

Moments of Bivariate Order Statistics for the Standard Normal Distribution

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Abstract

This technical report presents numerical values of moments of bivariate order statistics for the standard normal distribution under three certain cases, which occur in typical applications of ranked set sampling or judgement post-stratification.¹

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Let $(Z_{h1}, Z_{h2})_{h=1}^H$ be an iid random sample from a standard bivariate normal distribution with correlation ρ_{12} . Without loss of generality, we restrict attention to $\rho_{12} \geq 0$ in this report. Denote the order of Z_{h1} among $Z_{11}, \dots, Z_{H1}$ by $R_{h:H}$, and the order of Z_{h2} among $Z_{12}, \dots, Z_{H2}$ by $S_{h:H}$. The bivariate order statistics of (Z_1, Z_2) are denoted by $(Z_{1(r,s)}, Z_{2(r,s)})$, i.e., random variables (Z_1, Z_2) given the ranks $R_{h:H} = r$ and $S_{h:H} = s$.

Computing moments of $(Z_{1(r,s)}, Z_{2(r,s)})$ involves calculation of the bivariate rank distribution $\pi_{rs} (= Pr(R_{h:H} = r, S_{h:H} = s))$. Since $\pi_{rs} = \pi_{sr}$ and $\pi_{rs} = \pi_{\bar{r}\bar{s}}$ ($\bar{r} \equiv H + 1 - r$, $\bar{s} \equiv H + 1 - s$) hold here, Table 2 and Table 3 only list the necessary elements for the bivariate rank distribution for the case $H = 3$ and $H = 4$, respectively. Note for $H = 2$, we have

$$\pi_{11} = \frac{1}{4} + \frac{1}{2\pi} \sin^{-1} \rho_{12}.$$

Because $E(Z_{1(r,s)}) = E(Z_{2(s,r)})$ and $E(Z_1|r, s) + E(Z_1|\bar{r}, \bar{s}) = 0$, it is sufficient that Table 1, 2 and 3 present the elements in the upper triangular matrix of $[E(Z_{1(r,s)})]_{H \times H}$ for each case.

To obtain $\text{cov}(Z_{k(r,s)}, Z_{l(r,s)})$, $k = 1, 2$ and $l = 1, 2$, we now need $E(Z_{k(r,s)}Z_{l(r,s)})$ since $\text{Cov}(Z_{k(r,s)}, Z_{l(r,s)}) = E(Z_{k(r,s)}Z_{l(r,s)}) - E(Z_{1(r,s)})E(Z_{2(s,r)})$. It is easy to verify the following results: (1) $\text{Var}(Z_{1(r,s)}) = \text{Var}(Z_{2(s,r)})$ and $\text{Cov}(Z_{1(r,s)}, Z_{2(r,s)}) = \text{Cov}(Z_{1(s,r)}, Z_{2(s,r)})$; (2) $\text{Cov}(Z_{k(r,s)}, Z_{l(r,s)}) = \text{Cov}(Z_{k(\bar{r}, \bar{s})}, Z_{l(\bar{r}, \bar{s})})$ for any $k, l = 1, 2$. Table 1, 2 and 4 list the necessary elements of the covariance matrix of $(Z_{1(r,s)}, Z_{2(r,s)})$ for each case. Other elements can be calculated by applying the above two results.

All the entries in the tables were computed using Gaussian quadrature. The formulas are given below:

$$\pi_{rs} \approx \sum_{k=\mathcal{L}}^{\mathcal{U}} \frac{C_k}{\pi} \sum_{j=1}^M \sum_{i=1}^M \omega_i \omega_j \theta_1^k \theta_2^{r-1-k} \theta_3^{s-1-k} \theta_4^{H-r-s+1+k}$$

$$E(Z_{1(r,s)}) \approx \frac{\sqrt{2}}{p_{rs}} \sum_{k=\mathcal{L}}^{\mathcal{U}} \frac{C_k}{\pi} \sum_{j=1}^M \sum_{is=1}^M \{\omega_i \omega_j t_i \theta_1^k \theta_2^{r-1-k} \theta_3^{s-1-k} \theta_4^{H-r-s+1+k}\}$$

$$E(Z_{k(r,s)}Z_{l(r,s)}) \approx \frac{2}{p_{rs}} \sum_{k=\mathcal{L}}^u \frac{C_k}{\pi} \sum_{j=1}^M \sum_{i=1}^M \left\{ \omega_i \omega_j t_i^{I(k=1)+I(l=1)} \left(\rho_{12} t_i + \sqrt{1 - \rho_{12}^2} t_j \right)^{I(k=2)+I(l=2)} \theta_1^k \theta_2^{r-1-k} \theta_3^{s-1-k} \theta_4^{H-r-s+1+k} \right\}$$

where

$$\begin{aligned} \theta_1 &= Pr(Z_1 < z_1, Z_2 < z_2) \quad , \quad \theta_2 = Pr(Z_1 < z_1, Z_2 > z_2), \\ \theta_3 &= Pr(Z_1 > z_1, Z_2 < z_2) \quad , \quad \theta_4 = Pr(Z_1 > z_1, Z_2 > z_2), \\ z_1 &= \sqrt{2}t_i \quad \quad \quad , \quad z_2 = \sqrt{2}\rho_{12}t_i + \sqrt{2(1 - \rho_{12}^2)}t_j; \end{aligned}$$

t_i is the i -th zero of the Hermite polynomial $H_M(t)$ and ω_i is the i -th weight factor; $I(.)$ is the indicator function. Tables of t_i and ω_i are available for $M = 1$ to 20 in Salzer et al. (1952).

References

Salzer, H., Zucker, R., and Capuano, R. (1952). Table of the zeros and weight factors of the first twenty hermite polynomials. *Journal of Research of the National Bureau of Standards*, 48:111–116.

Table 1: The Bivariate Standard Normal Case for H=2

ρ_{12}	$E(Z_{1(r,s)})$		$Var(Z_{1(r,s)})$		$Cov(Z_{1(r,s)}, Z_{2(r,s)})$		ρ_{12}	$E(Z_{1(r,s)})$		$Var(Z_{1(r,s)})$		$Cov(Z_{1(r,s)}, Z_{2(r,s)})$	
	(1,1)	(1,2)	(1,1)	(1,2)	(1,1)	(1,2)		(1,1)	(1,2)	(1,1)	(1,2)	(1,1)	(1,2)
0	5642	5642	6817	6817	0	0	0.5	6347	4231	7005	6142	3039	2656
0.025	5692	5590	6838	6795	142	141	0.525	6364	4135	7001	6096	3203	2780
0.05	5741	5536	6858	6771	285	281	0.55	6380	4035	6997	6049	3369	2903
0.075	5788	5481	6877	6746	429	420	0.575	6393	3931	6990	6000	3537	3026
0.1	5834	5424	6894	6720	574	559	0.6	6404	3823	6983	5950	3706	3148
0.125	5878	5365	6910	6693	720	696	0.625	6412	3710	6975	5900	3876	3269
0.15	5921	5304	6925	6665	866	833	0.65	6418	3593	6965	5848	4049	3389
0.175	5962	5242	6939	6635	1014	968	0.675	6421	3471	6954	5795	4223	3509
0.2	6001	5177	6952	6604	1163	1103	0.7	6421	3343	6942	5740	4398	3628
0.225	6039	5111	6963	6572	1313	1237	0.725	6418	3208	6929	5685	4576	3746
0.25	6075	5043	6973	6539	1464	1370	0.75	6412	3066	6914	5628	4756	3864
0.275	6110	4972	6982	6505	1616	1502	0.775	6401	2915	6899	5571	4938	3981
0.3	6143	4900	6989	6469	1770	1634	0.8	6386	2754	6883	5512	5123	4097
0.325	6174	4825	6996	6432	1924	1764	0.825	6365	2582	6866	5452	5311	4212
0.35	6204	4748	7001	6394	2080	1894	0.85	6338	2396	6848	5391	5501	4327
0.375	6232	4669	7005	6355	2236	2023	0.875	6303	2192	6830	5328	5695	4441
0.4	6259	4587	7007	6315	2394	2151	0.9	6258	1965	6812	5265	5893	4554
0.425	6284	4502	7009	6273	2553	2278	0.925	6199	1705	6795	5200	6097	4666
0.45	6307	4415	7009	6231	2714	2405	0.95	6119	1395	6780	5135	6308	4778
0.475	6328	4325	7008	6187	2876	2531	0.975	5999	989	6772	5068	6532	4890

Note: All the values should be multiplied by 10^{-4} . The values of $E(Z_{1(r,s)})$ should be negative in the above table.

Table 2: The Bivariate Standard Normal Case for H=3

ρ_{12}	π_{rs}		$E(Z_{1(r,s)})$				$Var(Z_{1(r,s)})$					$Cov(Z_{1(r,s)}, Z_{2(r,s)})$			
	(1,1)	(1,2)	(1,1)	(1,2)	(1,3)	(2,1)	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(1,1)	(1,2)	(1,3)	(2,2)
0	1111	1111	8463	8463	8463	0	5595	5595	5595	4487	4487	0	0	0	0
0.025	1141	1111	8533	8461	8390	37	5622	5594	5566	4486	4487	103	93	102	88
0.05	1171	1110	8602	8457	8314	73	5648	5590	5535	4486	4488	207	186	203	177
0.075	1202	1109	8668	8450	8237	110	5673	5585	5503	4484	4490	313	279	302	265
0.1	1233	1107	8731	8439	8156	147	5696	5577	5470	4482	4492	420	373	401	354
0.125	1264	1105	8793	8426	8073	184	5718	5568	5435	4479	4495	528	466	498	443
0.15	1295	1102	8852	8409	7988	221	5738	5556	5399	4476	4498	637	559	594	532
0.175	1327	1099	8909	8390	7899	259	5756	5542	5361	4472	4503	748	652	689	622
0.2	1359	1096	8963	8368	7808	297	5773	5526	5322	4467	4507	860	746	784	712
0.225	1391	1091	9016	8342	7714	335	5789	5507	5281	4462	4512	973	839	877	802
0.25	1424	1087	9066	8313	7617	374	5803	5487	5239	4456	4518	1088	933	969	893
0.275	1457	1081	9114	8281	7517	413	5815	5464	5196	4450	4523	1204	1026	1060	985
0.3	1491	1076	9159	8245	7413	453	5826	5440	5151	4443	4530	1322	1120	1149	1077
0.325	1525	1069	9203	8206	7307	494	5835	5413	5105	4435	4536	1441	1214	1238	1170
0.35	1560	1062	9244	8163	7196	535	5843	5384	5057	4426	4542	1562	1308	1326	1264
0.375	1596	1054	9283	8117	7082	577	5849	5352	5008	4417	4549	1684	1402	1414	1358
0.4	1631	1046	9319	8067	6965	621	5854	5319	4958	4407	4555	1808	1497	1500	1454
0.425	1668	1037	9353	8012	6843	665	5857	5284	4906	4397	4562	1934	1591	1585	1550
0.45	1705	1027	9385	7954	6717	710	5858	5246	4853	4386	4568	2061	1686	1669	1648
0.475	1743	1016	9414	7891	6586	756	5858	5206	4798	4374	4574	2190	1781	1753	1747
0.5	1782	1005	9440	7824	6451	804	5856	5164	4742	4361	4579	2320	1876	1835	1847
0.525	1822	992	9464	7752	6310	853	5853	5120	4685	4348	4584	2453	1971	1917	1948
0.55	1862	979	9485	7675	6164	904	5848	5074	4626	4333	4589	2587	2067	1998	2051
0.575	1904	964	9502	7592	6012	957	5841	5026	4566	4318	4592	2723	2163	2078	2155
0.6	1947	948	9517	7503	5854	1011	5832	4975	4505	4302	4595	2862	2259	2157	2261
0.625	1991	931	9528	7409	5688	1069	5822	4923	4442	4286	4597	3002	2356	2235	2369
0.65	2037	913	9536	7307	5515	1128	5811	4868	4378	4268	4598	3145	2453	2313	2479
0.675	2084	893	9540	7198	5334	1191	5797	4812	4312	4249	4597	3290	2551	2390	2590
0.7	2133	871	9540	7080	5144	1257	5783	4753	4245	4230	4595	3437	2649	2466	2704
0.725	2183	848	9536	6954	4943	1327	5766	4692	4177	4209	4592	3587	2748	2542	2821
0.75	2237	822	9527	6817	4730	1401	5748	4630	4107	4188	4587	3740	2847	2616	2940
0.775	2292	793	9512	6669	4504	1481	5729	4565	4036	4165	4580	3896	2947	2691	3062
0.8	2351	761	9490	6507	4262	1566	5708	4498	3964	4142	4571	4055	3048	2764	3188
0.825	2414	726	9462	6329	4002	1659	5685	4430	3890	4117	4560	4217	3149	2837	3317
0.85	2482	686	9424	6132	3719	1762	5662	4359	3815	4091	4546	4384	3252	2909	3451
0.875	2555	640	9376	5911	3407	1876	5638	4287	3738	4064	4531	4555	3356	2981	3590
0.9	2636	586	9314	5659	3059	2006	5613	4214	3660	4036	4513	4731	3462	3053	3735
0.925	2728	522	9233	5363	2660	2158	5589	4139	3581	4007	4493	4914	3570	3123	3888
0.95	2838	440	9122	5000	2180	2344	5566	4064	3500	3977	4473	5107	3680	3194	4051
0.975	2982	324	8957	4510	1548	2598	5550	3991	3417	3948	4454	5314	3796	3264	4231

Note: All the values should be multiplied by 10^{-4} . The values of $E(Z_{1(r,s)})$ should be negative in the above table.

Table 3: The Bivariate Standard Normal Case for H=4

ρ_{12}	π_{rs}				$E(Z_{1(r,s)})$							
	(1,1)	(1,2)	(1,3)	(2,3)	(1,1)	(1,2)	(1,3)	(1,4)	(2,1)	(2,2)	(2,3)	(2,4)
0	625	625	625	625	10294	10294	10294	10294	2970	2970	2970	2970
0.025	647	631	618	627	10375	10316	10268	10209	3008	2981	2959	2932
0.05	670	637	612	629	10454	10335	10239	10122	3045	2992	2948	2893
0.075	693	643	604	632	10530	10350	10206	10032	3081	3003	2937	2853
0.1	716	648	597	635	10603	10361	10169	9939	3117	3013	2925	2812
0.125	740	653	589	639	10674	10369	10129	9843	3152	3024	2913	2770
0.15	764	657	581	643	10742	10374	10085	9743	3187	3035	2901	2728
0.175	788	662	573	647	10808	10375	10038	9641	3221	3046	2888	2684
0.2	813	666	564	652	10871	10372	9986	9534	3255	3056	2875	2639
0.225	838	669	555	657	10931	10366	9931	9425	3289	3067	2862	2592
0.25	864	672	545	663	10989	10356	9871	9312	3322	3078	2847	2544
0.275	890	675	536	670	11044	10342	9807	9195	3356	3089	2833	2495
0.3	916	677	525	677	11096	10324	9739	9074	3389	3100	2817	2444
0.325	943	679	515	685	11146	10303	9667	8949	3422	3111	2800	2392
0.35	971	681	504	694	11193	10277	9590	8820	3455	3122	2783	2337
0.375	999	682	493	703	11238	10247	9508	8687	3488	3133	2764	2281
0.4	1028	682	481	714	11280	10213	9421	8548	3521	3144	2745	2222
0.425	1057	682	469	725	11319	10174	9329	8405	3555	3155	2724	2161
0.45	1088	682	457	737	11355	10131	9231	8256	3589	3166	2701	2098
0.475	1119	681	444	751	11388	10082	9128	8102	3623	3177	2677	2032
0.5	1150	679	431	765	11418	10029	9018	7942	3657	3187	2651	1963
0.525	1183	677	417	781	11445	9970	8903	7776	3693	3198	2623	1891
0.55	1216	674	402	799	11469	9905	8780	7602	3728	3207	2593	1815
0.575	1251	670	388	818	11489	9835	8649	7421	3765	3217	2560	1736
0.6	1286	665	372	839	11506	9758	8511	7233	3803	3226	2525	1652
0.625	1323	660	356	862	11519	9673	8364	7035	3842	3234	2486	1564
0.65	1361	653	340	887	11528	9582	8208	6828	3882	3242	2444	1470
0.675	1401	645	322	914	11533	9482	8042	6611	3924	3249	2397	1370
0.7	1442	636	304	945	11533	9373	7864	6382	3968	3255	2346	1264
0.725	1485	625	286	979	11528	9253	7673	6139	4014	3260	2290	1150
0.75	1530	612	266	1017	11517	9123	7468	5882	4062	3263	2227	1028
0.775	1578	598	246	1059	11500	8979	7247	5608	4114	3265	2157	895
0.8	1629	580	225	1107	11475	8820	7007	5313	4170	3264	2079	750
0.825	1683	560	202	1162	11443	8644	6744	4995	4231	3262	1989	591
0.85	1741	536	179	1226	11400	8446	6453	4649	4299	3256	1887	414
0.875	1805	508	154	1300	11344	8221	6129	4266	4374	3247	1768	214
0.9	1876	472	128	1389	11273	7962	5760	3836	4461	3233	1626	15
0.925	1957	427	100	1499	11180	7653	5329	3340	4563	3212	1452	287
0.95	2054	368	70	1644	11053	7270	4803	2743	4691	3181	1228	624
0.975	2182	278	37	1857	10863	6742	4094	1951	4870	3131	906	1087

Note: All the values should be multiplied by 10^{-4} . The values of $E(Z_{1(r,s)})$ should be negative in the above table.

Table 4: The Bivariate Standard Normal Case for H=4 (CONT.)

ρ_{12}	$Var(Z_{1(r,s)})$								$Cov(Z_{1(r,s)}, Z_{2(r,s)})$							
	(1,1)	(1,2)	(1,3)	(1,4)	(2,1)	(2,2)	(2,3)	(2,4)	(1,1)	(1,2)	(1,3)	(1,4)	(2,2)	(2,3)	(2,4)	
0	4917	4917	4917	4917	3605	3605	3605	3605	0	0	0	0	0	0	0	
0.025	4948	4925	4907	4885	3609	3606	3603	3599	83	72	71	81	67	67	71	
0.05	4977	4931	4894	4852	3613	3608	3602	3594	167	143	142	161	133	133	142	
0.075	5004	4934	4880	4816	3617	3609	3601	3587	252	216	213	240	200	199	213	
0.1	5030	4934	4862	4780	3620	3611	3599	3581	338	288	283	318	268	266	283	
0.125	5054	4933	4843	4742	3622	3614	3598	3573	426	361	353	394	335	332	353	
0.15	5077	4929	4821	4702	3624	3616	3597	3565	516	434	423	469	403	398	423	
0.175	5098	4922	4797	4661	3626	3619	3595	3557	607	508	493	544	471	465	493	
0.2	5117	4914	4771	4619	3626	3622	3594	3548	699	582	562	617	540	531	562	
0.225	5135	4902	4742	4575	3627	3626	3592	3538	793	656	631	688	609	598	631	
0.25	5151	4889	4711	4529	3627	3629	3589	3528	888	731	700	759	679	664	700	
0.275	5166	4873	4678	4482	3626	3633	3587	3517	985	806	769	829	749	731	769	
0.3	5179	4854	4642	4434	3625	3638	3584	3505	1084	882	838	897	820	799	838	
0.325	5190	4833	4604	4384	3623	3642	3580	3493	1184	958	906	965	891	866	906	
0.35	5199	4810	4564	4333	3620	3647	3576	3480	1286	1034	974	1032	963	934	974	
0.375	5207	4784	4522	4280	3618	3652	3572	3467	1389	1111	1042	1097	1036	1002	1042	
0.4	5213	4756	4478	4226	3614	3657	3566	3452	1495	1189	1110	1162	1109	1071	1110	
0.425	5217	4725	4431	4171	3610	3662	3560	3437	1602	1266	1178	1226	1184	1140	1178	
0.45	5219	4691	4382	4114	3605	3668	3554	3421	1711	1345	1245	1289	1259	1209	1246	
0.475	5220	4655	4331	4056	3600	3673	3546	3405	1822	1424	1313	1350	1336	1279	1313	
0.5	5219	4617	4278	3996	3594	3678	3537	3387	1935	1504	1381	1412	1414	1349	1381	
0.525	5216	4576	4222	3935	3588	3683	3528	3369	2050	1584	1448	1472	1492	1420	1448	
0.55	5211	4533	4165	3873	3581	3688	3517	3350	2167	1665	1515	1531	1573	1492	1515	
0.575	5205	4487	4105	3809	3573	3693	3506	3330	2286	1746	1583	1590	1654	1565	1583	
0.6	5196	4438	4043	3744	3565	3697	3493	3309	2407	1828	1650	1648	1737	1638	1650	
0.625	5186	4387	3979	3677	3556	3700	3479	3287	2531	1911	1718	1705	1822	1712	1718	
0.65	5174	4334	3913	3609	3546	3703	3463	3264	2657	1995	1785	1761	1909	1787	1785	
0.675	5160	4278	3845	3540	3536	3704	3447	3240	2786	2079	1853	1817	1997	1863	1853	
0.7	5144	4219	3774	3469	3524	3705	3428	3214	2918	2165	1921	1872	2088	1941	1921	
0.725	5126	4157	3702	3396	3513	3704	3408	3188	3052	2251	1989	1926	2182	2019	1989	
0.75	5107	4094	3627	3323	3500	3702	3386	3160	3190	2339	2057	1980	2278	2099	2058	
0.775	5085	4027	3551	3247	3486	3698	3362	3131	3330	2428	2126	2034	2377	2181	2126	
0.8	5062	3958	3472	3170	3472	3692	3337	3100	3475	2518	2195	2087	2479	2264	2196	
0.825	5038	3887	3392	3092	3457	3684	3309	3067	3623	2610	2265	2139	2585	2349	2265	
0.85	5011	3813	3310	3012	3440	3674	3279	3034	3775	2703	2335	2191	2696	2437	2336	
0.875	4984	3736	3225	2931	3424	3661	3247	2998	3933	2798	2406	2243	2813	2527	2407	
0.9	4955	3658	3140	2848	3406	3645	3213	2960	4097	2896	2478	2294	2935	2621	2479	
0.925	4926	3578	3052	2763	3388	3626	3176	2921	4267	2997	2552	2345	3067	2719	2553	
0.95	4899	3497	2964	2677	3369	3604	3137	2879	4448	3103	2627	2396	3209	2822	2628	
0.975	4878	3417	2876	2589	3353	3582	3099	2836	4645	3216	2706	2448	3370	2934	2708	

Note: All the values should be multiplied by 10^{-4} . The values of $E(Z_{1(r,s)})$ should be negative in the above table.