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ARMA MODEL IDENTIFICATION USING THE GENERALIZED PARTIAL AUTOCORRELATION ARRAY

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ABSTRACT

The generalized partial autocorrelation (GPAC) array was introduced by Woodward and Gray (1981) as a method of identifying the order of an ARMA(p,q) process. The GPAC identification technique is a generalization of the model identification method popularized by Box and Jenkins (1970). Davies and Petrucelli (1984) provide simulation evidence to argue that in fact the GPAC is of limited usefulness due to the fact that the sample GPAC array is unstable when applied to time series of only moderate length.

In this paper we address the findings of Davies and Petrucelli and show that in general they are not valid. Essentially they have concentrated on examining variability between GPAC arrays when the model identification capabilities of the GPAC array depend on variability within an array. Through a simulation study it is shown that GPAC *patterns* are more stable than Davies and Petrucelli suggest and that the W-statistic provides model identification results comparable to those of AIC in a fraction of the time. The W-statistic is defined for purposes of measuring the patterns in a sample GPAC array automatically and providing a quantitative means of assessing the model identification information in an array. We also examine sample GPAC arrays based on the estimates of Tsay and Tiao (1984). These are shown to perform better than the Yule-Walker based arrays examined previously, particularly when near nonstationary components are in the model. In certain cases the GPAC results using the Tsay and Tiao estimates are far superior to those of AIC. We briefly examine the use of an overfitting procedure proposed by Gray and Woodward (1986) to automatically perform model identification in the presence of nonstationary components.

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1. Introduction

The generalized partial autocorrelation (GPAC) function was introduced by Woodward and Gray (1981) for purposes of model identification in the ARMA(p,q) setting. The GPAC function is an extension of the partial autocorrelation function used by Box and Jenkins (1975) in ARMA model identification. Woodward and Gray (1981) used an array to present the information in the GPAC function, and this array was shown to be related to the S-array of Gray, Kelley, and McIntire (1978). Woodward and Gray (1981) showed that the GPAC array uniquely determines p and q when the true autocorrelation is known, a property it shares with the S-array. Unique identification of p and q when the true autocorrelation function is known is only assured using the Box-Jenkins approach when either p=0 or q=0. Woodward and Gray (1981) discussed the use of the GPAC based on single, finite length realizations, and showed examples in which the model identifying pattern in the GPAC was clearly discernible. Davies and Petruccelli (1984) presented simulation evidence and real data examples to argue that the sample GPAC array is unstable when applied to time series of only moderate length and that its use in detecting MA components is limited.

In this paper we discuss the findings of Davies and Petrucelli (1984) and show that their conclusions are unfounded, largely due to the fact that they are essentially confusing variability between GPAC arrays with variability within an array. In Section 2 we define the GPAC function and associated array. In Section 3 we discuss the estimation of the GPAC array from sample data and present alternative estimation approaches. We also discuss an ad hoc quantification of the pattern in the GPAC as a means of assessing the model identification capabilities of the GPAC. In that section we also briefly discuss model identification using GPAC arrays in the nonstationary case. Finally, in Section 4 we re-examine the results of Davies and Petrucelli (1984) and further explore the model identification capabilities of the GPAC array by comparing it to the AIC method of Akaike (1974). Even though we still believe that the GPAC is best applied when inspected visually by the investigator, a statistic is introduced to measure the pattern in the GPAC array so that the comparison with AIC is strictly quantitative. Even in this automated form, which may be able to be improved upon, GPAC compares very favorably with the AIC and in fact in many circumstances it is much better. The automated form of GPAC model identification requires far less computation time than AIC.

2. The Generalized Partial Autocorrelation Function

Consider the univariate ARMA(p,q) process given by

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} \tag{2.1}$$

for $t = 0, \pm 1, \pm 2, \ldots$ where a_t is assumed to be white noise, with the autoregressive coefficients ϕ_i, i=1,...,p and moving average coefficients θ_i, i=1,...,q being real constants. We often write (2.1) in the form $\phi(B)X_t = \theta(B)a_t$ where

$$\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$$

$$\theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$$

and where for a function of t, $f(t)$, the backshift operator B^k is defined by $B^k f(t) = f(t-k)$. The generalized partial autocorrelation is defined to be

$$\phi_{kk}^{(j)} = \rho_{j+1}/\rho_j \quad \text{if } k=1$$

$$= |A(k,j)|/|B(k,j)| \quad \text{if } k>1 \tag{2.2}$$

where $B(s,t)$ is the sxs matrix defined by

$$B(s,t) = \begin{bmatrix} \rho_t & \rho_{t-1} & \cdots & \rho_{t-s+1} \\ \rho_{t+1} & \rho_t & \cdots & \rho_{t-s+2} \\ \vdots & \vdots & \vdots & \vdots \\ \rho_{t+s-1} & \rho_{t+s-2} & \cdots & \rho_t \end{bmatrix} \tag{2.3}$$

and $A(s,t)$ is the matrix composed of the first s-1 columns of $B(s,t)$ with the sth column given by $(\rho_{t+1}, \ldots, \rho_{t+s})'$. The GPAC element $\phi_{kk}^{(j)}$ is thus the solution of the extended Yule-Walker equations for the kth autoregressive coefficient of an ARMA(k,j). More generally, in this paper we will use the notation $\phi_{kp}^{(q)}$ to denote the kth autoregressive coefficient corresponding to an ARMA(p,q) model. Woodward and Gray (1981) suggest displaying the GPAC elements as an array whose $(k,j)th$ element is

$$\phi_{kk}^{(j)}, \quad (j=0,\ 1,\ \ldots;\ \ k=1,\ 2,\ \ldots\ ) \tag{2.4}$$

The model identification pattern in this array is based on the fact that if the process is ARMA(p,q)

$$\phi_{kk}^{(q)} = 0, \quad k > p, \quad \phi_{pp}^{(q)} \neq 0 \tag{2.5}$$

$$\phi_{pp}^{(j)} = \phi_p, \quad j \geq q$$

This pattern uniquely identifies the order of a stationary ARMA(p,q) process. The form of the GPAC array when X_t is an ARMA(p,q) process is shown in Table 2.1. In order to identify p and q given a GPAC array, one searches for a column p in which constant behavior oocurs associated with a row q in which the elements are zero for columns k > p. Woodward and Gray (1981) show that the GPAC elements can be obtained as a ratio of elements of the S-array proposed by Gray, Kelley and McIntire (1978). The behavior exhibited by (2.5) derives from the fact that the autocorrelation function from a stationary ARMA(p,q) process given in (2.1) satisfies the difference equation

$$\rho_k = \phi_1\rho_{k-1} + \cdots + \phi_p\rho_{k-p} , \quad k \geq q + 1. \tag{2.6}$$

Table 2.1 GPAC Array for an ARMA(p,q) Process

Autoregressive Order

		1	$\cdots$	p-1	p	p+1	p+2	$\cdots$
	0	$\phi_{11}^{(0)}$	$\cdots$	$\phi_{p-1,p-1}^{(0)}$	$\phi_{pp}^{(0)}$	$\phi_{p+1,p+1}^{(0)}$	$\phi_{p+2,p+2}^{(0)}$	$\cdots$
Moving	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	
Average	q-1	$\phi_{11}^{(q-1)}$		$\phi_{p-1,p-1}^{(q-1)}$	$\phi_{pp}^{(q-1)}$	$\phi_{p+1,p+1}^{(q-1)}$	$\phi_{p+2,p+2}^{(q-1)}$	$\cdots$
Order	q	$\phi_{11}^{(q)}$		$\phi_{p-1,p-1}^{(q)}$	ϕ_p	0	0	$\cdots$
	q+1	$\phi_{11}^{(q+1)}$	$\cdots$	$\phi_{p-1,p-1}^{(q+1)}$	ϕ_p	u^*	u	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

* u = undefined

Special care must be exercised when dealing with processes which are nonstationary or very nearly so. Before considering this case we provide the following definition.

Definition 2.1 The k complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ will be said to approach the unit circle uniformly if $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_k|$ as $|\lambda_i| \to 1$.

Findley (1978) and Quinn (1980) show that if:

(a) X_t is an ARMA($s+d,q$) process where d roots of the characteristic equation approach the unit circle uniformly

(b) Of the d roots in (a)
  (i) m are distinct
  (ii) j of these have highest multiplicity h

(c) ρ_m^* is the limiting value of ρ_m as the d roots approach the unit circle uniformly

then, ρ_m^* satisfies a linear homogeneous difference equation of order j, of the form

$$\psi(B)\ \rho_m^* = 0 \tag{2.7}$$

for all integer m, where $\psi(B) = 1 - \psi_1 B - \cdots - \psi_j B^j$ and $|\psi_j|=1$. Actually, it can be shown that $\psi(B)$ in (2.7) is that operator formed from the product of the nonstationary factors of highest multiplicity in $\phi(B)$. The following result is a consequence of the above remarks.

Theorem An ARMA(p,q) process is nonstationary if and only if for some $k \le p$, $\phi_{kk}^{(0)} = \phi_{kk}^{(i)} = \pm 1$, $i = 1, 2, \ldots$.

This "dropping back" in the order of the difference equation being satisfied by ρ_m^* holds approximately for ρ_m whenever some of the roots of $\phi(B)$ are close to the unit circle. When this occurs a column of the GPAC array prior to the pth column will be "nearly constant" with these values being real numbers near 1 or -1. The implications of these results when modeling a realization from a nonstationary or nearly nonstationary ARMA process will be discussed in Section 3.

3. Using the GPAC Array with Data

(a) Estimation Techniques

Obviously, the fact that the GPAC pattern uniquely identifies the order of a stationary process is only useful in practice if the patterns can be identified when GPAC elements have been estimated. Woodward and Gray (1981) suggest estimating the GPAC elements by replacing the autocorrelations in A(s,t) and $B(s,t)$ with their corresponding sample estimates, i.e. ρ_k is estimated by

$$r_k = \sum_{t=1}^{n-k}(X_t-\bar{X})(X_{t+k}-\bar{X})/\sum_{t=1}^{n}(X_t-\bar{X})^2 \quad ,k\geq 0. \tag{3.1}$$

In other words, the GPAC array elements are estimated by $\hat{\phi}_{kk}^{(j)}$, the extended Yule Walker estimate of ϕ_k , if the process is assumed to be ARMA(k,j). Woodward and Gray (1981) show that these estimates can be obtained as simple ratios of elements in the sample S-array. However, Yule-Walker estimates can be poor, especially in the presence of roots near the unit circle.

Clearly, any technique for estimating the autoregressive coefficients of an ARMA(p,q) will yield a sample GPAC array. We will investigate the use of the estimates of Tsay and Tiao (1984) as alternatives to the Yule-Walker approach suggested originally by Woodward and Gray (1981). Tsay and Tiao (1984) introduced estimates of the autoregressive parameters of an ARMA(p,q) process based on iterated AR(k) regressions. Their estimates are least squares estimates obtained from recursively adding MA type terms to an AR(k). These estimates can be obtained recursively using the formula

$$\tilde{\phi}_{mk}^{(j)} = \tilde{\phi}_{m,k+1}^{(j-1)} - \frac{\tilde{\phi}_{m-1,k}^{(j-1)}\ \tilde{\phi}_{k+1,k+1}^{(j-1)}}{\tilde{\phi}_{kk}^{(j-1)}} \tag{3.2}$$

where $m = 1, \ldots, k$; $k \geq 1$ and $j \geq 1$ where

$$\tilde{\phi}_{0k}^{(j-1)} = -1$$

$$\tilde{\phi}_{mk}^{(0)} = \text{ordinary least squares estimate for AR}(p)$$

and for an ARMA(p,q) process

$$\tilde{\phi}_{mp}^{(q)} = \tilde{\phi}_{m}$$

We will refer to estimates using this approach as TT estimates. Tsay and Tiao (1984) showed that their estimates are consistent for the autoregressive parameters when initialized by the ordinary least squares estimates, whether or not the model contains nonstationary components and that TT estimates are asymptotically equivalent to YW estimates when X_t is stationary. On the other hand, Findley (1980) has reportedly shown that when the process is nonstationary, the sample autocorrelation approaches ρ^* so that the Yule-Walker estimates approach the ψ_i in Equation 2.7. In this paper we will use the implementation of the TT estimation procedure utilized by Gray and Woodward (1986) and initialize with Burg (1975) estimates.

Example 3.1 Consider a realization of length 300 from the ARMA(3,2) process

$$X_t - 1.5X_{t-1} + 1.21X_{t-2} - .46X_{t-3} = a_t + .2a_{t-1} + .9a_{t-2} \tag{3.3}$$

where a_t is normal white noise with zero mean and unit variance. It can be easily seen that this ARMA(3,2) model is stationary with none of its roots close to the unit circle. In Table 3.1 we display the GPAC arrays for this realization using TT and YW estimates. Notice that there is very little difference in the arrays of Table 3.1, and that for each array the identification as an ARMA(3,2) is clear since $\phi_{33}^{(j)} \approx .52$ for $j \geq 2$ and $\phi_{kk}^{(2)} \approx 0$ for $k > 3$.

A parametric procedure for estimating coefficients is to use maximum likelihood (ML). These estimates are obtained using an iterative scheme which is

Table 3.1 GPAC Arrays Based on a Realization of Length 300
From the ARMA(3,2) Model of Example 3.1

(a) Using TT Estimates

Autoregressive Order

		1	2	3	4	5	6
	0	.851	-.705	.430	.353	-.347	-.125
Moving	1	.624	-.444	.902	.723	-.459	-.809
Average	2	.433	.024	.512	.123	-.046	-.063
Order	3	.450	-9.586	.509	.281	-.199	-.042
	4	1.295	-.136	.536	-.179	-.003	-.343
	5	1.492	4.881	.538	-.192	23.770	-.343

(b) Using YW Estimates

Autoregressive Order

		1	2	3	4	5	6
	0	.851	-.702	.417	.355	-.323	-.144
Moving	1	.623	-.445	.911	.687	-.465	-.688
Average	2	.433	.022	.514	.111	-.019	-.078
Order	3	.449	-10.393	.512	.186	-.449	-.073
	4	1.298	-.134	.527	-.641	.219	-.218
	5	1.493	4.896	.532	.159	-.730	-.117

sometimes very slow to converge, especially if some of the model components are nonstationary or nearly nonstationary. Thus, a GPAC technique based on ML estimates may require substantially more computation time than YW or TT, especially when roots of the characteristic equation are near the unit circle. This is the reason the ML estimates were not used by Gray and Woodward (1981) where the GPAC was introduced and will not be considered any further now. Akaike's Information Criterion (AIC), see Akaike (1974), is theoretically based on ML estimation and consequently also requires much more computation time than either the YW or TT GPAC. This will be discussed more in Section 4.

(b) Nonstationary Components

The results of Gray, Kelley and McIntire (1978), Findley (1978) and Quinn (1980) suggest that the sample autocorrelation function of an ARMA(p,q) process with stationary and "nearly nonstationary" components will approximately satisfy a difference equation of order less than p. In essence, these nearly nonstationary components tend to dominate other components. For this reason, Box and Jenkins (1975) recommend differencing a series whose sample autocorrelation damps slowly, i.e. which approximately satisfies the first order difference equation $\rho_k - \psi_1\rho_{k-1} = 0$ where ψ_1 is near 1. Box and Jenkins (1975) defined the ARIMA(p,d,q) model to accomodate these unit roots. Gray and Woodward (1981) further discuss the problem of model identification in the nonstationary case, taking into consideration any roots near the unit circle (± 1 and complex roots), and they showed that these nonstationary components can be detected using the S-array and GPAC array. Nonstationary components manifest themselves in the GPAC array as a nearly constant column whose values are "near" 1 or -1. The important point here is that when such a pattern is detected, this should serve as a warning that other stationary components may be present in the model which may not be detectable until the data is transformed by the nonstationary factors. Tsay and Tiao (1984) show that TT estimates are consistent whether or not the model has nonstationary components, and that convergence to the components of nonstationary factors is like $1/n$ instead of $1/\sqrt{n}$ as in the stationary case. They claim that use of the TT estimates removes the need for preliminary transformation. This is true to an extent. That is, TT estimates vastly improve the estimates in the near nonstationary case, but there can be problems with their use also (see Gray and Woodward 1986).

Example 3.2 Consider a realization of length 300 from the ARMA(4,1) process

$$X_t - 1.8X_{t-1} + 2.29X_{t-2} - 1.292X_{t-3} + .495X_{t-4} = a_t - .7a_{t-1}$$

where again a_t is normal white noise with zero mean and unit variance. We display the AR factors of this model in Table 3.2.

Table 3.2 Factor Table for the ARMA(4,1) Model of Example 3.2

Factor	Roots	Absolute Reciprocal	System Frequency
$1 - B + .99B^2$	$.5 \pm .87i$	.995	.17
$1 - .8B + .5B^2$	$.8 \pm 1.17i$	.707	.15

From Table 3.2 it can be easily seen that this model contains two roots close to the unit circle and two roots relatively far removed from it. The GKM-Findley-Quinn result suggests that the GPAC arrays, based on YW estimates will tend to show a second order behavior where the second column is nearly constant with $|\phi_{22}^{(j)}| \approx 1$, $j=0, 1, \ldots$ However, the preceding discussion indicates that the GPAC arrays based on TT estimates may indicate the proper ARMA(4,1) model due to the Tsay-Tiao consistency result. In Table 3.3 we display the GPAC arrays for this realization using TT and YW estimates. Inspection of these two arrays indicates that the second order behavior due to the nearly nonstationary component, i.e. the nearly constant second column with $|\phi_{22}^{(j)}| \approx 1$, is by far the most pronounced pattern in both arrays. However, the YW GPAC shows no indication of an ARMA(4,1) pattern but instead suggests an ARMA(2,0). The GPAC array based on TT estimates also shows a very constant second column and also suggests an ARMA(2,0) although the zero behavior is not strong. Notice also that in the TT GPAC array the fourth

Table 3.3 GPAC Arrays Based on a Realization of Length 300 From the ARMA(4,1) Model of Example 3.2

(a) Using TT Estimates

Autoregressive Order

		1	2	3	4	5	6
	0	.505	-.995	.185	-.377	-.320	-.117
Moving	1	-.965	-.994	-1.767	-.512	-.196	.142
Average	2	2.036	-.993	1.267	-.755	-.340	-.068
Order	3	.519	-.989	.250	-.484	-.451	.524
	4	-.900	-.989	-1.288	-.568	-.267	1.074
	5	2.102	-.988	1.393	-.672	-1.198	.024

(b) Using YW Estimates

Autoregressive Order

		1	2	3	4	5	6
	0	.502	-.974	-.136	-.076	-.110	-.084
Moving	1	-.949	-.978	.400	.119	-.053	-.111
Average	2	2.029	-.976	1.550	.574	-.184	-.091
Order	3	.519	-.972	.512	-1.318	.522	-.124
	4	-.875	-.970	-1.165	-.650	-.683	-.263
	5	2.105	-.969	.609	-.030	-.105	-.378

column shows a certain degree of constancy, and that there is a slight indication of an ARMA(4,1). From this example we see that even when TT estimates are used, the nonstationarity pattern is the strongest pattern indicated by the GPAC array and may therefore mask the true pattern or destroy it completely. We therefore recommend a dynamic prefiltering technique introduced by Gray and Woodward (1986) for identifying and removing nonstationary components. Later we show that this can be done automatically so that two passes through the data are not necessary.

(c) Measuring the GPAC Pattern

Davies and Petrucelli (1984) examined the model identification capabilities of the GPAC by performing a simulation study in which 1000 realizations from the ARMA(3,2) model of Example 3.1 were generated. Empirical means and standard deviations of GPAC elements involved in the ARMA(3,2) pattern were found across the 1000 realizations. Davies and Petrucelli questioned the model identification capabilities of the GPAC because of what they perceived to be high variability across realizations. However, the fact that a particular GPAC element is .275 for one realization and .655 for another may not be relevant for model identification. The question for model identification is "What is the *pattern* of array values within a single realization?" and not "What are particular array values for a realization or across realizations?" The actual results of the Davies and Petrucelli (1984) simulation relate more to the performance of Yule-Walker estimators as coefficient estimators than they do to model identification via the GPAC.

In an effort to examine the actual model identification capabilities of the GPAC, we investigate measuring the pattern within a given sample GPAC array. Here, the strength of the ARMA(k,j) pattern is measured by the W-statistic defined by

$$W(k,j)=w_c\ C(k,j) + w_z\ Z(k,j) \tag{3.4}$$

where

$$C(k,j) = \sqrt{\frac{\sum_{i=0}^{3} c_i \left[\hat{\phi}_{kk}^{(j+i)} - \bar{\phi}(k,j) \right]^2}{\sum_{i=0}^{3} c_i}}$$

$$Z(k,j) = \sqrt{\frac{\sum_{i=1}^{3} (z_i \; \hat{\phi}_{k+i,k+i}^{(j)})^2}{\sum_{i=1}^{3} z_i}} \Bigg/ |\bar{\phi}(k,j)|$$

$$\bar{\phi}(k,j) = \frac{\sum_{i=0}^{3} c_i \; \hat{\phi}_{kk}^{(j+i)}}{\sum_{i=0}^{3} c_i}$$

and w_c , w_z , c_i , i=0,...,3 and z_i , i=1,...3 are non-negative constants which have to be chosen. In essence $C(k,j)$ measures the strength of a column constancy behavior for the kth column beginning in row j while $Z(k,j)$ measures zero behavior in the jth row beginning in column k+1 relative to the size of the constants. The order selected is (p,q) such that

$$W(p,q) = \min_{\substack{0 \le k \le K \\ 1 \le j \le J}} W(k,j) \tag{3.5}$$

Unless some of the weights are taken to be zero, the W-statistic measures the GPAC pattern by looking for four "constants" down a column such that the row containing the first of these constants has three "zeros" immediately following that constant. In Section 4 we present results of simulations investigating the model identification capabilities of the GPAC array on the basis of the W statistic.

It should be noted that the procedure described in (3.4) and (3.5) applies to the stationary case. If nonstationary or nearly nonstationary components are present in the data, then direct application of the W-statistic would be expected to often select p too small based on the dicussion in Sections 2 and 3, especially when using Yule-Walker estimates. A procedure for automatically detecting nonstationary behavior using overfitting as described by Gray and Woodward (1986), removing the nonstationary components and modeling the stationary model components is feasible and will be discussed in Section 4.

4. Rejoinder to Davies and Petrucelli Results

The conclusions of Davies and Petrucelli (1984) concerning the performance of the GPAC were that in practice, the GPAC array has minimal model identification capability especially when $q > 0$. Their findings in support of these conclusions can be summarized as follows:

(a) Simulation results showed lack of stability of GPAC elements for the ARMA(3,2) model in (3.3) even with realization lengths of 300 and 500

(b) Marked failure of the GPAC to identify the following following models based on realizations of various lengths

$$\text{(i)}\ (1-.4B)X_t = (1-.7B)a_t$$

$$\text{(ii)}\ (1-.5B)X_t = (1+B-.4B^2)a_t \tag{4.1}$$

(c) Failure of the GPAC to provide "correct" model identification for several classical data sets for which mixed models have been obtained using other techniques

These findings will be addressed individually in this section, and it will be shown that in each case their conclusions are not valid. Before proceeding, we mention that simulation results quoted in this paper are based on realizations generated using N(0,1) white noise with the normal deviates being generated using IMSL routine

GGNPM on the IBM 3081-D24 computer at Southern Methodist University. For each simulated model to be examined, 100 realizations were generated for each realization length under consideration. Both the YW GPAC and TT GPAC arrays were computed for each realization. AIC is also calculated for each realization using the 1974 TIMSAC program. The W-statistic defined in Section 3 was calculated using $w_c = 1$, $w_z = 1$, $c_0 = c_1 = 1$, $c_2 = .8$, $c_3 = .6$ and $z_1 = 1$, $z_2 = .8$, $z_3 = .6$ to allow for slight deterioration of the pattern, i.e. beginning with c_1 and z_1 the constants were taken as simply linear weights. For each realization, the identification procedures selected from the rectangular array of possible orders, $p = 1, \ldots, 6$ and $q = 0, \ldots, 3$. The top 3 models as indicated by an identification procedure were identified for each realization. For simulation from a particular model we table the percentage of realizations for which the true model was correctly identified and, in parentheses, the percentage of realizations for which the true model was among the top three choices for each of the identification techniques.

(a) ARMA(3,2) Simulation Results

Woodward and Gray (1981) presented a YW GPAC array for a realization of length 300 from the ARMA(3,2) model considered earlier in Example 3.1 and showed that the GPAC pattern indicating an ARMA(3,2) was discernible. Based upon the simulation study described in Section 3, Davies and Petrucelli (1984) claimed that realizations from the model in Example 3.1 for which the ARMA(3,2) identification can be made are "the exception rather than the rule." They concluded that even for realizations of length 500, the corresponding standard deviations were comparatively high. As we have already pointed out, the standard deviations they spoke of are not really relevant as this section will demonstrate. We re-examined the model identification capabilities of the GPAC in this setting using the W-statistic to measure the pattern.

In Table 4.1 we display the results of our simulation investigation. In the table we indicate the percent of realizations for which an ARMA(3,2) was selected as the first choice by the W-statistic and the percent for which an ARMA(3,2) was in the top three choices for various realization lengths. For comparison, we also display the model identification results using AIC. From the table it can be seen that in fact with n=300 the W-statistic correctly identified the model about 70% of

Table 4.1 Percent Correct Classification for

$$(1-1.5B + 1.21B^2 - .46B^3)X_t = (1 + .2B + .9B^2)a_t$$

Realization Length		500	300	200	100	50
TT GPAC	1st Choice	81	68	62	30	16
	Top 3	92	82	73	38	27
YW GPAC	1st Choice	79	71	56	20	9
	Top 3	94	83	75	33	23
AIC	1st Choice	73	80	76	45	15
	Top 3	88	91	91	74	39

the time while with n=500 the W-statistic gave approximately 80% correct classification using the TT and YW GPAC. The GPAC based procedures picked an ARMA(3,2) as one of the top three choices slightly over 80% of the time for n=300 and over 90% of the time at n=500. These results dispell the notions that the realization given by Woodward and Gray (1981) was an "exception" and that the "comparatively high" variability in GPAC elements for n=500 is too large for model identification. The GPAC results for n=300 and n=500 were similar to the AIC results shown in the table with the GPAC showing a slight advantage at n=500 and AIC having somewhat better identification at n=300. Also shown in Table 4.1 are the identification results for n=200, 100 and 50. There we see that n=50 is too small for identification of this model by either AIC or GPAC. At n=200 and n=100 AIC does better at identifying the ARMA(3,2) and at picking the ARMA(3,2) as one of the top three models. As we shall see in the sections that follow, this is not a general result, i.e. AIC will not always outperform the W-statistic on small samples.

Before terminating the discussion here, we consider the computation times involved in performing the identification routines. It should be noted that in calculating computation times, we have not included the computation of sample autocorrelations and autocovariances. Including these calculations would have increased the times for YW GPAC and AIC. In the simulations summarized in Table 4.1, the GPAC procedures proved to be much faster than AIC. The YW GPAC required approximately .02 seconds per realization, independent of realization length. The TT GPAC required about .1 seconds for every 50 observations in a realization, i.e. the computation time for n=50 was about .1 seconds, for n=300 about .6 seconds, etc. On the other hand, the TIMSAC-74 version of AIC required around 25 seconds per realization for the simulations described in Table 4.1.

It is informative at this point to compare the use of GPAC and AIC for model identification. Both procedures essentially involve the calculation of a statistic based upon the parameter estimates for each candidate model. However, the GPAC using either TT or YW, does not require estimation of the moving average parameters while AIC does. Estimation of the moving average parameters is difficult, especially near the noninvertible region. Since GPAC avoids this estimation, satisfactory parameter estimates can be obtained without the necessity of using time consuming maximum likelihood and approximate maximum likelihood routines.

(b) Simulations from ARMA Models

(i) Re-examination of the Davies and Petrucelli Models

Davies and Petrucelli (1984) report extensive simulations based on the ARMA models of (i) and (ii) in (4.1). They report that model identification was not possible for realization lengths even as large as 200. They did not suggest alternative approaches. First it should be pointed out that examination of these two models reveals the source of the model identification difficulties. Model (i) is obviously nearly white noise since there is near cancellation of the two operators. The true autocorrelations, ρ_k, for Model (i) are shown in Table 4.2 for k= 1, 2, ..., 8, and they confirm our observations.

Table 4.2 True Autocorrelations for the ARMA(1,1) Model

$$(1 - .4B)X_t = (1 - .7B)a_t$$

k	ρ_k	k	ρ_k
1	-.232	5	-.006
2	-.093	6	-.002
3	-.037	7	-.001
4	-.015	8	-.000

It is clear that realizations from (i) will be difficult to distinguish from white noise unless realization lengths are exceptionally long.

Model (ii) also has some "near cancellation" which is evident when $(1+B-.4B^2)$ is factored as $(1-.3062B)(1+1.3062B)$. Another problem with this model which is clear in this factored form is that this ARMA(1,2) model is actually non-invertible. The invertible process with the same autocorrelations as the Davies and Petrucelli model has the moving average operator

$$\theta(B) = (1-.3062B)(1+\frac{1}{1.3062}B)$$

$$= (1-.3062B)(1+.7656B) \ .$$

$$= 1 + .4594B - .2344B^2$$

Thus we will consider realizations from the invertible ARMA(1,2) model

$$(1-.5B)X_t = (1 + .4594B - .2344B^2)a_t \tag{4.2}$$

rather than the one in (4.1). Realizations from this model should behave much like those of an MA(1) or MA(2) due to the nearly cancelling factor. Thus it is doubtful that any of the existing techniques could claim a better model for this data than an MA(1) or MA(2) without a fairly large sample. The true autocorrelations for the model in (4.2) (which are the same as those for Model 4.1(ii)) are shown in Table 4.3. These indicate that only ρ_1 is substantially different from zero but that it is too large for a true MA(1) but not for an MA(2), demonstrating why an MA(1) or MA(2) is likely to be chosen (and it is not a poor model for this data) unless the sample size is quite large.

Table 4.3 True Autocorrelations for the ARMA(1,2) Model
$(1 - .5B)X_t = (1 + .4594B - .2344B^2)a_t$

k	ρ_k	k	ρ_k
1	.617	5	.024
2	.191	6	.012
3	.096	7	.006
4	.048	8	.003

Realizations of length n=100 and n=200 were simulated from Model 4.1(i) and (4.2). As would be expected from Table 4.2, many realizations from model 4.1(i) are not distinguishable from white noise. Specifically, applying the Ljung-Box-Pierce statistic (see Ljung and Box 1978) to these realizations using a 5% level of significance and 25 sample autocorrelations, we failed to reject white noise 73% of the time for n=100 and 45% of the time for n=200. Actually, the analysis of any time series should begin with a preliminary test for white noise. Appropriate model identification would involve finding models for only the realizations for which white noise is rejected. Also, Model 4.2 is poorly identified as an ARMA(1,2) by both GPAC and AIC procedures as demonstrated by the fact that neither procedure identified an ARMA(1,2) model as the first choice more than 15% of the time for n=100 or n=200. Although the GPAC array can be used to detect pure MA models, it has always been recommended by the authors that one should inspect the sample autocorrelations before examining the GPAC. If this is done, in the case of the MA process there is no need to consider the GPAC. Thus a more proper use of these methods would probably have resulted in selection of an MA(1) or MA(2) model. Allowing pure MA models using AIC may have resulted in AIC model identification as an MA(1) or MA(2).

(ii) Simulations from Other ARMA Models

From the preceding observations we conclude that the conclusions drawn by Davies and Petrucelli on the ARMA(3,2) model of Example 3.1 are incorrect and that the criticism of the GPAC on the basis of the ARMA(1,1) and ARMA(1,2) models in (4.1) are probably valid for any current model identification method. Thus we believe their results provide little or no evidence concerning the value of the GPAC as a method for determining a proper ARMA model. We conclude this section by examining the GPAC array using more reasonably selected ARMA models. In Table 4.4 are listed 10 ARMA models on which this simulation analysis will be based. In this table we show the factors of the autoregressive and moving average operators. In Table 4.5 we show the results of the simulation. As before, for each model we table the percent of the realizations for which each procedure identified the true model and the percent for which the true model was one of the top three choices.

Several observations can be made from this table. First, the consistency of the GPAC results is demonstrated by the fact that in all cases, better identification was obtained using n=200 than for n=100 or n=50. Due to the consistency properties

Table 4.4 ARMA Models Used in Simulations - Showing Factors

	Model	AR Factors	MA Factors
(a)	$(1-1.3B + .7B^2)X_t = (1 + .7B)a_t$	$1-1.3B + .7B^2$	$1 + .7B$
(b)	$(1-1.3B + .7B^2)X_t = (1 - .7B)a_t$	$1-1.3B + .7B^2$	$1 - .7B$
(c)	$(1 - .1B - .56B^2)X_t = a_t$	$1-.8B$ $1+.7B$	
(d)	$(1 - 1.5B + 1.21B^2 - .46B^3)X_t = a_t$	$1-.79B+.65B^2$ $1-.71B$	
(e)	$(1 - .96B^2 + .64B^4)X_t = a_t$	$1-1.6B + .8B^2$ $1+1.6B + .8B^2$	
(f)	$(1 - .96B^2 + .64B^4)X_t = (1 + .8B^2)a_t$	$1-1.6B + .8B^2$ $1+1.6B + .8B^2$	$1 + .8B^2$
(g)	$(1 + B + .99B^2)X_t = a_t$	$1+B+.99B^2$	
(h)	$(1 + B + .99B^2)X_t = (1-.7B)a_t$	$1+B+.99B^2$	$1 - .7B$
(i)	$(1-2.5B+2.96B^2-1.75B^3+.49B^4)X_t = a_t$	$1-1.3B+.7B^2$ $1-1.2B+.7B^2$	
(j)	$(1-2.5B+2.96B^2-1.75B^3+.49B^4)X_t = (1+.7B)a_t$	$1-1.3B+.7B^2$ $1-1.2B+.7B^2$	$1 + .7B$

Table 4.5 Simulation Results for Models of Table 4.4

Percentage of Realizations Correctly Identified

Model	n = 50			n=100			n= 200		
	TT	YW	AIC	TT GPAC	YW GPAC	AIC	TT GPAC	YW GPAC	AIC
(a)	33	13	27	53	31	59	71	50	52
	(65)*	(56)	(81)	(78)	(73)	(83)	(89)	(81)	(75)
(b)	24	25	50	35	40	61	64	63	60
	(60)	(50)	(77)	(68)	(67)	(81)	(86)	(88)	(75)
(c)	43	44	51	73	69	54	83	82	58
	(66)	(65)	(76)	(90)	(90)	(74)	(93)	(94)	(71)
(d)	44	38	47	54	45	51	78	77	58
	(55)	(53)	(76)	(67)	(69)	(74)	(91)	(91)	(72)
(e)	82	80	72	92	95	64	98	97	53
	(98)	(96)	(85)	(100)	(100)	(79)	(100)	(100)	(78)
(f)	29	8	17	58	36	56	85	73	76
	(72)	(48)	(43)	(94)	(89)	(86)	(99)	(98)	(91)
(g)	92	70	44	98	74	44	99	84	40
	(100)	(99)	(65)	(100)	(99)	(56)	(100)	(100)	(58)
(h)	18	33	42	33	29	45	72	38	54
	(45)	(81)	(98)	(79)	(80)	(91)	(94)	(88)	(93)
(i)	51	3	4	69	1	5	81	8	6
	(70)	(6)	(19)	(88)	(8)	(19)	(95)	(17)	(18)
(j)	6	3	6	16	5	6	48	5	4
	(26)	(13)	(13)	(36)	(11)	(18)	(80)	(12)	(13)

* The values in parentheses are the percentage of realizations for which the true model was selected as one of the top three choices.

of the Tsay-Tiao (for both stationary and nonstationary series) and of Yule-Walker estimates (for stationary time series), the GPAC arrays are "consistent" for the true pattern which results in a W-statistic of zero, the minimum value attainable by the W-statistic. However, the AIC is well known to not be consistent, a property which is demonstrated in the table by the fact that its identification results are not generally any better at n=200 than at n=100.

Secondly, the GPAC techniques clearly tend to outperform AIC for purely autoregressive processes. For ARMA processes in which there is no near cancellation of operators (such as (a), (h) and (j)), the GPAC procedures tend to perform better than AIC at n=200, although for model (h) the AIC performance is substantially better at n=50. On the other hand, when there is near cancellation, the AIC tends to outperform GPAC. Model (a) in Table 4.4 is an ARMA(2,1) example used by Brockwell and Davis (1987) to demonstrate the performance of AIC. Clearly, there is no near cancellation in this model. For model (a) the TT GPAC and AIC perform similarly for n=50 or n=100 with the TT GPAC providing better identification at n=200. Model (b) in Table 4.4 is a modification of the Brockwell model in which the moving average factor is changed so that the factors more nearly cancel. This can be seen by noticing that the roots of $1\text{-}1.3B\text{+}.7B^2\text{=}0$ are $.93\pm.75i$, i.e. they have positive real part. The roots of $1+.7B=0$ and $1-.7B=0$ are -1.43 and 1.43 respectively so that the root of $1-.7B=0$ is closer in the complex plane to $.93\pm.75i$. When models (a) and (b) are inverted into their infinite order autoregressive form, it can easily be seen that model (b) is more nearly an AR(1) model. In Table 4.5 we see that for n=100, the AIC results are better than those of GPAC while for n=200 the GPAC provides identification comparable to or better than AIC. For most of the ARMA(p,q) models considered with q>0, the GPAC procedures compared favorably with AIC.

Another point of interest from the tables is that TT GPAC results are typically better than those for the YW GPAC. The difference seems to be large in the presence of roots near the unit circle, i.e. (g) and (h), as would be expected. Also of interest are models (i) and (j) which contain nearly repeated roots. There we see the TT GPAC clearly outperforming AIC and YW GPAC although n=100 seemed to be too small for identification in model (j).

In models (e), (g), (i) and (j) the TT GPAC clearly outperforms AIC. Although the ARMA(4,0) was the most popular choice by AIC for realizations from the AR(4) model in (e), several realizations were mismodeled as an ARMA(4,q), $q \neq 0$ or ARMA(5,q). For the nearly nonstationary AR(2) model in (g), AIC tended to model the series as an ARMA(2,1) almost as often as it did as an AR(2). Similarly, when the YW GPAC did not identify an AR(2) for these realizations, it usually selected an ARMA(2,1). For the AR(4) model of (i) with nearly repeated roots, AIC tended to identify realizations of length 50 or 100 as having autoregressive order 2 or 3, and when n=200 AIC tended to identify AR(5) or ARMA(4,q) models for varying values of q. For realizations of length 50 or 100 from the ARMA(4,1) model of (j), none of the techniques gave satisfactory results. However, for n=200 the TT GPAC had results far superior to those of AIC with AIC selecting ARMA(4,q) and ARMA(5,q) models for varying values of q. Simulations from model (j) were also run for n=300, in which case an ARMA(4,1) was identified as the top model for 71% of the realizations using the TT GPAC while YW GPAC and AIC performance was no better than it was at n=100 and n=200. It is obvious that the approximate MLE routine used by Akaike in TIMSAC-74 is not performing well in the presence of the nearly repeated roots. Improved MLE routines in more recent implementations of AIC may provide better AIC results. The YW GPAC tended to select second order models for the models of (i) and (j).

Finally we consider the ARMA(4,1) model considered previously in Example 3.2, and in Table 4.6 we show the model identification results analogous to those of Table 4.5 following the row heading "No Prefiltering." As in Table 4.5, the percent correct classification is given in the table with the number of times the correct model was among the top three choices given in parentheses. It can be seen that none of the model identification procedures performed well, as would be expected due to the near nonstationary factor. The TT GPAC results were poor but were better than those for the YW GPAC. This is consistent with the discussion in Example 3.2. Gray and Woodward (1986) proposed a method for removing nonstationarities by fitting the data with a high order autoregressive model, calculating a factor table associated with this model and then transforming the data by factors associated with roots near the unit circle. In order to demonstrate that this procedure can be used effectively in an automated mode in this setting, we re-examined these simulations. For each of the 100 realizations reported above, we overfit the data with a 10th order autoregressive model and then transformed the data by any factors for which the absolute value of the reciprocal of the associated root(s) was greater than a threshold chosen here to be .95. The resulting transformed series was then modeled. The purpose of this procedure was to remove the nonstationary component, leaving a stationary ARMA(2,1) process to be modeled. When a second order nonstationarizing filter was used and the transformed data was modeled as an ARMA(2,1), then the procedure models the data as an ARMA(4,1). The results of the simulation are shown in Table 4.6 in the section titled "Prefiltering." Note that although we have explained this procedure in a stepwise manner, the prefiltering and subsequent modeling was done in a single pass through the data. There it can be seen that the prefiltering improved the identification performance markedly. Notice that after the prefiltering, the TT GPAC and YW GPAC results are very similar. The AIC routine was not run with the prefiltering, but such a procedure could also be applied with it.

Table 4.6 Percent Correct Classification for

$(1 - 1.8B + 2.29B^2 - 1.292B^3 + .495B^4)X_t = (1 - .7B)a_t$

n=300

	TT GPAC	YW GPAC	AIC
No Prefiltering	25 (61)	1 (10)	9 (32)
Prefiltering	57 (83)	58 (83)	--

(c) GPAC on Real Data

Due to the concern expressed by Davies and Petrucelli that the GPAC patterns are not clear for real data, especially when "known" models have moving average terms, we compare the performance of the GPAC array as measured by the W-statistic on real data. We have examined the following series considered by Davies and Petrucelli, i.e. Series A, Series C, Series E (sunspot data) and Series J (output series) from Box and Jenkins (1975), and the metals data from Makridakis (1978).

In Table 4.7 we show models identified by Box and Jenkins, AIC and the W-statistic for Series A, C and E. The AIC results are those obtained using TIMSAC-74 with the rectangular search of model orders. The AIC and W-statistic results in Table 4.6 are based on model selection for $p \leq 8$ and $q \leq 2$. Box and Jenkins identified two possible models for Series A, C, and E. These are displayed as BJ1 and BJ2. Whenever they differenced the data to form an ARIMA(p,d,q) model, we have expressed this in Table 4.6 as an ARMA(p+d,q) model for notational consistency. We also show the top three choices found by AIC and with the W-statistic (on the TT GPAC array). These are denoted AIC1 - AIC3 and W1 - W3 respectively and are given in order of preference. The series were also modeled using overfitting with a 10th order autoregressive and prefiltering on the basis of a threshold of .95. The model identification results based on the W-statistic after the prefiltering are denoted W1P-W3P.

Table 4.7 Modeling of Real Data Series

	BJ1	BJ2	AIC1	AIC2	AIC3	W1	W2	W3	W1P	W2P	W3P
Series A	(1,1)	-----	(1,1)	(2,1)	(1,2)	(7,0)	(1,1)	(7,1)	(7,0)	(1,1)	(7,1)
Series C	(2,0)	(2,2)	(3,1)	(2,2)	(4,1)	(2,0)	(1,0)	(2,1)	(2,0)	-----	-----
Series E	(2,0)	(3,0)	(2,0)	(2,1)	(3,0)	(2,0)	(8,0)	(2,1)	(8,0)	(5,0)	(7,2)

On the basis of these results it appears that the GPAC array obtains models consistent with those of previous techniques. A few comments are in order here. Davies and Petrucelli (1984) quote AIC models obtained by Ozaki (1977). These differ in some cases from the AIC models obtained here. For example, for Series A, Ozaki shows the ARMA(7,0) model to be the optimal for our range of orders which is consistent with the W-statistic choice. Our implementation of AIC chooses the ARMA(7,0) as best of the pure AR models considered. CAT (see Parzen 1974) and FPE (see Akaike 1969) also select an AR(7) model for series A. It should be noted that our overfitting detected no near nonstationarities so that the data was not prefiltered. Thus W1P-W3P are the same as W1-W3.

Ozaki finds the ARMA(2,0) model for Series C to be second best. Overfitting as we did for Series A results in a second order prefilter with essentially the same coefficients as the ARMA(2,0) model chosen by the W-statistic. Applying the Ljung-Box-Pierce test to the transformed data, we failed to reject white noise at the 5% level, and thus the modeling procedure is terminated. This is indicated in Table 4.7 by the fact that no models are given for second and third choice after prefiltering. However, it is interesting to note that examination of Series C reveals a seeming discontinuity around the 60th data value. Since this is data collected every minute, it seems that there is some indication that an adjustment may have been made after the first hour causing an ARMA model to not be an appropriate model for the data.

It should be noted that the ARMA(2,0) model for Series E is a poor model (see Woodward and Gray 1978) which arises from the fact that the ARMA model which best fits the sunspot data has a pair of complex roots close to the unit circle. The ARMA(2,0) behavior is reflecting this near nonstationarity rather than the complete model. Overfitting results in a second order near nonstationary prefilter. As indicated in Table 4.7, applying this prefiltering to the data results in an ARMA(8,0) as the first choice of the W-statistic. Ozaki finds the ARMA(8,0) as optimal among our range of orders.

The 13th order models suggested by Parzen (1979) and Woodward and Gray (1981) for the metals data are excluded from consideration in the model ranges considered for the models in Table 4.7. If the range is changed to $p \leq 15$ and $q \leq 2$, the ARMA(13,1) model considered by Woodward and Gray (1981) is the second W-statistic choice. It should be pointed out that overfitting did not identify any near nonstationary components.

For Series J, Box and Jenkins obtained an ARMA(4,2) model while Kitigawa (1977) used AIC to obtain an ARMA(3,2) model. Davies and Petrucelli claimed that the inspection of the GPAC array for Series J did not "confirm either model as being appropriate." In Table 4.8 we show the YW GPAC array to which Davies and Petrucelli referred.

Table 4.8 Yule-Walker GPAC for Series J

Autoregressive Order

		1	2	3	4	5	6
	0	.971	-.804	.188	.260	.059	-.063
Moving	1	.923	-.723	1.259	.220	.332	-.076
Average	2	.885	-.563	.541	-.074	.101	.164
Order	3	.858	-.374	.497	.912	.185	.109
	4	.845	.009	.553	-.427	.630	.431
	5	.845	-53.828	.552	.770	-.076	.392

The ARMA(3,2) behavior in this array is very clear with $\phi_{33}^{(j)} \approx .54$ for $j \geq 2$ and with $\phi_{kk}^{(2)}$ small for $k \geq 4$. Thus, one would clearly pick an ARMA(3,2) as the only possible chioce from this table. Moreover, the W-statistic overwhelmingly selects an ARMA(3,2). Table 4.9 shows the W-statistics associated with model orders p=1, ... ,6 and q=0, ... ,5. The W-statistic picked an ARMA(3,2) as the first model choice by a wide margin. It should be mentioned that the "constant" behavior in column one may have caused some confusion. Since there is no accompanying zero behavior, this should lead to no difficulty although it does suggest the presence of a root in the neighborhood of the unit circle. However, using a 10th order overfit autoregressive model with threshold of .95, we detect no components sufficiently close to the unit circle to warrant filtering the data. The point is that this is a detectable pattern which can easily be measured and identified using statistics such as the W-statistic, contradicting the claim of Davies and Petrucelli.

Table 4.9 Array of W-statistic Values for Series J
Based on the GPAC Array of Table 4.8

Autoregressive Order

		1	2	3	4	5	6
	0	.641	.540	.932	1.304	.940	8.130
Moving	1	1.020	2.347	.773	3.906	1.211	3.125
Average	2	.577	20.833	.253	2.398	1.647	1.067
Order	3	.701	23.256	1.179	1.866	1.942	1.560
	4	.464	25.000	1.155	2.008	10.309	3.289
	5	40.000	24.390	1.506	10.526	71.429	16.667

5. Concluding Remarks

In this paper we have examined the potential of the GPAC array for ARMA model identification. We have shown that its performance can be expected to be much better than that indicated by the results of Davies and Petrucelli. This is also consistent with the empirical experience of the present authors using GPAC arrays. It should be pointed out that the GPAC method, as well as most other model identification methods, should be used with care when some roots of the characteristic equation are close to the unit circle. We recommend a prefiltering procedure such as the one we applied in Table 4.6 as a preliminary step in any identification procedure so that the nonstationary components of the model do not mask the stationary portion.

Although we feel that the best use of the GPAC is via inspection by the analyst, we have introduced the W-statistic here to eliminate the subjectivity of the method for comparison purposes. The choice of constants in the simulation analyses reported here have not been shown to be optimal and may be able to be improved. In fact, the general form of the W-statistic in (3.4) may not be the best way to measure the pattern in the array. Our results simply show that recognizable patterns do exist in GPAC arrays and the W-statistic is a measure of these patterns. As such, the W-statistic should be viewed as a guide for helping the analyst to consider the most suggestive patterns. The results in Table 4.5 indicate that the GPAC can be used successfully to identify a wide variety of ARMA models. In many instances the identification is far superior to that using the TIMSAC-74 version of AIC. Another important point is that a GPAC analysis can be performed in a fraction of the time required by AIC. We have picked the TIMSAC-74 version of AIC as an example of an automatic model identification procedure in wide use. Improvements have been made in AIC resulting from better maximum likelihood estimation, consistency has been attained using BIC, etc. Our purpose in comparing our results with this version of AIC has not been to claim that our method is necessarily superior, but that it indeed does identify ARMA models competitively with the established techniques in far less computing time.

Finally, we now recommend using Tsay-Tiao estimates of the autoregressive coefficients for purposes of calculating GPAC arrays from data. Arrays based on unconditional ML estimates have not been examined in this report. Such ML GPAC arrays based on good starting values (Tsay-Tiao, etc.) may be the optimal approach if time is not a factor. However, the fact that the current implementations of GPAC using TT and YW estimates do not require estimation of the moving average parameters is very appealing.

In conclusion, we believe that the assertion by Davies and Petrucelli that the GPAC is not useful for ARMA(p,q) models with $q>0$ is unfounded and that GPAC performs well on real data.

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