

SOME INFERENCE PROBLEMS ASSOCIATED WITH THE

COMPLEX MULTIVARIATE NORMAL DISTRIBUTION

by

John C. Young

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DEPARTMENT OF STATISTICS
Southern Methodist University

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using the same assumptions as given in (1.1.0), has also derived the complex

It appears that N. R. Goodman (1963), working independently and

where $\bar{\Gamma}$ is some arbitrary complex parameter.

$$\exp\left(-\frac{1}{4}\bar{\Gamma}^*L\bar{\Gamma}\right)$$

function for the above distribution and found it to be given by

$\bar{\xi}$ is assumed to be 0. Wooding (1956) also derived the characteristic represents the transpose of the conjugate of $\bar{\xi}$. The mean of the vector where L is the non-singular Hermitian variance-covariance matrix and $\bar{\xi}^*$

$$\Pi^{-p}|L|^{-1}\exp(-\bar{\xi}^*L^{-1}\bar{\xi}) \quad (1.1.1)$$

as

then the p-variate complex normal density function could be represented

$$E(x_m x_n) = E(y_m y_n) ; E(x_m y_n) = -E(x_n y_m) \quad (1.1.0)$$

components of $\bar{\xi}$ had the following form

x_j and y_j . It was found that if the covariance relation between the

complex random variable with real and imaginary components given by

ξ_j ($j = 1, 2, \dots, p$). Then each ξ_j ($j = 1, 2, \dots, p$) represents a

bution. Let $\bar{\xi}$ denote a complex normal random vector with components

R. A. Wooding (1956) derived the p-variate complex normal distri-

1. Introduction

CHAPTER I

multivariate normal distribution. He extended the development further by deriving the complex Wishart distribution, denoting it as the complex Wishart distribution. Also, Goodman derived the distributions of the complex analogues for the sample multiple and sample partial correlation coefficients. In the complex case, these are denoted as the sample multiple coherence and the sample partial coherence. All developments that have followed in this area of complex multivariate analysis have been based on the assumption of the density function specified by Wooding (1956) or Goodman (1963).

Other authors who made early contributions in this area of complex multivariate analysis have been M. S. Srivastava (1965), N. Giri (1965), D. G. Kabe (1966b), and C. G. Khatri (1965). D. G. Kabe (1965a) gave a simplified method of deriving the distribution of the sample multiple coherence and the sample partial coherence. He represented the density functions in a finite series while Goodman (1963) had presented these as infinite series. Kabe (1966b) developed the complex analogue some results in classical multivariate normal regression theory. Srivastava (1965) gives a direct and simplified method of deriving the Wishart distribution. Khatri's (1965) contributions were in many areas. He obtained the maximum likelihood estimates of the parameters of the complex multivariate normal given by Wooding (1956) and Goodman (1963); he then derived the distributions of these estimates. Also included in his work are procedures for transforming from densities of complex random variables to other complex random variables. Using these techniques, Khatri (1965) derived the distributions of the sample multiple coherence.

Khatri (1965) also derives the distribution of Wilks' Λ in the complex case. He gives a representation of Wilks' Λ in the complex

case) as the product of real independent beta variables. It is this development that proves useful to the work given in this dissertation. Through the work of the above authors the foundation for complex multivariate analysis, based on the assumption of a complex multivariate density as given in (1.1.1), has been established. It is the purpose of this dissertation to extend this development by examining Wilks' Λ which appears as a solution to some inference problems associated with this particular complex multivariate normal distribution.

This extension is covered in Chapter III by an examination of Wilks' Λ as presented in the multivariate analysis of variance of complex normal data, the regression analysis of complex normal variables upon real dummy variables, and discrimination among several complex normal populations with the same variance-covariance matrix but with different means. Also the goodness of fit of a hypothetical discriminant function is considered.

In Chapter II, some basic theorems dealing with the complex multivariate normal are established along with the Bartlett decomposition of a complex Wishart matrix. These results are used in establishing some different expressions for the sample coherence and sample multiple coherence. These results prove useful in working with Wilks' Λ in Chapter III. Chapter IV presents a summary of the work covered in this dissertation along with possible extensions of this research into other areas.

2. Definitions and Notation

Let ξ denote a $p \times 1$ complex random vector,

$$\xi' = [\xi_1, \xi_2, \dots, \xi_p]_{1 \times p}$$

(1.2.0)

where $\xi_j = x_j + iy_j$. Denote the $(2p \times 1)$ vector of real and imaginary components by

$$\bar{\eta}' = [x_{1,Y1}, x_{2,Y1}, \dots, x_{p,Y1}, x_{1,Y2}, x_{2,Y2}, \dots, x_{p,Y2}]_{1 \times 2p} \quad (1.2.1)$$

and let this random vector have a $2p$ -variate normal distribution with

$$\bar{\delta}' = E(\bar{\eta}') = [\mu_{1,Y1}, \mu_{2,Y1}, \dots, \mu_{p,Y1}, \mu_{1,Y2}, \mu_{2,Y2}, \dots, \mu_{p,Y2}]$$

and the variance-covariance matrix given by

$$\Sigma_{\bar{\eta}} = E[(\bar{\eta} - \bar{\delta})(\bar{\eta} - \bar{\delta})']$$

where the 2×2 submatrices of $\Sigma_{\bar{\eta}}$ have the following special form

$$\begin{bmatrix} E[x_j - E(x_j)][x_k - E(x_k)] & E[x_j - E(x_j)][y_k - E(y_k)] \\ E[y_j - E(y_j)][x_k - E(x_k)] & E[y_j - E(y_j)][y_k - E(y_k)] \end{bmatrix}$$

$$= \begin{cases} \frac{\sigma_k^2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } j = k \\ \frac{2}{\sigma_j \sigma_k} \begin{bmatrix} \alpha_{jk} & -\beta_{jk} \\ \beta_{jk} & -\alpha_{jk} \end{bmatrix} & \text{if } j \neq k \end{cases} \quad (1.2.2)$$

Then the density function of $\bar{\eta}$ is given by

$$p(\bar{\eta}) = \frac{1}{(2\pi)^p |\Sigma_{\bar{\eta}}|^{1/2}} \exp \left(-\frac{1}{2} (\bar{\eta} - \bar{\delta})' \Sigma_{\bar{\eta}}^{-1} (\bar{\eta} - \bar{\delta}) \right) \quad (1.2.3)$$

provided that $|\Sigma_{\bar{\eta}}| \neq 0$.

It has been shown by Wooding (1956) and Goodman (1963) that if

condition (1.2.2) is satisfied for $\bar{\eta}$, then $\bar{\xi}$ is distributed as a p-variate complex normal vector with probability density function given by

$$P(\bar{\xi}) = (|\Sigma_{\bar{\xi}}|)^{-1} \exp[-(\bar{\xi} - \bar{\eta})^* \Sigma_{\bar{\xi}}^{-1} (\bar{\xi} - \bar{\eta})] \quad (1.2.4)$$

where the mean $\bar{\eta}$ is given as

$$\bar{\eta}' = E(\bar{\xi}') = [\mu_1 + i\nu_1, \mu_2 + i\nu_2, \dots, \mu_p + i\nu_p]$$

and the variance-covariance structure is given by the Hermitian positive definite matrix $\Sigma_{\bar{\xi}}$, where

$$\Sigma_{\bar{\xi}} = E[(\bar{\xi} - \bar{\eta})(\bar{\xi} - \bar{\eta})^*]$$

and the elements of $\Sigma_{\bar{\xi}}$ are given as

$$\sigma_{jk} = \begin{cases} \sigma_k^2 & \text{if } j = k \\ (\alpha_{jk} + i\beta_{jk}) & \text{if } j \neq k \end{cases}$$

The density function (1.2.4) will be denoted as $CN^p(\bar{\eta}, \Sigma)$.

Let $F(\bar{\xi})$ denote the distribution function of the complex random

vector $\bar{\xi}$ with density given as $CN^p(\bar{\eta}, \Sigma)$. By the probability that $\bar{\xi}$ is less than $\bar{\xi}_0$, the following is meant.

$$F(\bar{\xi}_0) = P(\bar{\xi} < \bar{\xi}_0) = P(\bar{\xi}_R < \bar{\xi}_{0R}; \bar{\xi}_I < \bar{\xi}_{0I}) \quad (1.2.5)$$

where $\bar{\xi}_R$ denotes the real part of $\bar{\xi}$ and $\bar{\xi}_I$ denotes the real random variable attached to the imaginary quantity.

The complex Wishart density function with parameter Z and degrees of freedom $n-1$ will be denoted as $CW^p(S|n-1|Z)$. Let $\tilde{\xi}$, where $\tilde{\xi}' = [\xi_1, \xi_2, \dots, \xi_p]$, be distributed as a $CN^p(\tilde{\mu}, Z)$. As in real multivariate analysis, the relationship between the component ξ_p and a linear combination of $\xi_1, \xi_2, \dots, \xi_{p-1}$ has a very important role in complex multivariate analysis. In the real case this relationship is measured by the multiple correlation coefficient which is defined as the maximum correlation between one component and a linear combination

$$r^p(n) = \frac{1}{2} P(p-1) \left\{ \prod_{j=1}^p \Gamma(n-j+1) \right\}.$$

where

$$P(S) = [\Gamma^p(n)]^{-1} |Z|^{-n} \exp[-\text{tr } Z^{-1} S]$$

has been shown by Goodman (1963) and Khatri (1965) to be distributed as a complex Wishart with probability density function given by

$$\tilde{\xi} = \frac{1}{n} \sum_{i=1}^n \tilde{\xi}_i$$

where

$$S = \sum_{i=1}^n (\tilde{\xi}_i)(\tilde{\xi}_i)' - n(\tilde{\xi})(\tilde{\xi})'$$

Then the random matrix S , defined by

buted p -variate complex normal vectors with density given by $CN^p(\tilde{\mu}, Z)$.

Let $\tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_n$ denote n independent and identically distributed

of the others. In complex multivariate analysis, Goodman (1963) has termed this quantity as the multiple coherence and shown it to be given by

$$(1.2.6) \quad \xi_p^2(\xi_1, \xi_2, \dots, \xi_{p-1}) = \frac{\xi_{12}^* \xi_{11} \xi_{12}}{\xi_{21} \xi_{11} \xi_{12}} = \frac{\sigma_{pp}}{\sigma_{pp}}$$

where $\xi_{11}, \xi_{12}, \xi_{21}$ are obtained from the variance-covariance matrix Σ which is partitioned as

$$\Sigma = \begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{pp} \end{bmatrix}$$

Goodman (1963) and Khatri (1965) have given the distribution of the sample estimate, $\hat{\xi}_p^2$, of $\xi_p^2(\xi_1, \xi_2, \dots, \xi_{p-1})$ as

$$f(\hat{\xi}_p^2) = \frac{\Gamma(n)}{\Gamma(p-1)\Gamma(n-p+1)} (1-\hat{\xi}_p^2)^{n-p} \hat{\xi}_p^{2(p-1)} f(n, n; p-1, p-1, \hat{\xi}_p^2)$$

where p denotes (1.2.6), the population multiple coherence of ξ_p with $\xi_1, \xi_2, \dots, \xi_{p-1}$. $\hat{\xi}_p^2$ denotes the sample estimate and $f(\cdot, \cdot, \cdot, \cdot)$ denotes the hypergeometric function, i.e.,

$$f(n, n; p-1, p-1, \hat{\xi}_p^2) = \sum_{j=0}^{\infty} \frac{\Gamma(n)}{\Gamma(n+j)} \cdot \frac{\Gamma(p)}{\Gamma(p+j)} \cdot \frac{\Gamma(p-1)}{\Gamma(p-1+j)} \cdot \frac{\Gamma(1)}{\Gamma(1+j)}$$

$$P(\xi) = (W|Z|)^{-1} \exp[-(\xi - \bar{\mu})^* Z^{-1} (\xi - \bar{\mu})] .$$

function given by

Let $\xi_{p \times 1}$ denote a complex random vector with probability density

freedom.

where χ^2_f stands for a chi-square random variable with f degrees of

$$Y = ZZ^* \sim \frac{\chi^2_{(2)}}{2} . \quad (2.1.0)$$

clear that

dent $N(0, 1/2)$ random variables. From real univariate theory, it is defined by $Y = ZZ^* = X^2 + Y^2$ which is the sum of squares of two independent variables with $\mu = 0$, and $\sigma^2 = 1/2$. Consider the real random variable in Chapter I, Z is composed of two real, independently distributed normal random variable times its conjugate. Let $Z \sim CN_1(0, 1)$. As pointed out random variable. The analogy of this in the complex case is the complex of great importance to establish the distribution of the square of the In the case of real normally distributed random variables, it is

presented in latter chapters.

formal proofs in order to provide a firm foundation for the material similarity, it seems worthwhile to include the following results with of complex random variables and real random variables. Despite this There is a great deal of similarity between the multivariate analysis

1. Basic Results

CHAPTER II

Consider the transformation given by

$$\bar{y} = A \bar{x} \quad \bar{x} = A^{-1} \bar{y}$$

where A is a real or complex matrix of rank $r \leq p$. If $r = p$, then the probability density function of the random vector $\bar{y}$ is given by

$$P(\bar{y}) = \frac{1}{(2\pi)^p} \exp \left\{ -\frac{1}{2} (\bar{y} - A \bar{\mu})^* (A^{-1} A^{-1})^{-1} (\bar{y} - A \bar{\mu}) \right\} \cdot |J|$$

where $J(x \rightarrow y)$ is given by Khatri (1965)

$$J = |A|^{-1} |A^*|^{-1}$$

Thus the density function of $\bar{y}$ becomes

$$P(\bar{y}) = \frac{1}{(2\pi)^p} \exp \left\{ -\frac{1}{2} (\bar{y} - A \bar{\mu})^* (A^{-1} A^{-1})^{-1} (\bar{y} - A \bar{\mu}) \right\} \cdot |J|$$

This result is stated formally as the following theorem.

Theorem 2.1. If $\bar{x}$ is distributed according to $CN_p(\bar{\mu}, \Sigma)$, then $\bar{y} = A \bar{x}$ is distributed according to $CN_p(A \bar{\mu}, A \Sigma A^*)$, where A is a $p \times p$ real or

complex matrix of rank p .

Now consider the transformation

$$\bar{y} = A \bar{x} \quad \bar{x} = A^{-1} \bar{y}$$

where the matrix A is of rank $r < p$. The mean of $\bar{y}$ is given by

$$E(\bar{y}) = E(A \bar{x}) = A E(\bar{x}) = A \bar{\mu}$$

and the covariance is given by

$$E[\bar{y} - E(\bar{y})][\bar{y} - E(\bar{y})]^* = A E[(\bar{x} - E(\bar{x}))(\bar{x} - E(\bar{x}))^*] A^*$$

$$= A \Sigma A^*$$

Since $A_{r \times p}$ is of rank r , it is known that there exists a matrix

$C_{(p-r) \times p}$ such that the transformation

$$\begin{pmatrix} \bar{Y}_{r \times 1} \\ \bar{Z}_{(p-r) \times 1} \end{pmatrix} = \begin{pmatrix} A \\ C \end{pmatrix} \bar{\xi}$$

is a non-singular transformation. Then $\bar{Y}$ and $\bar{Z}$ will have a joint com-

plex normal distribution and the marginal distribution of $\bar{Y}$ will be

$CN_r(\bar{A}\bar{\mu}, A\bar{\Sigma}A')$. This result will be stated as the following theorem.

Theorem 2.2. If $\bar{\xi}$ is distributed accordingly to $CN_p(\bar{\mu}, \bar{\Sigma})$, then $\bar{Y} = A\bar{\xi}$

is distributed accordingly to $CN_r(\bar{A}\bar{\mu}, A\bar{\Sigma}A')$, where A is a $r \times p$ matrix

of real or complex elements of rank $r \leq p$.

Now let $\bar{\xi}_{p \times 1} \sim CN_p(0, I_{p \times p})$, and make the transformation

$$\bar{Y}_{p \times 1} = U \bar{\xi}_{p \times 1}$$

where U is a unitary matrix, i.e., $U^*U = UU^* = I_p$. Then by Theorem 2.1,

$\bar{Y}_{p \times 1} \sim CN_p(0, I_p)$. It should be noted that the elements of U must be

constants. If the elements are random variables, the transformation is

a random unitary transformation and the distribution of $\bar{Y}$ is not necessarily

that of a $CN_p(0, I_p)$. In either case

$$\bar{Y}^* \bar{Y} = \bar{\xi}^* U^* U \bar{\xi} = \bar{\xi}^* \bar{\xi} \quad (2.1.1)$$

Thus, $\bar{\xi}^* \bar{\xi}$ remains invariant under a unitary transformation. Now consider

the transformation

$$\bar{Y}_{r \times 1} = A_{r \times p} \bar{\xi}_{p \times 1} \quad (2.1.2)$$

where the matrix A is of rank $r < p$ and has the additional property

$A_{A^*} = I_r$. This transformation will be referred to as a semi-unitary

transformation. As in Theorem 2.2, it is known that the matrix A may

be augmented with a matrix B such that the transformation

$$\begin{pmatrix} Y \\ \bar{Z} \end{pmatrix}_{r \times 1} = \begin{pmatrix} A \\ B \end{pmatrix} \begin{pmatrix} \bar{\xi} \end{pmatrix}_{(p-r) \times 1}$$

is of full rank and furthermore it is unitary. Thus

$$\begin{pmatrix} Y \\ \bar{Z} \end{pmatrix}_{r \times 1} \sim \text{CN}_{\bar{p}}^p(0, I_{\bar{p}})$$

and from the invariance property (2.1.1)

$$\xi_1^* \xi_1 + \xi_2^* \xi_2 + \dots + \xi_p^* \xi_p = Y^* Y + Y^* \bar{Z} + \bar{Z}^* Y + \bar{Z}^* \bar{Z} + \dots + \bar{Z}^* \bar{Z}$$

and the variables $Y_1, Y_2, \dots, Y_r, Z_{r+1}, Z_{r+2}, \dots, Z_p$ are distributed

as independent $\text{CN}_1(0, 1)$ as are the $\xi_1, \xi_2, \dots, \xi_p$.

From (2.1.1) we have

$$\bar{Y}^* \bar{Y} = Y^* Y + Y^* \bar{Z} + \bar{Z}^* Y + \bar{Z}^* \bar{Z}$$

and since $Y_1, Y_2, \dots, Y_r$ for an independent set of $\text{CN}_1(0, 1)$, then from

(2.1.0)

$$\chi_{2r}^2 \sim Y^* Y + Y^* \bar{Z} + \bar{Z}^* Y + \bar{Z}^* \bar{Z}$$

Also

$$\chi_{2p}^2 \sim \xi_1^* \xi_1 + \xi_2^* \xi_2 + \dots + \xi_p^* \xi_p$$

and

$$A = \sum_{i=1}^r \beta_i \beta_i^{-1}$$

where the λ_i are the characteristic roots of A, and the β_i are the corresponding orthogonal characteristic vectors of A. Since A is idempotent of rank r, then $\lambda_i = 1$, for $i = 1, \dots, r$ and $\lambda_j = 0$ for $j = r + 1, \dots, p$, thus the spectral decomposition of A is given by

$$A = \sum_{i=1}^p \lambda_i \beta_i \beta_i^{-1}$$

used, A may be written as

Since A is a symmetric matrix, the spectral decomposition of A may be matrix of rank r. The distribution of XAX^* needs to be established. and consider matrix of Hermitian forms, XAX^* where A is a real idempotent

Let $X = [\xi_1, \xi_2, \dots, \xi_n]$ where the $\xi_i \sim$ Independent $CN(0, I)$

$\bar{Z}^* Z = \xi^* \xi - \bar{Y}^* \bar{Y} \sim \frac{\chi^2_{2(p-r)}}{2}$ and is independent of $\bar{Y}^* \bar{Y}$. where A is a semi-unitary transformation, then $\bar{Y}^* \bar{Y} \sim \frac{\chi^2_{2r}}{2}$. Moreover, Theorem 2.3. If $\xi \sim CN^p(0, I)$, then $\xi^* \xi \sim \frac{\chi^2_{2p}}{2}$ and if $\bar{Y}^* \bar{Y} = A \xi^* \xi$

stated as the following theorem.

This result is the complex analogue of Fisher's Lemma, and is formally

$$Z^* Z^{r+1} + \dots + Z^* Z^p = (\xi^* \xi^1 + \dots + \xi^* \xi^p) - (Y^* Y^1 + \dots + Y^* Y^r)$$

Since

$$\xi^* \xi^1 + \xi^* \xi^2 + \dots + \xi^* \xi^p - (Y^* Y^1 + \dots + Y^* Y^r) \sim \frac{\chi^2_{2(p-r)}}{2}$$

Then

n independent and identically distributed p -variate complex normal vectors

Let $\xi_1, \xi_2, \dots, \xi_n$ where $\xi_i' = [\xi_{i1}, \xi_{i2}, \dots, \xi_{ip}]$, denote

conclusion of this section.

procedure. The minor changes that are encountered are pointed out at the

achieved in the case of a complex Wishart matrix by following the above

angular matrix by using random orthogonal transformations. This has been

transpose and giving the distribution of the elements of the lower tri-

Wishart matrix as the product of a lower triangular matrix times its

Bartlett decomposition of a real Wishart matrix, i.e., expressing the

Kshirsagar (1959) has presented an elegant method of deriving the

2. Bartlett Decomposition of a Complex Wishart Matrix

the matrix of Hermitian forms XAX^* is distributed as a $CW^p(r|Z)$.

Independent $CN^p(\bar{0}, Z)$, then for A a real idempotent matrix of rank $r \leq p$,

Theorem 2.4. $X = [\xi_1, \xi_2, \dots, \xi_n]$ where ξ_i are distributed as

This result will be stated formally as the following theorem.

$$XAX^* = \sum_{i=1}^r X \beta_i \beta_i^* X^* = \sum_{i=1}^r \bar{Y}_i \bar{Y}_i^* \sim CW^p(XAX^* | r | Z).$$

$\beta_i \beta_i^* = 0$, thus

and the $\bar{Y}_1, \bar{Y}_2, \dots, \bar{Y}_r$ form an independent set of $CN^p(\bar{0}, Z)$ since

$i = 1, 2, \dots, r$ which is the same as $\bar{Y}_i \sim CN^p(\bar{0}, Z)$ since $\beta_i \beta_i^* = 1$

transformation $\bar{Y}_i = X \beta_i$, it is seen that $\bar{Y}_i \sim CN^p(\bar{0}, \beta_i \beta_i^* \otimes Z)$ for

by $CN^{p \times n}(\bar{0}, I \otimes Z)$ where $\otimes$ stands for the Kronecker product. Making the

Since $\xi_i \sim CN^p(\bar{0}, Z)$, then the distribution of the matrix X is given

$$XAX^* = X \sum_{i=1}^r \beta_i \beta_i^* X^* = \sum_{i=1}^r X \beta_i \beta_i^* X^*.$$

with probability density function given by $CN(\bar{q}, I^p)$. Denote the matrix

of the vectors by X , i.e., $X = [\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n]$ $n \times p$.

Let A denote the matrix defined by

$$A = XX^* = \sum_{i=1}^n \bar{x}_i \bar{x}_i^*$$

then it has been shown [Khatri (1965)] that the matrix A is distributed

as $CW_p(A|n|I^p)$.

Now consider the vectors defined by

$$\begin{aligned} \bar{z}_1 &= \begin{bmatrix} \xi_{11} \\ \xi_{12} \\ \vdots \\ \xi_{1n} \end{bmatrix} ; \quad \bar{z}_2 = \begin{bmatrix} \xi_{21} \\ \xi_{22} \\ \vdots \\ \xi_{2n} \end{bmatrix} ; \quad \dots ; \quad \bar{z}_p = \begin{bmatrix} \xi_{p1} \\ \xi_{p2} \\ \vdots \\ \xi_{pn} \end{bmatrix} \end{aligned} \quad (2.2.0)$$

which are the vectors of observations on each of the p -variables. Each of the variables ξ_{ir} is independent and distributed as $CN(0, 1)$.

Orthogonalize these vectors, $\bar{z}_1, \dots, \bar{z}_p$, by using the Gram-Schmidt

process such that the new vectors form an orthonormal basis. The new

vectors, $\bar{y}_1, \bar{y}_2, \dots, \bar{y}_p$, may be represented as

$$\begin{aligned} \bar{y}_1 &= b_{11} \bar{z}_1 \\ \bar{y}_2 &= b_{21} \bar{z}_1 + b_{22} \bar{z}_2 \\ &\vdots \\ \bar{y}_p &= b_{p1} \bar{z}_1 + b_{p2} \bar{z}_2 + \dots + b_{pp} \bar{z}_p \end{aligned}$$

The $b_{ii}^{(k)}$ for $i = 1, 2, \dots, p$ can be chosen to be real while the $b_{kj}^{(k)}$,

$k > j$, $k = 1, 2, \dots, p$, $j = 1, 2, \dots, p$ will in general be complex.

To see this consider [Hohn (1958)]

$$\bar{y}_1 = b_{11}^{(k)} \bar{z}_1$$

and since $b_{11}^{(k)}$ is to be chosen such that $\bar{y}_1^{(k)} \bar{y}_1 = 1$, this implies $b_{11}^{(k)} = \frac{1}{\sqrt{\bar{z}_1^{(k)} z_1^{(k)}}}$ which is real. Now

$$\bar{y}_2 = b_{21}^{(k)} \bar{z}_1 + b_{22}^{(k)} \bar{z}_2$$

and with the condition, $\bar{y}_1^{(k)} \bar{y}_2 = 0$, this implies

$$b_{21}^{(k)} b_{11}^{(k)} \bar{z}_1^{(k)} \bar{z}_1 + b_{22}^{(k)} \bar{z}_1^{(k)} \bar{z}_2 = 0.$$

This is a homogeneous equation in two unknowns, $b_{21}^{(k)}$ and $b_{22}^{(k)}$. A complete

solution is given by

$$\begin{aligned} b_{21}^{(k)} &= -(z_1^{(k)} \cdot z_2) t \\ b_{22}^{(k)} &= (z_1^{(k)} \cdot z_1) t \end{aligned}$$

where t is an arbitrary parameter. Continuing in this fashion, it can

be shown that $b_{33}^{(k)}$ is proportional to $(z_1^{(k)} z_1) (z_2^{(k)} z_2)$. So if t is chosen to be real, we have $b_{22}^{(k)}$ real, also $b_{33}^{(k)}$ real, and all other $b_{ii}^{(k)}$ will be real. Attention should be called to $b_{kj}^{(k)}$, $k > j$. For a real t , this

does not imply that $b_{kj}^{(k)}$ will be real. Thus the $b_{ii}^{(k)}$, $i = 1, 2, \dots, p$,

will be restricted to be real.

Then the above system may be written as

$$\begin{bmatrix} \bar{y}'_1 \\ \bar{y}'_2 \\ \vdots \\ \bar{y}'_p \end{bmatrix} = \begin{bmatrix} b_{11} & b_{21} & \dots & b_{p1} \\ 0 & b_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_{pp} \end{bmatrix} \begin{bmatrix} \bar{z}'_1 \\ \bar{z}'_2 \\ \vdots \\ \bar{z}'_p \end{bmatrix} \quad (2.2.1)$$

Denote (2.2.1) as

$$Y \bar{X} = \bar{B} \quad Y^{p \times n} = \bar{B}^{p \times p} \quad X^{n \times p}$$

Since

$$X = [\xi_1, \xi_2, \dots, \xi_n]^{n \times p} = \begin{bmatrix} \bar{z}'_1 \\ \bar{z}'_2 \\ \vdots \\ \bar{z}'_p \end{bmatrix}^{p \times n}$$

Let $B = [b_{ij}]$. Since the diagonal elements of B are real, then the diagonal elements of $B^{-1} = B$ must necessarily be real. Hence, b_{ii} for $i = 1, 2, \dots, p$ are real elements and the reciprocal relation may be written as

$$X^{p \times n} = B^{p \times p} X^{n \times p} \quad (2.2.2)$$

or as

$$\begin{bmatrix} \bar{z}'_1 \\ \bar{z}'_2 \\ \vdots \\ \bar{z}'_p \end{bmatrix} = \begin{bmatrix} b_{11} & b_{21} & \dots & b_{p1} \\ 0 & b_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_{pp} \end{bmatrix} \begin{bmatrix} \bar{y}'_1 \\ \bar{y}'_2 \\ \vdots \\ \bar{y}'_p \end{bmatrix}$$

which implies that

$$\bar{z}_k = b_{k1-1} + b_{k2-2} + \dots + b_{kk-k} \quad (2.2.3)$$

i.e., the $\bar{z}_k$ can be expressed in terms of the unitary vectors $\bar{y}_k$,

$$k = 1, \dots, p.$$

From (2.2.2) it is seen that the matrix X can be represented as

the matrix B times the semi-unitary matrix Y , then

$$XX^* = BYY^*B^* = BB^*, \quad \text{since } YY^* = I_p.$$

The distribution of XX^* is $CW^p(A|n|I_p)$ thus $BB^* \sim CW^p(A|n|I_p)$ or that

$A = BB^*$. Thus a complex Wishart matrix can be expressed in terms of a

lower triangular matrix, whose diagonal elements are real and the non-

zero off diagonal elements are complex, times the transpose of its conjugate.

Also note that

$$\begin{aligned} \bar{y}_j^* \bar{z}_k &= b_{kj} \\ k &= 1, \dots, p \\ j &= 1, 2, \dots, k \end{aligned} \quad (2.2.4)$$

this result follows from (2.2.3).

Now keep $\bar{z}_1, \bar{z}_2, \dots, \bar{z}_{k-1}$ fixed. This automatically fixes

$\bar{y}_1, \bar{y}_2, \dots, \bar{y}_{k-1}, b_{11}, \dots, b_{k-1,1}, b_{12}, \dots, b_{k-1,k-1}$ by virtue

of relation (2.2.4). Also recall that the $\bar{z}_{1t}$ are distributed independently

as $CN_1(0, 1)$. Consider, from (2.2.4), the b_{kt} , $t = 1, 2, \dots, k-1$, which

may be written as

(2.2.5)

$$\begin{aligned} b_{k1} &= \bar{y}_1^* \bar{z}_k \\ b_{k2} &= \bar{y}_2^* \bar{z}_k \\ &\vdots \end{aligned}$$

$$b_{k \ k-1} = \bar{y}_{k-1}^* \bar{z}_k$$

and by holding $\bar{z}_1, \bar{z}_2, \dots, \bar{z}_{k-1}$ fixed, this becomes a semi-unitary transformation from $\xi_{k1}, \xi_{k2}, \dots, \xi_{kn}$ to $b_{k1}, b_{k2}, \dots, b_{k \ k-1}$. Observing that

$$[\bar{z}_1, \bar{z}_2, \dots, \bar{z}_{k-1} \mid \bar{z}_k, \dots, \bar{z}_p] \quad \begin{matrix} \text{fixed} \\ \text{random} \end{matrix}$$

automatically requires that

$$\left[\begin{array}{ccc|ccc} \xi_{1n} & \dots & \xi_{2n} & \dots & \xi_{k-1 \ n} & \xi_{kn} \\ \vdots & & \vdots & & \vdots & \vdots \\ \vdots & & \vdots & & \vdots & \vdots \\ \xi_{12} & \dots & \xi_{22} & \dots & \xi_{k-1 \ 2} & \xi_{k2} \\ \vdots & & \vdots & & \vdots & \vdots \\ \xi_{11} & \dots & \xi_{21} & \dots & \xi_{k-1 \ 1} & \xi_{k1} \\ \vdots & & \vdots & & \vdots & \vdots \\ \xi_{p1} & \dots & \xi_{p2} & \dots & \xi_{pn} & \end{array} \right] \quad \begin{matrix} \text{fixed} \\ \text{random} \end{matrix}$$

Thus

$$b_{kj} = \bar{y}_j^* \bar{z}_k \quad \text{for } j = 1, 2, \dots, k-1$$

is distributed as a linear combination of the ξ_{kt} where $t = 1, 2, \dots, n$ and the ξ_{kt} are distributed as independently $CN_1(0, 1)$. Thus

$$b_{kj} \sim CN_1(0, 1) \quad (2.2.6)$$

since $\text{var}(b_{kj}) = \gamma_{jj}^* = 1$, and $E(b_{jk}^j) = 0$. Furthermore, the b_{kt} forms an independent set for $t = 1, 2, \dots, k-1$. Observe that for $t \neq t'$

$$\text{cov}(b_{kt}, b_{kt'}) = \text{cov}(\gamma_{*Z}^t, \gamma_{*Z}^{t'}) = \gamma_{*Y}^t \gamma_{*Y}^{t'} = 0.$$

Thus (2.2.6) is the conditional distribution of the b_{kj} with

$\bar{z}_1, \bar{z}_2, \dots, \bar{z}_{k-1}$ being fixed, the $\bar{z}_t$'s do not enter into the distribution, so (2.2.6) is also the unconditional distribution of the b_{kj} 's, for $j > k$ and the b_{kj} 's form a mutually independent set. Hence, the

result

$$(2.2.7) \quad b_{k-1}, \dots, b_{k-k-1} \sim \text{Independent } \text{CN}_1^1(0, 1).$$

As stated previously, the transformation (2.2.5) is a semi-unitary transformation from $\bar{z}_1, \dots, \bar{z}_{kn}$ to $b_{k1}, b_{k2}, \dots, b_{k-k-1}$. This transformation can be completed with $[n - (k-1)]$ new random variables in such a way as to maintain independence between $b_{k1}, b_{k2}, \dots, b_{k-k-1}$, and the $[n - (k-1)]$ random variables, so that the complete transformation is unitary. Since the transformation is of full rank and unitary, then

$$\bar{z}_1^* \bar{z}_1 + \bar{z}_2^* \bar{z}_2 + \dots + \bar{z}_{kn}^* \bar{z}_{kn} =$$

$$b_{k1}^* b_{k1} + \dots + b_{k-k-1}^* b_{k-k-1} + [n - (k-1)] \text{ random variables.}$$

Rewrite the above as

$$\bar{z}_k^* \bar{z}_k = b_{k1}^* b_{k1} + \dots + b_{k-k-1}^* b_{k-k-1} + [n - (k-1)] \text{ random variables} \quad (2.2.8)$$

and note the independence among the variables on the left once more.

Furthermore, (2.2.8) is distributed as a $\chi_{(2n)}^2$, but by (2.2.3), it

is seen that

$$\bar{Z}^* \bar{Z}^{-k} = b_{k1}^* b_{k1} + \dots + b_{k-k-1}^* b_{k-k-1} + b_{kk}^* .$$

Also, by virtue of (2.1.0), it is seen

$$b_{k1}^* b_{k1} + \dots + b_{k-k-1}^* b_{k-k-1} \sim \frac{\chi^2_{2(k-1)}}{2} .$$

Thus by Fisher's Lemma, it is seen that

$$b_{kk}^2 = \bar{Z}^* \bar{Z}^{-k} - (b_{k1}^* b_{k1} + \dots + b_{k-k-1}^* b_{k-k-1}) \sim \frac{\chi^2_{2[n-(k-1)]}}{2}$$

and b_{k2}^2 is independent of $b_{k1}, \dots, b_{k-k-1}$ and $\bar{Z}_1, \bar{Z}_2, \dots, \bar{Z}_{k-1}$.

This result is true for every $k = 1, 2, \dots, p$.

In summary, it is seen that a complex Wishart matrix A , may be

represented as the product of a lower triangular matrix, $B = [b_{kj}]$, where

$b_{kk}, k = 1, 2, \dots, p$, are real and $b_{kj}, k > j$, are complex times

the transpose of its conjugate where the

$b_{kj} \sim \text{independent } CN^1(0, 1) \text{ for } j < k, k = 1, 2, \dots, p$

and

$$b_{kk}^2 \sim \frac{\chi^2_{2[n-(k-1)]}}{2} .$$

These results are similar to that of the decomposition of a real

Wishart matrix in that the $b_{kj} \sim \text{independent } N(0, 1) \text{ for } j < k,$

$k = 1, 2, \dots, p$ and $b_{kk}^2 \sim \chi^2_{2[n-(k-1)]}$.

3. Coherence Between Two Complex Random Variables

Ruben (1966) obtained, in the case of real bivariate normal situation, a very good approximation to the distribution of the correlation coefficient. This was done by expressing the sample correlation coefficient as a ratio of a linear combination of a standardized normal variate and a chi-variate to another chi-variate, where all three were mutually independent. This work has been extended to the multivariate situation by Kshirsagar (1969), who gives a matrix representation of this result for correlation between two vectors.

Following Ruben's work in the bivariate case and Kshirsagar's work in the multivariate case, similar results have been obtained in the complex case. These will be presented here.

Consider a random sample $\xi_1, \xi_2, \dots, \xi_n$ of size n from a $CN_2(\bar{\mu}, \Sigma)$ where

$$\xi_k = \begin{bmatrix} \xi_{1k} \\ \xi_{2k} \end{bmatrix}; \quad \bar{\mu} = \begin{bmatrix} \mu_1 + i\nu_1 \\ \mu_2 + i\nu_2 \end{bmatrix}; \quad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

The Wishart matrix of sum of squares of products is given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \sim CW_2(A|n-1|\Sigma),$$

where

$$a_{ij} = \sum_{k=1}^n (z_{1j} - \bar{z}_1)(z_{1i} - \bar{z}_1)^*$$

and

$$\bar{z}_1 = \frac{\sum_{k=1}^n z_{1k}}{n} \quad \text{for } i = 1, 2.$$

From Theorem 2.1, the distribution of $\bar{Y}$ is given by $CN^2(0, CC' = I_2)$ and, furthermore, the Wishart matrix A has been reduced to its canonical form, i.e., $D = CAC^* \sim CW^2(D|n-1|I_2)$. These results are summarized in the following table

Variable	ξ	$\bar{Y} = C(\xi - \bar{\mu})$
mean	$\bar{\mu}$	$\bar{\mu}$
variance	Σ	I
Wishart matrix	A	$D = CAC^*$

The Wishart matrix $D = CAC^*$ has the form

$$D = \begin{bmatrix} \frac{a_{11}}{2} & -\frac{a_{11}}{2} \frac{a_{12}^*}{a_{11}^*} + \frac{a_{11}^* a_{22} (1-pp^*)}{2} \frac{1/2}{a_{12}^*} \\ \frac{a_{11}}{2} & -\frac{a_{11}}{2} \frac{a_{12}^*}{a_{11}^*} + \frac{a_{11}^* a_{22} (1-pp^*)}{2} \frac{1/2}{a_{12}^*} \end{bmatrix}$$

where

$$\delta = \frac{a_{11}^* a_{22} (1-pp^*)}{2} - \frac{a_{12}^* a_{22} (1-pp^*)}{a_{12}^*} + \frac{a_{12}^* a_{22} (1-pp^*)}{a_{12}^*} + \frac{a_{22}^2 (1-pp^*)}{a_{22}}$$

Since $D \sim CW^2(D|n-1|I_2)$, it may be written as $D = BB^*$ where B is a lower triangular matrix given by the Bartlett decomposition. BB^* is given by (observing the b_{ii} for $i = 1, 2$ are real)

$$\begin{bmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} b_{11}^* & b_{21}^* \\ 0 & b_{22}^* \end{bmatrix} = \begin{bmatrix} b_{11}^2 & b_{21} b_{11}^* \\ b_{21} b_{11}^* & (b_{21} b_{21}^* + b_{22}^2) \end{bmatrix}$$

and equating the elements of BB^* with the corresponding elements of $D = CAC^*$, the following system of equations is obtained.

$$\begin{aligned} \frac{a_{11}^2}{b_{11}^2} &= \frac{a_{11}^2}{b_{11}^2} \\ b_{11}^2 a_{11}^2 &= \frac{a_{12}^2}{a_{11}^2} - \frac{a_{11}^2 a_{22}^2 (1-p)^{1/2}}{a_{11}^2} \\ b_{12}^2 a_{11}^2 + b_{22}^2 &= \frac{a_{11}^2 a_{12}^2}{a_{11}^2} - \frac{a_{11}^2 a_{22}^2 (1-p)^{1/2}}{a_{11}^2} - \frac{a_{11}^2 a_{22}^2 (1-p)^{1/2}}{a_{11}^2} + \frac{a_{22}^2}{a_{22}^2} \end{aligned}$$

Solving the above system for the a_{ij} ($i = 1, 2$; $j = 1, 2$) the following

solutions are obtained:

$$\left\{ \begin{aligned} a_{11}^2 &= \frac{a_{11}^2}{b_{11}^2} \\ a_{12}^2 &= \frac{a_{11}^2 a_{22}^2 (1-p)^{1/2}}{b_{11}^2} + \frac{a_{11}^2 a_{22}^2 (1-p)^{1/2}}{b_{11}^2} \\ a_{22}^2 &= \frac{a_{11}^2 a_{22}^2 (1-p)^{1/2}}{b_{11}^2} + \frac{a_{11}^2 a_{22}^2 (1-p)^{1/2}}{b_{11}^2} + \frac{a_{11}^2 a_{22}^2 (1-p)^{1/2}}{b_{11}^2} \end{aligned} \right\} \quad (2.3.1)$$

Now, from Section 2, it has been shown that the

$$\begin{aligned} b_{11}^2 &\sim \chi_{2(n-1)}^2 \\ b_{22}^2 &\sim \chi_{2(n-2)}^2 \end{aligned}$$

$$b_{12}^2 \sim CN_1(0, 1)$$

and all are mutually independent. Upon substituting (2.3.1) into

(2.3.0), the following representation for $\tilde{r}$ is obtained

$$\tilde{r} = \frac{b_{22}}{b_{12} + \tilde{\rho} b_{11}}$$

which can be rewritten as

$$\tilde{r} = \frac{\frac{\chi^2_{2n-4}}{\sqrt{2}}}{\frac{\chi^2_{2n-2}}{\sqrt{2}} + \tilde{\rho} \frac{\chi^2_{2n-2}}{\sqrt{2}}} \quad (2.3.2)$$

where χ^2_v denotes a real chi-variable with v degrees of freedom. Attention

should be called to the distribution of $\tilde{r}$ in the null case, i.e., $\rho = 0$,

$$\tilde{r} = \frac{\chi^2_{2n-4}}{\chi^2_{2n-2}}$$

which has the appearance of what could be called a complex t .

Consider a probability statement about the complex random variable

defined by (2.3.2)

$$P(\tilde{r} > \tilde{r}_0) = P\left(\frac{\frac{\chi^2_{2n-4}}{\sqrt{2}}}{\frac{\chi^2_{2n-2}}{\sqrt{2}} + \tilde{\rho} \frac{\chi^2_{2n-2}}{\sqrt{2}}} > \tilde{r}_0\right)$$

or rewritten as

$$P(L > 0)$$

where

$$L = \text{CN}(0, 1) + \tilde{\rho} \frac{\chi^2_{2n-2}}{\sqrt{2}} - \tilde{\rho} \frac{\chi^2_{2n-4}}{\sqrt{2}} \quad (2.3.3)$$

Now following Kshirsagar's (1969) work, a matrix representation for coherence between two complex vectors will be derived. Let the variance-covariance of two complex vectors $\bar{X}$ $\bar{p} \times 1$ and $\bar{Z}$ $\bar{q} \times 1$ be given by

$$\bar{Z} = \begin{bmatrix} \bar{Z}_{11} & \bar{Z}_{12} \\ \bar{Z}_{21} & \bar{Z}_{22} \end{bmatrix} \quad \begin{matrix} \bar{p} & \bar{q} \\ (p+q) \times (p+q) \end{matrix} \quad (2.3.4)$$

and let the matrix of corrected sum of squares (s.s.) and sum of products (s.p.) of observations from a sample of size $n+1$ on these variables be

given by

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \quad \begin{matrix} p & q \\ (p+q) \times (p+q) \end{matrix} \quad (2.3.5)$$

which is based on n degrees of freedom. Then the following matrices can

be obtained:

$\bar{B}_0$ = matrix of regression coefficients $S_{12}^{-1} S_{11}'$, of $\bar{X}$ on $\bar{Z}$;

B = matrix of s.s. and s.p., due to regression $S_{12}^{-1} S_{22} S_{21}'$;

A = "residual" s.s. and s.p. matrix $S_{11} - S_{12} S_{22}^{-1} S_{21} = S_{11.2}$;

$A+B$ = "total" matrix S_{11} .

The corresponding matrices for the population are $\bar{B} = \bar{Z}_{12}^{-1} \bar{Z}_{11}'$, $\bar{B}_0$, $\bar{B}$, $\bar{A} + \bar{B}$ = $\bar{Z}_{11}$ and $\bar{Z}_{11.2} = \bar{Z}_{11} - \bar{Z}_{12} \bar{Z}_{22}^{-1} \bar{Z}_{21}'$, respectively.

If the vector $\bar{Y}^{(p+q) \times 1}$ defined by $\bar{Y}' = [\bar{X}' | \bar{Z}']$ is distributed as

$CN^{(p+q)}(\bar{\mu}, \Sigma)$ where $\bar{\mu}' = [\bar{\mu}_X' | \bar{\mu}_Z']$ and Σ is defined by (2.3.4), then the

matrix S , defined by (2.3.5) is distributed as $CW^{(p+q)}(S | n | \Sigma)$ and from

this density it has been shown by Kabe (1966) that

$$S_{11 \cdot 2} \sim CW^p \left(S_{11 \cdot 2} | n - q | \Sigma_{11 \cdot 2} \right) \quad (2.3.6)$$

and that $S_{11 \cdot 2}$ is distributed independently of B_0 and S_{22} . Also,

Kabe (1966) has shown that

$$S_{22} \sim CW^q (S_{22} | n | \Sigma_{22}) \quad (2.3.7)$$

and that B_0 , for fixed S_{22} , is distributed as

$$CN^p(\bar{B}, \Sigma_{11 \cdot 2}) \quad (2.3.8)$$

and is independent of $S_{11 \cdot 2}$.

Now let Φ , η , M , F , C , and K be lower triangular matrices such

that

$$\Sigma_{22} = \Phi \Phi'$$

$$\Sigma_{11 \cdot 2} = \eta \eta'$$

$$S_{22} = M M'$$

$$A = F F'$$

$$B = K K'$$

$$A+B = C C'$$

Define further

(2.3.9)

$$U = \eta^{-1} (B_0 - B) M$$

$$V = \phi^{-1} M$$

$$W = \eta^{-1} F$$

$$\tilde{F} = \eta^{-1} L_{\phi^{*-1}}^{12}$$

$$\tilde{R} = F^{-1} S_{M^{*-1}}^{12}$$

$$R = C^{-1} K$$

$$L = RR^* = C^{-1} B C^{*-1}$$

Transform to U , V , and W from B_0 , L^{22} , and $S^{11 \cdot 2}$, respectively, in (2.3.6), (2.3.7), and (2.3.8). Then it can be shown

$$(1) \quad u_{ij} \quad (i = 1, \dots, p; j = 1, 2, \dots, q), \text{ the } pq \text{ variables}$$

in U are independent $CN(0, 1)$;

$$(2) \quad v_{ij} \quad (j = 1, \dots, q), \text{ the diagonal elements of the lower triangular matrix } V \text{ are independent } \chi_{2(n-j+1)}^2 / \sqrt{2} \text{ and the off diagonal element } v_{kj} \text{ for } k > j \text{ are independent } CN(0, 1) \text{ and independent } v_{jj};$$

$$(3) \quad w_{ij} \quad (i = 1, 2, \dots, p), \text{ the diagonal elements of the lower triangular matrix } W \text{ are independent } \chi_{2(n-i+1)}^2 / \sqrt{2} \text{ varies, while } w_{ij}, i > j, \text{ are independent } CN(0, 1), \text{ independent of } w_{ii} \text{ also.}$$

Referring back to the transformations in (2.3.9), it can be seen $L = RR^*$ is the matrix generalization of r^2 , which is the ratio of

regression sum of squares to the total sum of squares and $\bar{R}\bar{R}'$ is the matrix generalization of $r^2/1-r^2$, the ratio of the regression sum of squares to the residual sum of squares. Observing that

$$U + \bar{P}V = \eta^{-1} S_{12} M^{*-1} - 1$$

where $\bar{P}$ is the population matrix corresponding to $\bar{R}$, then

$$\bar{R} = W^{-1}(U + \bar{P}V) \quad (2.3.10)$$

which is the matrix analogue to the result (2.3.2), i.e.,

$$\bar{r} = \frac{CN_1(0, 1) + \frac{\chi^2_{2n-2}}{\sqrt{2}}}{\frac{\chi^2_{2n-4}}{\sqrt{2}}}$$

which corresponds to Ruben's result in the real case. Then

$$\xi = U + \bar{P}V - W\bar{R}_0 \quad (2.3.11)$$

is the matrix analogue of result (2.3.3), and also corresponds to

Kshirsagar's result in the real case.

From this result (2.3.11), one can obtain results similar to Ruben's about every element of the matrix ξ , however in multivariate analysis, one prefers an overall criterion based on the whole matrix rather than indi-

vidual results on the elements. Hence, no attempt is made to pursue this further; instead, we shall be considering overall criteria such as Wilks'

Λ in the complex case.

4. Multiple Coherence

In the case of real multivariate normal analysis, an extensive

amount of work has been done relating to the multiple correlation

coefficient, i.e., the maximum correlation between a random variable

x_p and a linear combination of a set of variables $x_1, x_2, \dots, x_{p-1}$.

If , where

$$Z = \begin{bmatrix} z_{11} & z_{12} & \dots & z_{1p} \\ z_{21} & z_{22} & \dots & z_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ z_{p1} & z_{p2} & \dots & z_{pp} \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \\ z_{31} & z_{32} \\ z_{41} & z_{42} \end{bmatrix}$$

denotes the variance-covariance structure of this set of variables

$x_1, x_2, \dots, x_p$, it can be shown that the square of the maximum cor-

relation coefficient is given by

$$r^2_{x_p(x_1, \dots, x_{p-1})} = \frac{z_{21} z_{11}^{-1} z_{12}}{z_{22}}$$

and the same multiple correlation is given by

$$r^2_{x_p(x_1, \dots, x_{p-1})} = \frac{A_{21} A_{11}^{-1} A_{12}}{A_{22}}$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

is the sample variance-covariance matrix based on a sample of size n . Hodgson (1967), following Ruben's (1966) approach, has shown $R^2/(1-R^2)$ is distributed as

$$\left[\chi_{k-1}^2 + \left\{ u + \left(p(1-p_2) \right)^{-\frac{1}{2}} \chi_{n-1}^2 \right\}^2 \right] \chi_{n-k-1}^2$$

where the χ^2 and the unit normal variate u are mutually independent.

From this, Hodgson has obtained a normal approximation for the statistic $R^2/(1-R^2)$. These results have been achieved in a somewhat similar

fashion for the multiple coherence in a multivariate complex normal

situation.

Suppose $\bar{\xi} \sim \text{CN}_p(\bar{\mu}, \Sigma)$. Without loss of generality, it will be

assumed that $\bar{\mu} = \bar{0}$ and Σ has the following special form

$$\Sigma = \begin{bmatrix} 1 & 0 & \dots & 0 & p \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ p & 0 & \dots & 0 & 1 \end{bmatrix}$$

where p is the true multiple coherence between ξ_p and $\xi_1, \xi_2, \dots, \xi_{p-1}$, as defined in (1.2.6). This can be shown as below:

Let $\bar{\xi} \sim \text{CN}_p(\bar{\mu}, \Sigma)$, where

and

$$\xi_p^2(\xi_1, \xi_2, \dots, \xi_{p-1}) = \frac{\xi_{12} \xi_{11}^{-1} \xi_{12}^o}{\xi_{pp}^o}$$

formation, since

Furthermore, the multiple coherence remains invariant under this trans-

$$A\bar{A}^{*'} = \left[\begin{array}{c|c} \xi_{12} D^{*-1} - 1 & \frac{\xi_{12}^o}{1} \\ \hline D^{-1} \xi_{12} & \frac{\xi_{12}^o}{1} \end{array} \right] \begin{array}{c} 1 \\ \xi_{pp}^o \end{array}$$

then $E(\bar{Y}^{-p \times 1}) = \bar{\theta}$ and $\text{var}(\bar{Y}) = A\bar{A}^{*}$, observe

$$A = \left[\begin{array}{c|c} \bar{\theta}' & \frac{\xi_{12}^o}{1} \\ \hline \bar{\theta} & \xi_{pp}^o \end{array} \right]$$

form

Observing that ξ_{11} is an hermitian positive definite matrix, then there exists a D such that $\xi_{11} = D D^{*}$ and $D^{-1} \xi_{11} D^{*-1} = I_{p-1}$. Make the following transformation from $\bar{\xi} \rightarrow \bar{Y}$ by $\bar{Y} = A(\bar{\xi} - \bar{\mu})$, where A has the following

$$\bar{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_p \end{bmatrix} \quad \text{and} \quad \bar{\xi} = \left[\begin{array}{ccc|ccc} \xi_{11} & \xi_{12} & \dots & \xi_{p1} & \xi_{p2} & \dots & \xi_{pp} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right] = \left[\begin{array}{c|c} \xi_{11} & \xi_{21} \\ \hline \xi_{12} & \xi_{pp}^o \end{array} \right]$$

$$\begin{aligned}
 x_1 &= \frac{\beta_1}{\bar{y}_+} \\
 x_2 &= \frac{\bar{a}_2}{\bar{y}_+} \\
 &\vdots \\
 x_{p-1} &= \frac{\bar{a}_{p-1}}{\bar{y}_+}
 \end{aligned}$$

$$x_p = \bar{y}_p$$

where $\bar{a}_2, \bar{a}_3, \dots, \bar{a}_{p-1}$ are so chosen that

$$\bar{a}_1' \bar{\varepsilon} = \bar{a}_2' \bar{\varepsilon} = \bar{a}_3' \bar{\varepsilon} = \dots = \bar{a}_{p-1}' \bar{\varepsilon} = 0,$$

i.e., find $p-2$ vector orthogonal to $\bar{\varepsilon}$, and noting the following relation-

ships among the x 's.

$$\text{var}(x_i) = 1 \quad \text{for } i = 1, 2, \dots, p$$

$$\begin{aligned}
 \text{cov}(x_i, x_j) &= a_{ij}' a_j = 0 \quad \text{for } i \neq j; i = 1, 2, \dots, p-1, j = 1, 2, \dots, p-1 \\
 \text{cov}(x_i, x_1) &= \beta_{ij}' a_1 = 0 \quad \text{for } i = 2, \dots, p-1 \\
 \text{cov}(x_p, x_1) &= \beta_{pj}' a_1 = 0 \quad \text{for } i = 2, \dots, p-1
 \end{aligned}$$

and

$$\text{cov}(x_1, x_p) = \text{cov}\left(x_p, \frac{1}{\beta_{11}'} \bar{y}_+ - \frac{\sqrt{\beta_{11}'} \beta_{11}'}{\beta_{11}'}\right) = \frac{\sqrt{\beta_{11}'} \beta_{11}'}{\beta_{11}'} = \beta_{11}'$$

$$= \beta_{11}'$$

Thus we have

$$x \sim \text{CN}(\bar{0}, \frac{1}{\beta_{11}'})$$

where

$$Z = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}$$

(2.4.0)

Let $\xi_1, \xi_2, \dots, \xi_n$ denote a random sample of size n from a

$CN^p(0, Z)$ where Z is of the form given by (2.4.0). Let A_ξ denote the

complex Wishart matrix associated with this sample, i.e.,

$$A_\xi = \sum_{i=1}^n \xi_i \xi_i^* \sim CM^p(A|n|Z).$$

It is desired to look at the Bartlett decomposition of this Wishart

matrix, but the distribution of A_ξ is not in the canonical form. By

making a transformation of the original sample, it is possible to

express A_ξ as

$$A_\xi = CTT^*C^*$$

where T is a $p \times p$ triangular matrix whose elements are independent random

variables, the off diagonal elements being $CN^1(0, 1)$ and the diagonal

elements being χ variables. To see this, consider the following argument.

Let $\bar{\xi} \sim CN^p(0, Z)$, Z is hermitian positive definite, thus

$Z = CC^*$. Transform from A_ξ to $A_{\bar{\xi}}$ by

$$A_{\bar{\xi}} = C^{-1}A_\xi C^{*-1}$$

observing

$$\bar{s}'T + (1-p_2)^2 \bar{e}' = [p_1' 1, 0 \ 0 \dots 0] + \left[(1-p_2)^2 \bar{e}_{p_1}', (1-p_2)^2 \bar{e}_{p_2}', \dots, (1-p_2)^2 \bar{e}_{pp}' \right]$$

and letting X' represent this quantity, then

$$CT = \left[\begin{array}{c|c} T & X' \\ \hline 0 & (1-p_2)^2 \bar{e}_{pp}' \end{array} \right]$$

and CTT^*C^* is given as

$$CTT^*C^* = \left[\begin{array}{c|c} T T^* & \bar{X}' T^* \\ \hline \bar{T} \bar{X}^* & (1-p_2)^2 \bar{e}_{pp}' + \bar{X}' \bar{X}^* \end{array} \right]$$

observing that

$$\bar{X}' \bar{X}^* = [p_1' 1 + (1-p_2)^{1/2} \bar{e}_{p_1}'] [p_1' 1 + (1-p_2)^{1/2} \bar{e}_{p_1}']$$

$$+ (1-p_2)^{p-1} \sum_{i=2}^p \bar{e}_{p_i} \bar{e}_{p_i}'$$

Hence

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{pp} \end{array} \right] = \left[\begin{array}{c|c} T T^* & X' T^* \\ \hline T T^* & (1-p_2)^2 \bar{e}_{pp}' + X X^* \end{array} \right] \quad (2.4.2)$$

Interest lies with the quantity $R^2/(1-R^2)$. Observe that

$$\frac{|A|}{|A| \cdot |A|} = \frac{1}{|A|}$$

which may be rewritten

$$= \frac{\{ \tau_2^{(1-p-1)} \} \cdot | \tau_2^{(1-p-1)} \} + \{ \bar{\tau}_2^{(1-p-1)} \}}{\tau_2^{(1-p-1)}} =$$

then

$$\frac{R_2}{2} \frac{(1-p_2)^{pd}}{2} = \frac{| \{ p_1^{11} + (1-p_2)^{1/2} p_2 \} | + (1-p_2)^{1/2} p_2^{11}}{(1-p_2)^{pd}} =$$

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$$\frac{R_2^{1-R_2}}{R_2^{t_2} + (p/(1-p_2))^{1/2} |t_{11}|_2^{p-1} \sum_{i=2}^{1=2} t_{p1}^{p1} t_{*p1}^{p1}} = \frac{t_2^{t_2}}{(2.4.4)}$$

where $|t_{p1}|^2 + (p/(1-p_2))^{1/2} |t_{11}|^2$ represents the modulus of the quantity.

Let $\delta = \left(\frac{p}{(1-p_2)^{1/2}} t_{p_1} + t_{11} \right)$ and using the fact that $t_{p_1} \sim CN^1(0, 1)$

and t_{11} which is a chi-variable and can be approximated by a real

$N \left(\sqrt{\frac{4}{2(2n-2)-1}}, \frac{4}{1} \right)$ the δ can be approximated by

$$\delta \sim \text{CN} \left(\sqrt{\frac{4n-5}{4}} \frac{1}{p^2}, \frac{1}{p^2} + \frac{1}{(1-p^2)} \right)$$

then

$$\frac{R^2}{1-R^2} = \frac{(1 + \frac{1}{4}\lambda) \chi^2_{(2, \lambda)} + \chi^2_{\sum_{i=1}^{p-1} [n-(i-1)]}}{\chi^2_{2(n-p-1)}}$$

where the noncentrality parameter λ is given by

$$\lambda = \frac{p^2}{(1-p^2)}$$

If $p = 0$, i.e., the true multiple coherence coefficient is zero, then

$$\frac{R^2}{1-R^2} = F \left[2 \sum_{k=1}^{p-1} \{ [n-(k-1)], 2[n-(p-1)] \} \right] .$$

This is exact and follows from (2.4.4).

hypotheses about complex multivariate normal populations possesses optimum
Giri (1965) has shown that the likelihood ratio criterion for certain
means. Represent the alternate hypothesis by H_a .

against the alternate hypothesis that the populations have different

$$H_0: \bar{\mu}(1) = \bar{\mu}(2) = \dots = \bar{\mu}(k)$$

is desired to develop a test for the hypothesis of equal means, i.e.,
 $2, \dots, n_\alpha; \alpha = 1, 2, \dots, k$ on which the observations are made. It
the vector of responses for the λ^{th} member of the α population ($\lambda = 1,$
and the same variance covariance structure given by Σ . Let $\bar{\xi}_{\alpha\lambda}$ denote
complex p-variate normal populations with means $\bar{\mu}(1), \bar{\mu}(2), \dots, \bar{\mu}(k)$
Let samples of size $n_1, n_2, \dots, n_k$ be available from $k = q+1$
means of k complex p-variate normal populations.
statistic will be used to develop tests for distinguishing between the
procedure. It is for these reasons that the complex analogue of Wilks'
Thus it has the desirable properties associated with the likelihood ratio
is the simplest one and is related to the likelihood ratio criterion.
Pillai, and Wilks. Wilks' statistic, usually referred to as Wilks' Λ ,
lutions. Test statistics have been developed by Roy, Lawley, Hotelling,
developed for testing hypotheses related to the means of these k popu-
In the case of k real p-variate normal populations, tests have been

1. A Test for Equality of Means

CHAPTER III

properties which are characteristic of this test in the real case. These properties will not be investigated here, but since the likelihood ratio test has been found satisfactory for the problem in the real case, it will be used here in the complex case. The author feels that the desirable properties are maintained.

Letting $N = n_1 + n_2 + \dots + n_k$, then the likelihood function, Q , may be written as

$$Q = \frac{1}{N!} \exp \left\{ - \sum_{k=1}^K \sum_{\alpha=1}^n [\bar{\xi}_{\alpha k} - \bar{\eta}(\alpha)]^2 - \sum_{\alpha=1}^n [\bar{\xi}_{\alpha k} - \bar{\eta}(\alpha)]^2 \right\}.$$

The likelihood ratio statistic is given by

$$\lambda^c = \frac{\text{Max } Q}{\text{Max } Q_0}$$

where $\text{Max } Q_0$ is the maximum of the likelihood function under the assumption that the null hypothesis, H_0 , is true and $\text{Max } Q$ is the maximum value of the likelihood function over the entire parameter space, denoted by Ω . The test is given by rejecting H_0 when λ^c is less than some specified constant depending on the size of the test.

Using Giri's (1965) results, the $\text{Max } Q_0$ is obtained when

$$\hat{\eta} = \frac{1}{N} \sum_{k=1}^K \sum_{\alpha=1}^n \bar{\xi}_{\alpha k} = \bar{\xi}..$$

and

$$\hat{\xi} = \frac{1}{N} \sum_{k=1}^K \sum_{\alpha=1}^n \bar{\xi}_{\alpha k} \quad \text{where } \bar{\xi}_{\alpha k} = \frac{1}{n_{\alpha}} \sum_{\alpha=1}^n (\bar{\xi}_{\alpha k} - \bar{\xi}..) *'.$$

The Max Q is given when

$$\hat{\mu} = \frac{1}{n} \sum_{\alpha} \bar{\xi}_{\alpha\lambda} = \bar{\xi}_{\alpha\cdot}$$

and

$$\hat{\Sigma} = A = \sum_{\alpha=1}^k A_{\alpha} \quad \text{where} \quad A_{\alpha} = \sum_{\lambda=1}^n n_{\alpha} (\bar{\xi}_{\alpha\lambda} - \bar{\xi}_{\alpha\cdot})(\bar{\xi}_{\alpha\lambda} - \bar{\xi}_{\alpha\cdot})^{*'} .$$

After some simplification, it can be shown that

$$V^c = \frac{|A|}{|T|}$$

and observing that

$$T = A + B$$

where B is defined by

$$B = \sum_{\alpha=1}^k n_{\alpha} (\bar{\xi}_{\alpha\cdot} - \bar{\xi}_{\cdot\cdot})(\bar{\xi}_{\alpha\cdot} - \bar{\xi}_{\cdot\cdot})^{*'} .$$

Then the likelihood ratio statistic is given by

$$V^c = \frac{|A|}{|A+B|} .$$

The statistic V^c is the complex analogue of the result obtained by Wilks;

it will be referred to as Wilks' V in the complex case and denoted as V^c . Gupta (1971) has derived the exact distribution of V^c when the

alternate hypothesis is of unit rank. He gives the explicit expression

for the case of $p = 2, 3$ and general degrees of freedom for the hypothesis and error.

The results obtained by application of the likelihood ratio principle

may be summarized in the following complex multivariate analysis of variance

table (CMANOVA).

Source of Variation	d.f.	s.s. and s.p.
Between groups	q	$B = \sum_{k=1}^n \sum_{\alpha=1}^k (\bar{\bar{x}}_{\alpha\cdot} - \bar{\bar{x}}_{\cdot\cdot}) (\bar{\bar{x}}_{\alpha\cdot} - \bar{\bar{x}}_{\cdot\cdot})^*$
Within groups	n-q	$A = +$
TOTAL	n = N-1	$A+B = \sum_{k=1}^n \sum_{\alpha=1}^k (\bar{\bar{x}}_{\alpha\cdot} - \bar{\bar{x}}_{\cdot\cdot}) (\bar{\bar{x}}_{\alpha\cdot} - \bar{\bar{x}}_{\cdot\cdot})^*$

† (by subtraction).

That the degrees of freedom are indeed q for the between groups and n-q

for the within groups will now be shown by examining the distributions of

the matrices B and A.

Define a $p \times N$ matrix X by

$$X = [\bar{\bar{x}}_{11}, \bar{\bar{x}}_{12}, \dots, \bar{\bar{x}}_{1n_1}, \bar{\bar{x}}_{21}, \bar{\bar{x}}_{22}, \dots, \bar{\bar{x}}_{2n_2}, \dots, \bar{\bar{x}}_{k1}, \bar{\bar{x}}_{k2}, \dots, \bar{\bar{x}}_{kn_k}]$$

Let E_{NN} denote an $N \times N$ matrix composed of all ones and let $\bar{e}_{N \times 1}$ denote

a vector with all elements being unity. Also define $\bar{e}_\alpha$, an $N \times 1$ vector,

with zero everywhere except in the n_α positions; there are ones in all of

the n_α positions, i.e.,

$$\bar{e}_\alpha' = [0 \ 0 \ \dots \ 0 \ \dots \ 1 \ \dots \ 1 \ \dots \ 1 \ \dots \ 0]_{1 \times N} \quad \text{with } \overbrace{1 \ 1 \ \dots \ 1}^{n_\alpha \text{ ones}}$$

Then the matrices A, B, and A+B may be written as

$$(3.1.0) \quad B = X \left[\begin{array}{c} k \\ \sum_{\alpha=1}^n \frac{1}{e^{-\alpha}} \end{array} \right] E_{NN} \left[X^* \right] = XCX^* ;$$

$$(3.1.1) \quad A = X \left[\begin{array}{c} I_N - \sum_{k=1}^n \frac{1}{e^{-\alpha}} \end{array} \right] X^* = XDX^* ;$$

$$(3.1.2) \quad A+B = X \left[\begin{array}{c} I_N - \frac{1}{N} E_{NN} \end{array} \right] X^* = XEX^* .$$

It should be noted that the matrices C, D, and E are idempotent of ranks q, n-q, and n, respectively.

Observing that

$$(3.1.3) \quad E(X) = M = [\bar{p}(1), \bar{p}(1), \dots, \bar{p}(1), \dots, \bar{p}(k), \bar{p}(k), \dots, \bar{p}(k)] \quad (3.1.3)$$

and recalling that

$$\bar{\xi}_{\alpha\lambda} \sim CN^p(\bar{\mu}(\alpha), \Sigma)$$

then by Theorem 2.4,

$$(3.1.4) \quad (X-M)C(X-M)^* \sim CM^p \quad (X-M)C(X-M)^* | q | \Sigma) .$$

Thus the degrees of freedom associated with the between groups sum of squares (s.s.) and sum of products (s.p.) matrix B, is given by the rank of the idempotent matrix C which is q.

From (3.1.3), it can be shown that

$$E(XCX^*) = q\Sigma + MCM^*$$

and under the assumption of H_0 , i.e., equal means for the k-complex

p-variate populations, M can be written as

$$M = \bar{\mu} e_{1 \times N}$$

where $\bar{\mu}$ represents the common value. Observe that under H_0 ,

$$MC = \emptyset.$$

Then

$$(X-M)C(X-M)' = (X-M)C(X^* - M^*)'$$

$$= XCX^{*'} - XCM^{*'} - MCX^{*'} + MCM^{*'}.$$

$$= XCX^{*'}.$$

Thus

$$XCX^{*'} \sim CW^D(XCX^{*'} | q | \Sigma) \quad \text{if} \quad \bar{\mu}(1) = \bar{\mu}(2) = \dots = \bar{\mu}(k).$$

Now the distribution of $A = XCX^{*'}$ needs to be established. From

Theorem 2.4, it is seen that

$$(X-M)D(X-M)' \sim CW^D((X-M)C(X-M)' | n-q | \Sigma).$$

And the degrees of freedom associated with A are given by $n-q$, the rank

of the idempotent matrix D .

Now

$$(X-M)D(X-M)' = XCX^{*'} - XDM^{*'} - MDX^{*'} + MDM^{*'}.$$

and it can be shown that

$$MD = \emptyset.$$

Hence

$$(X-M)D(X-M)^* = XD(X-M)^* \sim \text{CW}^p(XDX^* | n-q | \Sigma) \quad .$$

Attention should be called to the fact that the distribution of $A = XD(X-M)^*$ is independent of the null hypothesis, i.e., regardless of whether H_0 is true or not, the distribution of A is still $\text{CW}^p(A | n-q | \Sigma)$.

Also, it should be noted that A and B are distributed independently. Since, as in the real case, two forms XCX^* and $XD(X-M)^*$ are independent if

$$CD = \emptyset \quad .$$

To summarize these results, it is seen that

$$B \sim \text{CW}^p(B | q | \Sigma) \quad \text{if} \quad \bar{n}(1) = \bar{n}(2) = \dots = \bar{n}(k)$$

$$A \sim \text{CW}^p(A | n-q | \Sigma) \quad \text{independent of } H_0$$

and A is distributed independently of B . The test statistic, V^c , is given by the ratio of the determinates of the complex Wishart matrices A and

$$A+B, \text{ i.e.,}$$

$$V^c = \frac{|A|}{|A+B|} \quad .$$

Khatri (1965) has shown that the distribution of V^c , where V^c is

defined as above, is the same as the product of p independent real beta

variables, i.e.,

$$V^c = \frac{|A|}{|A+B|} = \prod_{i=1}^p u_i$$

where $u_i \sim B(n-q-i+1, q)$ and all are independent for $i = 1, 2, \dots, p$.

Furthermore, he has shown

$$-m \log \frac{c}{2} \chi^2_{2pq}$$

where $m = (2n - q - p)$ and " $\hat{\cdot}$ " denotes "approximately distributed."

Hence if H_0 is true, $-m \log \frac{c}{2} \chi^2_{2pq}$ will be distributed as an approximate chi-square variable with degrees of freedom given by $2pq$.

To summarize the procedure for testing the hypothesis of equal means among $k = q + 1$ complex p -variate normal populations, one must first calculate the matrices A and $A+B$. Then, consider the statistic $\frac{c}{2} \chi^2_{2pq}$ given by

$$\frac{c}{2} \chi^2_{2pq} = \frac{|A|}{|A+B|}$$

and compute $-m \log \frac{c}{2} \chi^2_{2pq}$. If $-m \log \frac{c}{2} \chi^2_{2pq}$ is greater than χ^2_{2pq} at the desired α -level, then reject H_0 , otherwise do not reject H_0 .

2. A Test for the Dimensionality of the Mean Space

In the last section a test for equality of the means from k p -variate complex normal populations with the same variance covariance structure was developed. The hypothesis, H_0 , implies that the mean vectors of the k populations would lie in a complex space of zero dimension, i.e., represented by a point in the complex space. If H_0 is rejected in favor of H_a , then the means of the k populations may lie in a complex 1 dimensional space, that is they are collinear, or in a complex 2 dimensional space, that is they are coplanar, and so forth up to a complex k dimensional space. In real multivariate analysis, it is important to know the dimensionality of the space spanned by the mean vectors for this is equivalent

to knowing the number of linear discriminant functions that are necessary to distinguish among the k-populations. This is true also in the complex normal situation. In this section a test for the dimensionality of the complex space spanned by the mean vectors will be developed and will be shown to be equivalent to the number of linear discriminant functions needed to discriminate among the k populations.

Recalling the complex multivariate analysis of variance table that was developed in the previous section, and adding a column for the expectation of the s.s. and s.p. matrices, then the table is given by

Source of Variation	d.f.	s.s. and s.p.	E(s.s. and s.p.)
Between groups	q	$B = X'CX^*$	$E(B) = qL + MCM^*$
Within groups	n-q	$A = X'DX^*$	$E(A) = (n-q)L$
TOTAL	n	$A+B$	

Now M was defined by (3.1.4) as

$$E(X) = M = [\bar{\mu}(1) \dots \bar{\mu}(1) | \dots | \bar{\mu}(k) \dots \bar{\mu}(k)]^{p \times N}.$$

Denote the rank of M by r(M) and observe that

$$r(M) = r[\bar{\mu}(1), \bar{\mu}(2), \dots, \bar{\mu}(k)] = r(\mu)$$

which is equivalent to the number of linearly independent points representing the means of the k populations. Denote the k populations by $\pi_1, \pi_2, \dots, \pi_k$ and consider a linear combination of ξ , an observation from one of these populations, given by $\bar{\xi}^* \xi$. Suppose further that

$$\bar{\xi}^* \bar{\mu}(1) = \bar{\xi}^* \bar{\mu}(2) = \dots = \bar{\xi}^* \bar{\mu}(k) = \delta,$$

i.e., that $\bar{\xi}^*_{*1}$ has the same mean in $\pi_1, \pi_2, \dots, \pi_k$. Then this linear function given by $\bar{\xi}^*_{*1}$ would not be useful in discriminating among the populations, i.e., one could not determine which population ξ came from by examining $\bar{\xi}^*_{*1}$. Furthermore, observe that

$$\bar{\xi}^*_{*1} M = [\delta \ \delta \ \dots \ \delta] \cdot e_{1 \times N}$$

then

$$\bar{\xi}^*_{*1} M C = \delta \cdot e_{1 \times N} \quad C = \delta \cdot e_{1 \times N} \left(\sum_{\alpha=1}^k \frac{1}{n_{\alpha}} e_{\alpha-\alpha} - \frac{1}{N} E_{NN} \right)$$

$$= \delta \cdot \bar{0} = \bar{0}$$

and hence

$$\bar{\xi}^*_{*1} M C M^* \bar{\xi} = 0$$

Conversely, if $\bar{\xi}^*_{*1} M C M^* \bar{\xi} = 0$, this implies

$$\sum_{\alpha=1}^k n_{\alpha} (u_{\alpha} - \bar{u})^* = 0 \quad (3.2.0)$$

where $\bar{\xi}^*_{*1} \bar{u}(\alpha) = u_{\alpha}$ and $\bar{u} = \frac{\sum_{\alpha=1}^k n_{\alpha} \bar{\xi}^*_{*1} \bar{u}(\alpha)}{\sum_{\alpha=1}^k n_{\alpha}}$. But (3.2.0) implies that

each term of the summation must be zero or that

$$u_{\alpha} = \bar{u} \quad \text{for all } \alpha = 1, 2, \dots, k$$

From this it is seen that

$$\bar{\xi}^*_{*1} \bar{u}(1) = \bar{\xi}^*_{*1} \bar{u}(2) = \dots = \bar{\xi}^*_{*1} \bar{u}(k)$$

So if $\bar{\lambda}^* \bar{\xi}$ has the same mean value in $\pi_1, \pi_2, \dots, \pi_k$ then

$\bar{\lambda}^* \bar{\xi} \bar{\pi} = 0$ and conversely if $\bar{\lambda}^* \bar{\xi} \bar{\pi} = 0$, then $\bar{\lambda}^* \bar{\xi}$ will have the same

mean value in $\pi_1, \pi_2, \dots, \pi_k$. In either case $\bar{\lambda}^* \bar{\xi}$ will not be use-

ful as a discriminator.

Thus a linear function of $\bar{\xi}$, given by $\bar{\lambda}^* \bar{\xi}$, will not be a good

discriminator if

$$\bar{\lambda}^* \bar{\xi} \bar{\pi} = 0. \quad (3.2.1)$$

An adequate number of linear discriminators will be given by p minus the number of independent $\bar{\lambda}$'s which satisfy (3.2.1). For any $\bar{\lambda}$, such that

(3.2.1) is satisfied, $\bar{\lambda}^* \bar{\xi}$ will be called a covariate. Observe that

(3.2.1) is equivalent to

$$\bar{\lambda}^* \bar{\xi} \bar{\pi} = 0.$$

So the maximum number of such covariates is equal to the multiplicity of the zero characteristic roots of the matrix $\bar{\lambda}^* \bar{\xi}$. Denote this multiplicity by s . Now the rank of $\bar{\lambda}^* \bar{\xi}$ is equivalent to the number of non-zero characteristic roots of $\bar{\lambda}^* \bar{\xi}$, say r , which is the number of linear discriminant functions needed. Hence, we have the number of discriminant functions needed given by

$$r = p - s.$$

Let the s covariates be given by

$$\bar{\lambda}^* \bar{\xi} (1), \bar{\lambda}^* \bar{\xi} (2), \dots, \bar{\lambda}^* \bar{\xi} (s)$$

$$Q = \frac{1}{N} \exp \left(- \sum_{k=1}^n \sum_{\alpha=1}^n \left(\bar{\xi}_{\alpha k} - \bar{\eta}(\alpha) \right) \left(\bar{\xi}_{\alpha k} - \bar{\eta}(\alpha) \right)^{-1} \right)$$

Recall that the likelihood function, Q , was given as

by using the likelihood ratio criterion.

for some unspecified $\bar{v}$. A test for this hypothesis will now be developed

$$L = \bar{v}^T L_k$$

such that

is equivalent to the hypothesis that there exists a matrix L of rank s , i.e., $r(L) = s$. Thus the hypothesis of needing r discriminate functions

$\bar{\xi}(2), \dots, \bar{\xi}(s)$ are independent, this implies that the rank of L is

$\bar{\xi}^*(t) \bar{\eta}(\alpha) = v, (t = 1, 2, \dots, s; \alpha = 1, 2, \dots, k)$. Since $\bar{\xi}(1)$,

since $\bar{\xi}^*(t) \bar{\xi}(t) = 1, 2, \dots, s$ is a covariate and this implies that

$$L_{s \times n} \bar{v}^T L_k = \bar{v}^T L_k = [\bar{v}, \bar{v}, \dots, \bar{v}]_{s \times k}$$

then

$$L = \begin{bmatrix} \bar{\xi}^*(1) \\ \bar{\xi}^*(2) \\ \vdots \\ \bar{\xi}^*(s) \end{bmatrix} \quad s \times p \quad (3.2.2)$$

and let the matrix L be defined by

the following procedure.

zation under the assumption that H is true will be carried out through

be noted that the matrix L and vector $\bar{v}$ are not specified. The maxi-

which is equivalent to saying the $r(MCM^*) = r$. In all cases it should

$$H: L\bar{v} = \bar{v}E_{1k}$$

or

$$H: L\bar{v}(\alpha) = \bar{v} \quad \alpha = 1, 2, \dots, k$$

equivalent to the hypothesis that there exists a matrix L such that

ditions of the hypothesis of needing r discriminant functions. This is

Now the likelihood function must be maximized subject to the con-

$\bar{v}(\alpha) = \bar{v}(\alpha) \cdot$

$\bar{v}(2), \dots, \bar{v}(k)$. The maximum of $\bar{v}$ with respect to α is given when

The parameter space Ω consists of the $2pk$ parameters given by $\bar{v}(1)$,

$$\bar{v}(\alpha) = \bar{v}(\alpha) \cdot \exp \left(- \sum_{k=1}^{\alpha} n_k \left(\bar{v}(\alpha) - \bar{v}(k) \right) \cdot \sum_{k=1}^{\alpha} \bar{v}(\alpha) - \bar{v}(k) \right) \cdot \left(\bar{v}(\alpha) - \bar{v}(k) \right)$$

$$\bar{v}(\alpha) = \text{constant w.r.t. unknown parameters, } \bar{v}(\alpha)$$

Assuming $\bar{v}$ to be known, the $\bar{v}$ can be written as

$$\bar{v}(\alpha) = \frac{1}{N} \exp \left(- \sum_{k=1}^{\alpha} n_k \left(\bar{v}(\alpha) - \bar{v}(k) \right) \cdot \sum_{k=1}^{\alpha} \bar{v}(\alpha) - \bar{v}(k) \right) \cdot \left(\bar{v}(\alpha) - \bar{v}(k) \right)$$

written as

By adding and subtracting $\bar{v}(\alpha) = \frac{1}{N} \sum_{k=1}^{\alpha} \bar{v}(\alpha)$ and simplifying, $\bar{v}$ may be

Make the following transformation

$$L_{\bar{z}}^{\alpha}(\alpha) = \bar{y}(\alpha) \quad \alpha = 1, 2, \dots, k$$

$$L_{\bar{z}}^{\alpha}(\alpha) = \bar{z}(\alpha) \quad \alpha = 1, 2, \dots, k$$

choosing L_1 such that

$$L_{\bar{z}}^{\alpha}(\alpha) = \emptyset.$$

Letting R^2 denote the exponent of $\emptyset$, then

$$R^2 = \sum_{\alpha=1}^k n^{\alpha} (\bar{z}(\alpha) - \bar{y}(\alpha)) \cdot L_{\bar{z}}^{\alpha}(\alpha) - \bar{y}(\alpha)$$

$$= \sum_{\alpha=1}^k n^{\alpha} (L_{\bar{z}}^{\alpha}(\alpha) - L_{\bar{y}}^{\alpha}(\alpha)) \cdot L_{\bar{z}}^{\alpha}(\alpha) - L_{\bar{y}}^{\alpha}(\alpha)$$

$$+ \sum_{\alpha=1}^k n^{\alpha} (L_{\bar{z}}^{\alpha}(\alpha) - L_{\bar{y}}^{\alpha}(\alpha)) \cdot L_{\bar{z}}^{\alpha}(\alpha) - L_{\bar{y}}^{\alpha}(\alpha)$$

and under the assumption that the hypothesis is true, the above becomes

$$R^2 = \sum_{\alpha=1}^k n^{\alpha} (\bar{y}(\alpha) - \bar{y}(\alpha)) \cdot L_{\bar{z}}^{\alpha}(\alpha) - \bar{y}(\alpha)$$

$$+ \sum_{\alpha=1}^k n^{\alpha} (\bar{z}(\alpha) - L_{\bar{z}}^{\alpha}(\alpha)) \cdot L_{\bar{z}}^{\alpha}(\alpha) - L_{\bar{z}}^{\alpha}(\alpha)$$

and to maximize the likelihood, the exponent R^2 is minimized (observing

that R^2 is real). Hence

$$\min R^2 = \min \sum_{\alpha=1}^k n^{\alpha} (\bar{y}(\alpha) - \bar{y}(\alpha)) \cdot L_{\bar{z}}^{\alpha}(\alpha) - \bar{y}(\alpha)$$

since differentiation with respect to $L_1 \bar{h}(\alpha)$ gives $L_1 \bar{h}(\alpha) = L_1 \bar{h}(\alpha)$, so that the second term in R^2 vanishes. Thus

$$\min R^2 = \min \sum_{\alpha=1}^k n_{\alpha} [L_1 \bar{h}(\alpha) - \bar{v}]^* (L_1 \bar{h}(\alpha) - \bar{v})$$

$$= \min \sum_{\alpha=1}^k n_{\alpha} \text{tr}(L_1 \bar{h}(\alpha) - \bar{v}) [L_1 \bar{h}(\alpha) - \bar{v}]^* = \min \sum_{\alpha=1}^k n_{\alpha} \bar{h}(\alpha) - \bar{v}$$

$$= \min \sum_{\alpha=1}^k n_{\alpha} \text{tr}(L_1 \bar{h}(\alpha) - \bar{v}) [L_1 \bar{h}(\alpha) - \bar{v}]^* = \min \sum_{\alpha=1}^k n_{\alpha} \bar{h}(\alpha) - \bar{v}$$

where

$$\bar{h}(\alpha) = \frac{\sum_{\alpha=1}^k n_{\alpha} \bar{h}(\alpha)}{\sum_{\alpha=1}^k n_{\alpha}}$$

thus

$$\min R^2 = \min \text{tr}(L_1 \bar{h}(\alpha) - \bar{v})$$

$$\cdot \left\{ \sum_{\alpha=1}^k n_{\alpha} (L_1 \bar{h}(\alpha) - \bar{v}) [L_1 \bar{h}(\alpha) - \bar{v}]^* + \sum_{\alpha=1}^k n_{\alpha} [L_1 \bar{h}(\alpha) - \bar{v}] [L_1 \bar{h}(\alpha) - \bar{v}]^* \right\}$$

and minimizing this with respect to $\bar{v}$, i.e., $\bar{v} = L_1 \bar{h}$, then

$$\min R^2 = \min \text{tr}(L_1 \bar{h}(\alpha) - \bar{v}) [L_1 \bar{h}(\alpha) - \bar{v}]^* \left\{ \sum_{\alpha=1}^k n_{\alpha} [L_1 \bar{h}(\alpha) - \bar{v}] [L_1 \bar{h}(\alpha) - \bar{v}]^* \right\}$$

$$= \min \text{tr}(L_1 \bar{h}(\alpha) - \bar{v}) [L_1 \bar{h}(\alpha) - \bar{v}]^* (w.r.t. L)$$

where B is the between groups s.s. and s.p. matrix from the CMANOVA table.

Now let $T = I\Xi^{1/2}$, then $TT^* = I\Xi I^*$, and

$$\min R^2 = \min \operatorname{tr}(TT^*) (T\Xi^{-1/2} B\Xi^{-1/2} T^*)$$

$$= \min \operatorname{tr} Y (\Xi^{-1/2} B\Xi^{-1/2}) Y^*$$

where $Y = (TT^*)^{-1/2}$, observing $YY^* = I$. So the problem of minimizing R^2 with respect to T reduces to minimizing the $\operatorname{tr}[Y(\Xi^{-1/2} B\Xi^{-1/2}) Y^*]$

subject to the condition that $YY^* = I$. As in the case of real positive definite symmetric matrices, the minimum is given by the sum of the

smallest s roots of $\Xi^{-1/2} B\Xi^{-1/2}$ [Rao(1965)], i.e., the s smallest roots

of

$$|\Xi^{-1/2} B\Xi^{-1/2} - \lambda_i I| = 0 \quad i = 1, 2, \dots, s$$

or

$$|B - \lambda_i \Xi| = 0 \quad i = 1, 2, \dots, s$$

Thus the maximum value of Q under H is given as

$$(\text{constant}) \cdot \exp(-\text{sum of the smallest } s \text{ roots of } \Xi^{-1/2} B\Xi^{-1/2})$$

and the likelihood ratio statistic λ is given as

$$\lambda = e^{-(\text{sum of the smallest } s \text{ roots of } \Xi^{-1/2} B\Xi^{-1/2})}$$

and

$$-2 \ln \lambda = (\text{sum of } s \text{ smallest roots of } \Xi^{-1/2} B\Xi^{-1/2})$$

$$= 2(\lambda_{r+1} + \lambda_{r+2} + \dots + \lambda_p)$$

where $\lambda_{x+1}, \lambda_{x+2}, \dots, \lambda_p$ are the s smallest roots of $\Sigma^{-1/2} B \Sigma^{-1/2}$. The distribution of $-2 \ln \lambda$ is an asymptotic chi-square with degrees of freedom given by the number of parameters specified in Ω minus the number of restrictions of the parameters specified by the hypothesis.

Rao (1965), in the real case, has presented a geometrical argument for determining the degrees of freedom for this particular test. Wani and Kabe (1970) and Kshirsagar (1971) have presented an analytical argument which is somewhat easier to follow for the non-geometrician.

Kshirsagar's procedure will be followed here to determine the degrees of freedom for the chi-square test of dimensionality.

The number of parameters specified in Ω is given by $2pk$. The number of parameters restricted by the maximization of $\bar{Q}$ under H needs to be determined. Recall the procedure for maximizing $\bar{Q}$ under H . First the transformation by L and L_1 took

$$\bar{u}(\alpha) \rightarrow \begin{cases} L_1 \bar{u}(\alpha) \\ L_1 \bar{u}(\alpha) \end{cases} \quad \alpha = 1, 2, \dots, k$$

and the exponent R^2 was minimized, where R^2 was given as

$$R^2 = \sum_{\alpha=1}^k n_{\alpha} [\bar{y}(\alpha) - \bar{v}]^* (L L_1^* L_1)^{-1} [\bar{y}(\alpha) - \bar{v}]$$

$$+ \sum_{\alpha=1}^k n_{\alpha} [\bar{z}(\alpha) - \bar{z}]^* (L_1 \Sigma L_1^*)^{-1} [\bar{z}(\alpha) - \bar{z}] - L_1 \bar{u}(\alpha)]$$

= Part I + Part II .

This minimization was done in steps:

Step I: Min of Part II, i.e., min of Part II with respect to

$L_{II}^{-1}(\alpha)$. The number of parameters restricted was $2(p-s)k$

or $2rs$.

Step II: Min of Part I, i.e., min of Part I with respect to $\bar{y}$.

The number of parameters restricted was $2s$.

This left

$$\text{Min } R^2 = \text{Min tr}(L\bar{L}L^*)^{-1}(L\bar{B}L^*)^{-1} \quad (\text{w.r.t. } L)$$

Now L is an $s \times p$ matrix of rank s and the number of parameters restricted needs to be determined. Write

$$L = s \begin{bmatrix} H & G \\ & \end{bmatrix} \quad \begin{matrix} s \\ p-s \end{matrix}$$

and note that

$$H^{-1}L = \begin{bmatrix} I & \\ & H^{-1}G \end{bmatrix} \quad s$$

Returning to

$$\text{Min tr}(L\bar{L}L^*)^{-1}(L\bar{B}L^*)^{-1}$$

which can be written as

$$\text{Min tr}[L^*(L\bar{L}L^*)^{-1}(L\bar{B}L^*)^{-1}]$$

or as

$$\min \text{tr}[L^* P (P L L L^* P)^{-1} (P L B)]$$

where P is any $s \times s$ non-singular hermitian matrix. Choose P to be H^{-1} ,

thus

$$P L = [I_s \mid L_0] \quad \text{where} \quad L_0 = P^{-1} G$$

hence the minimum involves only functions of L_0 , where L_0 has dimension-

ality given by $s \times (p-s)$. Thus only $2[s \times (p-s)]$ parameters are restricted in the minimization of $\text{tr}(L L L^*)^{-1} (L B L^*)$. Therefore the number of

parameters restricted under H is given as

$$2rk + 2s + 2s(p-s)$$

and the degrees of freedom of the chi-square test is given by

$$\text{d.f.} = 2pk - [2rk + 2s + 2s(p-s)]$$

$$= 2(p-r)(q-r)$$

where $q = k+1$.

In summary, the test of the hypothesis that the number of linear

discriminant functions is r , or the number of covariates is $s = p-r$, is

given by the likelihood ratio criterion as

$$-2 \ln(\lambda) = 2(\lambda^{r+1} + \lambda^{r+2} + \dots + \lambda^p) \chi^2_{2(p-r)(q-r)}$$

where $\lambda^{r+1}, \lambda^{r+2}, \dots, \lambda^p$ are the s smallest roots of

$$|B - \lambda I| = 0$$

where B is the between groups s.s. and s.p. matrix obtained from the

CMANOVA table.

If λ is unknown, then the estimate, $\hat{\lambda}/n-q$, where $\hat{\lambda}$ is the within

s.s. and s.p. from the CMANOVA table, may be used to obtain an approxi-

mate test for the number of discriminant functions needed. The test will

be given as the s smallest roots of

$$\left| B - \frac{\hat{\lambda}}{n-q} A \right| = |B - \eta A| = 0.$$

The test statistic is given by

$$(n-q)(2) (\eta_{r+1} + \eta_{r+2} + \dots + \eta_p) \chi^2_{2(p-r)(q-r)}$$

where $\eta_{r+1}, \eta_{r+2}, \dots, \eta_p$ are the s smallest roots of

$$|B - \eta A| = 0.$$

Instead of considering the roots of $|B - \eta A| = 0$, consider the

roots of

$$|B - r^2(A+B)| = 0$$

or

$$\left| B - \frac{r^2}{2} A \right| = 0.$$

i.e.,

$$\eta = \frac{r^2}{2} \quad \text{or} \quad \frac{\hat{\lambda}}{n-q} = \frac{r^2}{2}$$

or

$$1 + \frac{\hat{\lambda}}{n} = \frac{1}{1-r^2}$$

for large n . Using the substitution

$$\log \left(1 + \frac{\lambda}{n} \right) = -\log(1-r^2)$$

then

$$\frac{\lambda}{n} = -\log(1-r^2)$$

and

$$\frac{n}{p} = \frac{\lambda_{x+1} + \lambda_{x+2} + \dots + \lambda_p}{-\log(1-r^2)}$$

or

$$-2n \log \prod_{i=x+1}^p (1-r_i^2) \sim \chi^2_{2(p-x)}(q-x)$$

where the r_i^2 are the roots of $|B - r(A+B)| = 0$. This gives a procedure for estimating the dimensionality of the mean space. The procedure is to test sequentially the following hypotheses.

dimensionality	space	criterion
H: $r=0$	0	1 pt. $-2n \log \prod_{i=1}^p (1-r_i^2) \sim \chi^2_{2pq}$
H: $r=1$	1	2 pts. (collinear) $-2n \log \prod_{i=2}^p (1-r_i^2) \sim \chi^2_{2(p-1)}(q-1)$
H: $r=2$	2	3 pts. (coplanar) $-2n \log \prod_{i=3}^p (1-r_i^2) \sim \chi^2_{2(p-2)}(q-2)$

If the means are known, then the number of discriminant functions can be determined exactly by examining the rank of $\bar{u}$ where $\bar{u}$ is defined as

$$\bar{u} = u \left(I - \frac{1}{k} E_{kk} \right)$$

To see this, consider L as defined by (3.2.2) and under H , the hypothesis that the number of discriminant functions is r , then

$$L_H = \bar{V} E_{1k} .$$

Observe

$$L_H = L_H \left(I - \frac{1}{k} E_{kk} \right) = \bar{V} E_{1k} \left(I - \frac{1}{k} E_{kk} \right) = \emptyset$$

or

$$(3.2.3) \quad L_H \bar{U}^{s \times p} \bar{U}^{p \times k} = \emptyset .$$

Now the rank of L is s since $\bar{L}(1)$, $\bar{L}(2)$, $\dots$, $\bar{L}(s)$ are linearly independent and this is the exact number of covariates that exist. From

(3.2.3), we have

$$(3.2.4) \quad L_H \bar{U}^{s \times p} \bar{U}^{p \times p} = 0$$

where $\bar{U}^{p \times p} = \bar{U}^{p \times k} \bar{U}^{k \times p}$, and from (3.2.3) we have

$$\begin{aligned} \bar{L}^{*'}(1)E &= 0 \\ \bar{L}^{*'}(2)E &= 0 \\ &\vdots \\ \bar{L}^{*'}(s)E &= 0 \end{aligned}$$

and there are no more non-trivial solutions to the system $E\bar{L} = 0$, thus we have the $r(E) = p-s$ which is the same as $r(\bar{U}) = p-s = r$.

Thus the rank of $\bar{U}$ is the same as the number of discriminant functions needed. Attention should be called to the fact that this is the exact number of discriminant functions needed and not an estimate.

In the first section, Wilks' Λ was defined as

$$\Lambda^c = \frac{|A|}{|A+B|}$$

and it was shown that the test statistic for the hypothesis of equal

means, i.e., $\bar{\mu}(1) = \bar{\mu}(2) = \dots = \bar{\mu}(k)$, was given by

$$-m \log \Lambda^c \sim \chi^2_{2pq} \quad (3.2.5)$$

The distribution and correction factor $m = 2n - p - q$ were derived by

Khatri (1965).

When the dimensionality of the mean space is zero, this is the same

as saying all of the means are equal and the test was given as

$$-2n \log \frac{|A|}{|A+B|} \sim \chi^2_{2pq} \quad (3.2.6)$$

where the r_i^2 are obtained from $|B - r_i^2(A+B)| = 0$. But the hypothesis

that the dimensionality is zero is the same as the hypothesis that

$\bar{\mu}(1) = \bar{\mu}(2) = \dots = \bar{\mu}(k)$. Thus the two test statistic (3.2.5) and (3.2.6)

should be equivalent. Since

$$\Lambda^c = \frac{|A|}{|A+B|} = \frac{|A|}{|A+B|} = \frac{|A|}{|A+B|} = \frac{|A|}{|A+B|}$$

it is seen that the two tests differ only in the constants $m = 2n - p - q$

and $2n$. Khatri's result has been corrected, so it will be used in (3.2.6)

and the correction will be adjusted in general without a formal proof.

Thus a test for dimensionality being r will be given as

To find the solution, consider the last s eigenvectors $\bar{y}$ satisfying

$$(3.2.7) \quad (\Sigma^{-1/2} B \Sigma^{-1/2} - \lambda I) \bar{y}^{\text{px}1} = 0$$

corresponding to the s smallest eigenvalues of

$$(3.2.8) \quad |\Sigma^{-1/2} B \Sigma^{-1/2} - \lambda I| = 0.$$

The condition $y y^* = I$ will be satisfied since the eigenvalues corresponding to distinct roots are orthogonal and eigenvalues corresponding to

repeated roots can be made orthogonal. We can take y^* to be y^* from the s orthogonal eigenvectors $[\bar{y}_1, \bar{y}_2, \dots, \bar{y}_s]$ corresponding to the s

smallest roots and

$$\hat{L}^* = \Sigma^{-1/2} \hat{y}^* = \Sigma^{-1/2} \hat{y}^*.$$

and this is the maximum likelihood estimator of L^* .

If we consider the largest $p-s$ eigenvectors corresponding (3.2.7)

and (3.2.8), these will provide the r discriminant functions. To show

that $L L^* = I$, consider

$$(\Sigma^{-1/2} B \Sigma^{-1/2} - \lambda I) \bar{y} = 0$$

or

$$(\Sigma^{-1/2} B \Sigma^{-1/2} - \lambda I) \bar{y} = 0$$

or

$$(B - \lambda \Sigma I) \Sigma^{-1/2} \bar{y} = 0$$

or

$$(B - \lambda \Sigma I) \bar{y} = 0$$

where $\bar{y} = \Sigma^{-1/2} \bar{y}$. Now the $\bar{y}$'s were mutually orthogonal, i.e.,

$$\bar{y}^{*'}(1)\bar{y}(j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

but $\bar{y} = Z^{-1/2} \bar{y}$, thus $\bar{y}^{*'}(1)\bar{y}(j) = 0$, which implies that $IX_{1*}' = \emptyset$.

Now consider a new observation $\bar{z}$ from one of the populations

π_α ($\alpha = 1, 2, \dots, k$). It is desired to place this observation into one of π_α ($\alpha = 1, 2, \dots, k$). The procedure for this discrimination

process is as follows:

Step 1: Find the matrices B and A by finding the CMANOVA from the sample observations.

Step 2: Obtain roots of

$$|B - r^2(A+B)| = 0 \quad r_1^2 > r_2^2 > \dots > r_p^2$$

Step 3: Determine the dimensionality sequentially by finding the smallest value of r such that

$$-m \log \prod_{i=r+1}^p (1-r_i^2) \sim \chi_{2(p-r)}^2(q-r)$$

is insignificant.

Step 4: Determine the eigenvectors

$$[B - r_i^2(A+B)]\bar{z}(1) = 0 \quad i = 1, 2, \dots, r$$

corresponding to $r_1^2, \dots, r_r^2$.

Step 5: Normalize the $\bar{z}(1)$ by

$$\bar{z}^{*'}(1)\bar{z}(j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

if z is not known use $\bar{z} = A/n - q$.

In the last section, a test of the hypothesis that r discriminant functions are needed to discriminate among the $k = q+1$ complex p -variate normal populations was developed. The following discussion will center

3. Goodness of Fit of a Single Hypothetical Discriminant Function

and assign π_j if minimum is $d_j^2, \bar{\xi}_0$. In case of a tie, one could randomize.

$$\left\{ d_1^2, \bar{\xi}_0, d_2^2, \bar{\xi}_0, \dots, d_k^2, \bar{\xi}_0 \right\}$$

Step 7: Determine the minimum of

$$d_{\alpha, \bar{\xi}_0}^2 = [\bar{\xi}_{*'}(1) \bar{\xi}_0 - \bar{\xi}_{*'}(1) \bar{\xi}(\alpha)]^2 + [\bar{\xi}_{*'}(1) \bar{\xi}_0 - \bar{\xi}_{*'}(1) \bar{\xi}(\alpha)]^2 + \dots + [\bar{\xi}_{*'}(r) \bar{\xi}_0 - \bar{\xi}_{*'}(r) \bar{\xi}(\alpha)]^2 + [\bar{\xi}_{*'}(r) \bar{\xi}_0 - \bar{\xi}_{*'}(r) \bar{\xi}(\alpha)]^2$$

The distance (modulus) of this point from the estimated mean of π_α is the quantity that will determine which population that $\bar{\xi}_0$ will go into, i.e.,

$$L_1 \bar{\xi}_0 \begin{bmatrix} \bar{\xi}_{*'}(1) \bar{\xi}_0 \\ \bar{\xi}_{*'}(r) \bar{\xi}_0 \end{bmatrix}$$

If a new observation is observed and the measured characteristics are recorded as $\bar{\xi}_0$, then

$$\begin{aligned} \bar{\xi}_{*'}(1) \bar{\xi}_0 &= Y_1 \\ \bar{\xi}_{*'}(2) \bar{\xi}_0 &= Y_2 \\ &\vdots \\ \bar{\xi}_{*'}(r) \bar{\xi}_0 &= Y_r \end{aligned}$$

Step 6: Then the discriminant functions are

around the case where the dimensionality is one, i.e., only one discriminant function is needed and with tests about this single discriminant function. In this section a test of the goodness of fit of a single hypothetical function will be developed. This hypothesis will consist of two parts, (i) whether a single hypothetical function is adequate, and (ii) whether the hypothetical function agrees with the true discriminant function. As in the real case, the hypothesis will be stated as the collinearity aspect and the direction aspect. The test statistic will be partitioned into two parts, one corresponding to the collinearity aspect and the other to the direction aspect. Independence between these factors will be shown.

The CMANOVA, as presented in the last sections, can be considered to be a regression of the complex vector $\bar{\xi}$ on the real dummy vector variable $\bar{Y}$. The dummy vector variables, $\bar{Y}_1, \bar{Y}_2, \dots, \bar{Y}_q$, have components of either 0 or 1 depending on which population the $\bar{\xi}$ comes from, i.e.,

$$\bar{Y}'_{\alpha} = [0 \ 0 \ \dots \ 0 \ \dots \ 0 \ \dots \ 1 \ \dots \ 1 \ \dots \ 0 \ \dots \ 0 \ \dots \ 0]$$

$$n_1 \quad n_{\alpha} \quad n_k$$

and the regression can be written as

$$E(\bar{\xi}) = \bar{\mu}_{\alpha} \quad \text{if } \bar{\xi} \in \pi_{\alpha} \quad \alpha = 1, 2, \dots, k$$

$$= \bar{\mu}_k + (\bar{\mu}_1 - \bar{\mu}_k) \bar{Y}_1 + \dots + (\bar{\mu}_q - \bar{\mu}_k) \bar{Y}_q$$

which can be rewritten as

$$E(X) = \bar{\mu}_{E1N} + \beta Y$$

where

$$x = [\bar{\epsilon}_{11} \bar{\epsilon}_{12} \dots \bar{\epsilon}_{1n} | \bar{\epsilon}_{k1} \bar{\epsilon}_{k2} \dots \bar{\epsilon}_{kn}]$$

and

$$y_{N \times q} = [\bar{y}_1, \bar{y}_2, \dots, \bar{y}_q]$$

with

$$\beta = [\bar{\mu}_1 - \bar{\mu}_k, \bar{\mu}_2 - \bar{\mu}_k, \dots, \bar{\mu}_q - \bar{\mu}_k] \cdot$$

With this in mind, the CMANOVA, may be written as

Source of Variation	d.f.	s.s. and s.p.
Regression (Between)	q	$B = C_{xy} C_{yy}^{-1} C_{yx}$
Error (Within)	n-q	$A = C_{xx} - C_{xx} C_{xy} C_{yy}^{-1} C_{yx}$
TOTAL	n	$A+B = C_{xx}$

where for $j = (I_N - \frac{1}{N} E_{NN})$, then

$$\begin{aligned} C_{yy} &= Y' J Y & C_{xy} &= X' J Y \\ C_{yx} &= Y' J X & C_{xx} &= X' J X \end{aligned}$$

Define A_1 as the matrix consisting of the first 1 rows and columns of A and likewise for B. With $A_0 = I$ and $B_0 = I$. Then

$$V^c = \frac{|A|}{\prod_{i=1}^p |A_i|} = \frac{|A+B|}{\prod_{i=1}^p |A_i+B_i|} \frac{\prod_{i=1}^p |A_i+B_i|}{\prod_{i=1}^p |A_i|} \frac{\prod_{i=1}^p |A_i|}{\prod_{i=1}^p |A_i+B_i|}$$

which can be written in terms of regression as

$$V^c = \frac{\prod_{i=1}^p |A_i|}{\prod_{i=1}^p |A_i+B_i|} = \frac{\prod_{i=1}^p |A_i|}{\prod_{i=1}^p |A_i+B_i|} \frac{\prod_{i=1}^p |A_i+B_i|}{\prod_{i=1}^p |A_i|} \frac{\prod_{i=1}^p |A_i|}{\prod_{i=1}^p |A_i+B_i|}$$

where $R_2^{\xi_1}(\xi_1, \xi_2, \dots, \xi_{i-1})$ is the multiple coherence of ξ_i with

$\xi_1, \xi_2, \dots, \xi_{i-1}$. This fact follows from

$$\frac{|A_i|}{\prod_{i=1}^p |A_i|} = \frac{|A_i+B_i|}{\prod_{i=1}^p |A_i+B_i|} \frac{\prod_{i=1}^p |A_i+B_i|}{|A_i|}$$

which is evident from (2.4.4). It can be shown that

$$\frac{1 - R_2^{\xi_1}(\xi_1, \xi_2, \dots, \xi_{i-1}, \bar{Y})}{1 - R_2^{\xi_1}(\xi_1, \xi_2, \dots, \xi_{i-1})} \sim \frac{\chi_2^2[n-(q+1-i)]}{\chi_2^2[n-(q+1-i)] + \chi_2^{2q}} = B(n-q+1-i, q) \cdot$$

Thus,

$$V^c = \frac{\prod_{i=1}^p |A_i|}{\prod_{i=1}^p |A_i+B_i|} = \frac{\prod_{i=1}^p |A_i|}{\prod_{i=1}^p |A_i+B_i|} \frac{\prod_{i=1}^p |A_i+B_i|}{\prod_{i=1}^p |A_i|} \frac{\prod_{i=1}^p |A_i|}{\prod_{i=1}^p |A_i+B_i|}$$

(3.3.1)

where $u_i \sim B(n-q+1-i, q)$. This is the result obtained by Khatri (1965),

who has shown that the u_i ($i = 1, 2, \dots, p$) form an independent set

of beta variables. Another definition of V is given by equation (3.3.1), i.e., if a statistic V is distributed as $\prod_{i=1}^p u_i$ where u_i ($i = 1, 2, \dots, p$) are independently distributed as $B(n-q+1-i, q)$, then V is said to have

a complex Wilks' V distribution with parameters n, p , and q , denoted by

$V(n, p, q)$. Thus

$$V = \prod_{i=1}^p u_i^c = \prod_{i=2}^p u_i^c$$

$$= u_1^c V(n-1, p-1, q)$$

$$= u_1 u_2^c V(n-2, p-2, q)$$

The hypothesis of equal means can be restated in terms of the regression of $\bar{z}$ on $\bar{y}$ as one of no association between $\bar{z}$ and $\bar{y}$. This hypothesis can then be broken into the following sub-hypotheses

$H_1: \xi_1$ has no association with $\bar{y}$

$H_2: \xi_2$ (eliminating ξ_1) has no association with $\bar{y}$

$H_p: \xi_p$ (eliminating $\xi_1, \xi_2, \dots, \xi_{p-1}$) has no association with $\bar{y}$.

If the overall hypothesis of no association is rejected, then at least one of the H_i ($i = 1, 2, \dots, p$) is rejected.

Observe that the criterion for the sub-hypothesis is given by

$$u_1 = \frac{\frac{|A_1|}{|A_1|} \frac{|A_1|}{|A_1|}}{\frac{|A_1|}{|A_1|} \frac{|A_1|}{|A_1|}} = \frac{1 - R_2^2(\xi_1, \xi_2, \dots, \xi_{1-1}, \bar{Y})}{1 - R_2^2(\xi_1, \xi_2, \dots, \xi_{1-1})} \frac{|A_1|}{|A_1|} \frac{|A_1|}{|A_1|}$$

It is desired to develop a goodness of fit test for a hypothetical

discriminant function. Denote this function by $\bar{y}_1^* \bar{\xi} = Z_1$ and consider a

non-singular transformation of $\bar{y} \rightarrow \bar{Z}$ by $L\bar{\xi} = \bar{Z}$ where the first row of

L is $\bar{y}_1^*$. It is now desired to test that Z_1 is the reason that the

dimensionality is not zero. Considering the population means of the

transformed variables, Z_1 is the one that differs from $Z_2, Z_3, \dots, Z_p$.

If this is true, then after eliminating Z_1 the set $Z_2, \dots, Z_p$ will have

no association with $\bar{Y}$.

Thus the test statistic is given by

$$\bar{y}_1^* = \frac{\bar{y}_1^*}{\bar{y}_1^*} = \frac{\bar{y}_1^*}{\bar{y}_1^*}$$

where $\bar{y}_1^* = |LAL^*| / |L(A+B)L^*|$, i.e., the transformed CMANOVA. But

observe

$$\bar{y}_1^* = \frac{|LAL^*|}{|A|} = \frac{|L(A+B)L^*|}{|A+B|} = \bar{y}_1^*$$

thus the test statistic is given by

$$\bar{y}_1^* = \frac{\bar{y}_1^*}{\bar{y}_1^*} = \frac{\bar{y}_1^*}{\bar{y}_1^*}$$

true, then $\bar{\lambda}^{*'}\bar{\xi}$ is the only variable that has a different mean among the

If the hypothesis that $\bar{\lambda}^{*'}\bar{\xi}$ is the only discriminant function is

alternate factorization can be achieved in a similar manner.

the direction and 'partial' collinearity aspects is presented. The

with extensive use of Kshirsagar's work. The factorization of $\tilde{A}$ into

The work presented here in the complex case is a combination of the two,

Kshirsagar (1970) gave an analytical argument for the same factorization.

was developed by Bartlett (1951), who gave a geometrical argument.

will be discussed later. In the real and general case this factorization

collinearity and 'partial' direction aspect. The use of the word 'partial'

or $\tilde{A}$ can be the factor into two independent factors measuring the

the direction aspect and the partial collinearity aspect can be achieved

Now a factorization of $\tilde{A}$ into two independent factors that measure

(3.3.3) an approximate χ^2 test can be constructed.

dent, since the r_i^2 ($i = 1, 2, \dots, p$) are not independent. From

given by $\bar{\lambda}^{*'}\bar{\xi}$. Observe further that the two factors are not indepen-

ality of one. The first factor tests that the discriminant function is

Observe that the second factor is the test statistic for the dimension-

$$(3.3.3) \quad \left\{ \frac{\sum_{i=1}^p \frac{u_i}{(1-r_i^2)}}{\left\{ \prod_{i=2}^p (1-r_i^2) \right\} \frac{\bar{\lambda}^{*'} \tilde{A} \bar{\lambda} / \bar{\lambda}^{*'} (A+B) \bar{\lambda}}{\left\{ \prod_{i=2}^p (1-r_i^2) \right\}}} \right\}$$

so the test statistic becomes

$$\tilde{A}^c = \prod_{i=1}^p (1-r_i^2)$$

but from

argument. Let D be the between groups matrix for the transformed
To show that the above is indeed the case, consider the following
factor.

where cV_4 is the collinearity factor and cV_5 is the 'partial' direction

$$\tilde{V}^c = {}^cV_4 \cdot {}^cV_5$$

The alternative factorization is given by

factor.

where cV_2 is the direction factor and cV_3 is the 'partial' collinearity

$$\tilde{V}^c = {}^cV_2 \cdot {}^cV_3$$

can be represented as

Now assuming that the factorization of $\tilde{V}^c$ can be achieved and $\tilde{V}^c$

$$\tilde{V}^c = \frac{{}^cV_1 \cdot \tilde{\Sigma}^{*'}(A+B)\tilde{\Sigma}}{{}^cV_1} = \frac{{}^cV_1 \cdot \tilde{\Sigma}^{*'}\tilde{\Sigma}}{{}^cV_1} \quad (3.3.4)$$

hypothesis has been developed and was shown to be on (3.3.1)
 $Z_1, \dots, Z_p$ all have the same mean in the k groups. The test for the
all uncorrelated with Z_1 , as $L\tilde{L}^{*'} = I$. Under the hypothesis,
Then Z_1 is the hypothetical discriminant function and $Z_2, \dots, Z_p$ are

$$\tilde{Z} = L\tilde{\xi}$$

Transform from $\tilde{\xi}$ to $\tilde{Z}$, where $\tilde{Z}' = [Z_1, Z_2, \dots, Z_p]$ by
 $L\tilde{L}^{*'} = I$. (There is no loss of generality for this assumption.)
singular matrix with $\tilde{\xi}^{*'}$ as its first row and let L be such that
 $\tilde{\xi}^{*'}\tilde{\xi}$, has the same mean in all the k groups. Let L be a $p \times p$ non-
k populations. Any other linear combination of $\tilde{\xi}$, uncorrelated with

variables Z , then

$$D = IBL^* = [d_{ij}]$$

and the within groups matrix is W , given by

$$W = IAL^* = [w_{ij}]$$

As in the real case, it is obvious that W has the Wishart density given by $CW^p(W|n-q|I)$ since $LBL^* = I$ and likewise $D = IBL^*$ is distributed as a non-central complex Wishart, with the non-centrality parameter due only to the difference in means of Z_1 alone in the k groups and thus affects b_{11} only, i.e., b_{11} is a non-central χ^2 with $2q$ degrees of freedom. So the density of D may be written as

$$CW^p(D|q|I)\xi(b_{11})$$

where $\xi(b_{11})$ is the contribution due to the non-centrality. An explicit expression for $\xi(b_{11})$ will not be necessary in this development.

Now $W \sim CW^p(W|n-q|I)$ where $W = [w_{ij}]$. From this we have

$$w_{11}, \frac{\sqrt{w_{11}}}{w_{11}} (i = 2, \dots, p) \text{ and } \tilde{w}_{ij} = w_{ij} - \frac{w_{i1}w_{j1}}{w_{11}} (i, j = 2, \dots, p)$$

are independently distributed. The $\frac{\sqrt{w_{11}}}{w_{11}}$ are complex standard normal and independent variables, while the matrix $\tilde{W} = [\tilde{w}_{ij}]$ is a complex

Wishart of order $p-1$. This follows, as in the real case, from the fact

that if $S \sim CW^p(S|n|I)$ and

$$\theta \sim \text{CN}^{p-1}(0, I)$$

$$\tilde{D} \sim \text{CW}^{p-1}(\tilde{D}|q-1|I)$$

$$\tilde{W} \sim \text{CW}^{p-1}(\tilde{W}|n-q-1|I)$$

Thus the following results are obtained

$\bar{\theta}$ denote the column vectors θ_i ($i = 2, 3, \dots, p$).

depend on w_{11} or b_{11} , hence it is the unconditional distributions. Let

density of θ_i for d_{11} and w_{11} fixed. This is $\text{CN}_1(0, I)$ but does not

which is a $\text{CN}_1(0, I)$ random variable. To see this consider the conditional

$$\theta_i = \sqrt{\frac{d_{11} w_{11}}{d_{11} + w_{11}}} \left(\frac{d_{11}}{d_{11} + w_{11}} - \frac{w_{11}}{d_{11} + w_{11}} \right) \quad \text{for } i = 2, \dots, p$$

Consider the variable

these elements are independent of $\tilde{W}$ and hence $\tilde{W}$.

are independent and $\tilde{D} = [d_{ij}]$ is distributed as $\text{CW}^{p-1}(\tilde{D}|q-1|I)$. All of

and are independently distributed. The $d_{11}/(d_{11})^{1/2}$ are $\text{CN}_1(0, I)$ and

$$d_{11}, \frac{d_{11}}{\sqrt{d_{11}}} \quad (i = 2, \dots, p) \quad \text{and} \quad d_{ij} = d_{ij} - \frac{d_{11} d_{ij}}{d_{11}} \quad (i, j = 2, \dots, p)$$

In a similar matter, $\tilde{D}$ can be split up as

$$\tilde{W} \sim \text{CW}^{p-1}(\tilde{W}|n-q-1|I) .$$

we have

Wishart and is independent of S_{11} and $S_{21}^{-1} S_{11}^{-1}$. So applying this result, then $S_{22 \cdot 1} = S_{22} - S_{21}^{-1} S_{11}^{-1} S_{12}$ is of order $p-k$ and distributed as a complex

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

and all are independent. Observe that $\bar{\theta}^*$ is distributed as a pseudo-complex Wishart with one degree of freedom.

Recall that Wilks' Λ^c with parameters n, p, q was defined as

$$\Lambda^c = \frac{|A|}{|A+B|} = \Lambda^c(n, p, q)$$

where $A \sim \text{CW}^p(A|n-q|Z)$ independent of $B \sim \text{CW}^p(B|q|Z)$. The parameter n

is the d.f. of A plus the degrees of freedom of B ; p is the order of

$A+B$; q is the degrees of freedom of B . Using this terminology, then

$$\frac{|W|}{|W+\theta\theta^*|} \sim \Lambda^c(n-q-1+1, p-1, 1) = \Lambda^c_5$$

$$\frac{|W+\theta\theta^*|}{|W+\tilde{D}+\theta\theta^*|} \sim \Lambda^c(n-1, p-1, q-1) = \Lambda^c_4$$

$$\frac{|W|}{|W+\tilde{D}|} \sim \Lambda^c(n-1, p-1, q-1) = \Lambda^c_3$$

$$\frac{|W+\tilde{D}|}{|W+\tilde{D}+\theta\theta^*|} \sim \Lambda^c(n-1, p-1, 1) = \Lambda^c_2 .$$

Furthermore, independence between Λ^c_2 and Λ^c_3 is obtained or independence between Λ^c_4 and Λ^c_5 is obtained. The same argument for the real case is

valid in the complex situation, i.e., if $A \sim \text{CW}^p(A|F|I)$ and $\tilde{E}_1 \sim \text{CN}^p(0, I)$

($i = 1, 2, \dots, n$), then

$$t = \frac{|A|}{|A + \sum_{i=1}^n \tilde{E}_i \tilde{E}_i^*|}$$

is independently distributed of $A + \sum_{i=1}^n \tilde{\varepsilon}_i \tilde{\varepsilon}_i^*$.

Now, it must be shown that $\tilde{A}$ is indeed given by the product

$\tilde{A}^2 \cdot \tilde{A}^3$. And then it will be shown that $\tilde{A}^2$ is the direction factor and $\tilde{A}^3$ is the 'partial' collinearity factor. The alternate factor-

ization, $\tilde{A} = \tilde{A}^4 \cdot \tilde{A}^5$, can also be shown. Consider

$$\tilde{A}^2 \cdot \tilde{A}^3 = \frac{|W|}{|\tilde{W} + \tilde{D} + \theta \theta^*|} = \frac{\frac{|W|}{\tilde{W}_{11}}}{\frac{|\tilde{W} + \tilde{D} + \theta \theta^*|}{\tilde{W}_{11}}} =$$

and observe that

$$|\tilde{W} + \tilde{D} + \theta \theta^*| = \frac{1}{p_{11} + w_{11}} |W + D| \quad (3.3.5)$$

then the above can be written as

$$= \frac{\frac{|W|}{\tilde{W}_{11}}}{\frac{|W + D|}{p_{11} + w_{11}}}$$

$$= \frac{[\tilde{\varepsilon}^* (A+B) \tilde{\varepsilon}]}{[\tilde{\varepsilon}^* A \tilde{\varepsilon}]} \cdot \frac{|\tilde{\varepsilon}^* (A+B) \tilde{\varepsilon}|}{|L(A+B) L^*|}$$

Since $w_{11} = \tilde{\varepsilon}^* A \tilde{\varepsilon}$ and $(w_{11} + p_{11}) = \tilde{\varepsilon}^* (A+B) \tilde{\varepsilon}$

(3.3.6)

$$|H| = |W_{11}| \tilde{W} + \tilde{D}$$

and

$$D + W = H + \bar{k} \bar{k}^*$$

then observe that

$$\bar{k}' = [\sqrt{a_{11}}, a_{12}/\sqrt{b_{11}}, \dots, a_{1p}]$$

and a vector k by

$$\left[\begin{array}{c|c} w_{p1} & w_{21} \\ \hline w_{11} & w_{12} \dots w_{1p} \end{array} \right] \begin{array}{l} \\ \\ w_{1j} + a_{1j} \\ (i,j = 2, \dots, p) \end{array}$$

define a matrix H by

and it is seen that (3.3.4) is verified.
Now it must be shown that $\sqrt{c_2}$ is the direction factor and $\sqrt{c_3}$ is the 'partial' collinearity factor. Consider $\sqrt{c_2}$ first. To see this,

$$\begin{aligned} \frac{c}{\sqrt{V}} &= \frac{c}{\sqrt{V}} \\ &= \frac{\frac{\bar{r}^* (A+B) \bar{r}}{\bar{r}^* A \bar{r}}}{\frac{|A+B|}{|A|}} \end{aligned}$$

and from (3.3.5), that

$$|H + \bar{k} \bar{k}^*| = (w_{11}^{11} + b_{11}^{11}) |w + \bar{b} + \bar{\theta} \bar{\theta}^*| \quad (3.3.7)$$

Then $\bar{A}_2^c$ may be written in terms of (3.3.6) and (3.3.7) as

$$\bar{A}_2^c = \frac{(b_{11}^{11} + w_{11}^{11})}{|H|} \cdot \frac{|H + \bar{k} \bar{k}^*|}{|H|}$$

then the above may be written as

$$= \frac{(b_{11}^{11} + w_{11}^{11})}{1} \cdot \frac{1 + \bar{k}^* H^{-1} \bar{k}}{1}$$

Since $|H + \bar{k} \bar{k}^*| = |H| \cdot |I + H^{-1} \bar{k} \bar{k}^*| = |H| \cdot |I + \bar{k}^* H^{-1} \bar{k}|$ and $\bar{A}_2^c$ can be further simplified to

$$= \frac{(b_{11}^{11} + w_{11}^{11})}{(b_{11}^{11} + w_{11}^{11})} \cdot [1 - \bar{k}^* (H + \bar{k} \bar{k}^*)^{-1} \bar{k}]$$

this can be seen by using the fact that

$$(I + P\bar{Q})^{-1} = I - P(I + \bar{Q}P)^{-1} \bar{Q}$$

Noting that

$$\bar{k}^* = \bar{k}^* I^{-1} \bar{k}^* / [\bar{k}^* (I^{-1} \bar{k}^* + \bar{k})]^{1/2}$$

then $\bar{A}_2^c$ may be written as

$$\bar{A}_2^c = \frac{(b_{11}^{11} + w_{11}^{11})}{(b_{11}^{11} + w_{11}^{11})} \left\{ 1 - \frac{\bar{k}^* (I^{-1} \bar{k}^* + \bar{k})}{\bar{k}^* (I^{-1} \bar{k}^* + \bar{k})} \right\}$$

and observing that

$$\frac{b_{11} + w_{11}}{w_{11}} = \frac{\tilde{\delta}' (L_{-1}^{-1} (W+D) L_{-1}^{-1}) \tilde{\delta}}{\tilde{\delta}' (L_{-1}^{-1} W L_{-1}^{-1}) \tilde{\delta}}$$

then

$$cV_2 = \frac{1 - \frac{\tilde{\delta}' (L_{-1}^{-1} D L_{-1}^{-1}) \tilde{\delta}}{\tilde{\delta}' (L_{-1}^{-1} W L_{-1}^{-1}) \tilde{\delta}}}{\frac{\tilde{\delta}' (L_{-1}^{-1} W L_{-1}^{-1}) \tilde{\delta}}{\tilde{\delta}' (L_{-1}^{-1} (W+D) L_{-1}^{-1}) \tilde{\delta}}}$$

and using the fact that $D = LBL'$ and $W = LAL'$, then

$$cV_2 = \frac{1 - \tilde{\delta}' B (A+B)^{-1} B \tilde{\delta} / \tilde{\delta}' B \tilde{\delta}}{\tilde{\delta}' A \tilde{\delta} / \tilde{\delta}' (A+B) \tilde{\delta}}$$

Then cV_3 can be represented as

$$cV_3 = \frac{cV}{cV_2}$$

$$= \frac{\frac{|A|}{|A+B|}}{1 - \tilde{\delta}' B (A+B)^{-1} B \tilde{\delta} / \tilde{\delta}' B \tilde{\delta}}$$

Returning to the CMANOVA for the transformed variable $Z = L\tilde{\xi}$, it

can be expressed as

Source of Variation	d.f.	s.s. and s.p.
Regression (Between)	q	$LBL' = D = C_{ZY} C_{YY}^{-1} C_{YZ}$
Error (Within)	n-q	$LAL' = W = C_{ZZ} \cdot V$
TOTAL	n	$L(A+B)L' = C_{ZZ}$

$$(3.3.11) \quad \tilde{V}^c = \frac{cV}{\bar{r}'A\bar{r}/\bar{r}'(A+B)\bar{r}} = \frac{|c_{yy \cdot t}|/|c_{yy}|}{cV} = \frac{|c_{yy \cdot t}|}{|c_{yy \cdot x}|} \quad (3.3.11)$$

Hence another expression for $\tilde{V}^c$ is given by

$$\frac{|c_{yy \cdot t}|}{|c_{yy}|} = \frac{|c_{tt}|}{|c_{tt \cdot y}|} = \frac{|c_{tt}|}{|c_{tt} - c_{ty}c_{yy}^{-1}c_{yt}|} = \frac{\bar{r}'c_{xx}\bar{r}}{\bar{r}'c_{xx}\bar{r} - \bar{r}'A\bar{r}} \quad (3.3.12)$$

Consider the first factor, using (3.3.9), this can be written as

$$\tilde{V}^c = \frac{|c_{yy \cdot t}|}{|c_{yy \cdot x}|} \cdot \frac{|c_{yy}|}{|c_{yy \cdot t}|} \quad (3.3.13)$$

the effect of $Z_1 = t$ removed, $\tilde{V}^c$ can be written as

Multiplying and dividing (3.3.10) by $|c_{yy \cdot t}|$, the residual of $\bar{y}$ with
and (3.3.10) is a measure of the "lack of relationship" between $\bar{y}$ and $\bar{L}_2^c$.
Now (3.3.9) is a measure of the "lack of relationship" between $\bar{L}_2^c$ and $\bar{y}$

$$(3.3.10) \quad \tilde{V}^c = \frac{|c_{yy \cdot x}|}{|c_{yy}|} \quad (3.3.10)$$

it is seen that another expression for $\tilde{V}^c$ is given by

$$(3.3.9) \quad \begin{vmatrix} c_{xx} & c_{xy} \\ c_{yx} & c_{yy} \end{vmatrix} = |c_{xx}| \cdot |c_{yy}| - c_{yx}c_{xy} = |c_{yy}| \cdot |c_{xx}| - c_{xy}c_{yx} = |c_{xx}| \cdot |c_{yy}| - c_{xy}c_{yx} \quad (3.3.9)$$

then using the fact that

$$(3.3.8) \quad \tilde{V}^c = \frac{|c_{zz \cdot y}|}{|c_{xx \cdot y}|} = \frac{|c_{zz}|}{|c_{xx}|} \quad (3.3.8)$$

The $\tilde{V}^c$ can be expressed as

which is the test statistic when $\bar{y}^* \bar{t} = t$ is eliminated, thus measuring

the association between $\bar{y}$ and $\bar{z}$, eliminating t . Now $\bar{V}^c$ must be factored into $\bar{V}_2^c$ and $\bar{V}_3^c$. This will be achieved by factoring the numerator and denominator of (3.3.11).

Let s denote the sample projection of t on the $\bar{y}$ space. The re-

gression of $\bar{t}$ on $\bar{y}$ is $C_{\bar{y}\bar{y}}^{-1} C_{\bar{y}t}$, thus the regression of $t = \bar{y}^* \bar{t}$ on $\bar{y}$ is $\bar{y}^* C_{\bar{y}\bar{y}}^{-1} C_{\bar{y}t}$. The sample projection s is given by $s = \bar{y}^* C_{\bar{y}\bar{y}}^{-1} C_{\bar{y}t}$ or

by $s = m^* \bar{y}$ where $m^* = \bar{y}^* C_{\bar{y}\bar{y}}^{-1} C_{\bar{y}t}$. Let $\bar{y}$ denote the remaining $q-1$ linear

functions of the $\bar{y}$'s. Then $|C_{\bar{y}\bar{y}}|$ can be factored into $|C_{\bar{y}\bar{y}}^{ss}| \cdot |C_{\bar{y}\bar{y}}^{zz}|$. This follows from (3.3.9). Likewise $|C_{\bar{y}t}|$ may be factored into

$|C_{\bar{y}t}| = |C_{\bar{y}\bar{y}}^{ss}| \cdot |C_{\bar{y}\bar{y}}^{zz}|$ and $\bar{V}^c$ may be expressed as

$$\bar{V}^c = \frac{|C_{\bar{y}\bar{y}}^{ss}| \cdot |C_{\bar{y}\bar{y}}^{zz}| \cdot |C_{\bar{y}\bar{y}}^{ss}|}{|C_{\bar{y}\bar{y}}^{ss}| \cdot |C_{\bar{y}\bar{y}}^{zz}|} = \frac{|C_{\bar{y}\bar{y}}^{ss}|}{|C_{\bar{y}\bar{y}}^{ss}|} \cdot \frac{|C_{\bar{y}\bar{y}}^{zz}|}{|C_{\bar{y}\bar{y}}^{zz}|}$$

or as

$$\bar{V}^c = \left\{ \frac{|C_{\bar{y}\bar{y}}^{ss}|}{|C_{\bar{y}\bar{y}}^{ss}|} \right\} \cdot \left\{ \frac{|C_{\bar{y}\bar{y}}^{zz}|}{|C_{\bar{y}\bar{y}}^{zz}|} \right\}$$

Now it will be shown that

$$\bar{V}_2^c = \frac{|C_{\bar{y}\bar{y}}^{ss}|}{|C_{\bar{y}\bar{y}}^{ss}|}$$

Now

$$\begin{aligned} C_{\bar{y}\bar{y}}^{ss} \cdot x &= m^* C_{\bar{y}\bar{y}}^{-1} C_{\bar{y}t} \\ &= \bar{y}^* C_{\bar{y}\bar{y}}^{-1} C_{\bar{y}t} - C_{\bar{y}\bar{y}}^{-1} C_{\bar{y}\bar{y}} C_{\bar{y}\bar{y}}^{-1} C_{\bar{y}t} \\ &= \bar{y}^* C_{\bar{y}\bar{y}}^{-1} C_{\bar{y}t} - \bar{y}^* C_{\bar{y}\bar{y}}^{-1} C_{\bar{y}\bar{y}} C_{\bar{y}\bar{y}}^{-1} C_{\bar{y}t} \\ &= \bar{y}^* C_{\bar{y}\bar{y}}^{-1} C_{\bar{y}t} - \bar{y}^* C_{\bar{y}\bar{y}}^{-1} C_{\bar{y}\bar{y}} C_{\bar{y}\bar{y}}^{-1} C_{\bar{y}t} \end{aligned}$$

and

$$C_{ss \cdot t}^m = m^* C_{yy \cdot t}^m$$

$$= \bar{\ell}^* C_{xy}^{-1} C_{yy}^{-1} [C_{yy}^{-1} C_{xy}^{-1} C_{yy}^{-1} C_{xy}^{-1} - C_{xy}^{-1} C_{yy}^{-1} C_{xy}^{-1} C_{yy}^{-1}]$$

$$= \bar{\ell}^* C_{xy}^{-1} C_{yy}^{-1} [C_{yy}^{-1} C_{xy}^{-1} C_{yy}^{-1} C_{xy}^{-1} - C_{xy}^{-1} C_{yy}^{-1} C_{xy}^{-1} C_{yy}^{-1}]$$

$$= \bar{\ell}^* C_{xy}^{-1} C_{yy}^{-1} [C_{yy}^{-1} C_{xy}^{-1} C_{yy}^{-1} C_{xy}^{-1} - C_{xy}^{-1} C_{yy}^{-1} C_{xy}^{-1} C_{yy}^{-1}]$$

$$= \bar{\ell}^* C_{xy}^{-1} C_{yy}^{-1} [C_{yy}^{-1} C_{xy}^{-1} C_{yy}^{-1} C_{xy}^{-1} - C_{xy}^{-1} C_{yy}^{-1} C_{xy}^{-1} C_{yy}^{-1}]$$

$$= \bar{\ell}^* C_{xy}^{-1} C_{yy}^{-1} [C_{yy}^{-1} C_{xy}^{-1} C_{yy}^{-1} C_{xy}^{-1} - C_{xy}^{-1} C_{yy}^{-1} C_{xy}^{-1} C_{yy}^{-1}]$$

Thus

$$\frac{|C_{ss \cdot x}^m|}{|C_{ss \cdot t}^m|} = \frac{\bar{\ell}^* B \bar{\ell} - \bar{\ell}^* B (A+B)^{-1} B \bar{\ell}}{(\bar{\ell}^* B \bar{\ell})^2} = \frac{\bar{\ell}^* B \bar{\ell} - \bar{\ell}^* B (A+B)^{-1} B \bar{\ell}}{\bar{\ell}^* B \bar{\ell}}$$

$$= \frac{\bar{\ell}^* B \bar{\ell} \bar{\ell}^* A \bar{\ell}}{\bar{\ell}^* B \bar{\ell} \bar{\ell}^* A \bar{\ell} - \bar{\ell}^* B (A+B)^{-1} B \bar{\ell}} = \frac{\bar{\ell}^* A \bar{\ell}}{\bar{\ell}^* B \bar{\ell} \bar{\ell}^* A \bar{\ell} - \bar{\ell}^* B (A+B)^{-1} B \bar{\ell}}$$

In a similar matter, C_{V_3} follows. Thus we have

$$C_{V_2} = \frac{|C_{ss \cdot x}^m|}{|C_{ss \cdot t}^m|} \quad \text{and} \quad C_{V_3} = \frac{|C_{yy \cdot sx}^m|}{|C_{yy \cdot st}^m|}$$

Now $\tilde{V}_2^c$ is a measure of the association of $\bar{s}$, the sample projection on the $\bar{y}$ space, with the $\bar{\xi}$ space with the effect of the discriminant

function, t , removed. Now $\bar{s} = m^* \bar{y}$, where $m^* = \bar{\xi}^* C_{xy}^{-1} C_{yy}$. Observe

that $\bar{m}$ is determined by $\bar{\xi}$, the given direction, thus one would expect no association to be present if $\bar{\xi}^* \bar{\xi} = t$ is eliminated. Since $\bar{y} = \bar{m}^* \bar{y}$ is considered first and then the $\bar{y}$'s are considered after eliminating $\bar{m}^* \bar{y}$, $\tilde{V}_3^c$ is called the 'partial' collinearity factor. Considering the $\bar{y}$'s first and then considering $\bar{s}$ after the elimination of $\bar{y}$ will give the alternate factorization, i.e.,

$$\tilde{V}^c = \tilde{V}_4^c \cdot \tilde{V}_5^c$$

where $\tilde{V}_4^c$ is the collinearity factor and $\tilde{V}_5^c$ is the 'partial' direction

factor.

$\tilde{V}_3^c$ is the collinearity factor since it depends on the number, but

not the direction, of the remaining $q-1$ variables.

In summary, we have the test for the hypothesis that $\bar{\xi}^* \bar{\xi}$ is the

only linear discriminant function needed to discriminate among the

$k = q+1$ complex p -variate normal population based on the statistic

$$\tilde{V}^c = \frac{\tilde{V}_2^c}{\tilde{V}_3^c} = \frac{\frac{\bar{\xi}^* (A+B) \bar{\xi}}{\bar{\xi}^* A \bar{\xi}}}{1 - \frac{\bar{\xi}^* B (A+B) \bar{\xi}}{\bar{\xi}^* A \bar{\xi}}} \cdot \frac{1 - \frac{\bar{\xi}^* B (A+B) \bar{\xi}}{\bar{\xi}^* A \bar{\xi}}}{\frac{|A|}{|A+B|}}$$

where $\lambda_2^c \sim \lambda(n-1, p-1, 1)$ independently of $\lambda_3^c \sim \lambda(n-1, p-1, q-1)$; where λ_2^c is the direction factor and λ_3^c is the 'partial' collinearity factor.

Through the work of Wooding (1956) and then Goodman (1963) the foundation for complex multivariate analysis was established. This work was extended by the major contribution of Khatri (1965) who developed much of the basic theory needed for the analysis of complex multivariate normal random variables. It was this author's intentions of further extending this development by examining the complex analogue of Wilks' statistic as used in the multivariate analysis of variance procedure and as used in the discrimination problem.

Chapter II of this paper contains the basic theorems necessary for the development of the analogue of Wilks' statistic as developed in Chapter III. Also included are results pertaining to the decomposition of a complex Wishart matrix. This decomposition was fundamental in establishing results about the coherence between two complex random variables and the coherence between two complex random vectors. One possible extension of this area is the investigation of the complex distribution which occurs as the distribution of the sample estimate when the true coherence between two complex normal variates is zero. This is in direct analogy with the real case.

The development pertaining to Wilks' statistic was presented in Chapter III. In Section 1 of Chapter III, a test for the equality of means of k complex p -variate normal populations was developed. This was

SUMMARY

CHAPTER IV

accomplished by use of the likelihood ratio criterion. A possible extension of this section would be to examine the properties of this test procedure when applied to complex random variables. Some work has been done in this area by Giri (1965). In Section 2, a test for dimensionality of the mean space was developed and it was shown that this was equivalent to knowing the number of linear discriminant functions needed to discriminate among the k -populations. A needed extension is to determine the exact correction factor that should be used for the chi-square test. In Section 3, a test for the goodness of fit of a single hypothetical discriminant function was developed. The test statistic was factored into two independent parts, one for the direction of the hypothetical discriminant function and the other to test the 'partial' collinearity of the means. A possible extension would be the goodness of fit of more than one discriminant function. Another extension would be the goodness of fit of a single discriminant function from the vector space of dummy variables. For the p -variate complex normal as defined by Wooding and Goodman, Khatri (1964) has noted "that one can handle all the classical problems of point estimation and testing hypotheses concerning the parameters of complex multivariate normal populations much as one handles those for multivariate populations in real variables." This is true for the problems that have been considered in this paper. As a matter of fact, all the work so far with the p -variate complex normal as defined by Wooding and Goodman has been done by paralleling the real case. Even so, this type of development is important because complex multivariate analysis has possible use and applications in stochastic processes and spectral analysis of multivariate time series and point processes. N. R. Goodman and M. R. Dubman (1968) have considered the theory of time-varying

spectral analysis and complex Wishart matrix processes and the assumption of stationarity and normality. They have considered complex Gaussian processes, Wishart processes and time-varying spectral estimates along with distributional results associated with them. This author feels that the work reported in this dissertation will be helpful, indirectly at least, in such investigations.

David Brillinger (1968) has considered the canonical analysis of stationary time series. The distributional results about canonical correlation of complex normal vectors are likely to be useful in this area.

1. Bartlett, M. S. (1951). "Goodness of fit of a single hypothetical discriminant function in the case of several groups," Annals of Eugenics, 16, 199-214.
2. Brillinger, D. R. (1968). "The canonical analysis of stationary time series," Multivariate Analysis II, P. R. Krishnaiah, Ed., Academic Press, New York and London, 331-349.
3. Giri, N. (1965). "On the complex analogues of T^2 and R^2 tests," The Annals of Mathematical Statistics, 36, 664-671.
4. Gupta, A. K. (1971). "Non-null distribution of Wilks' likelihood ratio criterion." (To appear in Annals of Mathematical Statistics.)
5. Goodman, N. R. (1963). "Statistical analysis based on a certain multivariate complex Gaussian distribution," The Annals of Mathematical Statistics, 34, 152-178.
6. _____, and Dubman, M. R. (1968). "Theory of time-varying spectral analysis and complex Wishart matrix processes," Multivariate Analysis II, P. R. Krishnaiah, Ed., Academic Press, New York and London, 351-365.
7. Hodgson, V. (1967). "On sampling distribution of the multiple correlation coefficient." (Presented at IMS meeting in Washington, D.C., December 1967.)
8. Hohn, F. E. (1958). Elementary Matrix Algebra. The Macmillan Company, New York.
9. Kabe, D. G. (1966a). "Complex analogue of some classical non-central multivariate distributions," Australian Journal of Statistics, 8, 99-103.
10. _____ (1966b). "Complex analogues of some results in classical normal multivariate regression theory," Australian Journal of Statistics, 8, 87-91.
11. Khatri, C. G. (1964). "Distribution of the largest or the smallest characteristic root under null hypothesis concerning complex multivariate normal populations," Annals of Mathematical Statistics, 35, 1807-1810.
12. _____ (1965). "Statistical analysis based on a certain multivariate complex distribution," Annals of Mathematical Statistics, 34, 98-114.

LIST OF REFERENCES

13. Kshirsagar, A. M. (1959). "Bartlett decomposition and Wishart distribution," Annals of Mathematical Statistics, 30, 238-241.
14. . (1969). "Correlation between two vector variables," Journal of Royal Statistical Society, 31, 477-485.
15. . (1970). "Direction and collinearity statistics in discriminant analysis." (To appear in Calcutta Statistical Association Bulletin.)
16. . (1971). "Degree of freedom for the chi-square test of dimensionality." (To appear in the American Statistician.)
17. Rao, C. R. (1965). Linear Statistical Inference and Its Application. Wiley and Sons, New York.
18. Ruben, H. (1966). "Some new results on the distribution of the sample correlation," Journal of Royal Statistical Society, 31, 477-485.
19. Srivastava, M. S. (1965). "On the complex Wishart distribution," Annals of Mathematical Statistics, 36, 313-315.
20. Want, J. K. and Kabe, D. G. (1970). "On a certain minimization problem," The American Statistician, 24, 29-30.
21. Wooding, R. A. (1956). "The multivariate distribution of complex normal variates," Biometrika, 43, 212-225.

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13. ABSTRACT

Consider $k = q+1$ complex multivariate normal populations with the same variance-covariance matrix but with different means. The problem of the number of discriminant functions needed to discriminate among the k -populations is considered and shown to be equivalent to the rank of the mean-space. A test for this dimensionality being r is presented. Also developed is the statistic for testing the goodness of fit of a single hypothetical discriminant function. As in the case of the real normal populations, the test statistic for this single hypothetical function is presented as the product of two independent factors, whose distributions are given; one measuring the direction aspect of the hypothetical function, and the other measures the collinearity aspect that is the necessity of only one function. Also included is the Bartlett decomposition of a complex Wishart matrix and some results pertaining to the coherence of complex random variables.

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