Statistical Analysis of Adaptive Type-II Progressively Hybrid Censored Data

H. K. T. Ng*, D. Kundu†, P. S. Chan**

Abstract

In this paper a mixture of Type-I censoring and Type-II progressively censoring schemes, called adaptive progressively hybrid censoring scheme, is introduced for life-testing or reliability experiments. For this censoring scheme, the number of effective sample size m is fixed in advance and the progressive censoring scheme $(R_1, R_2, \ldots, R_m)$ is provided but the values of R_i may change during the experiment. If the experimental time exceeds a prefixed time T but the number of observed failures does not reach m, we would want to terminate the experiment as soon as possible by adjusting the R_i's. Computational formulas for the expected total test time are provided. Point and interval estimations of the mean lifetime under exponential distribution are discussed with this censoring scheme. Different methods have been compared using Monte Carlo simulation.

Keywords: Life-testing; maximum likelihood estimator; asymptotic methods; coverage probability; exponential distribution; chi-square distribution.

Postal address: *Department of Statistical Science, Southern Methodist University, 3225 Daniel Avenue, Dallas, Texas, USA 75275-0332

Postal address: † Department of Mathematics & Statistics, Indian Institute of Technology, Kanpur 208016, India.

Postal address: ** Department of Statistics, The Chinese University of Hong Kong, Shatin, Hong Kong

1 Introduction

In life-testing and reliability studies, the experimenter may not always obtain complete information on failure times for all experimental units. Data obtained from such experiments are called *censored data*. Saving the total time on test and the cost associated with it are some of the major reasons for censoring. A censoring scheme, which can balance between (i) total time spent for the experiment; (ii) number of units used in the experiment; and (iii) the efficiency of statistical inference based on the results of the experiment, is desirable.

The most common censoring schemes are Type-I (time) censoring, where the life-testing experiment will be terminated at a pre-fixed time T; and Type-II (item) censoring, where the life-testing experiment will be terminated as soon as the r-th (r is pre-fixed) failure is observed. However, the conventional Type-I and Type-II censoring schemes do not have the flexibility of allowing removal of units at points other than the terminal point of the experiment. Because of that, a more general censoring scheme called *progressive Type-II right censoring* has been introduced. Briefly, it can be described as follows: Consider an experiment in which n units are placed on a life-test. At the time of the first failure, R_1 units are randomly removed from the remaining $n-1$ surviving units. Similarly, at the time of the second failure, R_2 units from the remaining $n-2-R_1$ units are randomly removed. The test continues until the m-th failure at which time, all the remaining $R_m = n - m - R_1 - R_2 - \ldots - R_{m-1}$ units are removed. The $R_i^{'}s$ are fixed prior to the study. Readers may refer to Balakrishnan and Aggarwala [2] and Balakrishnan [1] for extensive reviews of the literature on progressive censoring.

Recently, Kundu and Joarder [14] proposed a censoring scheme called Type-II progressively hybrid censoring scheme, in which the life-testing experiment with progressive Type-II right censoring scheme $(R_1, \ldots, R_m)$ is terminated at a prefixed time T. However, the drawback of the Type-II progressive hybrid censoring, similar to the conventional Type-I censoring (time censoring), is that the effective sample size is random and it can turn out to be a very small number (even equal to zero), and therefore the standard statistical inference procedures may not be applicable or they will have low efficiency. In this paper we suggest an adaptive Type-II progressively hybrid censoring, where we allow $R_1, R_2, \ldots, R_m$ to be dependent on the failure times so that the effective sample size is always m, which is fixed in advance. A properly planned hybrid progressively censored life-testing experiment can save both the total time on test and the cost induced by failure of the units and increase the efficiency of statistical analysis. For application of different hybrid progressive censoring schemes in Bayesian variable sampling plans, one can refer to Lin, Hwang and Balakrishnan [15] for more details.

The rest of the paper is organized as follows. In Section 2, we first introduce the notation and describe the adaptive Type-II progressive hybrid censoring scheme. In Section 3, we derive the MLE for the parameter and discuss the construction of confidence interval for the parameter by different methods, when the underlying distribution is exponential. Section 4 provides the computation formulas for the expected total test time which will be useful for experimental planning purpose. In Section 5, the efficiency of the MLEs based on the proposed censoring scheme with the Type-II progressively hybrid censoring scheme proposed by Kundu and Joarder [14] are compared. Confidence intervals obtained by different methods are also compared in term of their coverage probabilities and conditional expected widths by means of extensive Monte Carlo simulations. Suggestions and comments are made based on these simulation results.

2 Model Description

Suppose n independent units are placed on a life-test with the corresponding lifetimes X_1, X_2, ..., X_n being identically distributed. We assume that X_i, $i = 1, 2, \ldots, n$ are independently and identically distributed with probability density function (PDF) $f_X(x; \boldsymbol{\theta})$ and cumulative distribution function (CDF) $F_X(x; \boldsymbol{\theta})$, where $\boldsymbol{\theta}$ denotes the vector of parameters. Prior to the experiment, a number $m < n$ is determined and the progressively Type-II censoring scheme $(R_1, R_2, \ldots, R_m)$ with $R_i > 0$ and $\sum\limits_{i=1}^{m} R_i + m = n$ is specified. During the experiment, i-th failure is observed and immediately after the failure, R_i functioning items are randomly removed from the test. We denote the m completely observed (ordered) lifetimes by $X_{i:m:n}^{(R_1,\ldots,R_m)}$, $i = 1, 2, \ldots, m$, which are the observed progressively Type-II right censored sample. For convenience, we will suppress the censoring scheme in the notation of the $X_{i:m:n}$'s.

We also denote the observed values of such a progressively Type-II right censored sample by $x_{1:m:n} < x_{2:m:n} < \ldots < x_{m:m:n}$.

As noted by Ng, Chan and Balakrishnan [17], it is expected that a progressive censoring plan has a longer test duration than a single (conventional Type-II) censoring plan in return for the gain in efficiency. The value of R_i at the time of the i-th failure $X_{i:m:n}$ may be determined depending on the objective of the experimenter. The objective may be controlling the total time on test or having higher chance to observe some extreme failures (usually leading to a gain in efficiency for statistical inference). Suppose the objective is to control the total time on test, a reasonable design to control the total time on test is to terminate the experiment at a prefixed time. This problem is considered in [14] for a fixed progressive censoring scheme $(R_1, R_2, \ldots, R_m)$ and they called this type of censoring as *Type-II progressively hybrid censoring.* The drawback of this censoring scheme is that the effective sample size is random and it can turn out to be a very small number (even equal to zero) so that usual statistical inference procedures will not be applicable or they will have low efficiency. Therefore, we suggest considering an adaptive censoring scheme in which the number of effective sample size m is fixed in advance and the progressive censoring scheme $(R_1, R_2, \ldots, R_m)$ is provided, but the values of some of the R_i may change accordingly during the experiment.

Suppose the experimenter provides a time T, which is an ideal total time on test, but we still allow the experiment to run over time T. If the m-th progressively censored observed failure occurs before time T (i.e. $X_{m:m:n} < T$), the experiment stops at the time $X_{m:m:n}$ (see Figure 1). Otherwise, once the experimental time passed time T but the number of observed failures has not reached m, we would want to terminate the experiment as soon as possible. This setting can be viewed as a design in which we are assured of getting m observed failure times for efficiency of statistical inference plus the total time on test will not be too far away from the ideal time T.

Theorem. *Suppose $X_{1:n}$ is the first order statistic of a random sample from a continuous distribution F with sample size n. For $n_1 < n_2$, $E(X_{1:n_1}) \geq E(X_{1:n_2})$.*

Proof: *The distribution function of the first order statistic of a random sample from a continuous distribution F with sample size n is* $\Pr(X_{1:n} \leq x) = 1 - [1 - F(x)]^n$. *Let the first order*

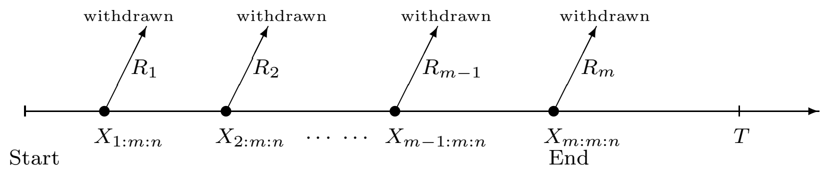

Figure 1: Experiment terminates before time T (i.e. $X_{m:m:n} < T$)

statistic of a random sample of size n_1 and n_2 be Y_1 and Y_2, and their distribution functions to be $G_1(y) = 1 - [1 - F(y)]^{n_1}$ and $G_2(y) = 1 - [1 - F(y)]^{n_2}$, respectively. For $n_1 < n_2$, we have $G_1(y) \leq G_2(y)$, i.e. Y_2 is stochastically smaller than Y_1. From David and Nagaraja [9] (Section 4.4), if $E(Y_1)$ and $E(Y_2)$ exists, then $E(Y_1) \geq E(Y_2)$. $\odot$

Suppose J is the number of failures observed before time T, i.e.

$$X_{J:m:n} < T < X_{J+1:m:n}, \quad J = 0, 1, \ldots, m$$

where $X_{0:m:n} \equiv 0$ and $X_{m+1:m:n} \equiv \infty$. According to the above theorem, after the experiment passed time T, we set $R_{J+1} = \cdots = R_{m-1} = 0$ and $R_m = \left(n - m - \sum_{i=1}^{J} R_i\right)$. This formulation leads us to terminate the experiment as soon as possible if the $(J+1)$-th failure time is greater than T for $(J+1) < m$. Figure 2 gives the schematic representation of this situation. The value of T plays an important role in the determination of the values of R_i and also as a compromise between a shorter experimental time and a higher chance to observe extreme failures. One extreme case is when $T \to \infty$, which means time is not the main consideration for the experimenter, then we will have a usual progressive Type-II censoring scheme with the pre-fixed progressively censoring scheme $(R_1, \ldots, R_m)$. Another extreme case can occur when $T = 0$, which means we always want to end the experiment as soon as possible, then we will have $R_1 = \cdots = R_{m-1} = 0$ and $R_m = n - m$ which results in the conventional Type-II censoring scheme.

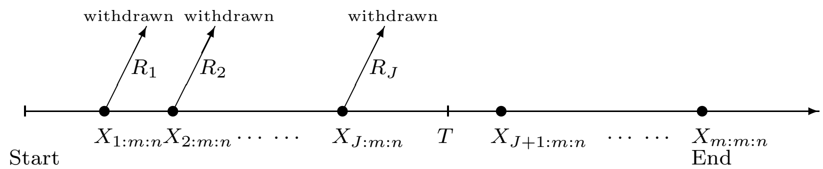

Figure 2: Experiment terminates after time T (i.e. $X_{m:m:n} \geq T$) with adaptive Type-II progressive hybrid censoring

3 Statistical Inference for Exponential Distribution

3.1 Point Estimation

Given $J = j$, the likelihood function is given by

$$\begin{aligned} L(\boldsymbol{\theta}|J=j) &= c\left[\prod_{i=1}^{m} f(x_{i:m:n};\boldsymbol{\theta})\right]\left\{\prod_{i=1}^{j}[1-F(x_{i:m:n};\boldsymbol{\theta})]^{R_i}\right\} \\ &\quad \times [1-F(x_{m:m:n};\boldsymbol{\theta})]^{\left(n-m-\sum_{i=1}^{j} R_i\right)} \end{aligned}$$

where

$$\begin{aligned} c &= n(n-R_1-1)\cdots(n-R_1-\ldots-R_j-1) \\ &\quad \times(n-R_1-\ldots-R_j-2)\cdots(n-R_1-\ldots-R_j-m+1). \end{aligned}$$

The exponential distribution is one of the most widely used life-time models in the areas of life testing and reliability. The volume by Balakrishnan and Basu [3] (see also Chapter 19 of [12]) provides an extensive review of the genesis of the distribution and its properties, including several characterization results. For exponential distribution with PDF $f(x) = \lambda e^{-\lambda x}, x > 0$ and CDF $F(x) = 1 - e^{-\lambda x}, x > 0$ (denote as $\text{Exp}(\lambda)$), the log-likelihood function is

$$\begin{aligned} \ln L(\lambda|J=j) &= \textit{constant} + m\ln\lambda \\ &\quad -\lambda\left[\sum_{i=1}^{m} x_{i:m:n} + \sum_{i=1}^{j} R_i x_{i:m:n} + \left(n-m-\sum_{i=1}^{j} R_i\right)x_{m:m:n}\right]. \end{aligned}$$

We have

$$\frac{\partial L(\lambda|J=j)}{\partial\lambda} = \frac{m}{\lambda} - \sum_{i=1}^{m} x_{i:m:n} - \sum_{i=1}^{j} R_i x_{i:m:n} - \left(n-m-\sum_{i=1}^{j} R_i\right)x_{m:m:n},$$

$$\frac{\partial^2 L(\lambda|J=j)}{\partial\lambda^2} = -\frac{m}{\lambda^2},$$

therefore, the maximum likelihood estimator of λ is given by

$$\hat{\lambda} = \frac{m}{\sum_{i=1}^{m} x_{i:m:n} + \sum_{i=1}^{j} R_i x_{i:m:n} + \left(n - m - \sum_{i=1}^{j} R_i\right) x_{m:m:n}}$$

and an estimate of asymptotic variance is $\widehat{var}(\hat{\lambda}) = \frac{\hat{\lambda}^2}{m}$. For brevity we denote

$$\delta = \delta(\mathbf{x}, j) = \sum_{i=1}^{m} x_{i:m:n} + \sum_{i=1}^{j} R_i x_{i:m:n} + \left(n - m - \sum_{i=1}^{j} R_i\right) x_{m:m:n}.$$

3.2 Construction of Confidence Interval for λ

1. Conditional exact confidence interval (EX)

We consider $\theta = 1/\lambda$ and the MLE of θ is $\hat{\theta} = \frac{1}{\hat{\lambda}}$. Let Y be a Gamma(α, β), i.e., a gamma random variable with shape parameter α and scale parameter β. Then the PDF of Y is given by

$$g(y; \alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y}, \quad y > 0.$$

where $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$ is the gamma function.

Conditional on $J = j > 0$, the exact conditional distribution of $\hat{\theta}$ is given by (see Appendix B)

$$f_{\hat{\theta}}(t|J=j) = \frac{\sum_{i=0}^{j} c_{i,j}(\mathbf{R}) e^{-Tb^*_{i,j}(\mathbf{R})/\theta} g\left(t - \frac{Tb^*_{i,j}(\mathbf{R})}{m}; m, \frac{m}{\theta}\right)}{\sum_{i=0}^{j} c_{i,j}(\mathbf{R}) e^{-Tb^*_{i,j}(\mathbf{R})/\theta}}.$$

where $\mathbf{R} = (R_1, R_2, \ldots, R_j)$,

$$c_{i,j}(\mathbf{R}) = \frac{(-1)^i}{\left\{\prod_{l=1}^{i} \sum_{k=j-i+1}^{j-i+l} (R_k + 1)\right\} \left\{\prod_{l=1}^{j-i} \sum_{k=l}^{j-i} (R_k + 1)\right\}},$$

$$b^*_{i,j}(\mathbf{R}) = \sum_{k=j-i+1}^{j} (R_k + 1) + \left(n - j - \sum_{k=1}^{j} R_k\right),$$

in which we use the usual conventions that $\prod_{l=1}^{0} d_l \equiv 1$ and $\sum_{l=i}^{i-1} d_l \equiv 0$. For $J=0$, the conditional PDF of $\hat{\theta}$, given $J=0$, is

$$f_{\hat{\theta}}(t|J=0) = g\left(t-\frac{nT}{m}; m, \frac{m}{\theta}\right).$$

We can show that the probability $\Pr(\hat{\theta} \geq w|J=j)$ is increasing function of θ. This assumption guarantees the invertibility of the pivotal quantities and construct exact confidence intervals of θ based on the exact distribution of $\hat{\theta}$. Several articles including [5, 8, 7, 11, 13], have used this approach for constructing exact confidence intervals in different contexts. For $J=j>0$, the conditional $100(1-\alpha)\%$ confidence interval for θ, (θ_L, θ_U) can be obtained as the solutions of the following nonlinear equations:

$$\begin{aligned}
\sum_{i=0}^{j} c_{i,j}(\mathbf{R}) e^{-Tb^*_{i,j}(\mathbf{R})/\theta_L}\left[\frac{\alpha}{2} - \Gamma\left(m, \frac{m}{\theta_L}\cdot \max\left(0, \hat{\theta} - \frac{Tb^*_{i,j}(\mathbf{R})}{m}\right)\right)\right] &= 0,\\
\sum_{i=0}^{j} c_{i,j}(\mathbf{R}) e^{-Tb^*_{i,j}(\mathbf{R})/\theta_U}\left[1-\frac{\alpha}{2} - \Gamma\left(m, \frac{m}{\theta_U}\cdot \max\left(0, \hat{\theta} - \frac{Tb^*_{i,j}(\mathbf{R})}{m}\right)\right)\right] &= 0,
\end{aligned}$$

where $\Gamma(m,z) = \frac{1}{\Gamma(m)}\int_z^\infty t^{m-1}\exp(-t)dt$. For $J=0$, the conditional $100(1-\alpha)\%$ confidence interval for θ, can be obtained as the solutions of the following nonlinear equations:

$$\begin{aligned}
\frac{\alpha}{2} &= \Gamma\left(m, \frac{m}{\theta_L}\cdot \max\left(0, \hat{\theta}-\frac{nT}{m}\right)\right),\\
1-\frac{\alpha}{2} &= \Gamma\left(m, \frac{m}{\theta_U}\cdot \max\left(0, \hat{\theta}-\frac{nT}{m}\right)\right).
\end{aligned}$$

The corresponding conditional $100(1-\alpha)\%$ confidence interval for λ, (λ_L, λ_U), can be obtained as

$$\lambda_L = \frac{1}{\theta_U} \text{ and } \lambda_U = \frac{1}{\theta_L}.$$

2. Normal approximation of the MLE (NA)

Based on the normal approximation of the MLE, we can say that $\frac{\hat{\lambda}-\lambda}{\sqrt{Var(\hat{\lambda})}}$ is asymptotically normally distributed with zero mean and unit variance, i.e.

$$\frac{\hat{\lambda}-\lambda}{\sqrt{Var(\hat{\lambda})}} \dot{\sim} N(0,1).$$

If we replace the variance $var(\hat{\lambda})$ by its estimate, we can obtain a $100(1-\alpha)\%$ confidence interval for λ as

$$\hat{\lambda} \pm z_{1-\alpha/2} \frac{\hat{\lambda}}{\sqrt{m}},$$

where z_q is the q-th upper percentile of a standard normal distribution.

3. Normal approximation of the log-transformed MLE (NL)

The problem with applying normal approximation of the MLE is that when the sample size is small, the normal approximation may be poor. However a different transformation of the MLE can be used to correct the inadequate performance of the normal approximation. Based on the normal approximation of the log-transformed MLE, a $100(1-\alpha)\%$ confidence interval for λ is

$$\exp\left[\ln(\hat{\lambda}) \pm z_{1-\alpha/2}\sqrt{Var(\ln\hat{\lambda})}\right],$$

where $var(\ln\hat{\lambda})$ can be approximated by delta method as

$$\widehat{Var}(\ln\hat{\lambda}) = \frac{1}{m}.$$

A $100(1-\alpha)\%$ confidence interval for λ is

$$\exp\left[\ln(\hat{\lambda}) \pm z_{1-\alpha/2}\sqrt{\frac{1}{m}}\right].$$

4. Likelihood ratio-based confidence interval (LR)

A likelihood ratio-based conditional confidence interval is constructed using the likelihood ratio statistic [16] for testing the hypothesis $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda \neq \lambda_0$. In our case, the likelihood ratio statistic is given by

$$2[\ln L(\hat{\lambda}|J = j) - \ln L(\lambda_0|J = j)].$$

The asymptotic distribution of the likelihood ratio statistic is chi-square with one degree of freedom (χ^2_1). An approximate $100(1-\alpha)\%$ confidence interval for λ is the region

$$\left\{\lambda : 2[\ln L(\hat{\lambda}|J = j) - \ln L(\lambda|J = j)] \leq \chi^2_{1,1-\alpha}\right\}.$$

5. Bootstrap confidence interval

We construct confidence intervals based on the parametric bootstrap, using the percentile bootstrap method and bootstrap t method (see, for example, [10]). To obtain the percentile bootstrap confidence intervals for λ, we use the following algorithm:

Parametric percentile bootstrap confidence interval (PB):

1. Based on the original sample $\mathbf{x} = (x_{1:m:n}, x_{2:m:n}, \ldots, x_{m:m:n})$, obtain $\hat{\lambda}$, the MLE of λ.

2. Simulate the adaptive Type-II hybrid progressive censored sample, say $(y_{1:m:n}, \ldots, y_{m:m:n})$, with the underlying distribution as $\text{Exp}(\hat{\lambda})$ (simulation algorithm is described in the following section) with censoring scheme $(R_1, \ldots, R_m)$ and pre-fixed T.

3. Compute the MLEs of λ based on $y_{1:m:n}, y_{2:m:n}, \ldots, y_{m:m:n}$, say $\hat{\lambda}^*$.

4. Repeat Steps 2 - 3 B times and obtain $\hat{\lambda}^{*(1)}$, $\hat{\lambda}^{*(2)}$, $\ldots$, $\hat{\lambda}^{*(B)}$.

5. Arrange $\hat{\lambda}^{*(1)}$, $\hat{\lambda}^{*(2)}$, $\ldots$, $\hat{\lambda}^{*(B)}$ in ascending order and obtain $\hat{\lambda}^{*[1]}$, $\hat{\lambda}^{*[2]}$, $\ldots$, $\hat{\lambda}^{*[B]}$.

A two-sided $100(1-\alpha)\%$ percentile bootstrap confidence interval of λ, say $[\lambda_L^*, \lambda_U^*]$, is then given by

$$\lambda_L^* = \hat{\lambda}^{*([B\alpha/2])}, \qquad \lambda_U^* = \hat{\lambda}^{*([B(1-\alpha/2)])}.$$

To obtain the bootstrap-t confidence intervals for λ, we use the following algorithm:

Parametric bootstrap-t confidence interval (TB):

1. Based on the original sample $\mathbf{x} = (x_{1:m:n}, x_{2:m:n}, \ldots, x_{m:m:n})$, obtain $\hat{\lambda}$, the MLE of λ.

2. Simulate the adaptive Type-II hybrid progressive censored sample, say $(y_{1:m:n}, \ldots, y_{m:m:n})$, with the underlying distribution as $\text{Exp}(\hat{\lambda})$ (simulation algorithm is described in the following section) with censoring scheme $(R_1, \ldots, R_m)$ and pre-fixed T.

3. Compute the MLEs of λ based on $y_{1:m:n}, y_{2:m:n}, \ldots, y_{m:m:n}$, say $\hat{\lambda}^*$.

4. Compute the t-statistic

$$T = \frac{\sqrt{m}(\hat{\lambda}^* - \hat{\lambda})}{\hat{\lambda}^*}$$

5. Repeat Steps 2 - 4 B times and obtain $T^{(1)}$, $T^{(2)}$, $\ldots$, $T^{(B)}$.

6. Arrange $T^{(1)}, T^{(2)}, \ldots, T^{(B)}$ in ascending order and obtain $T^{[1]}, T^{[2]}, \ldots, T^{[B]}$.

A two-sided $100(1-\alpha)\%$ bootstrap-t confidence interval of λ, say $[\lambda_{t,L}, \lambda^*_{t,U}]$, is then given by

$$\lambda_{t,L} = \hat{\lambda} + T^{([B\alpha/2])}\frac{\hat{\lambda}}{\sqrt{m}}, \qquad \lambda_{t,U} = \hat{\lambda} + T^{([B(1-\alpha/2)])}\frac{\hat{\lambda}}{\sqrt{m}}.$$

6. Bayesian Analysis

Bayesian inference provides an alternative way to estimate the parameter λ, especially when prior information about the sampling distribution of λ is available. A reasonable choice of prior distribution for Λ is a gamma prior with parameters a and b (denote by Gamma(a, b)). Given the data, the posterior density of λ is proportional to $\lambda^{a+m-1}\exp(-\lambda\delta)$, which is the kernel of Gamma$(a+m, b+\delta)$. Therefore, if we assume the commonly used squared error loss function, the Bayesian estimator of λ is given by

$$\tilde{\lambda} = \frac{a+m}{b+\delta}.$$

A $100(1-\alpha)\%$ Bayesian credible interval, say $(\tilde{\lambda}_L, \tilde{\lambda}_U)$, can be obtained as the solutions of the following equations:

$$\begin{aligned}
\int_0^{\tilde{\lambda}_L} \frac{(b+\delta)^{(a+m)}}{\Gamma(a+m)}\lambda^{a+m-1}\exp(-(b+\delta)\lambda)d\lambda &= \frac{\alpha}{2},\\
\int_0^{\tilde{\lambda}_U} \frac{(b+\delta)^{(a+m)}}{\Gamma(a+m)}\lambda^{a+m-1}\exp(-(b+\delta)\lambda)d\lambda &= 1-\frac{\alpha}{2}.
\end{aligned}$$

If a is an integer, a $100(1-\alpha)\%$ Bayesian credible interval can be obtained as

$$\left(\frac{\chi^2_{(a+m),1-\alpha/2}}{2(b+\delta)}, \frac{\chi^2_{(a+m),\alpha/2}}{2(b+\delta)}\right).$$

In the simulation study presented in Section 4 below, we use the non-informative prior with $a = b = 0$ (say, BN) and prior distribution Gamma(0.1, 0.1) (say, BA).

3.3 Expected Total Test Time

As we mentioned before, the difference between the proposed adaptive Type-II progressively hybrid censoring scheme and the one proposed by Kundu and Joarder [14] is that the maximum total time on test is not fixed in advance. In practical applications, it is useful to have an idea on the total test time for a particular life-testing plan. When an adaptive Type-II progressively hybrid censoring scheme is used, one can obtain the expected total test time (ETT) by

$$ETT = E(X_{m:m:n}) \;=\; \sum_{j=1}^{m} \Pr(J=j)E(X_{m:m:n}|J=j). \tag{1}$$

For exponential distribution, we can show that the probability mass function of the value J for a pre-fixed value of T is (see Appendix A)

$$\begin{aligned}\Pr(J=j) &= \Pr(X_{j:m:n} < T \le X_{j+1:m:n}) \\ &= c_{j-1}\exp\left(-r_{j+1}\lambda T\right)\sum_{i=1}^{j}\frac{a_{i,j}}{(r_i - r_{j+1})}\left\{1-\exp\left[-(r_i - r_{j+1})\lambda T\right]\right\},\end{aligned} \tag{2}$$

$j = 0, 1, 2, \ldots, m$, where

$$\begin{aligned} r_j &= m - j + 1 + \sum_{i=j}^{m} R_i,\, j = 1, 2, \ldots, m, \text{ with } r_{m+1} \equiv 0, \\ c_{j-1} &= \prod_{i=1}^{j} r_i,\, j = 1, 2, \ldots, m, \text{ with } c_0 \equiv 1, \\ a_{i,j} &= \prod_{\substack{k=1 \\ k\neq i}}^{j} \frac{1}{r_k - r_i},\, 1 \le i \le j \le m. \end{aligned}$$

Based on the memoryless property of exponential distribution and the properties of exponential order statistics, we can obtain the conditional expectation of $X_{m:m:n}$ for $j = 0, 1, \ldots, m-1$ as

$$\begin{aligned} E(X_{m:m:n}|J=j) &= T + E(Y_{m^*:n^*}) \\ &= T + \lambda \sum_{k=n^*-m^*+1}^{n^*} \frac{1}{k}, \end{aligned} \tag{3}$$

where $Y_{m^*:n^*}$ denotes the m^*-th order statistic from a sample of size n^* with $n^* = n - j - \sum_{i=1}^{j} R_i$ and $m^* = m - j$. For $j = m$, the conditional expectation of $X_{m:m:n}$ is

$$\begin{aligned} E(X_{m:m:n}|J=m) &= \int_0^T \frac{x f_{X_{j:m:n}}(x)}{F_{X_{j:m:n}}(T)} dx \\ &= \frac{c_{m-1}\sum_{i=1}^{m}\frac{a_{i,m}}{r_i}\left[\frac{1-\exp(-r_i\lambda T)}{r_i\lambda} - T\exp\left(-r_i\lambda T\right)\right]}{\Pr(J=m)}. \end{aligned} \tag{4}$$

Usually, some information on λ from past data or prior experience should be available. Therefore, from Eqs. (2)–(4), we can approximate the expected test length provided that the values of n and m, the progressive censoring scheme $(R_1, \ldots, R_m)$ and T are specified.

4 Monte Carlo Simulation and Numerical Comparisons

We first describe the procedure to generate Type-II progressively hybrid censored data (from any distribution F) for given values of n, m, T and $(R_1, \ldots, R_m)$:

1. Generate an ordinary Type-II progressive censored sample $X_{1:m:n}$, $X_{2:m:n}$, $\ldots$, $X_{m:m:n}$ with censoring scheme $(R_1, \ldots, R_m)$ based on the method proposed in [6].

2. Determine the value of J, where $X_{J:m:n} < T < X_{J+1:m:n}$, and discard the sample $X_{j+1:m:n}, \ldots, X_{m:m:n}$.

3. Generate the first to $(m-j-1)$-th order statistics from a truncated distribution $f(x)/[1-F(x_{j+1:m:n})]$ with sample size $\left(n - \sum_{i=1}^{j} R_i - j - 1\right)$ as $X_{j+2:m:n}$, $X_{j+3:m:n}, \ldots, X_{m:m:n}$.

For exponential distribution, a more convenient algorithm based on spacing of progressively censored exponential order statistics (Balakrishnan and Aggarwala, 2000, Section 3.3) can be used in place of Step 1 above.

4.1 Comparison of two progressively hybrid censoring schemes

In this subsection, we compare the efficiency of the MLE based on the adaptive Type-II progressively hybrid censoring scheme with the hybrid censoring scheme proposed by Kundu and Joarder [14]. If the censoring scheme proposed in [14] is employed, the MLE is given by

$$\hat{\lambda}_{KJ} = \begin{cases} \dfrac{J}{\sum\limits_{i=1}^{J}(R_i+1)x_{i:m:n} + T\left(n-J-\sum\limits_{i=1}^{J} R_i\right)} & x_{m:m:n} > T, \\ \dfrac{m}{\sum\limits_{i=1}^{m}(R_i+1)x_{i:m:n}} & x_{m:m:n} \leq T. \end{cases}$$

The expected total time on test and expected number of failure based on the censoring scheme proposed in [14] can be computed respectively by

$$ETT_{KJ} = \Pr(J=m)E(X_{m:m:n}|J=m) + [1-\Pr(J=m)]T$$

$$\text{and } EM_{KJ} = \sum_{j=1}^{m} j \Pr(J = j).$$

Monte Carlo simulation is used to compare the efficiency of the MLE coming from two different hybrid censoring schemes. Different values of n, m and T and three progressive censoring schemes for each setting are considered. For brevity, for example, the censoring scheme (0,0,1,1,1,1,1,0,0,0) is denoted by (0*2,1*5,0*3). Without loss of generality, we set $\lambda = 1$. The biases and mean squares errors (MSEs) are estimated based on 10,000 simulations and they are reported in Tables 1–3. For the sake of comparison, based on exact calculation, the expected total time on test for both censoring schemes, the expected effective sample size for scheme proposed in [14] and the probability of getting no observation (i.e. $\Pr(J = 0)$) are also presented in Tables 1–3.

From Tables 1–3, we observed that the MLEs based on the adaptive Type-II progressively hybrid censoring schemes give larger biases but smaller MSEs compare to those based on the hybrid censoring scheme proposed in [14]. Although the proposed censoring scheme gives better performance in estimation in terms of MSE, the trade-off is a longer experimental time and a larger effective sample size. Therefore, the proposed censoring scheme will be useful to obtain a higher efficiency in estimation of parameter when the length of the experiment is not a major concern.

In studying the effect of different censoring schemes, we observed that the MSEs are not much difference for the three chosen censoring schemes for each set of n and m, however, the expected total time on test can be very different for different censoring scheme. For example, in Table 1, (n, m) = (15,10) with $T = 0.25$, the MSEs for censoring schemes (0*9,5), (5,0*9) and (0*2,1*5,0*3) are 1.670, 1.694 and 1.664 and their corresponding expected total time on test are 1.0396, 2.8566 and 1.2494, respectively. This suggested that a significantly reduction in the experimental time without sacrificing much in efficiency of estimation, one should use the conventional Type-II censoring scheme and avoid the use of censoring schemes with heavy censoring at the early stages of the experiment.

Table 1: Biases and MSEs of the MLEs for different sample sizes and censoring schemes for $m = 5$

(n, m)	Scheme	$Bias$	MSE	ETT	$Bias_{KJ}$	MSE_{KJ}	ETT_{KJ}	EM_{KJ}	$\Pr(J=0)$
(15, 5)					$T = 0.25$				
	(0,0,0,0,10)	0.2536	0.6210	0.3916	0.1507	0.6765	0.2360	3.1848	0.0235
	(10,0,0,0,0)	0.2336	0.5365	2.1109	0.2855	1.0113	0.2499	1.6455	0.0235
	(2,2,2,2,2)	0.2501	0.5690	0.6023	0.1882	0.6634	0.2478	2.6382	0.0235
					$T = 0.50$				
	(0,0,0,0,10)	0.2455	0.6442	0.3893	0.2217	0.6664	0.3570	4.6313	0.0006
	(10,0,0,0,0)	0.2477	0.5357	2.1491	0.1934	0.6936	0.4983	2.4002	0.0006
	(2,2,2,2,2)	0.2619	0.6070	0.7113	0.2084	0.6282	0.4609	3.8843	0.0006
					$T = 1.00$				
	(0,0,0,0,10)	0.2513	0.5788	0.3893	0.2509	0.5793	0.3888	4.9946	0.0000
	(10,0,0,0,0)	0.2615	0.6118	2.1500	0.2007	0.6479	0.9651	3.4234	0.0000
	(2,2,2,2,2)	0.2451	0.5573	0.7580	0.2304	0.5669	0.6821	4.7511	0.0000
(25, 5)					$T = 0.25$				
	(0,0,0,0,20)	0.2480	0.5924	0.2183	0.2094	0.6264	0.1919	4.4330	0.0019
	(20,0,0,0,0)	0.2389	0.5366	2.1198	0.2561	0.9535	0.2499	1.7534	0.0019
	(4,4,4,4,4)	0.2436	0.5457	0.4068	0.1837	0.5803	0.2382	3.5675	0.0019
					$T = 0.50$				
	(0,0,0,0,20)	0.2526	0.5577	0.2182	0.2518	0.5588	0.2175	4.9853	0.0000
	(20,0,0,0,0)	0.2448	0.6059	2.1233	0.1911	0.7457	0.4979	2.4728	0.0000
	(4,4,4,4,4)	0.2472	0.6198	0.4513	0.2231	0.6328	0.3810	4.5896	0.0000
					$T = 1.00$				
	(0,0,0,0,20)	0.2496	0.5793	0.2182	0.2496	0.5793	0.2182	5.0000	0.0000
	(20,0,0,0,0)	0.2584	0.5938	2.1233	0.2009	0.6297	0.9618	3.4672	0.0000
	(4,4,4,4,4)	0.2544	0.5877	0.4566	0.2524	0.5897	0.4500	4.9663	0.0000
(50, 5)					$T = 0.25$				
	(0,0,0,0,45)	0.2469	0.5931	0.1043	0.2463	0.5939	0.1040	4.9897	0.0000
	(45,0,0,0,0)	0.2593	0.5705	2.1033	0.2964	1.0349	0.2499	1.8212	0.0000
	(9,9,9,9,9)	0.2545	0.5861	0.2255	0.2318	0.5984	0.1905	4.5896	0.0000
					$T = 0.50$				
	(0,0,0,0,45)	0.2497	0.5724	0.1043	0.2497	0.5724	0.1043	5.0000	0.0000
	(45,0,0,0,0)	0.2511	0.5645	2.1033	0.1999	0.7010	0.4976	2.5244	0.0000
	(9,9,9,9,9)	0.2440	0.5437	0.2283	0.2417	0.5459	0.2250	4.9663	0.0000
					$T = 1.00$				
	(0,0,0,0,45)	0.2464	0.5307	0.1043	0.2464	0.5307	0.1043	5.0000	0.0000
	(45,0,0,0,0)	0.2419	0.5590	2.1033	0.1820	0.5919	0.9591	3.4985	0.0000
	(9,9,9,9,9)	0.2560	0.6227	0.2283	0.2560	0.6227	0.2283	4.9998	0.0000

Table 2: Biases and MSEs of the MLEs for different sample sizes and censoring schemes for $m = 10$

(n,m)	Scheme	$Bias$	MSE	ETT	$Bias_{KJ}$	MSE_{KJ}	ETT_{KJ}	EM_{KJ}	$\Pr(J=0)$
(15, 10)					$T = 0.25$				
	(0*9,5)	0.1123	0.1670	1.0396	0.0652	0.3177	0.2500	3.3180	0.0235
	(5,0*9)	0.1159	0.1694	2.8566	0.1254	0.4738	0.2500	2.4817	0.0235
	(0*2,1*5,0*3)	0.1111	0.1664	1.2494	0.1079	0.3837	0.2500	3.2071	0.0235
					$T = 0.50$				
	(0*9,5)	0.1145	0.1683	1.0350	0.0501	0.2062	0.4979	5.8921	0.0006
	(5,0*9)	0.1097	0.1640	2.8947	0.0822	0.3017	0.5000	4.1511	0.0006
	(0*2,1*5,0*3)	0.1102	0.1659	1.7346	0.0791	0.2526	0.5000	5.2861	0.0006
					$T = 1.00$				
	(0*9,5)	0.1091	0.1610	1.0349	0.0828	0.1737	0.8837	8.9857	0.0000
	(5,0*9)	0.1083	0.1644	2.8956	0.0743	0.2064	0.9984	6.4526	0.0000
	(0*2,1*5,0*3)	0.1094	0.1665	2.4272	0.0917	0.1924	0.9950	7.4378	0.0000
(25, 10)					$T = 0.25$				
	(0*9,15)	0.1087	0.1697	0.4979	0.0266	0.2133	0.2488	5.5127	0.0019
	(15,0*9)	0.1090	0.1621	2.8645	0.1054	0.4505	0.2500	2.6975	0.0019
	(1*5,2*5)	0.1104	0.1640	0.6550	0.0642	0.2544	0.2500	4.8865	0.0019
					$T = 0.50$				
	(0*9,15)	0.1176	0.1718	0.4977	0.0883	0.1878	0.4359	8.9511	0.0000
	(15,0*9)	0.1158	0.1695	2.8690	0.0785	0.2959	0.5000	4.3138	0.0000
	(1*5,2*5)	0.1108	0.1634	0.8385	0.0782	0.1845	0.4946	7.5501	0.0000
					$T = 1.00$				
	(0*9,15)	0.1112	0.1746	0.4977	0.1110	0.1748	0.4973	9.9928	0.0000
	(15,0*9)	0.1082	0.1664	2.8690	0.0790	0.2096	0.9981	6.5511	0.0000
	(1*5,2*5)	0.1188	0.1710	0.9884	0.1091	0.1753	0.8422	9.4527	0.0000
(50, 10)					$T = 0.25$				
	(0*9,40)	0.1135	0.1758	0.2207	0.0936	0.1893	0.2041	9.3057	0.0000
	(40,0*9)	0.1087	0.1614	2.8490	0.1140	0.4267	0.2500	2.8477	0.0000
	(4*10)	0.1121	0.1670	0.4541	0.0801	0.1960	0.2487	7.1350	0.0000
					$T = 0.50$				
	(0*9,40)	0.1115	0.1684	0.2207	0.1115	0.1685	0.2206	9.9985	0.0000
	(40,0*9)	0.1103	0.1587	2.8490	0.0837	0.2728	0.5000	4.4298	0.0000
	(4*10)	0.1096	0.1619	0.5654	0.0959	0.1675	0.4480	9.1792	0.0000
					$T = 1.00$				
	(0*9,40)	0.1078	0.1584	0.2207	0.1078	0.1584	0.2207	10.0000	0.0000
	(40,0*9)	0.1171	0.1714	2.8490	0.0869	0.2123	0.9979	6.6215	0.0000
	(4*10)	0.1134	0.1680	0.5856	0.1116	0.1690	0.5725	9.9326	0.0000

Table 3: Biases and MSEs of the MLEs for different sample sizes and censoring schemes for $m = 15$

(n, m)	Scheme	$Bias$	MSE	ETT	$Bias_{KJ}$	MSE_{KJ}	ETT_{KJ}	EM_{KJ}	$\Pr(J=0)$
(25, 15)					$T = 0.25$				
	(0*14,10)	0.0693	0.0912	0.8872	0.0210	0.1868	0.2500	5.5300	0.0019
	(10,0*14)	0.0674	0.0909	3.2871	0.0590	0.2976	0.2500	3.6417	0.0019
	(0*2,1*10,0*3)	0.0659	0.0903	1.1128	0.0464	0.2216	0.2500	5.2300	0.0019
					$T = 0.50$				
	(0*14,10)	0.0717	0.0942	0.8870	0.0256	0.1173	0.4983	9.8210	0.0000
	(10,0*14)	0.0734	0.0919	3.2916	0.0483	0.1861	0.5000	6.1548	0.0000
	(0*2,1*10,0*3)	0.0765	0.0968	1.5211	0.0535	0.1445	0.5000	8.4838	0.0000
					$T = 1.00$				
	(0*14,10)	0.0720	0.0929	0.8870	0.0623	0.0983	0.8355	14.3913	0.0000
	(10,0*14)	0.0681	0.0919	3.2916	0.0478	0.1255	0.9999	9.6351	0.0000
	(0*2,1*10,0*3)	0.0665	0.0912	2.4247	0.0549	0.1073	0.9987	11.7043	0.0000
(50, 15)					$T = 0.25$				
	(0*14,35)	0.0708	0.0915	0.3681	0.0216	0.1093	0.2616	10.9217	0.0000
	(35,0*14)	0.0683	0.0936	3.2716	0.0718	0.2917	0.2500	3.8743	0.0000
	(2*5,3*5,2*5)	0.0743	0.0964	0.6125	0.0482	0.1398	0.2500	8.5478	0.0000
					$T = 0.50$				
	(0*14,35)	0.0702	0.0938	0.3600	0.0678	0.0956	0.3565	14.8721	0.0000
	(35,0*14)	0.0690	0.0933	3.2716	0.0506	0.1872	0.5000	6.3353	0.0000
	(2*5,3*5,2*5)	0.0723	0.0912	0.8577	0.0585	0.1023	0.4979	12.0303	0.0000
					$T = 1.00$				
	(0*14,35)	0.0673	0.0895	0.3646	0.0673	0.0895	0.3646	15.0000	0.0000
	(35,0*14)	0.0732	0.0928	3.2716	0.0545	0.1245	0.9999	9.7446	0.0000
	(2*5,3*5,2*5)	0.0734	0.0942	1.0486	0.0672	0.0969	0.8787	14.3384	0.0000

4.2 Comparison of methods for confidence interval construction

In this subsection, Monte Carlo simulation was employed to investigate the performance of different confidence interval construction method. Criteria appropriate to the evaluation of the various methods under scrutiny include: closeness of the coverage probability to its nominal value; and expected interval width. For each simulated sample under a particular setting, we computed 95% confidence intervals and checked whether the true value lay within the interval and recorded the length of the confidence interval. This procedure was repeated 10,000 times. The estimated probability coverage was computed as the number of confidence intervals that covered the true values divided by 10,000 while the estimated conditional expected width of the confidence interval was computed as the sum of the lengths for all intervals covered the true values divided by the number of confidence intervals that covered the true values. The coverage probabilities and the conditional expected widths for different sample sizes, censoring schemes and $T = 0.25, 0.5$ and 1.0 are presented in Tables 4–6.

When comparing in terms of coverage probabilities, EX, NA, LR, PB, BN and BA are maintaining the coverage probabilities close to or above the nominal level in all the situations considered here. We observed that the exact confidence interval (EX) has coverage probabilities always above the nominal level, however, its conditional expected width is the largest among all the interval estimation procedures considered here. Among these methods, BA has the shortest conditional expected widths and followed by BN. On the other hand, we observed that the method TB produces the shortest conditional expected width among all the methods but its coverage probabilities may not be maintained at the nominal level in some situations.

For the Bayesian credible interval, we tried different prior distribution with different values of a and b and found that a prior distribution with correct information (for example, $a = 1$, $b = 1$ have $E(\Lambda) = 1 =$ true value) about the true value of λ improves the performance of the Bayesian credible interval compare to the one using non-informative prior ($a = b = 0$) (BN), however, a prior distribution not matched with the true value of λ (for example, $a = 2, b = 4$) worsens the performance of the Bayesian credible interval. We presented here the results for prior distribution with $a = b = 0$ and $a = b = 0.1$ for illustrative purpose.

Table 4: Coverage probabilities and conditional expected width of 95% confidence intervals based on different methods for $(n, m) = (15, 5)$ and (25,5)

			$T = 0.25$		$T = 0.5$		$T = 1.00$	
(n, m)	Scheme	Method	Coverage	Length	Coverage	Length	Coverage	Length
(15, 5)	(0,0,0,0,10)	EX	97.70	5.097	95.55	3.009	95.03	2.093
		NA	95.72	2.147	95.67	2.136	95.44	2.153
		NL	93.25	2.092	93.31	2.087	93.11	2.081
		LR	94.83	1.998	94.62	1.982	94.59	1.984
		PB	99.54	2.549	95.00	2.388	94.90	2.399
		TB	88.65	1.262	88.30	1.218	87.47	1.193
		BN	95.22	1.970	95.01	1.956	94.96	1.961
		BA	95.30	1.966	95.05	1.951	95.00	1.956
	(10,0,0,0,0)	EX	95.41	2.509	96.48	2.891	95.94	3.004
		NA	95.69	2.123	95.94	2.152	95.66	2.167
		NL	93.75	2.086	93.35	2.093	92.98	2.088
		LR	94.94	1.972	94.82	1.993	94.41	1.988
		PB	100.00	2.684	100.00	2.677	99.49	2.599
		TB	93.85	1.466	92.44	1.384	89.33	1.273
		BN	95.17	1.944	95.12	1.969	94.97	1.971
		BA	95.20	1.938	95.15	1.963	95.05	1.967
	(2,2,2,2,2)	EX	95.20	3.355	97.31	2.979	95.86	2.357
		NA	95.91	2.154	95.50	2.163	95.67	2.142
		NL	93.37	2.094	92.86	2.091	93.12	2.080
		LR	94.58	1.979	94.26	1.991	94.42	1.977
		PB	99.98	2.646	98.20	2.510	94.85	2.406
		TB	91.38	1.338	88.00	1.243	87.61	1.198
		BN	95.05	1.954	94.74	1.972	94.75	1.948
		BA	95.08	1.949	94.81	1.968	94.78	1.943
(25, 5)	(0,0,0,0,20)	EX	96.91	3.486	95.70	2.101	95.52	2.039
		NA	95.37	2.139	95.60	2.159	95.56	2.146
		NL	93.16	2.075	92.78	2.077	93.32	2.095
		LR	94.33	1.968	94.49	1.987	94.75	1.991
		PB	94.61	2.387	94.91	2.395	95.02	2.423
		TB	87.65	1.215	87.09	1.189	87.53	1.187
		BN	94.67	1.945	94.99	1.967	95.00	1.961
		BA	94.70	1.940	95.08	1.964	95.05	1.955
	(20,0,0,0,0)	EX	95.28	2.450	96.24	2.759	95.40	2.931
		NA	95.46	2.130	95.33	2.131	95.86	2.160
		NL	93.39	2.081	93.23	2.066	93.40	2.100
		LR	94.69	1.978	94.41	1.966	94.74	1.995
		PB	100.00	2.689	99.99	2.716	99.52	2.590
		TB	93.57	1.458	91.84	1.362	89.64	1.283
		BN	95.11	1.957	94.64	1.933	95.14	1.972
		BA	95.19	1.953	94.69	1.928	95.22	1.969
	(4,4,4,4,4)	EX	96.78	3.344	96.02	2.629	95.11	2.147
		NA	95.70	2.140	95.63	2.135	95.25	2.151
		NL	93.44	2.087	93.63	2.091	93.08	2.087
		LR	94.42	1.971	94.74	1.981	94.34	1.987
		PB	99.16	2.570	95.71	2.434	94.53	2.383
		TB	89.30	1.272	88.23	1.215	87.14	1.195
		BN	94.94	1.949	95.09	1.951	94.56	1.958
		BA	95.02	1.945	95.14	1.947	94.66	1.955

Table 5: Coverage probabilities and conditional expected width of 95% confidence intervals based on different methods for $(n, m) = (50, 5)$ and (15,10)

			$T = 0.25$		$T = 0.5$		$T = 1.00$	
(n, m)	Scheme	Method	Coverage	Length	Coverage	Length	Coverage	Length
(50, 5)	(0,0,0,0,45)	EX	95.00	2.146	95.45	2.035	95.51	2.020
		NA	95.49	2.134	95.62	2.152	95.79	2.148
		NL	93.16	2.076	92.89	2.068	92.86	2.072
		LR	94.54	1.979	94.23	1.969	94.58	1.985
		PB	94.81	2.408	94.65	2.414	94.94	2.413
		TB	87.84	1.191	87.68	1.192	87.29	1.189
		BN	95.00	1.956	94.74	1.946	95.08	1.968
		BA	95.10	1.953	94.80	1.942	95.13	1.963
	(45,0,0,0,0)	EX	96.19	2.394	96.43	2.828	97.70	3.106
		NA	95.95	2.171	95.82	2.152	95.24	2.135
		NL	93.02	2.092	93.20	2.085	93.21	2.075
		LR	94.47	1.995	94.60	1.986	94.35	1.970
		PB	100.00	2.665	99.99	2.684	99.47	2.628
		TB	93.59	1.464	92.02	1.374	89.28	1.269
		BN	94.90	1.967	95.13	1.969	94.56	1.942
		BA	94.98	1.963	95.20	1.965	94.67	1.940
	(9,9,9,9,9)	EX	95.56	2.641	95.00	2.147	95.48	2.040
		NA	95.69	2.153	95.65	2.139	95.94	2.155
		NL	93.04	2.078	93.43	2.089	93.20	2.089
		LR	94.28	1.971	94.70	1.984	94.82	1.996
		PB	95.61	2.418	95.02	2.406	95.17	2.416
		TB	87.51	1.211	87.80	1.208	87.79	1.201
		BN	94.89	1.958	95.13	1.958	95.29	1.970
		BA	94.96	1.954	95.17	1.952	95.38	1.966
(15, 10)	(0*9,5)	EX	95.60	2.128	96.29	2.250	96.80	2.329
		NA	95.52	1.328	95.85	1.335	95.87	1.330
		NL	93.96	1.319	93.79	1.314	94.38	1.324
		LR	94.69	1.283	94.62	1.279	95.20	1.289
		PB	100.00	1.548	99.91	1.522	96.15	1.413
		TB	97.42	1.191	94.85	1.102	92.08	1.021
		BN	94.95	1.273	94.93	1.271	95.43	1.280
		BA	94.98	1.271	95.01	1.271	95.49	1.279
	(5,0*9)	EX	95.56	1.547	96.39	1.650	96.00	1.871
		NA	95.53	1.332	95.56	1.326	95.27	1.323
		NL	93.90	1.319	94.26	1.320	94.06	1.316
		LR	94.90	1.288	95.00	1.284	94.57	1.278
		PB	100.00	1.506	100.00	1.530	99.70	1.522
		TB	97.75	1.224	97.03	1.164	94.57	1.087
		BN	95.19	1.280	95.25	1.274	94.82	1.269
		BA	95.20	1.279	95.31	1.274	94.87	1.269
	(0*2,1*5,0*3)	EX	96.22	1.714	96.00	1.963	96.02	1.959
		NA	95.46	1.328	95.48	1.328	95.54	1.325
		NL	94.23	1.323	94.06	1.320	94.29	1.320
		LR	95.05	1.289	94.70	1.280	94.94	1.282
		PB	100.00	1.514	99.98	1.519	99.07	1.502
		TB	97.52	1.198	95.51	1.124	93.89	1.065
		BN	95.34	1.280	95.06	1.273	95.07	1.272
		BA	95.37	1.279	95.07	1.271	95.08	1.271

Table 6: Coverage probabilities and conditional expected width of 95% confidence intervals based on different methods for $(n, m) = (25, 10)$ and (50,10)

			$T = 0.25$		$T = 0.5$		$T = 1.00$	
(n, m)	Scheme	Method	Coverage	Length	Coverage	Length	Coverage	Length
(25, 10)	(0*9,15)	EX	98.10	2.708	96.80	2.374	94.37	1.350
		NA	95.44	1.322	95.71	1.336	95.23	1.322
		NL	94.03	1.313	94.02	1.324	94.08	1.319
		LR	94.91	1.281	94.97	1.293	94.55	1.279
		PB	99.93	1.536	95.59	1.396	94.62	1.398
		TB	95.40	1.105	91.50	1.017	91.54	0.999
		BN	95.03	1.270	95.25	1.283	94.82	1.270
		BA	95.04	1.269	95.29	1.282	94.84	1.269
	(15,0*9)	EX	97.98	1.589	97.00	1.737	97.10	1.905
		NA	95.76	1.325	95.78	1.334	95.49	1.322
		NL	94.72	1.328	93.78	1.316	94.07	1.313
		LR	95.31	1.288	94.67	1.282	94.78	1.276
		PB	100.00	1.510	100.00	1.525	99.70	1.524
		TB	97.88	1.218	96.62	1.161	94.39	1.082
		BN	95.30	1.273	95.07	1.276	94.86	1.264
		BA	95.33	1.272	95.12	1.275	94.90	1.263
	(1*5,2*5)	EX	95.01	2.039	94.00	2.066	95.12	1.681
		NA	94.97	1.326	95.66	1.328	95.89	1.341
		NL	93.94	1.319	94.16	1.318	93.71	1.317
		LR	94.51	1.282	95.04	1.289	94.85	1.290
		PB	99.98	1.540	98.90	1.487	95.83	1.409
		TB	95.95	1.136	93.38	1.058	91.20	1.008
		BN	94.64	1.273	95.24	1.277	95.11	1.281
		BA	94.67	1.271	95.24	1.275	95.15	1.279
(50, 10)	(0*9,40)	EX	98.00	2.183	95.32	1.329	95.02	1.275
		NA	95.11	1.322	95.11	1.323	95.47	1.326
		NL	94.04	1.321	93.82	1.312	94.16	1.322
		LR	94.93	1.287	94.52	1.277	94.84	1.283
		PB	95.19	1.392	94.68	1.401	94.84	1.406
		TB	91.76	1.011	91.49	0.997	91.68	1.004
		BN	95.04	1.274	94.72	1.269	95.02	1.275
		BA	95.06	1.273	94.75	1.268	95.07	1.274
	(40,0*9)	EX	97.23	1.538	97.00	1.659	98.11	2.018
		NA	95.41	1.326	95.89	1.335	95.63	1.338
		NL	94.41	1.324	94.28	1.324	93.72	1.319
		LR	94.93	1.283	95.02	1.289	94.64	1.288
		PB	100.00	1.512	100.00	1.529	99.72	1.513
		TB	97.75	1.216	96.83	1.161	94.34	1.087
		BN	94.94	1.270	95.20	1.279	94.93	1.280
		BA	94.95	1.269	95.22	1.278	94.96	1.279
	(4*10)	EX	94.88	2.207	96.11	1.751	96.61	1.445
		NA	95.17	1.326	95.53	1.328	95.61	1.331
		NL	93.74	1.316	94.21	1.319	93.86	1.314
		LR	94.52	1.283	95.16	1.288	94.61	1.277
		PB	99.26	1.499	96.53	1.432	94.97	1.407
		TB	93.54	1.064	92.11	1.013	91.36	0.998
		BN	94.67	1.271	95.24	1.278	95.03	1.271
		BA	94.70	1.270	95.26	1.276	95.06	1.270

In terms of computational effort, the normal approximation confidence interval (NA) and the Bayesian credible intervals (BN and BA) are easy to compute with hand calculator and statistical tables while computer programs are required for the computation of the other confidence intervals.

Overall speaking, for interval estimation, the Bayesian credible interval provide a good balance between the coverage probabilities as well as the conditional expected widths. Therefore, we would recommend to use the Bayesian credible interval with non-informative prior in general if no prior information about the parameter is available, otherwise, Bayesian credible interval with informative prior should be used when reliable prior information about the parameter is available. If one wants to guarantee the coverage probability is above the nominal level and the width of the confidence interval is not the major concern, then the exact confidence interval (EX) should be used.

5 Concluding Remarks

In this paper, we proposed a adaptive Type-II progressive hybrid censoring scheme and discussed the statistical inference based on exponential lifetime data. We compared different statistical inference procedures and the performance of the MLE with the hybrid censoring scheme proposed by [14].

Based on our results, the Bayesian posterior mean for point estimation and Bayesian credible interval are recommened when reliable prior information about the unknown parameter is available, otherwise, MLE for point estimation and Bayesian credible interval with non-informative prior for interval estimation should be used in general.

From this study, once again, we can see that experimenter needs to compromise in between (i) total time on test; (ii) saving experimental units; and (iii) efficiency in estimation, and there is always trade-off between these three concerns. The computation formulas and results provided in this paper give a guideline on planning an experiment to compromise these three concerns. Further investigation on obtaining optimal experimental designs for given values of ideal total test time (T), number of units available for test (n) and the number of failures allowed for the experiment (m) would be of interest in experimental planning.

Reference

References

[1] Balakrishnan, N. (2007). Progressive censoring methodology: An appraisal, *Test*, **16**, 211-296 (with discussion).

[2] Balakrishnan, N., and Aggarwala, R. (2000). *Progressive Censoring: Theory, Methods and Applications.* Boston, MA: Birkhäuser.

[3] Balakrishnan, N. and Basu, A. P. (1995). *The Exponential Distribution: Theory, Methods and Applications*. Langhorne, Pennsylvania: Gordon and Breach.

[4] Balakrishnan, N., Childs, A. and Chandrasekar, B. (2002). An efficient computational method for moments of order statistics under progressive censoring, *Statistics and Probability Letters*, **60**, 359 - 365.

[5] Balakrishnan, N., Kundu, D., Ng, H. K. T. and Kannan, N. (2007). Point and Interval Estimation for a Simple Step-Stress Model with Type-II Censoring, *Journal of Quality Technology*, **39**, 35 - 47.

[6] Balakrishnan, N. and Sandhu, R. A. (1995). A simple simulational algorithm for generating progressive Type-II censored samples, *The American Statistician*, **49**, 229 - 230.

[7] Chen, S. M. and Bhattacharya, G. K. (1988). Exact confidence bound for an exponential parameter under hybrid censoring, *Communications in Statistics - Theory and Methods*, **16**, 1857 - 1870.

[8] Childs, A., Chandrasekar, B., Balakrishnan, N. and Kundu, D. (2003), Exact likelihood inference based on Type-I and Type-II hybrid censored samples from the exponential distribution, *Annals of the Institute of Statistical Mathematics*, **55**, 319 - 330.

[9] David, H. A. and Nagaraja, H. N. (2003). *Order Statistics*, Third Edition. New York, NY: Wiley.

[10] Efron, B. and Tibshirani, R. (1993). *An Introduction to the Bootstrap*, New York, NY: Chapman & Hall.

[11] Gupta, R.D. and Kundu, D. (1998). Hybrid censoring schemes with exponential failure distribution, *Communications in Statistics- Theory and Methods*, **27**, 3065 - 3083.

[12] Johnson, N. L. , Kotz, S. and Balakrishnan, N. (1994). *Continuous Univariate Distributions - Vol. 1*, 2nd Edition, New York, NY: John Wiley & Sons.

[13] Kundu, D. and Basu, S. (2000). Analysis of incomplete data in presence of competing risks, *Journal of Statistical Planning and Inference*, **87**, 221-239.

[14] Kundu, D. and Joarder, A. (2006). Analysis of Type-II Progressively Hybrid Censored Data, *Computational Statistics and Data Analysis*, **50**, 2509 - 2258.

[15] Lin, C. T., Hwang, Y. L. and Balakrishnan, N. (2008). Exact Bayesian variable sampling plans for the exponential distribution with adaptive progressive hybrid censoring, *Technical Report for NSC, Taiwan.*

[16] Neyman J. and Pearson E. S. (1928). On the use and interpretation of certain test criteria for purpose of statistical inference. *Biometrika*, **20**, 175 - 247.

[17] Ng, H. K. T., Chan, P. S. and Balakrishnan, N. (2004). Optimal Progressive Censoring Plan for The Weibull Distribution, *Technometrics*, **46**, 470 - 481.

A Appendix A. Probability Mass Function of J

For exponential distribution, the probability density function of $X_{j:m:n}$ is given by (see Balakrishnan and Aggawala, 2000)

$$f_{X_{j:m:n}}(x) = c_{j-1} \sum_{i=1}^{j} a_{i,j} \lambda \exp\left(-r_i \lambda x\right), \quad 0 < x < \infty,$$

where

$$\begin{aligned} r_j &= m - j + 1 + \sum_{i=j}^{m} R_i, j = 1, 2, \ldots, m, \\ c_{j-1} &= \prod_{i=1}^{j} r_i, j = 1, 2, \ldots, m, \\ a_{i,j} &= \prod_{\substack{k=1 \\ k \neq i}}^{j} \frac{1}{r_k - r_i}, 1 \leq i \leq j \leq m. \end{aligned}$$

Given $X_{j:m:n} = x_j$, $X_{j+1:m:n}$ is distributed as the first order statistic of a random sample of size $\left(n - j - \sum_{i=1}^{j} R_i\right) = r_{j+1}$ from a truncated exponential distribution with CDF

$$F_{X_{j+1:m:n}}(x | X_{j:m:n} = x_j) = 1 - \exp\left[-r_{j+1} \lambda (x - x_j)\right], x_j < x < \infty.$$

First, the probability that $J = 0$ and $J = m$ are

$$\begin{aligned} \Pr(J = 0) &= \Pr(X_{1:m:n} > T) = \exp\left(-n \lambda T\right), \\ \Pr(J = m) &= \Pr(X_{m:m:n} < T) = 1 - c_{m-1} \sum_{i=1}^{m} \frac{a_{i,m}}{r_i} \exp\left(-r_i \lambda T\right), \end{aligned}$$

respectively.

The probability mass function of J, $J = 1, 2, \ldots, m-1$ is

$$\begin{aligned} \Pr(J = j) &= \Pr(X_{j:m:n} < T \leq X_{j+1:m:n}) \\ &= \int_0^{\infty} \Pr(x < T \leq X_{j+1:m:n} | X_{j:m:n} = x) f_{X_{j:m:n}}(x) \, dx \\ &= \int_0^{T} \left[1 - F_{X_{j+1:m:n}}(T | X_{j:m:n} = x_j)\right] f_{X_{j:m:n}}(x) \, dx \\ &= c_{j-1} \exp\left(-r_{j+1} \lambda T\right) \lambda \left\{ \sum_{i=1}^{j} a_{i,j} \int_0^{T} \exp\left[-\lambda (r_i - r_{j+1}) x\right] dx \right\} \\ &= c_{j-1} \exp\left(-r_{j+1} \lambda T\right) \left\{ \sum_{i=1}^{j} \frac{a_{i,j}}{(r_i - r_{j+1})} \left[1 - \exp\left(-\lambda (r_i - r_{j+1}) T\right)\right] \right\}. \end{aligned}$$

Therefore, we can write the PMF of J as

$$\Pr(J = j) \quad = \quad c_{j-1} \exp\left(-r_{j+1}\lambda T\right) \sum_{i=1}^{j} \frac{a_{i,j}}{(r_i - r_{j+1})} \left[1 - \exp\left(-(r_i - r_{j+1})\lambda T\right)\right],$$

for $j = 0, 1, \ldots, m$, with $r_{m+1} \equiv 0$ and $c_{-1} \equiv 1$.

B Appendix B. Exact conditional distribution of $\hat{\theta}$ given $J = j$

First, we consider the case for $J = j > 0$. From Lemma 1 of Balakrishnan, Childs and Chandrasekar [?] (p.361), we have the following result for $j > 0$:

$$\begin{aligned}\int_0^T \int_0^{x_{j:m:n}} \cdots \int_0^{x_{3:m:n}} \int_0^{x_{2:m:n}} & \prod_{i=1}^{j} f(x_{i:m:n})[1 - F(x_{i:m:n})]^{R_i} dx_{1:m:n} dx_{2:m:n} \cdots dx_{j-1:m:n} dx_{j:m:n} \\ = & \sum_{i=0}^{j} c_{i,j}(\mathbf{R})[1 - F(T)]^{b_{i,j}(\mathbf{R})}, \qquad \text{(B.1)}\end{aligned}$$

where $\mathbf{R} = (R_1, R_2, \ldots, R_j)$,

$$c_{i,j}(\mathbf{R}) = \frac{(-1)^i}{\left\{\prod_{l=1}^{i} \sum_{k=j-i+1}^{j-i+l} (R_k + 1)\right\} \left\{\prod_{l=1}^{j-i} \sum_{k=l}^{j-i} (R_k + 1)\right\}} \text{ and } b_{i,j}(\mathbf{R}) = \sum_{k=j-i+1}^{j} (R_k + 1),$$

in which we use the usual conventions that $\prod_{l=1}^{0} d_l \equiv 1$ and $\sum_{l=i}^{i-1} d_l \equiv 0$.

We can show that the integral

$$\begin{aligned}& \int_T^{\infty} \int_{x_{j+1:m:n}}^{\infty} \cdots \int_{x_{m-1:m:n}}^{\infty} \left\{\prod_{i=j+1}^{m} f(x_{i:m:n})\right\} \\ & \qquad \times [1 - F(x_{m:m:n})]^{\left(n-m-\sum_{k=1}^{j} R_k\right)} dx_{m:m:n} dx_{m-1:m:n} \cdots dx_{j+1:m:n} \\ = & \quad C_j [1 - F(T)]^{\left(n-j-\sum_{k=1}^{j} R_k\right)}, \qquad \text{(B.2)}\end{aligned}$$

where

$$C_j = \left[\prod_{l=j}^{m-1} \left(n - \sum_{k=1}^{j} R_k - l\right)\right]^{-1}.$$

Combining the above results, we have

$$\int \cdots \int_{\Omega} L(\boldsymbol{\theta}|J=j) dx_{1:m:n} \cdots dx_{m:m:n} = C_j \sum_{i=0}^{j} c_{i,j}(\mathbf{R})[1-F(T)]^{b^*_{i,j}(\mathbf{R})} \tag{B.3}$$

where

$$\begin{aligned} \Omega &= \{(x_{1:m:n}, \ldots, x_{m:m:n}) | x_{1:m:n} < \ldots < x_{j:m:n} < T < x_{j+1:m:n} < \ldots < x_{m:m:n}\} \\ L(\boldsymbol{\theta}|J=j) &= \left[\prod_{i=1}^{m} f(x_{i:m:n})\right] \left\{\prod_{i=1}^{j} [1-F(x_{i:m:n})]^{R_i}\right\} [1-F(x_{m:m:n})]^{\left(n-m-\sum_{i=1}^{j} R_i\right)} \end{aligned}$$

and

$$b^*_{i,j}(\mathbf{R}) = \left[\sum_{k=j-i+1}^{j} (R_k+1)\right] + \left(n-j-\sum_{k=1}^{j} R_k\right).$$

Therefore, the joint distribution of $(X_{1:m:n}, \ldots, X_{m:m:n})$ based on adaptive hybrid progressive censoring is given by

$$\begin{aligned} & f_{\mathbf{X}}(x_{1:m:n}, \ldots, x_{m:m:n} | J=j) \\ = & \frac{\left[\prod_{i=1}^{m} f(x_{i:m:n})\right] \left\{\prod_{i=1}^{j} [1-F(x_{i:m:n})]^{R_i}\right\} [1-F(x_{m:m:n})]^{\left(n-m-\sum_{i=1}^{j} R_i\right)}}{C_j \sum_{i=0}^{j} c_{i,j}(\mathbf{R})[1-F(T)]^{b^*_{i,j}(\mathbf{R})}}, \\ & -\infty < x_{1:m:n} < \ldots < x_{j:m:n} < T < x_{j+1:m:n} < \ldots < x_{m:m:n} < \infty. \end{aligned}$$

For exponential distribution, the joint distribution of $(X_{1:m:n}, \ldots, X_{m:m:n})$ becomes

$$f_{\mathbf{X}}(x_{1:m:n}, \ldots, x_{m:m:n} | J=j) = \frac{\lambda^m e^{-\lambda \delta(\mathbf{x},j)}}{C_j \sum_{i=0}^{j} c_{i,j}(\mathbf{R}) e^{-\lambda T b^*_{i,j}(\mathbf{R})}},$$

where

$$\delta(\mathbf{x}, j) = \sum_{i=1}^{m} x_{i:m:n} + \sum_{i=1}^{j} R_i x_{i:m:n} + \left(n-m-\sum_{i=1}^{j} R_i\right) x_{m:m:n}.$$

Let $\theta = \frac{1}{\lambda}$ and the MLE of θ is $\hat{\theta} = \frac{\delta(\mathbf{x},j)}{m}$. The moment generating function of $\hat{\theta}$, conditional on $J=j$ is given by

$$E(e^{\omega\hat{\theta}}|J=j) = E\left(e^{\frac{\omega\delta(\mathbf{x},j)}{m}} \middle| J=j\right)$$

$$
\begin{aligned}
&= \int\cdots\int_{\Omega} e^{\frac{\omega\delta(\mathbf{x},j)}{m}} f_{\mathbf{X}}(x_{1:m:n},\ldots,x_{m:m:n})dx_{1:m:n}\cdots dx_{m:m:n} \\
&= \frac{\theta^{-m}\int\cdots\int_{\Omega} e^{-\left(\frac{1}{\theta}-\frac{\omega}{m}\right)\delta(\mathbf{x},j)}dx_{1:m:n}\cdots dx_{m:m:n}}{C_j\sum_{i=0}^{j} c_{i,j}(\mathbf{R})e^{-\lambda T b^*_{i,j}(\mathbf{R})}} \\
&= \frac{\theta^{-m}\left(\frac{m-\omega\theta}{m\theta}\right)^{-m}\int\cdots\int_{\Omega}\left(\frac{m-\omega\theta}{m\theta}\right)^{m} e^{-\left(\frac{1}{\theta}-\frac{\omega}{m}\right)\delta(\mathbf{x},j)}dx_{1:m:n}\cdots dx_{m:m:n}}{C_j\sum_{i=0}^{j} c_{i,j}(\mathbf{R})e^{-T b^*_{i,j}(\mathbf{R})/\theta}}.
\end{aligned}
$$

From Eq. (B.3), we can obtain

$$
\begin{aligned}
E(e^{\omega\hat{\theta}}|J=j) &= \frac{\left(1-\frac{\omega\theta}{m}\right)^{-m} C_j\sum_{i=0}^{j} c_{i,j}(\mathbf{R})e^{-\left(\frac{1}{\theta}-\frac{w}{m}\right)T b^*_{i,j}(\mathbf{R})}}{C_j\sum_{i=0}^{j} c_{i,j}(\mathbf{R})e^{-T b^*_{i,j}(\mathbf{R})/\theta}} \\
&= \frac{\left(1-\frac{\omega\theta}{m}\right)^{-m}\sum_{i=0}^{j} c_{i,j}(\mathbf{R})e^{-T b^*_{i,j}(\mathbf{R})/\theta}e^{wT b^*_{i,j}(\mathbf{R})/m}}{\sum_{i=0}^{j} c_{i,j}(\mathbf{R})e^{-T b^*_{i,j}(\mathbf{R})/\theta}}.
\end{aligned}
$$

Let Y be a Gamma(α,β), i.e., a gamma random variable with shape parameter α and scale parameter β. Then the PDF of Y is given by

$$
g(y;\alpha,\beta)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}y^{\alpha-1}e^{-\beta y}, \quad y>0.
$$

where $\Gamma(\alpha)=\int_0^{\infty}x^{\alpha-1}e^{-x}dx$ is the gamma function. For an arbitrary constant A, it can be shown that the moment generating function of $Y+A$ is (Johnson, Kotz and Balakrishnan [12], p. 338)

$$
M_{Y+A}(\omega)=e^{\omega A}\left(1-\frac{\omega}{\beta}\right)^{-\alpha}.
$$

It follows that the conditional PDF of $\hat{\theta}$, given $J=j>0$, is

$$
f_{\hat{\theta}}(t|J=j) = \frac{\sum_{i=0}^{j} c_{i,j}(\mathbf{R})e^{-T b^*_{i,j}(\mathbf{R})/\theta} g\left(t-\frac{T b^*_{i,j}(\mathbf{R})}{m};m,\frac{m}{\theta}\right)}{\sum_{i=0}^{j} c_{i,j}(\mathbf{R})e^{-T b^*_{i,j}(\mathbf{R})/\theta}}.
$$

For $J=0$, we can show that the joint distribution of $(X_{1:m:n},\ldots,X_{m:m:n})$ based on adaptive hybrid progressive censoring is

$$
f_{\mathbf{X}}(x_{1:m:n},\ldots,x_{m:m:n}|J=0)=\frac{n!}{(n-m)!}\frac{\left\{\prod_{i=1}^{m} f(x_{i:m:n})\right\}[1-F(x_{m:m:n})]^{(n-m)}}{[1-F(T)]^{n}},
$$
$$
T<x_{1:m:n}<\ldots<x_{m:m:n}<\infty.
$$

Following the same steps as above, we can obtain the moment generating function of $\hat{\theta}$, conditional on $J = 0$ as

$$E(e^{\omega\hat{\theta}}|J=0) = \left(1 - \frac{\omega\theta}{m}\right)^{-m} e^{\left(\frac{nT}{m}\right)\omega}$$

and the conditional PDF of $\hat{\theta}$, given $J = 0$, is

$$f_{\hat{\theta}}(t|J=0) = g\left(t - \frac{nT}{m}; m, \frac{m}{\theta}\right).$$