_{c4})\|$
λ=5									
1	.84748	.61308(-2)	.61308(-2)	.14874(-2)	.18005(-2)	.12793(-2)	.69265(-3)	.13732(-2)	.26875(-3)
2	.95902	.12716(-1)	.19261(-1)	.65096(-2)	.90198(-2)	.18682(-2)	.71665(-2)	.15153(-2)	.45366(-2)
3	.99069	.90768(-2)	.26038(-1)	.93090(-2)	.70594(-2)	.93095(-2)	.21499(-1)	.93096(-2)	.72143(-2)
4	.99812	.18809(-2)	.36991(-1)	.18809(-2)	.18693(-2)	.18809(-2)	.12354(0)	.18809(-2)	.18809(-2)
λ=15									
1	.84362	.22738(-2)	.22738(-2)	.36536(-3)	.39985(-3)	.21133(-3)	.88814(-4)	.19152(-3)	.85482(-4)
2	.96528	.56120(-2)	.60537(-2)	.22168(-2)	.19658(-2)	.36105(-3)	.79572(-3)	.22590(-3)	.32300(-3)
3	.99442	.41810(-2)	.48199(-2)	.50596(-2)	.16289(-2)	.55836(-2)	.11927(-2)	.55836(-2)	.63491(-3)
4	.99930	.69867(-3)	.26756(-2)	.69867(-3)	.49185(-3)	.69867(-3)	.95280(-3)	.69867(-3)	.36710(-3)
λ=25									
1	.84276	.14132(-2)	.14132(-2)	.18425(-3)	.19453(-3)	.86663(-4)	.33470(-4)	.68747(-4)	.57344(-5)
2	.96763	.36940(-2)	.35562(-2)	.12387(-2)	.93135(-3)	.15276(-3)	.28769(-3)	.81848(-4)	.91387(-4)
3	.99552	.25361(-2)	.23362(-2)	.31369(-2)	.72758(-3)	.44021(-2)	.36628(-3)	.44144(-4)	.16540(-4)
4	.99955	.44925(-3)	.90065(-3)	.44925(-3)	.21073(-3)	.44925(-3)	.19700(-3)	.44925(-3)	.94041(-4)
λ=100									
1	.84172	.38344(-3)	.38344(-3)	.33594(-4)	.19503(-4)	.13703(-4)	.91285(-5)	.34894(-5)	.66678(-5)
2	.97214	.10816(-2)	.85277(-3)	.21194(-3)	.11754(-3)	.12776(-4)	.17940(-4)	.41355(-5)	.29336(-5)
3	.99725	.49156(-3)	.38441(-3)	.45278(-3)	.74042(-4)	.27269(-3)	.17000(-4)	.27254(-3)	.40761(-5)
4	.99984	.15653(-3)	.71556(-4)	.16044(-3)	.15658(-4)	.16107(-3)	.49828(-5)	.16107(-3)	.15754(-5)

$$G^{(k+1)}(t;\lambda) = -\{(\lambda+2k-1)tG^{(k)}(t;\lambda) + (k-1)(\lambda+k-1)G^{(k-1)}(t;\lambda)\}/(\lambda+t^2) \quad (33)$$

Both approximations are again easily obtained and are compared in Table II.

TABLE II

$F=F(x;\lambda)$

x	F	$\lvert F-\Psi(\hat{u}_2)\rvert$	$\lvert F-\Psi(u_{c2})\rvert$	$\lvert F-\Psi(\hat{u}_4)\rvert$	$\lvert F-\Psi(u_{c4})\rvert$
$\lambda=8$					
1	.85923	.88435(-3)	.10474(-1)	.10775(-2)	.27054(-2)
2	.97513	.15229(-2)	.55286(-2)	.35956(-3)	.18162(-1)
3	.99574	.22587(-1)	.26138(-1)	.23719(-2)	.41849(-2)
4	.99914	.85731(-3)	.22577(0)	.85731(-3)	.14138(0)
$\lambda=15$					
1	.85014	.21427(-3)	.19545(-2)	.13003(-3)	.13038(-3)
2	.97579	.41616(-3)	.11051(-2)	.56756(-4)	.11922(-2)
3	.99719	.62148(-3)	.19386(-2)	.37046(-3)	.88182(-3)
4	.99968	.31776(-3)	.20971(-2)	.31776(-3)	.24880(-3)
$\lambda=25$					
1	.84644	.71744(-4)	.59943(-3)	.25405(-4)	.18419(-3)
2	.97629	.14734(-3)	.35461(-3)	.12537(-3)	.17872(-3)
3	.99779	.20748(-3)	.45913(-3)	.43588(-4)	.14348(-3)
4	.99984	.15087(-3)	.20138(-3)	.15980(-3)	.48919(-4)
$\lambda=100$					
1	.84257	.41168(-5)	.31827(-4)	.30888(-6)	.19349(-5)
2	.97699	.90464(-5)	.19873(-4)	.20117(-6)	.19630(-5)
3	.99845	.11062(-4)	.18600(-4)	.11585(-4)	.15303(-5)
4	.99995	.38694(-5)	.32512(-5)	.30888(-5)	.34374(-6)

4. THE NEW INVERSE APPROXIMATION

Now in (4) let us consider the inverse problem of approximating x with an expansion in terms of u. Under certain conditions [See Whittaker & Watson (1963), Hill & Davis (1968), Bol'shev (1959) & (1963), McCune (1974) or Nagao (1973).] we can use Lagrange's inversion formula to obtain

$$x = x_n + O(\lambda^{-(n+1)/2}) \quad (34)$$

where

$$x_n = u + \sum_{r=1}^{n} (-1)^r (\Psi^{(1)}(u))^{1-r} (D_u^{r-1}\{[F(u;\lambda)$$

$$- \Psi(u)]^r / \Psi^{(1)}(u)\})/r! \quad . \tag{35}$$

Equation (35) motivates the following definition.

Definition 3. Let $\hat{x}_n$ be defined by

$$\hat{x}_n = u + \sum_{r=1}^{n} A_r(u) E_{r,n}(u;\lambda) \tag{36}$$

where $A_r(u) = (-1)^r (\Psi^{(1)}(u))^{1-r}/r!$ and where $E_{r,n}(u;\lambda)$ is obtained by expanding $D_u^{r-1}\{[F(u;\lambda) - \Psi(u)]^r/\Psi^{(1)}(u)\}$ and then substituting $\hat{D}_n(u;\lambda)$ for $F(u;\lambda) - \Psi(u)$.

The following theorem establishes the asymptotic equivalence of x_{cn} and $\hat{x}_n$.

Theorem 2. If (9), (13) and (34) are valid, then

$$x - \hat{x}_n = O(\lambda^{-(n+1)/2}) \tag{37}$$

as $\lambda \to \infty$.

Proof: Expression (34) can be written as

$$x = u + \sum_{r=1}^{n} A_r(u) B_r(u;\lambda) + O(\lambda^{-(n+1)/2}) \tag{38}$$

where

$$B_r(u;\lambda) = \sum_{i=1}^{k_r} h_i(u)[F(u;\lambda)$$

$$- \Psi(u)]^{P_{i(1)}} (z^{(1)})^{P_{i(2)}} \cdots (z^{(r-1)})^{P_{i(r)}} \tag{39}$$

and where $z^{(t)} = F^{(t)}(u;\lambda) - \Psi^{(t)}(u)$, $t = 1,\ldots,r-1$, $h_i(u)$ is a function of u independent of λ, and $\sum_{\ell=1}^{r} P_{i(\ell)} = r$. Now consider

$$F(u;\lambda) - \Psi(u) = F(u;\lambda) - \hat{F}_n(u;\lambda) + \hat{D}_n(u;\lambda) \quad . \tag{40}$$

Substituting (40) into (39) and simplifying yields

$$B_r(u;\lambda) = E_{r,n}(u;\lambda) + O(\lambda^{-(n+1)/2}) \quad . \tag{41}$$

Now (38) and (41) yield

$$x = \hat{x}_n + O(\lambda^{-(n+1)/2}) \quad .$$

Q.E.D.

TABLE IV

	λ	.7500	.9500	.9750	.9975	.9995
t	10	.700	1.812	2.228	3.581	4.587
t_{c2}		.699	1.797	2.197	3.430	4.264
$\hat{t}_2$		.715	1.823	2.248	3.395	4.131
t_{c4}		.700	1.811	2.225	3.559	4.525
$\hat{t}_4$		.700	1.813	2.232	3.510	4.718
t	20	.687	1.725	2.086	3.153	3.850
t_{c2}		.687	1.721	2.079	3.119	3.777
$\hat{t}_2$		.690	1.727	2.090	3.109	3.744
t_{c4}		.687	1.725	2.086	3.151	3.842
$\hat{t}_4$		.687	1.725	2.086	3.144	3.861
t	40	.681	1.684	2.021	2.971	3.551
t_{c2}		.681	1.683	2.019	2.963	3.534
$\hat{t}_2$		.681	1.684	2.022	2.960	3.526
t_{c4}		.681	1.684	2.021	2.971	3.550
$\hat{t}_4$		.681	1.684	2.021	2.970	3.553
t	60	.679	1.671	2.000	2.915	3.460
t_{c2}		.679	1.670	2.000	2.912	3.453
$\hat{t}_2$		.679	1.671	2.000	2.910	3.449
t_{c4}		.679	1.671	2.000	2.914	3.460
$\hat{t}_4$		.679	1.671	2.000	2.914	3.461

5. CONCLUSIONS

Evaluation of the expansions u_{cn} or x_{cn} requires explicit knowledge of the first n + 2 cumulants of $F(\ ;\lambda)$, when evaluating up to terms of order $O(\lambda^{-(n+1)/2})$. On the other hand, evaluation of $\hat{u}_n$ or $\hat{x}_n$ requires knowledge of the first k derivatives (see (11)) of $F(\ ;\lambda)$. From the results of Tables I - IV, we can conclude that $\hat{u}_n$, $\hat{x}_n$, u_{cn}, and x_{cn} are very comparable in the sense of being good approximations. Hence, $\hat{u}_n$ and $\hat{x}_n$ may be valuable alternatives when the cumulants are difficult to evaluate as compared with obtaining the k derivatives of $F(\ ;\lambda)$.

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