EXACT TESTS OF BINOMIAL p = 1/2 WHEN ONE BINOMIAL EVENT POSSIBLY HAS DIFFERENT PROBABILITY

by

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EXACT TESTS OF BINOMIAL $p = \frac{1}{2}$ WHEN ONE BINOMIAL EVENT POSSIBLY HAS DIFFERENT PROBABILITY

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ABSTRACT

The data are n independent random binomial events, each resulting in success or failure. The event outcomes are believed to be trials from a binomial distribution with success probability p, and tests of $p = \frac{1}{2}$ are desired. However, there is the possibility that some unidentified event has a success probability different from the common value p for the other $n - 1$ events. Then, tests of whether this common p equals $\frac{1}{2}$ are desired. Fortunately, two-sided tests can be obtained that simultaneously are applicable for both situations. That is, the significance level for a test is same when one event has a different probability as when all events have the same probability. These tests are the usual equal-tail tests for $p = \frac{1}{2}$ (based on n independent trials from a binomial distribution).

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DISCUSSION AND RESULTS

Considered are n random binomial events that are statistically independent. Each event results in success or failure and the event outcomes are believed to represent trials from a binomial distribution (with probability of success denoted by p). Significance tests of the null hypothesis $p = \frac{1}{2}$ are desired. A complication occurs, however, in that one (unidentified) event might have a success probability different from the common probability for the other $n - 1$ events. When this happens, tests of whether the common value p for the other $n - 1$ events satisfies $p = \frac{1}{2}$ are desired.

The null hypothesis of $p = \frac{1}{2}$ is often encountered. For example, the bias of a coin with regard to flipping might be investigated (no bias occurs when heads and tails are equally likely). Since the probability of a coin (say, for heads) can be influenced by a poor flipping technique, there can be a definite possibility that one flip is associated with a different probability than the other flips. With care, however, this should occur (in a nontrivial sense) for at most one flip. Very often, which flip, if any, has the different probability is very difficult (if not outright impossible) to identify. The coin remains the same, however, and the interest is in its bias when acceptable kinds of flipping techniques are used.

Suppose that identification could be made of the one binomial event that possibly could have a different probability. If this identification is completely accurate and also independent of the outcome for that event, the event could be eliminated from consideration.

The remaining $n - 1$ events would then be independent trials from a binomial distribution for which $p = \frac{1}{2}$ could be tested. Unfortunately, a completely accurate identification that is independent of the observed outcome frequently is difficult to accomplish. Other types of identifications are possibilities but they impose a conditional effect on the joint distribution of the outcomes and/or can fail to make an accurate identification.

The various difficulties in identification can be ignored if tests are obtainable that simultaneously are usable when all events have the same probability and when all events but one have the same probability. More specifically, the significance level of a test is the same for both of these situations. Two-sided tests with this property, fortunately, do exist. These are the customary equal-tail tests where $p = \frac{1}{2}$ is rejected if and only if either the observed number of successes for the n events is at most i or is at least $n + 1 - i$, where $0 \leq i \leq n/2 - 1$. When all events have the same probability,

$$(\tfrac{1}{2})^{n-1} \sum_{j=0}^{i} \binom{n}{j} \tag{1}$$

is the significance level. Verification that this is also the significance level when exactly one event has a different probability is given in the next section.

Use of a test which allows for the possibility that one event has a different probability can be motivated from virtually any consideration. Examination of the data source and collection method could be the motivation. On the other hand, a desire to be careful could be the only motivation.

VERIFICATION

Under the null hypothesis, $n - 1$ of the events have probability $\frac{1}{2}$ and the other event has probability p', $(0 \le p' \le 1)$. For given i, $(0 \le i \le n/2 - 1)$, the contribution to the significance level from the lower tail is

P($i - 1$ or less events with prob. $\frac{1}{2}$ are successes and other event success)

+P(i or less events with prob. $\frac{1}{2}$ are successes and other event failure)

$$= p'(\tfrac{1}{2})^{n-1} \sum_{j=0}^{i-1} \binom{n-1}{j} + (1 - p')(\tfrac{1}{2})^{n-1} \sum_{j=0}^{i} \binom{n-1}{j} \quad ,$$

where a summation to $i - 1$ is zero when $i = 0$. Also, the contribution to the significance level from the upper tail is

P($n - i$ or more events with prob. $\frac{1}{2}$ are successes and other event success)

+P($n + 1 - i$ or more events with prob. $\frac{1}{2}$ are success and other event failure)

$$= p'(\tfrac{1}{2})^{n-1} \sum_{j=n-i}^{n-1} \binom{n-1}{j} + (1 - p')(\tfrac{1}{2})^{n-1} \sum_{j=n+1-i}^{n-1} \binom{n-1}{j}$$

$$= p'(\tfrac{1}{2})^{n-1} \sum_{j=0}^{i} \binom{n-1}{j} + (1 - p')(\tfrac{1}{2})^{n-1} \sum_{j=0}^{i-1} \binom{n-1}{j} \quad ,$$

where a summation starting at $n + 1 - i$ is zero when $i = 0$.

Thus, the significance level, which equals the sum of the two contributions, is

$$(\tfrac{1}{2})^{n-1} \sum_{j=0}^{i-1} \binom{n-1}{j} + (\tfrac{1}{2})^{n-1} \sum_{j=0}^{i} \binom{n-1}{j} \quad , \qquad (2)$$

since p' cancels out when the contribution from the lower tail is added to the contribution from the upper tail.

With $\binom{n-1}{-1}$ defined to be zero, and since $\binom{n-1}{0} = \binom{n}{0}$, the value of (2) is

$$(\tfrac{1}{2})^{n-1} \sum_{j=0}^{i} \left[\binom{n-1}{j} + \binom{n-1}{j-1} \right] .$$

However, for j = 0, 1, ..., i,

$$\binom{n-1}{j} + \binom{n-1}{j-1} = \binom{n}{j} .$$

Thus, (2) has the same value as (1) and the verification is completed. Incidentally, the contribution from the lower tail nearly equals that from the upper tail when i is of at least moderate size.

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