Analysis of Data from Censored Samples

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CHAPTER 11

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11.1 INTRODUCTION

In this chapter we consider a variety of techniques which are appropriate as tests of fit when only a certain portion of the random sample from a continuous underlying distribution is available. The censoring or deletion of observations can occur in several ways. The type or manner of censoring determines the appropriate method of analysis.

The most common and simple censoring schemes involve a planned limit either to the magnitude of the variables or to the number of order statistics which can be observed. These are called singly Type 1 and Type 2 censored data, respectively. The number of small (or large) order statistics which will be observed in Type 1 censoring is a random variable. In life testing applications it is quite common for an experiment to produce a Type 1 right censored sample by having n items placed on test and recording the values $0 < Y_{(1)} \ldots < Y_{(r)}$ of the failure times which are observed up to a fixed test time. (In this chapter observations will be referred to as Y, rather than X, since in plotting techniques we shall wish to plot observations on the vertical, or y-axis). Data arising from such a procedure

are occasionally also referred to as being truncated. If the life test is planned to continue until a fixed number, r, of failures occur, then the resulting failure data are Type 2 right censored. As another example, if one records only the 10 largest independent competitive bids on an oil lease, the observed sample is singly Type 2 censored on the left. Types 1 and 2 censoring are sometimes referred to as time censoring and failure censoring respectively.

In the more complicated situation in which the variables are subject to different censoring limits the sample is said to be multiply censored. If the different censoring limits are preplanned, as would result from placing items on a life test at different starting times with a single fixed termination time for the test, the data are progressively censored (Type 1). Samples which are progressively censored (Type 2) occur less often in practice but could result, again in life testing, if the units are put on test at the same time and then selected fixed numbers of (randomly chosen) unfailed items are removed from test immediately after different pre-planned numbers of failures have occurred.

The unplanned type of censored data which arises most often in practice is randomly time censored or arbitrary right censored data. The larger values (again usually in life testing) are not observed due to random censoring times which are statistically independent of the variable of interest (usually failure times). If some of the units are accidentally lost, destroyed or removed from the study prior to the measurement of the variable (failure time) and if these independent censoring times are

recorded then the data can still be analyzed for goodness of fit. In certain situations competing modes of failure will produce randomly censored data (see Example 11.2.3.2.) Combinations of multiply right and left censored data can also arise in practice (see Section 11.2.4)

The graphical technique of examining probability plots (Chapter 2) adapts quite easily to the censored sample situation. Subjective impressions should be formed with somewhat more caution than in the complete sample case, but the computational aspects are essentially unchanged. Probability plots are discussed in Section 11.2.

When the null distribution is completely specified, the probability integral transformation (see Section 4.2.3) may be employed to reduce the problem to a test for uniformity. Section 11.3 presents a number of examples of standard EDF (Chapter 4) goodness-of-fit statistics which have been modified in a straight-forward fashion to accommodate a censored uniform sample. Adaptations for correlation (Chapter 5) and spacings (Chapter 8) statistics are also discussed. For Type 2 censored samples a transformation of the uniform order statistics is described which makes it possible to analyze the data as if it were a complete random sample.

In testing fit, it is a common situation for the null hypothesis to be composite; the hypothesized parent population is not completely specified, but only the form $F(x|\theta)$ of the cumulative distribution function (cdf) is given. Here θ is an indexing parameter; it may be a vector of several components, some known and some unknown. One very natural approach which has been taken in the complete sample case is to replace the unknown

components in θ by efficient estimators (for example, the m.l.e.$\hat{\theta}$) and then to calculate a statistic based on $F(x|\hat{\theta})$ as if it were the completely specified distribution function. This has been done, for example, in many of the tests in Chapters 4 and 5. Censoring presents an extra complication for this approach simply because of increased complexity of efficient estimators of θ. A variety of results for the composite hypothesis problem are examined in the final Section 11.4. Adaptations of the chi-square procedure are not covered in this chapter. For some discussion on this topic, see Section 3.4.2.

11.2. PROBABILITY PLOTS.

Probability plotting has been described in Chapter 2 as a valuable technique for assessing goodness of fit with complete samples. This extends naturally to incomplete samples for most types of censoring. Even in the case of multiple censoring a probability plot can often be constructed quickly using only ordinary graph paper and a hand calculator.

In Section 11.2.1, the construction of probability plots for complete samples is reviewed. The method is extended to singly-censored samples in Section 11.2.2, to multiply right-censored samples in Section 11.2.3, and to other types of censoring in Sections 11.2.4-11.2.6. An easy-to use summary of the steps required in constructing a probability plot is given in Section 11.2.7.

11.2.1 Complete Samples

Let $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ be a complete ordered random sample of size n and let $F(y|\mu,\sigma)$ be the corresponding cdf where μ and σ are unknown location and scale parameters respectively. (Note that μ and σ are not necessarily the mean and standard deviation.) When there is no ambiguity $F(y|\mu,\sigma)$ will be shortened to $F(\cdot)$ or F.

Since μ and σ are location and scale parameters, we can write (as was done in Formula (2.9))

$$F(y|\mu,\sigma) = G\left(\frac{y-\mu}{\sigma}\right) = G(z) , \qquad (11.1)$$

where $Z = (Y-\mu)/\sigma$ is referred to as the standardized variable and $G(z)$, also referred to as $G(\cdot)$ or G, is the cdf of the standardized random variable. Using obvious notation, it follows that, using E for expectation or mean,

$$E\{Y_{(i)}\} = \mu + \sigma E\{Z_{(i)}\} = \mu + \sigma m_i ,$$

where $Z_{(i)}$ is the ith order statistic from the standardized distribution, and m_i is $E\{Z_{(i)}\}$. Similarly, for $0 \leq p_i \leq 1$,

p_i-th quantile of $F(y;\mu,\sigma) = \mu + \sigma \{p_i$-th quantile of $G(z)\}$

$$= \mu + \sigma \lfloor G^{-1}(p_i) \rfloor ,$$

where G^{-1} is the inverse function of G.

We can regard $Y_{(i)}$ as an estimate of its mean, or of the p_i-th quantile of $F(y;\mu,\sigma)$, where p_i is an appropriate probability. In constructing a probability plot we could plot the $Y_{(i)}$ on the y-axis versus m_i on the x-axis. If the sample is in fact from $F(y;\mu,\sigma)$ then the points will tend to fall on a straight line with intercept μ and slope σ. We then test our distributional assumption by visually judging the degree of linearity of the plotted points. Methods based on regression and correlation are discussed in Chapter 5.

It should be noted that if the null hypothesis is simple, that is, the values of all distributional parameters are specified beforehand, we can plot the $Y_{(i)}$ against their hypothesized means and then judge whether the plotted points fall near a straight line with intercept 0 and slope 1.

A drawback to using means of order statistics is that they are often difficult to compute. Quantiles, on the other hand, are easy to compute as long as F is easy to invert. A plot of the sample quantiles $Y_{(i)}$ versus theoretical quantiles of G is a probability plot as defined in Chapter 2; it is also called a quantile-quantile or Q-Q plot (Wilk and Gnanadesikan, 1968). However, the plots will be different from those in Chapter 2 where the observations were plotted on the horizontal or x-axis; here they are plotted on the vertical or y-axis. Special probability plotting paper is available for many families of distributions, but as was stated in Chapter 2 no special graph paper is required if F can be inverted

in closed form or if standard quantiles are available from tables or approximations. Often a scientific calculator and ordinary graph paper is all that are needed.

Table 11.1 lists the cdf's of some common families of distributions along with the formulas required to construct probability plots. The reader is referred to Chapter 2 for further discussion of these distributions. In this context the p_i will be referred to as <u>quantile</u> <u>probabilities</u>.

There is much discussion in the literature over the best choice of quantile probabilities for Q-Q plots (see Kimball (1960) and Barnett (1975)). A frequently used formula is given by $p_i = (i-c)/(n-2c+1)$, where c is some constant satisfying $0 \leq c \leq 1$. The choices $c=0$ and $c=0.5$ (see Chapter 2) are both popular. Here we use $c=0.3175$ since the resulting probabilities closely approximate medians of uniform $(0,1)$ order statistics (Filliben, 1975). This choice has the attractive invariance property that if p_i is the median of the ith order statistic from the uniform $(0,1)$ distribution, then $G^{-1}(p_i)$ is the median of $Z(i)$ and $F^{-1}(p_i)$ is the median of $Y(i)$, for any continuous F. Medians may also be preferred as measures of central tendency since the distributions of most order statistics are skewed. In the examples that follow we will adhere to the convention of choosing $c=0.3175$ unless stated otherwise. Thus we will plot the points

$$\{G^{-1}(p_i) , Y_{(i)}\} , \qquad (11.2)$$

Table 11.1
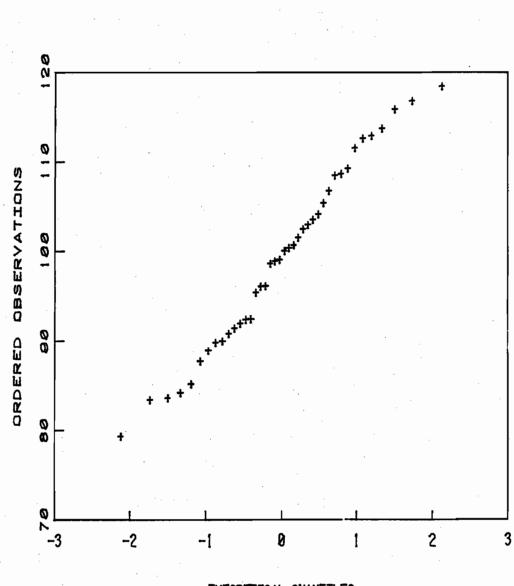### CDFs and Plotting Formulas for Selected Families of Distributions

Distribution*	$F(y)$	Abscissa	Ordinate
Uniform	$\frac{y-\mu}{\sigma}$	p_i	$Y_{(i)}$
Normal	$\Phi\left[\frac{y-\mu}{\sigma}\right]$	$\Phi_i^{-1}(p_i)$	$Y_{(i)}$
Lognormal	$\Phi\left[\frac{\log(y)-\mu}{\sigma}\right]$	$\Phi_i^{-1}(p_i)$	$\log\{Y_{(i)}\}$
Exponential	$1-\exp\left[-\left[\frac{y-\mu}{\sigma}\right]\right]$	$\log[1/(1-p_i)]$	$Y_{(i)}$
Extreme-value	$1-\exp\left[-\exp\left[\frac{y-\mu}{\sigma}\right]\right]$	$\log\{\log[1/(1-p_i)]\}$	$Y_{(i)}$
Weibull	$1-\exp\left[\left[\frac{y}{\sigma}\right]^m\right]$	$\log\{\log[1/(1-p_i)]\}$	$\log(Y_{(i)})$
Laplace	$\begin{cases} \frac{1}{2}\cdot\exp\left[\frac{y-\mu}{\sigma}\right], & y\leq\mu \\ 1-\frac{1}{2}\exp\left[-\frac{y-\mu}{\sigma}\right], & y>\mu \end{cases}$	$\begin{cases} \log(2\cdot p_i), & p_i\leq\frac{1}{2} \\ \log[1/(2-2p_i)], & p_i\geq\frac{1}{2} \end{cases}$	$Y_{(i)}$
Logistic	$1/\left[1+\exp-\left[\frac{y-\mu}{\sigma}\right]\right]$	$\log[p_i/(1-p_i)]$	$Y_{(i)}$
Cauchy	$\frac{1}{2}+\frac{1}{\pi}\cdot\arctan\left[\frac{y-\mu}{\sigma}\right]$	$\tan[\pi\cdot(p_i-\frac{1}{2})]$	$Y_{(i)}$

*Support of each distribution is $(-\infty<y<\infty)$ except for the uniform $(\mu<y<\mu+\sigma)$, lognormal $(y>0)$, exponential $(y>\mu)$, and Weibull $(y>0)$.

where $p_i = (i-0.3175)/(n+0.365)$. The particular choice of quantile pro-
babilities is not crucial since for any reasonably large sample different
choices will have little effect on the appearance of the main body of the
plot. There may be some noticeable differences, however, for extreme order
statistics from long-tailed distributions. (The reader should note that in
Chapter 2 the p_i of (11.2) was symbolized by $F_n(y)$, the empirical distri-
bution function.)

E 11.2.1.1. Uncensored Normal Example. Data for this example consist of
the first 40 values from the NOR data set which were simulated from the
normal distribution with $\mu=100$ and $\sigma=10$. A normal probability plot is
shown in Figure 11.1. The normal distribution provides a good fit to the
data. Note that the intercept and slope of a straight line drawn through
the points provide estimates of the theoretical mean and standard devia-
tion. (The reader should compare Figure 11.1 to Figure 2.15 where the full
NOR data set is plotted with X and Y axes interchanged from Figure 11.1.)

11.2.2 Singly-Censored Samples.

The method of the previous section can be applied directly in any
situation where the data consist of some known subset of order statistics
from a random sample. This is because the available $Y_{(i)}$ are still sample
quantiles from the complete sample and appropriate quantiles of G can be
calculated as before. Although only a portion of the observations from the
hypothetical complete sample can be plotted, the plotted positions of the
uncensored points are the same as when the complete sample is available.

Figure 11.1

Normal Probability Plot of the First 40 Observations
from the NOR Data Set

The only difference is that points corresponding to censored observations do not appear. The simple example of this is the case of a singly-censored sample.

E 11.2.2.1. Right-censored Normal Example. Data for this example consist of the smallest 20 values among the first 40 values listed in the NOR data set. A normal probability plot is shown in Figure 11.2. This plot is merely an enlargement of the lower portion of the plot shown in Figure 11.1.

The plotting procedure is the same for Type 1 as for Type 2 singly-censored samples; however, with Type 1 censoring there is one additional piece of information, namely the censoring time, that can be represented graphically. Suppose we observe the r smallest observations from a random sample of size n. For a location and scale family we plot the points $(G^{-1}(p_i), Y_{(i)})$ for i=1,2,...,r. Now suppose that the censoring is Type 1 and that the observations are all those that are less than some predetermined value t; thus $Y_{(r+1)}$ must be greater than t. This additional information can be given by plotting the point $\{G^{-1}(p_{r+1}),t\}$ with a symbol such as an arrow pointing up, thus indicating the range of possible values for $Y_{(r+1)}$. Nelson (1973) illustrates this technique.

11.2.3 Multiply Right-Censored Samples

The method of probability plotting extends easily to multiply right-censored samples; however, the computation of quantile probabilities is more complicated. For ease of explanation we will first consider the special case of progressive Type 1 censoring, but the methodology can be
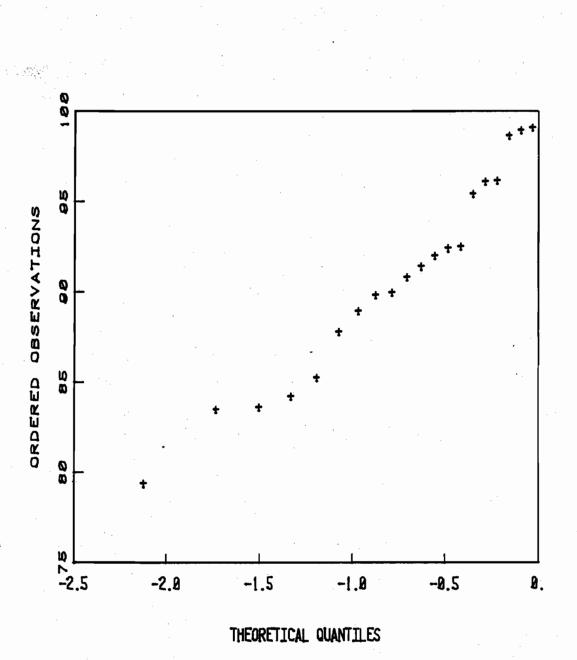
Figure 11.2

Normal Probability Plot of the Smallest 20 of the
First 40 values listed in the NOR Data Set

applied to any multiply right-censored sample. Suppose we place n units on test, using several different starting times, and terminate the experiment at time t. Now let $Y_{(1)} \leq Y_{(2)} \leq \ldots \leq Y_{(n)}$ denote the ordered lifetimes of the n units, some of which are failure times and some of which may be censoring times. If we observe r failures, then (n-r) units are still operating at time t. In this case the observed time to failure $Y_{(i)}$ does not necessarily represent the ith largest observation from the hypothetical complete sample, and $Y_{(i)}$ cannot be regarded as a sample quantile from the complete sample (unless $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ are all failure times).

We still wish to plot the r failure times against theoretical quantiles from G. The question now becomes, what proportion of the population falls below $Y_{(i)}$, or equivalently, what is the value of $F(Y_{(i)}|\mu, \sigma)$. Kaplan and Meier (1958) discuss the maximum likelihood nonparametric estimator of F for the case of a multiply right-censored (Type 1) sample. If S is the set of subscripts corresponding to those units which fail during the course of the experiment, then the Kaplan-Meier (K-M) estimator is given by

$$K\text{-}M\hat{F}_n(y) = 1 - \prod_{\substack{j \varepsilon S \\ j:Y_j \leq y}} \frac{n-j}{n-j+1} \quad .$$

(This estimator is undefined for $y > Y_{(n)}$ if $Y_{(n)}$ is not a failure time).
In the case of a complete sample the K-M estimator reduces to the familiar
EDF $F_n(y)$ = (the number of $Y_{(j)} \leq y$)/n, discussed in Chapter 4. The
estimated probability at the point Y_i provided by the Kaplan-Meier esti-
mator is given by

$$p_i(K-M) = 1 - \prod_{\substack{j \varepsilon S \\ j \leq i}} \frac{n-j}{n-j+1} \tag{11.3}$$

for $i \varepsilon S$. Herd (1960) and Johnson (1964) propose the similar quantile
probabilities

$$p_i(H-J) = 1 - \prod_{\substack{j \varepsilon S \\ j \leq i}} \frac{n-j+1}{n-j+2} \tag{11.4}$$

for $i \varepsilon S$. Implicit in the work of Nelson (1972) are the quantile pro-
babilities

$$p_i(N) = 1 - \prod_{\substack{j \varepsilon S \\ j \leq i}} \exp\left[-\frac{1}{n-j+1} \right] \tag{11.5}$$

for $i \varepsilon S$. Nelson refers to his method as (cumulative) <u>hazard</u> <u>plotting</u>, but
it is equivalent to probability plotting with the above special choice of
quantile probabilities. An algebraic comparison reveals that $p_i(K-M) >$

$p_i(N) > p_i(H-J)$ for all $i\epsilon S$. For a discussion of the properties of the Kaplan-Meier estimator see Peterson (1977). Results by Breslow and Crowley (1974) apply to the Kaplan-Meier estimator and the estimator implicit in the work of Nelson. See Gaver and Miller (1983) for a discussion of the jackknife technique for approximate confidence intervals in this setting. For a complete sample the formulas (11.3), (11.4) and (11.5) for quantile probabilities reduce to i/n, $i/(n+1)$ and $|1 - \exp(-s_i)|$, respectively, where $s_i = \Sigma_{j=1}^{i}(n-j+1)^{-1}$. The choice of probabilities given by

$$p_i(c) = 1 - \frac{n-c+1}{n-2c+1} \prod_{\substack{j\epsilon S \\ j\leq i}} \frac{n-j-c+1}{n-j-c+2} \qquad (11.6)$$

reduces to $(i-c)/(n-2c+1)$ with a complete sample. As a special case, $p_i(c) = p_i(H-J)$ when $c=0$. In the examples that follow we will remain consistent with Section 11.2.1 and use (11.6) with $c=0.3175$ unless stated otherwise. Again for purposes of assessing goodness of fit the particular formulation for quantile probabilities is of little consequence.

E 11.2.3.1 Multiply Right-Censored Example.

Data for this example consist of the 100 observations from the WE2 data set which were simulated from the Weibull distribution with $\sigma=1$ and $m=2$. The data were censored as follows: observations among the first, second, third, and fourth sets of 25 were recorded that were less than 1, 0.75, 0.50, and 0.25, respectively. This type of progressive Type 1 censoring could have occurred if four sets of 25 devices were placed on

test at times 0, 0.25, 0.50, and 0.75 with the experiment terminating at time 1. The 100 values (censored and failed) were ranked from smallest to largest. The 33 failure times are listed in Table 11.2 along with four different choices of quantile probabilities. Of the 67 censored devices, 23, 17, 17, and 10 devices had censoring times of 0.25, 0.50, 0.75, and 1.00, respectively. One purpose of this example is to show how close the agreement can be for different choices of quantile probabilities. Note also the relationship $p_i(K-M) > p_i(N) > p_i(H-J)$. A Weibull probability plot for the data using $p_i(c)$ with $c=0.3175$ is shown in Figure 11.3.
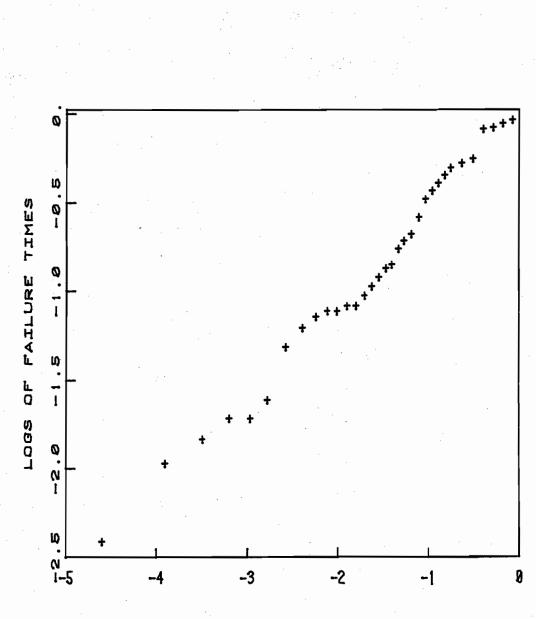
The remarks made in Section 11.2.1 and Chapter 2 concerning the interpretation of probability plots with complete samples hold also for the case of multiple censoring. However, in the case of a multiply right censored sample, the effect of censoring is to increase the variability on the right-hand side of the plot.

For Type 2 multiple right censoring consider the following simple situation. We place n units on a life test and when the rth unit fails we remove all but a fraction ϕ of the remaining working units. We then observe the failure time of those units not removed. In this situation the $p_i(c)$ values can be obtained from (11.6) where $Y_{(1)} < \ldots < Y_{(r)}$ are the first r failure times, $Y_{(r+1)} = \ldots = Y_{(r+(n-r)(1-\phi))}$ are the censoring times of the removed items and $Y_{(r+(n-r)(1-\phi)+1)}, \ldots, Y_{(n)}$ are the failure times of the items that were not removed. The set S in formula (11.6) consists of the indices of the first r failure times and the last $(n-r)\phi$

Table 11.2

Progressively censored Data From the WE2 Data Set

			Quantile Probabilities		
$\underline{i}$	Failure time	K-M	$\underline{N}$	H-J	c = 0.3175
1	0.09	0.010	0.010	0.010	0.007
2	0.14	0.020	0.020	0.020	0.017
3	0.16	0.030	0.030	0.030	0.027
4	0.18	0.040	0.040	0.040	0.037
5	0.18	0.050	0.050	0.050	0.047
6	0.20	0.060	0.060	0.059	0.057
30	0.27	0.073	0.073	0.072	0.070
31	0.30	0.086	0.086	0.086	0.083
32	0.32	0.100	0.099	0.099	0.096
33	0.33	0.113	0.112	0.112	0.109
34	0.33	0.126	0.125	0.125	0.122
35	0.34	0.139	0.139	0.138	0.136
36	0.34	0.153	0.152	0.151	0.149
37	0.36	0.166	0.165	0.164	0.162
38	0.38	0.179	0.178	0.177	0.175
39	0.40	0.192	0.191	0.190	0.188
40	0.42	0.206	0.204	0.203	0.201
41	0.43	0.219	0.218	0.216	0.215
42	0.47	0.232	0.231	0.229	0.228
43	0.49	0.245	0.244	0.242	0.241
61	0.51	0.264	0.262	0.261	0.260
62	0.56	0.283	0.281	0.279	0.278
63	0.62	0.302	0.300	0.298	0.297
64	0.65	0.321	0.318	0.316	0.316
65	0.68	0.340	0.337	0.335	0.334
66	0.71	0.359	0.356	0.353	0.353
67	0.74	0.377	0.375	0.372	0.371
85	0.76	0.416	0.412	0.409	0.409
86	0.78	0.455	0.450	0.446	0.447
87	0.92	0.494	0.488	0.483	0.485
88	0.93	0.533	0.526	0.520	0.522
89	0.95	0.572	0.564	0.556	0.560
90	0.97	0.611	0.602	0.593	0.598

Figure 11.3

Weibull Probability Plot of Progressively Censored
Data from the WE2 Data Set

failure times, and a probability plot can be drawn as described above.
More elaborate Type 2 multiply right-censored samples are handled in the
obvious manner.

E 11.2.3.2 Competing Modes Example.

Data for this example consist of the lifetimes, measured in millions
of operations, of 40 mechanical devices. The devices were placed on test
at different times, and 3 were still working at the end of the experiment.
The data are presented in Table 11.3. Only two modes of failure were
observed: either component A failed or component B failed. These two
components are identical in construction, but they are subject to different
stresses when the device is operated. Thus their life distributions need
not be identical. Quantile probabilities are given in Table 11.3 for the
device as a whole, component A, and component B under the columns headed
"device," "A," and "B," respectively. The data for the device are multiply
censored since the 35th, 36th and 40th ordered lifetimes are incomplete. In
addition, observations on component A are censored by failures of component
B and vice versa. This is an example of random censoring caused by
competing modes of failure.

Probability plots for the individual components were constructed using
several common life distributions. The lognormal distribution seemed to
offer the best fit. Lognormal probability plots are shown for components A
and B in Figure 11.4. The intercepts and slopes of the two lines suggested
by the plots appear to be different. This raises the possibility that,
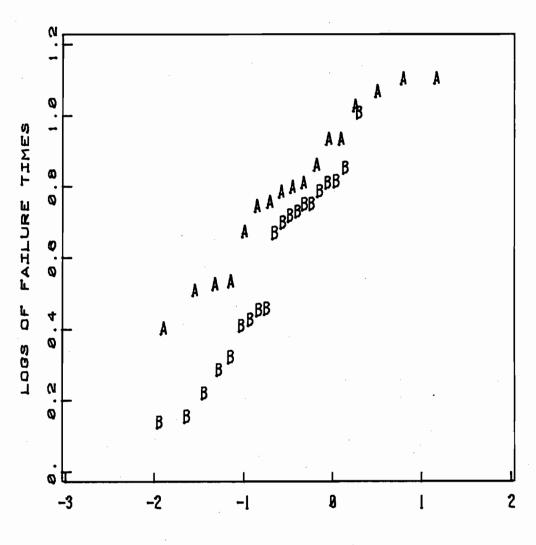while the life distributions of the two components may be of the same

Table 11.3

Life Data for Mechanical Device

_i_	Time	Failure Mode A	Failure Mode B	Quantile Probablities Device	Quantile Probablities A	Quantile Probablities B
1	1.151		✓	0.017		0.017
2	1.170		✓	0.042		0.042
3	1.248		✓	0.066		0.066
4	1.331		✓	0.091		0.091
5	1.381		✓	0.116		0.116
6	1.499	✓		0.141	0.020	
7	1.508		✓	0.166		0.141
8	1.534		✓	0.190		0.167
9	1.577		✓	0.215		0.192
10	1.584		✓	0.240		0.218
11	1.667	✓		0.265	0.052	
12	1.695	✓		0.289	0.084	
13	1.710	✓		0.314	0.116	
14	1.955		✓	0.339		0.246
15	1.965	✓		0.364	0.149	
16	2.013		✓	0.389		0.276
17	2.051		✓	0.413		0.305
18	2.076		✓	0.438		0.334
19	2.109	✓		0.463	0.187	
20	2.116		✓	0.488		0.365
21	2.119		✓	0.512		0.396
22	2.135	✓		0.537	0.228	
23	2.197	✓		0.562	0.269	
24	2.199		✓	0.587		0.430
25	2.227	✓		0.611	0.313	
26	2.250		✓	0.636		0.466
27	2.254	✓		0.661	0.360	
28	2.261		✓	0.686		0.505
29	2.349		✓	0.711		0.544
30	2.369	✓		0.735	0.415	
31	2.547	✓		0.760	0.470	
32	2.548	✓		0.785	0.524	
33	2.738		✓	0.810		0.597
34	2.794	✓		0.834	0.586	
35	2.883	(Working)				
36	2.883	(Working)				
37	2.910	✓		0.870	0.675	
38	3.015	✓		0.905	0.763	
39	3.017	✓		0.941	0.851	
40	3.793	(Working)				

Figure 11.4

Lognormal Probability Plots for Components A and B

family, the distribution parameters may be different. A distracting
feature is the noticeable gap near the center of the plots. The natural
tendency is to expect too much orderliness and to declare that something
unusual has occurred. But such anomalies frequently arise by chance and
should not be taken too seriously. The reader is referred to Hahn and
Shapiro (1967), page 264-265, for an example of a plot in which the same
unusual feature has arisen by chance.

If the life distributions for components A and B are independent and
lognormal, then the life of the device is distributed as the minimum of two
lognormal random variables. For illustration we assume the equality of
parameters. The cdf of the device is then given by

$$F(y|\mu,\sigma) = 1 - \left\{ 1 - \Phi \left[\frac{\log(y)-\mu}{\sigma} \right] \right\}^2 .$$

A probability plot for this distribution is constructed by plotting the
points

$$\{\Phi^{-1}(1 - \sqrt{1-p_i}) , \log(Y_{(i)})\}. \tag{11.7}$$

Such a plot is shown in Figure 11.5. If it is desired to fit different
sets of parameters to the individual components, we can always estimate
them using, say, the method of maximum likelihood. The estimated cdf of the
device, however, would then be difficult to invert. One way around this is
to estimate the probability integral transformation with $F(\cdot|\hat{\mu}_1,\hat{\mu}_2,\hat{\sigma}_1,\hat{\sigma}_2)$
and plot the $F(y_i|\hat{\mu}_1,\hat{\mu}_2,\hat{\sigma}_1,\hat{\sigma}_2)$ versus the p_i. This approach is
described more fully in Section 11.3. Note finally that the derivation of

special theoretical quantiles given in (11.7) would not have been necessary if we had modeled the lifetimes of the components as exponential, extreme-value, or Weibull random variables. This is because the minimum of any number of independent identically distributed random variables from one of these families is also of the same family.

Although the development in the last example is somewhat speculative in nature, it does serve to illustrate the versatility and usefulness of probability plotting, as well as its subjective and limited interpretability.

11.2.4 Other Types of Multiple Censoring

There are other more complicated types of multiple censoring which can arise in practice. A few of these will be discussed below. The thought to keep in mind is that a meaningful probability plot can always be constructed as long as the parent cdf can be estimated.

Occasionally, data arise which are multiply left-censored. If the observations are all multiplied by -1 then the resulting values can be viewed as being multiply right censored. We can now determine quantile probabilities using the formulas of Section 11.2.3. In terms of the subscripts of the original observations, the probabilities $p_i(c)$ given by the formula

$$p_i(c) = \frac{n-c+1}{n-2c+1} \prod_{\substack{j \in S \\ j \geq i}} \frac{j-c}{j-c+1} \quad , \tag{11.8}$$

reduce to $(i-c)/(n-2c+1)$ with complete samples.

A more complicated situation can occur when the data are both <u>multiply right- and multiply left-censored</u>. If all of the left-censored observations are not less than all of the right-censored observations, then quantile probabilities can no longer be calculated using a simple formula. But appropriate probabilities can still be determined as long as the cdf can be estimated nonparametrically. Turnbull (1976) shows how to calculate the maximum likelihood nonparametric estimate of the cdf when the data are arbitrarily right and left-censored, grouped and truncated.

Quantal response data occurs when <u>each observation is either right or left-censored</u>. In the following example the sample size is so small that firm inferences cannot be drawn; however, the example does show how quantal response data can arise, and does serve to illustrate how to construct a probability plot with such data.

El1.2.4.1 Quantal Response Example.

It is desired to investigate the nature of the distribution of the shelf life of a certain electronic set. A total of 47 sets are involved in the study. After $Y_{(i)}$ days on the shelf the ith set is tested and is found to be either good or bad. The set is never observed again. Thus a good set constitutes a right censored observation whereas a bad set constitutes a left censored observation. The number of days on the shelf at the times of test are as follows with failures indicated by an asterisk: 20,22, 23,25,26,27,28,29*,30,31,37,37,37,41,42,43,62,69,69,78,92,92,93,114,117, 124*,128*,130,136,151,211,226,231,242,244,244,244,244,245*,245,245,250,
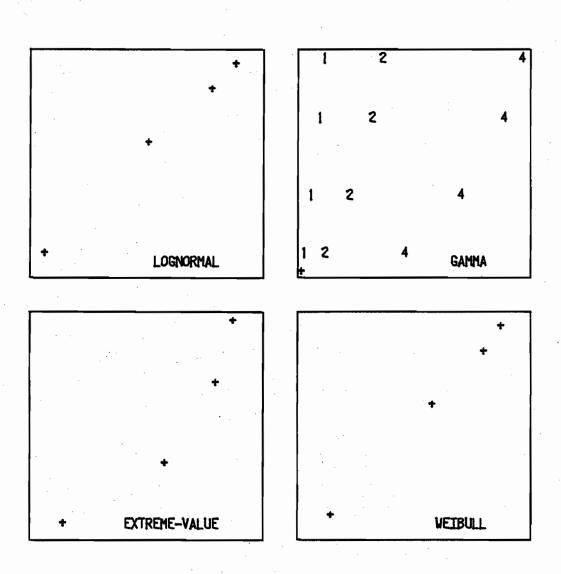
259*,259,287,317, and 340 days. Using the recursive algorithm given by Turnbull (1976), the maximum likelihood nonparametric estimate of the cdf is found to be

$$
F_n(y) = \begin{cases}
0 & -\infty < y < 28 \ , \\
\text{undefined} & 28 \leq y \leq 29 \ , \\
.056 & 29 < y < 117 \ , \\
\text{undefined} & 117 \leq y \leq 124 \ , \\
.143 & 124 < y < 244 \ , \\
\text{undefined} & 244 \leq y \leq 245 \ , \\
.222 & 245 < y < 340 \ , \\
\text{undefined} & 340 \leq y < \infty \ .
\end{cases}
$$

Four values of y were selected for purposes of probability plotting: 28.5, 120.5, 244.5, and 340. The first three are the midpoints of the three closed intervals which are assigned probability, and the last is the largest value for which $F_n(y)$ is defined. The four probabilities used are 0.028, 0.099, 0.127, and 0.222. The first three are the midpoints of the jumps and the fourth is equal to $\hat{F}_n(340)$. Probability plots are shown in Figure 11.5 for four families commonly used to model lifetimes. The lognormal, gamma (with origin 0 and shape near 1) and Weibull distributions all appear to fit the data well. These results are not inconsistent since the gamma distribution described (exponential distribution with origin zero) is a member of the Weibull family, and the lower portions of the Weibull and lognormal cdfs are very similar.

Figure 11.5

Probability Plots for the Shelf Life of Electronic Sets
(gamma shape parameter = 1,2, and 4; origin for gamma
plots is shown as "+")

Any conclusions, however, are highly tentative because of the small sample size and the severity of the censoring. If we use a jackknife technique or the theory of m.l.e. (see Turnbull, 1976) to estimate the variances of probabilities assigned to each of the four intervals, it then appears that none of the models considered in Figure 11.5 can be soundly rejected.

Grouping is perhaps the most common form of censoring encountered in practice. Each grouped observation is both right and left-censored. Quantile probabilities can be calculated using the formula for a complete sample. One approach to constructing a probability plot is to represent each observation with the endpoint (or midpoint) of the interval in which it falls. The resulting plot will have a stairstep appearance with the number of steps equal to the number of groups. One advantage of this approach is that the sample size is evident. A simplification is to plot only one point per group.

11.2.5 Proportional Hazards

A quasi-nonparametric method for analyzing survival data was proposed by Cox (1972,1975). The method is parametric in that it is assumed that the hazard functions for the observations are all proportional. But the method is nonparametric in that no prior restrictions are placed on the form of the hazards (and hence the cdfs). The cdf for a particular observation is estimated using all the data. This estimate then provides the appropriate quantile probabilities for purposes of probability plotting.

The Weibull (or exponential) family is the only family for which it makes sense to construct a probability plot after having assumed the proportional hazards model. This is because, for the Weibull family, a multiplicative change in the hazard function is equivalent to a change in the scale parameter. Thus it does not matter which cdf is estimated since the resulting probability plots will differ only in the labeling of their axes, and not in the degree of linearity of the plotted points.

11.2.6 Superposition of Renewal Processes

Finally, a very different situation will be described which perhaps stretches the definition of the term "censoring." Suppose we have n units that all begin operation at the same time. If a unit fails, it is instantly replaced with a new unit. It is assumed that the lifetimes of the original and replacement units are independent and identically distributed with cdf F. The exact times of failures are known but not the identities of the failed units. Except for the first failure, then, we cannot be sure for the ages of the failed units. We thus observe a superposition of renewal processes. The failure times are not censored here, but the identities and therefore the ages of the failed units are censored in a sense.

Trindade and Haugh (1979) describe a method for the nonparametric estimation of F in the above situation. The renewal function, M, is estimated using a straightforward nonparametric method. The parent cdf is then estimated by exploiting the relationship of F to M through the fundamental renewal equation. For any particular set of points in time,

the estimate of F provides appropriate probabilities for determining

corresponding theoretical standard quantiles for purposes of probability

plotting. Again we will emphasize that a meaningful probability plot can

always be constructed as long as the parent CDF can be estimated using a

nonparametric method.

11.2.7 Summary of Steps in Constructing a Probability Plot

Below are given the steps required in constructing a probability plot

with uncensored, singly-censored, multiply right-censored, and multiply

left-censored data. The user must provide a value of the constant c with

$0 \leq c \leq 1$. The values c=0.3175 and c=.5 are popular.

A.   Let $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ denote n ordered observations, some of which

   may be censored, and let S be the set of subscripts corresponding to

   the observations in the ordered list that are not censored.

B.   Determine quantile probabilities for each iϵS using one of the

   following formulas:

$$(i-c)/(n-2c+1) \qquad , \text{ for complete or singly-censored samples,}$$

$$1 - \frac{n-c+1}{n-2c+1} \prod_{\substack{j \epsilon S \\ j \leq i}} \frac{n-j-c+1}{n-j-c+2} \text{ , for multiply right-censored samples,}$$

$$\frac{n-c+1}{n-2c+1} \prod_{\substack{j \epsilon S \\ j \geq i}} \frac{j-c}{j-c+1} \text{ , for multiply left-censored samples.}$$

(11.9)

C.   Enter Table 11.1 and find the line corresponding to the hypothesized

family of distributions. Plot the entry under "abscissa" versus the

entry under "ordinate" for each iϵS.

11.3 TESTING A SIMPLE NULL HYPOTHESIS.

For this section it is assumed that the hypothesis of interest is H_0:

the sampled population has the completely specified absolutely continuous

cdf $F(y)$. As in other Chapters, this situations is called Case 0. For most

of the discussion the data at hand will consist of a singly right-censored

sample (Type 1 or Type 2), that is, the set of r smallest order statistics

$Y_{(1)}, \ldots, Y_{(r)}$. The probability integral transformation $U_{(i)} =$

$F(Y_{(i)}), i=1, \ldots, r$, can be applied, and an equivalent test of fit is that

the $U_{(1)} \leq \ldots \leq U_{(r)}$ are the r smallest order statistics of a random

sample of size n from the uniform (0,1) distribution. If the data are Type

1 censored at y = y* and if t = $F(y*)$, then r is a random variable giving

the number of order statistics for the uniform random sample which are less

than t; if the data are Type 2 censored, then r is fixed in advance.

Many of the methods which have been discussed in earlier Chapters have

been adapted to accommodate censoring of both types. These include

censored versions of EDF statistics (Section 4.7), correlation-type tests

(Section 5.3) and tests based on spacings (Section 8.9). Later we examine

procedures in which the order statistics $U_{(i)}, i=1, \ldots, r$, are transformed to

new values which under H_0 are distributed as a complete uniform sample.

Then any of the many tests for uniformity for a complete sample (Chapter 8)

may be applied to the transformed values.

11.3.1 EDF Statistics.

In Chapter 4, censored versions of EDF statistics were introduced. We will now illustrate the use of these statistics by applying them directly to a censored sample.

E11.3.1.1 Exponential Example. This is an example is of Type 1 censoring. Barr and Davidson (1973) give the smallest 7 observations in a Type 1 censored sample of size n=20. The hypothesized null distribution is the exponential distribution $F(y) = 1 - \exp(-y/10)$, y>0, with a censoring value at y*=2.2. Table 11.4 gives the values $Y_{(i)}$, with the values $U_{(i)}$ given by $U_{(i)} = F\{Y_{(i)}\}$, i=1,...,7. The Type 1 censoring value for u is then t=F(2.2)=0.1975. The $Z_{(i)}$ values shown here are first discussed in Section 11.3.3.3.

Table 11.4

Hypothetical Survival Data and Transformed Observations

i	$Y_{(i)}$	$U_{(i)}$	$Z_{(i)}$
1	.1	.00995	.03979
2	.2	.01980	.07918
3	.3	.02955	.11815
4	.4	.03921	.15681
5	.7	.06761	.27038
6	1.0	.09516	.38056
7	1.4	.13064	.52245

In Section 4.7.2. the <u>Kolmogorov-Smirnov</u> statistics $_1D_{t,n}$ and $_2D_{r,n}$ for censored data of Types 1 and 2, respectively, are defined; it is also shown how these may be transformed and referred to the asymptotic distribution tabulated in Table 4.3.1. Alternatively, the exact tables for finite n, given by Barr and Davidson (1973) may be used. Working from the values $U_{(i)}$ of Table 11.4, the statistic $_1D_{t,n}$ is found to be 7/20 - 0.131 = 0.219, with t = 0.1975 and n = 20. Direct interpolation in the tables of Barr and Davidson gives the approximate significance level p=0.11. Alternatively, use of the formulas of Dufour and Maag (1978) (Section 4.7) yields the modified statistic D_t^* = 4.472 (0.219)+ 0.19/4.472 = 1.022. Reference to the asymptotic points in Table 4.3.1 then gives a p-value of approximately 0.10. If the data had been Type 2 censored, the formulas would give a modified statistic D_r^* = 1.033 with p-value approximately 0.095. Percentage points of the asymptotic distribution are derived and tabled for the Kuiper Statistic, $V_{t,n} = {_1D^+_{t,n}} + {_1D^-_{t,n}}$, under Type 1 right censoring by Koziol (1980a).

In Section 4.7.3 two types of <u>Cramér-von Mises</u> statistics for censored data are given. The first type is denoted by $_1W^2_{t,n}$ for right censored data of Type 1. The second is for Type 2 censoring. Both were derived by Pettit and Stephens (1976a,b) by adapting the complete-sample definitions of these statistics. For the $U_{(i)}$ of Table 11.4, we have $_1W^2_{0.2,20}$ = 0.104 and $_1A^2_{0.2,20}$ = 1.214. Referring to Table 4.3.2 with t = 0.2 gives p-values 0.008 and 0.005, respectively. If the data are treated

as Type 2 censored, the test statistics become $_2W^2_{7,20} = 0.057$ and $_2A^2_{7,20}$ = 0.863; referring to Table 4.3.3 with $p = r/n = .35$ gives approximate p-values of 0.25 and 0.08.

E11.3.1.2. Uniform Example. This is an example of Type 2 censoring. Consider the first n=25 values from the UNI data set, rescaled to the unit interval. Suppose that the smallest r=15 values from this set are available and are to be tested for fit as uniform order statistics. The observed value of $_2D_{r,n}$ = 0.216 and $(\sqrt{n})_2D_{r,n}$ = 1.08; then D^*_r (Section 4.5) is $1.08 + .24/5 = 1.128$ and reference to the tabulated asymptotic distribution gives a p-value of 0.1296.

The p-value may also be computed from the formula for the asymptotic distribution given by Schey (1977) and quoted in Section 4.7.2. The value of $t = 15/25 = 0.60$; this gives $A_t = 2.041$, $B_t = 0.408$ in the notation of Section 4.7.2., and then $G_t(1.128)$ is 0.9362. The observed significance level for the two-sided test is then approximately $2(1-0.936) = 0.128$.

Comment A. Use of the Censoring Information.

When the observations $U_{(i)}$ are to be tested for uniformity, the value of t, or of r (whichever is given) is important, in addition to the values $U_{(i)}$. Thus, for example, in Table 11.4, there are 7 observations out of 20 below $t = 0.1975$, a number larger than expected. If the sample had been Type 2 censored, we could observe that the largest observation $U_{(7)}$ is 0.131, much smaller than the expected value 0.333. These facts are

implicitly used in calculating the EDF statistics. Also, the value of $U_{(r)}$, although not the value of t, is used in the censored version of spacings statistics (Section 8.9).

Comment B. Random Censoring.

Extensions of EDF statistics to situations involving randomly censored data generally involve a Kaplan-Meier estimator for the true distribution function. For versions of the Kolmogorov-Smirnov, Kuiper, and Cramér-von Mises see Koziol (1980b), Nair (1981) or Fleming et al (1980), who obtain asymptotic distributions and examine the adequacy of small sample approximations.

11.3.2. Correlation Statistics.

The statistics in this class, as discussed in Chapter 5, basically focus upon the strength of the pattern of linear association which is present in probability plots (see Chapter 2 and Section 11.2). Suppose all the observations $U_{(i)}$ are known between $U_{(s)}$ and $U_{(r)}$. These may be plotted against $m_i = i/(n+1)$, i=s,...r, and the coefficient R(X,m) may be calculated as described in Section 5.1.2. Because R^2 is scale-free, the value obtained is the same as if the $U_{(i)}$ were correlated with i, from 1 to r-s+1. The $U_{(i)}$ are a subset of order statistics from (0,1) and will themselves be uniform between limits which may or may not be known. In either case, the distribution of R^2 so calculated is the same under H_0, as that of R^2 for a full sample of size r-s+1. Thus Table 5.1 may be used to make a test. The weakness in this test procedure is that it does not make use of any Type 1 censoring values. In Chapter 5 it is shown how this may

be overcome: by including the censoring limits in the observed sample. The value of R^2 calculated from these values will have the same null distribution as R^2 calculated from a complete uniform sample of r-2+2, size so that again Table 5.1 can be used.

E11.3.2.1. <u>Exponential Example Revisited</u>. For the $r = 7$ values of $U_{(i)}$ in Table 11.4 the correlation coefficient $R(X,m)=0.964$, which yields $T = r(1-R^2(X,m)) = 0.49$. Reference to Table 5.1 shows that this value is not significant even at the 50% level. If the endpoints $s=0$ and $t=0.1975$ are included, then $T=1.071$, which is significant at the .20 level approximately.

11.3.3. Transformations to Enable Complete-Sample Tests.

11.3.3.1. Conditioning on the Censoring Values.

When all the values from $U_{(s)}$ to $U_{(r)}$ ($s<r$) are available, the test of H_o: that these are a subset from an ordered uniform sample, can be changed to a test for a complete sample. There are several ways to do this. The simplest method is as follows. For Type 1 censoring suppose the lower censoring value is A and the upper censoring value is B, and let $R = B-A$; then under H_0 the values $V_{(i)} = (U_{(i)}-A)/R$, $i=1,\ldots,r-s+1$, will be a complete ordered uniform sample on the unit interval and can be so tested.

For Type 2 censoring, under H_0 the values $U_{(s+1)},\ldots,U_{(r-1)}$ will be distributed as a complete ordered sample from the uniform distribution between limits $A=U_{(s)}$ and $B=U_{(r)}$. The transformation $V_{(i)} = (U_{(s+i)}-U_{(s)})/R$, can be made for $i=1,\ldots,n^*$, where $n^*=r-1-s$ and $R=B-A$; the $V_{(i)}$, $i=1,\ldots,n^*$ can then be tested for uniformity between 0 and 1.

E11.3.3.1. <u>Exponential Example Again</u>.   Consider, again, the data of Table
11.4.   The upper (Type 1) censoring value is t=0.1975; thus we can first
transform the $V_{(i)}$ as $V_{(i)} = U_{(i)}/0.1975$ and then test the $V_{(i)}$ for
uniformity between 0 and 1. The $V_{(i)}$ are then 0.050, 0.100, 0.150, 0.199,
0.342, 0.482 and 0.661. The EDF statistics are $D^+ = 0.375$, $D^- = 0.050$, $D =$
0.375, $V = 0.426$, $W^2 = 0.413$, $U^2 = 0.085$ and $A^2 = 2.107$. Reference to Case
0 tables (Table 4.2.1) gives p-values of 0.21 for D, 0.07 for W^2 and 0.08
for A^2.

11.3.3.2. Handling Blocks of Missing Observations.

Suppose censoring occurs in a uniform sample other than at the ends;
for example, $U_{(r)}$ and $U_{(r+q)}$ might be known, but the q-1 observations in
between are not known. A spacing $U_{(r+q)}-U_{(r)}$ is called a q-spacing. Now
suppose that S is a q-spacing covering unknown observations, and let its
length be d. Keeping in mind the exchangeability of uniform spacings
(see Chapter 8) we exchange S with the set of all spacings to the right of
S. Under H_0 the new sample $U_{(1)},U_{(2)},\ldots,U_{(r)},U^*_{(r+1)},\ldots,U^*_{(n-q+1)}$,
where $U^*_{(j)} = U_{(j+q)} - d$, for j=r+1,...,n-q+1, will be distributed as an
ordered uniform sample which is right-censored at U^*_{n-q+1}. The process may
be repeated if there is more than one such spacing. The method can be used
only if it is known how many values are missing in the spacings. Thus a
uniform (0,1) sample with known blocks of missing observations can be
transformed to behave like a right (or left) censored sample. Techniques
for this simpler kind of censoring can then be used. .

E 11.3.3.2.1. <u>Example from Chapter 4</u>. In Table 4.5.3, a set of 15 values

for Z is given which are distributed uniformly on (0,1) under the null

hypothesis that the original set X (also given there) is exponential.

Suppose the four values 0.237, 0.252, 0.252, 0.381 are lost from the set

Z_1^*. Then 0.434 - 0.229 is a 5-spacing of length d = 0.205. We subtract

d from all the values of Z starting with .446 to obtain 0.113, 0.189,

0.229, 0.241, (= 0.446 - d), 0.298, 0.317, 0.578, 0.757, 0.774, 0.778,

(= 0.993 - d), 0.795 (=1.0-d). These 11 values can be analyzed as being

right censored (Type 2) at 0.795, and thus can be tested by any of the

methods of Section 11.3.1. Alternatively, they can be transformed to be a

complete sample, as in Section 11.3.3.1 above, or by another method to be

described after the next example.

E11.3.3.2.2. <u>Exponential Example Modified</u>. These various techniques may be

combined to handle blocks of missing observations within, say, right-

censored data. Thus suppose the values in the U-set of Table 11.4, which

are Type 1 right censored at t = 0.1975, are, in fact, the values $U_{(1)}$,

$U_{(2)}$, $U_{(3)}$, $U_{(4)}$, $U_{(8)}$, $U_{(9)}$, $U_{(10)}$ - that is, values $U_{(5)}$, $U_{(6)}$, $U_{(7)}$,

constitute a block of missing observations. First the set of U's may be

transformed to a uniform sample as described in Section 11.3.3.1 above to

give a new set $V_{(i)} = U_{(i)}/0.1975$. The values are those given in Example

E11.3.3.1.1, but they now represent the order statistics with indices

1,2,3,4,8,9,10. There is thus a 4-spacing of length d = 0.342 - 0.199 =

0.143 between $V_{(4)}$ and $V_{(8)}$. Following the steps of this section, new

values V* are found to be $V_{(5)}^* = V_{(9)} - 0.143 = 0.339$, $V_{(6)}^* = 0.518$,

$V^*_{(7)} = 1 - d = 0.857$. These 7 values are to be treated as a right-censored sample of size 7, now of Type 2, with n = 10. (The 7 given values plus the 3 missing in the 4-spacing). Since the lower end-point of the distribution is known to be zero, the values 0.050, 0.100, 0.150, 0.199, 0.339, 0.518 can be divided by 0.857 and then tested for uniformity on the unit interval.

11.3.3.3. More Powerful Transformations.

A disadvantage of the above method of transforming to a complete sample for Type 2 censoring is that the resulting test examines the values of the $U_{(i)}$ relative to $U_{(s)}$ and $U_{(r)}$ but takes no account of whether these values themselves are too large or too small. (See Comment A in Section 11.3.1).

Michael and Schucany (1979) propose a modification of the above technique by which a subset of r uniform U(0,1) order statistics can be transformed montonically to behave like a complete U(0,1) sample of size r from the U(0,1) distribution. For definiteness the result is presented here in terms of right censorship, however, the technique can be applied to any kind of Type 2 censoring. For example, a q-spacing representing a block (q-1) missing observations in a sample size of size n can be shrunk to a 1-spacing in a "complete" sample of size n-q+1. The relative spacings between consecutive order statistics are not affected by this transformation.

Let $U_{(1)}, U_{(2)}, \ldots, U_{(r)}$ be the smallest r order statistics from a random sample of size n from the uniform (0,1) distribution, and let B(*) denote the cdf of $U_{(r)}$, which is known to have the Beta (r,n-r+1) distri-

bution (see Section 8.8.2). If $Z_{(1)}, Z_{(2)}, \ldots, Z_{(r)}$ are defined by

$$Z_{(i)} = U_{(i)} \cdot h_{r,n}(U_{(r)}) , \qquad (11.10)$$

where $h_{r,n}(x) = \{B^*(x)\}^{1/r}/x$, then the $Z_{(i)}$, $i=1,\ldots,r$, are distributed like a complete uniform $(0,1)$ sample of size r. The proof is straightforward by change of variable.

The computations for the transformation are easily performed on a scientific calculator since the beta cdf can be expressed as the binomial sum

$$B(u) = \sum_{i=r}^{n} \binom{n}{i} u^i (1-u)^{n-i} \qquad .$$

Any standard goodness-of-fit test for uniformity (Chapter 8) may now be applied to the transformed observations. The Anderson-Darling statistic is recommended because of its sensitivity to departures from uniformity in the tails of the distribution. The reason why this is important is best presented by illustration.

E11.3.3.1 Artifical Uniform Right-Censored Sample. Three artificial but informative examples of the transformation are shown in Figure 11.6 where the smallest 5 of 9 observations are plotted both before and after the transformation. In each example the values of the $U_{(i)}$ were artificially chosen to satisfy $U_{(i)}/U_{(5)} = i/5 = E(U_{(i)}/U_{(5)}|U_{(5)})$. The values for $U_{(5)}$ were chosen to be .500, .103, and .897 which correspond, respectively, to

the .500, .001, and .999 quantiles of the beta (5,5) distribution, which is the null distribution of $U_{(5)}$ when testing the hypothesis of uniformity for the $U_{(i)}$. Note the manner in which small and large values of $U_{(5)}$ affect the appearance of the transformed points. Small values of $U_{(5)}$ lead to small values for $Z_{(5)}$ which, in turn, will inflate most reasonably formulated goodness-of-fit statistics. But if $Z_{(5)}$ is large, the departure from uniformity may appear less pronounced ; however, $Z_{(5)}$ will be very close to 1 and this will inflate a statistic like the Anderson-Darling statistic which is sensitive to such an apparent departure from uniformity.

a. $u_{(5,9)} = .500$

```
u  |---X---X---X---X----X--------------------|

z  |-------X-----X------X------X------X-------|
```

b. $u_{(5,9)} = .103$

```
u  |XXXXX------------------------------------|

z  |----X----X----X----X----X---------------|
```

c. $u_{(5,9)} = .897$

```
u  |------X------X------X------X------X------|

z  |------X-------X------X-------X----------X|
```

Figure 11.6
Examples of the Transformation with $r = 5$, $n = 9$

E 11.3.3.3.2. Exponential Example. Consider again the values $U_{(i)}$, $i=1,\ldots,7$ given in Table 11.4. Using the transformation above, we first compute the scale factor $h = h_{7,20}(U_{(7)})$ as

$$h = \lfloor B*(0.13064) \rfloor^{1/7}/0.13064 = 3.9991 \qquad ,$$

where B*(.) is the Beta (7,14) distribution. The values $Z_{(i)} = hU_{(i)}$ are given in Table 11.4. The Cramér-von Mises statistic, W^2, calculated from the 7 values of $Z_{(i)}$ by the full-sample formulas (Equation 4.2) is 0.673, and the Anderson-Darling statistic is $A^2 = 3.404$. These have approximate p-values of 0.035 and 0.02. The Kolmogorov-Smirnov statistic is D = 0.47755 which has a p-value of approximately 0.056. The p-values using the transformed Z-values are lower than those using the statistics directly adapted for censoring (see Section 11.3.1).

COMMENT. This transformation technique for goodness-of-fit analysis of censored samples has some advantages over the other procedures which have been proposed for this problem. No new or additional tables of critical points are required. Any subset of order statistics can be analyzed. The power of the Anderson-Darling statistic based on the transformed sample appears to be generally greater than that of existing methods in the presence of left or right censorship. A minor disadvantage is the slight increase in computation to evaluate the scaling factor, $h_{r,n}(U_{(r)})$. The technique can be extended to all kinds of Type 2 censoring, even progressive censoring. For details and asymptotic results see Michael and Schucany (1979).

11.4 TESTING A COMPOSITE HYPOTHESIS

In this section the hypothesis of interest is that the sampled population has an absolutely continuous cdf $F(y|\theta)$, where θ is a vector of unknown (nuisance) parameters. Typically the censored data at hand must be a singly-censored sample if published tables of critical points are to be used. For more complicated types of censoring, such as multiple right censoring, little work has been done. For a particular set of data, it may be possible to modify a standard statistic and then estimate certain percentiles, or the observed significance level, using simulation techniques. When the censoring is Type 2, test statistics can often be constructed which have parameter-free null distributions. When the censoring is Type 1, statistics with asymptotically parameter-free distributions are a possibility.

11.4.1 Omnibus Tests

Turnbull and Weiss (1978) present an omnibus test for a composite null hypothesis based on the generalized likelihood ratio statistic. Their procedure is appropriate for discrete or grouped data and accommodates multiple censoring by employing the Kaplan-Meier estimate to maximize the alternative likelihood. In less complicated cases of Type 1 or 2 censoring several standard goodness-of-fit statistics have been modified to test a composite null hypothesis.

11.4.1.1. EDF Statistics for Censored Data with Unknown Parameters

Modifications of EDF statistics which accommodate certain types of censoring when the null hypothesis is simple were discussed in Section 11.3.1.1. Similar modifications for use in testing normality with unknown parameters, or exponentiality with unknown scale, are given in Sections 4.8.4 and 4.9.6.

E11.4.1.1.1 Normal   Example   The data consist of the smallest 20 values among the first 40 values listed in the NOR data set. We wish to test that the underlying distribution is normal. Gupta's estimates (Gupta, 1952) here are $\hat{\mu} = 98.233$ and $\hat{\sigma} = 9.444$. Relevant calculations are given in Table 11.5. The value of the Cramér-von Mises statistic is found to be, using Section 4.7.3,

$$2\hat{W}^2_{20,40} = \frac{20}{12(40)^2} + 0.02512 - \frac{40}{3}(0.5 - 0.53741)^3$$
$$= 0.02686.$$

Referring to Table 4.4.5, we find that the observed value is smaller than the .15 point which, by interpolation, is approximately 0.03. The value of $\hat{A}^2_{20,40}$ is 0.233; which is significant at about the .10 level.

EDF Tests for exponentiality with an unknown scale parameter are set out in Section 4.9. Note that use of the N transformation of Chapter 10 (see Section 10.5.6) converts a right-censored exponential sample to a complete sample of exponentials, with the same scale, and then any of the tests of Chapter 10 can be used. This property is explored in a test based on leaps, in Section 11.4.1.3 below.

Table 11.5

Steps in Calculating $\hat{W}^2$ for the Smallest 20 Order Statistics Among
the First 40 Observations in the NOR Data Set

i	$Y_{(i)}$	$\dfrac{i-0.5}{40}$	$\hat{U}_{(i)}$	$\left[\hat{U}_{(i)} - \dfrac{i-0.5}{n}\right]^2$
1	79.43	0.0125	0.02323	0.00012
2	83.53	0.0375	0.05974	0.00049
3	83.67	0.0625	0.06152	0.00000
4	84.27	0.0875	0.06962	0.00032
5	85.29	0.1125	0.08525	0.00074
6	87.83	0.1375	0.13531	0.00000
7	89.00	0.1625	0.16411	0.00000
8	89.90	0.1875	0.18878	0.00000
9	90.03	0.2125	0.19252	0.00040
10	91.87	0.2375	0.21778	0.00039
11	91.46	0.2625	0.23662	0.00067
12	92.02	0.2875	0.25529	0.00104
13	92.45	0.3125	0.27014	0.00179
14	92.55	0.3375	0.27364	0.00408
15	95.45	0.3625	0.38411	0.00047
16	96.13	0.3875	0.41188	0.00059
17	96.20	0.4125	0.41477	0.00001
18	98.70	0.4375	0.51972	0.00676
19	98.98	0.4625	0.53152	0.00476
20	99.12	0.4875	0.53741	0.00249
				0.02512

11.4.1.2. Correlation Statistics.

Consider again the sample correlation coefficient between the $Y_{(i)}$ and a set of constants K_i, denoted, as in Chapter 5, by $R(Y,K)$. Because $R(Y,K)$ is invariant with respect to a linear transformation of the $Y_{(i)}$, it follows that its null distribution does not depend on location or scale parameters of the distribution. This makes it a useful statistic for testing fit. Suppose $F(y)$ is the cdf of Y for location parameter 0 and scale parameter 1, and let $F^{-1}(\cdot)$ be its inverse. Suitable sets of constants are then $K_i = m_i$, where m_i is the expected value of the i-th order statistic of a sample of size n from $F(y)$, or $K_i = H_i = F^{-1}(i/(n+1))$. The statistic $Z = n\{1 - R^2(Y,K)\}$ has been discussed in Chapter 5, and percentage points have been given for censored versions of Z.

Chen (1984) presents a correlation statistic as an omnibus test for the composite hypothesis of exponentiality in the presence of random censoring. Asymptotic distributions are derived under a particular censorship model, which is quite robust provided that less than 40% of the observations are censored.

E11.4.1.2.1. Normal Example Revisited. Consider again the smallest 20 values among the first 40 values in the NOR data set. When testing for normality using $R(Y,K)$, we obtain $Z = 0.035$ which falls just below the .50 point. Thus on the basis of the statistic Z we cannot reject the hypothesis of normality at the usual levels. Another example of a correlation type statistic is given in the next section.

11.4.1.3.   Statistics Based on Spacings and on Leaps.   Spacings between
ordered uniforms were defined in Section 10.9.3.   Similarly, spaces can be
defined between order statistics $Y_{(i)}$ of a sample from any distribution. If
the distribution has no lower limit, the first spacing will be $D_1 =$
$Y_{(2)} - Y_{(1)}$, and so on.   Similarly leaps l_i can be defined by $l_i = D_i / E(D_i)$;
see Section 10.9.   An important property of leaps is that, for continuous
distributions, they will (under regularity conditions which may exclude the
extremes) be asymptotically independent, each with the exponential
distribution with mean 1.   Then, for distributions with unknown location
and scale parameters, a test for a distribution is reduced to a test for
exponentiality for the l_i (Section 10.9.4).   We illustrate the technique
with an example given by Mann, Scheuer and Fertig (1973) who created a test
for the extreme-value distribution by using leaps. The test applies very
well to right-censored data and can be adapted to a test for the Weibull
distribution. Both of these features are illustrated in the example.
E11.4.1.3.1 Weibull   Example.   The smallest 15 order statistics from a
sample of size of 22 are:15.5, 15.6, 16.5, 17.5, 19.5, 20.6, 22.8, 23.1,
23.5, 24.5, 26.5, 26.5, 32.7,33.8, 33.9. The null hypothesis here is that
the sample comes from a two-parameter Weibull distribution (Section
10.4.4); a three-parameter Weibull distribution is the alternative. The
steps in making the test are as follows. First find $X_{(i)} = \log t_{(i)}$,
$i=1,\ldots,15$. H_o then reduces to a hypothesis that the $X_{(i)}$ are from an
extreme-value distribution (Equation 5.22) with unknown parameter.

Suppose the $W_{(i)}$ are the order statistics of a sample of size 22 from an extreme-value distribution with location 0 and scale 1; let $m_i = E(W_{(i)})$. Define <u>modified leaps</u> l_i' (see Section 10.9.4), as follows:

$$l_i' = \{X_{(i+1)} - X_{(i)}\}/(m_{i+1} - m_i) , \quad i=1,\ldots14;$$

then, under H_o, the l_i' are (approximately i.i.d.) $Exp(0,\beta)$, with β unknown. The test statistic suggested by Mann, Scheuer and Fertig is $S = \Sigma_{i=s}^{r-1} l_i'/\Sigma_{i=1}^{r-1} l_i'$, where $s = (r+1)/2$ if r is odd and $s = (r+2)/2$ if r is even. The procedure can be regarded as applying the J transformation of Chapter 10 to the values l_i', to give a set of r-2 (approximate) Uniforms U; S is then the median and has approximately the Beta distribution $B(p,q)$ with $p = (r-1)/2$, $q = (r-1)/2$, if r is odd, and $p = (r-2)/2$ and $q = r/2$ if r is even. Mann, Scheuer and Fertig give values of $m_{i+1} - m_i$ from samples of size n=3 to n=25 for the calculation of l_i', and percentage points of S. For the present data, with r=15 and s=8 the value of S is given as 0.660; this is significant at about the 11% level. Although an extreme leap is used here, Monte Carlo studies by the above authors show that the Beta distribution for S is obtained for quite low values of n, and the technique might well be valuable applied to other distributions with unknown location/scale parameters.

Mann and Fertig (1975) consider ratios of other sums of leaps as well as ratios of weighted sums of leaps, and describe how their approach can be extended to progressively censored samples. We can use this example also to illustrate the use of correlation statistics for the extreme value

distribution. The test is for distribution (5.22), which has a short tail to the right and a long left tail. The r=15 values of $X_{(i)}$ are tested to correlate with $H_i = \log[-\log\{1-i/(n+1)\}]$, i=1,...15, n=22; then R=0.9446 and $Z = n\{1-R^2(X,H)\} = 1.616$. Interpolation in Table 5.10, for n=22 and $p = r/n = 0.68$, shows Z to be significant at approximately the 0.25 level.

It may be valuable to recognize a danger that exits in testing for the extreme-value distribution. For a full sample, it does not matter whether one takes X = log t, where t is Weibull data, and tests that X is from (5.22), or takes X'= -log t and tests that X' is from (5.21); the same value of the correlation coefficient is obtained by both methods, and both recommendations are seen in the literature. However, for a censored sample, it is important to follow the correct procedure: for right-censored Weibull data, take X = log t and test for right-censored data from (5.22) (as was done above), and for left-censored Weibull data, take X' = -log t and test for right-censored data from (5.21). This second test for Weibull is probably less likely to occur in practice, and the Mann-Scheuer-Fertig test is not set up for this case, although it could be adapted.

Two tests for the two-parameter exponential that can be used with doubly censored samples have been presented by Brain and Shapiro (1983). These tests combine the properties of spacings and of the correlation statistic to have good sensitivity to alternatives with monotone and nonmonotone hazard functions, respectively. Still other related work on

statistics based on spacings may be found in Mehrotra (1982). Some statistics based on modified leaps have also been studied by Tiku (1980, 1981).

11.4.2 Alternative Families of Distributions.

Typically when testing for goodness of fit we assume only that the underlying cdf is absolutely continuous. Occasionally we may wish to limit our choices to, say, two families of distributions. In particular we may wish to test the composite null hypothesis

$$H_1: F(y) = F_1(y|\theta_1)$$

against the composite alternate hypothesis

$$H_2: F(y) = F_2(y|\theta_2) \quad ,$$

where θ_1 and θ_2 are unknown (nuisance) parameters. Because we have narrowed the set of alternate distributions considerably, we should be able to tailor tests to the specific hypothesis of interest which are more powerful than omnibus goodness-of-fit tests. There have been several approaches to this problem.

Let $f_i(y|\theta)$ be the probability density function for family i, i=1,2. We will denote by L_i the sample likelihood under H_i after θ_i has been replaced with its maximum likelihood estimator, $\hat{\theta}_i$. This maximized likelihood is then

$$L_i = \prod_{j=1}^{n} f_i(Y_{(j)}; \hat{\theta}_i)$$

for the complete sample $Y_{(1)}, \ldots, Y_{(n)}$. We will denote the ratio of maximum likelihoods by

$$RML = L_1/L_2 .$$

Cox (1961,1962) formulates a test of H_1 versus H_2 which is based upon the statistic

$$T = \ln(RML) - E\lfloor \ln(RML) \rfloor ,$$

where E is the expectation under the null hypothesis, H_1. For complete samples the large sample distribution of T is approximated using maximum likelihood theory. Hoadley (1971) extends maximum likelihood theory to situations which include censoring. Thus valid approximations to the distribution of T are also possible with censored samples.

For location-scale families with pdfs $\sigma_1^{-1} f_1((y-\mu_1)/\sigma_1)$, Lehmann (1959) shows that the uniformly most powerful invariant (under linear transformations) test is based upon the (Lehmann) ratio of integrals

$$LRI = I_1/I_2 ,$$

where

$$I_1 = \int_{-\infty}^{\infty} \int_0^{\infty} v^{n-2} f_1(vy_1+u, \ldots, vy_n+u) dv du .$$

The RML statistic and some modified versions are discussed in a series of papers: Antle (1972,1973,1975); Dumonceaux, Antle, and Haas (1973); Dumonceaux and Antle (1973); Klimko and Antle (1975);Kotz (1973). Percentage points are given for the null distribution of RML for comparisons involving a number of different families of distributions. In some cases, the LRI and RML tests coincide. In others, the RML test is almost

as powerful as the LRI test. The authors make use of the fact that the distribution of the RML statistic is parameter-free whenever the families to be compared are both location-scale families. This result appears to hold for any Type II censored sample. The only tables of critical points which have been constructed for use with censored samples appear in Antle (1975) and apply to the situation where one is testing the null hypothesis that the underlying distribution is Weibull (or extreme-value) against the alternate hypothesis of lognormality (or normality), and vice versa.

E11.4.2.1. Lognormal vs. Weibull Example. We once more consider the smallest 20 values among the first 40 values listed in the NOR data set. We first exponentiated the data, and then proceded to test the lognormal against the Weibull family. An interactive procedure was used to determine the value of RML. Entries in Table IX of Antle (1975) must be compared to $RML^{1/20}$ which here was determined to be 1.063. This value is just above the 95 percent point and so we have the surprising result that we can reject the (true) hypothesis of normality in favor of the extreme-value distribution at the 0.05 level of significance.

Finally, a somewhat different approach to this general problem deserves mention. Farewell and Prentice (1977) construct a three-parameter family of generalized gamma distributions which includes the Weibull, lognormal and gamma families as special cases. Likelihood ratio tests using asymptotic likelihood results are recommended which can accommodate censoring as well as regression variables.

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This report is a survey of current techniques in goodness-of-analysis when the data set has been subjected to censoring. It is to appear as a chapter in forthcoming book to be published by Marcel Dekker, Inc. and edited by Ralph B. DpAgostino and Michael A. Stephens.