APPROXIMATE INTERVALS AND TESTS FOR PRODUCTS

AND RATIOS OF BINOMIAL PROBABILITIES

by

John E. Walsh

Technical Report No. 89
Department of Statistics ONR Contract

December 14, 1970

DEPARTMENT OF STATISTICS
Southern Methodist University

# APPROXIMATE INTERVALS AND TESTS FOR PRODUCTS

# AND RATIOS OF BINOMIAL PROBABILITIES

John E. Walsh*

Southern Methodist University**

## ABSTRACT

The data are the observed numbers of successes for sets of independent trials from binomial distributions (success and failure are the possible outcomes for each binomial event). Desired are confidence intervals and tests for the product of the success probabilities for these binomial distributions. Also desired are intervals and tests for the product of a specified set of these probabilities divided by the product of the remaining probabilities. The situation of interest is that where, for all distributions, the success probabilities are of at least moderate size (say, at least 2/3) and the numbers of trials are at least moderately large. Approximate intervals and tests are developed by use of logarithms, expansions, and normal approximations. These results have application in investigating and comparing system reliabilities.

---

## INTRODUCTION AND RESULTS

Consider independent data from N binomial distributions of success and failure, where $N \geq 2$. The i-th distribution has unknown probability $p_i$ for a success, involves $n_i$ trials, and its observed fraction of successes is denoted by $p_i'$ $(i=1,\ldots,N)$. Confidence intervals and significance tests are desired for $\prod_{i=1}^{N} p_i$ and for

$$\left( \prod_{i=1}^{m} p_i \right) \Big/ \left( \prod_{i=m+1}^{N} p_i \right),$$

where m is specified and such that $1 \leq m \leq N - 1$. Of course, by suitable indexing, any m of the binomial probabilities could be $p_1, \ldots, p_m$.

Investigation of such products of probabilities can be helpful in studies of system reliability, where the ratio occurs when a comparison of the reliabilities of two systems is of interest. Here, a success could represent satisfactory performance of a subsystem while failure represents unsatisfactory performance. Very frequently, all the $p_i$ are at least 2/3 for reliability situations. Otherwise, unacceptable system reliabilities would occur.

For brevity only confidence intervals are explicitly considered. Significance tests are obtainable from the intervals in the usual manner. The null value investigated depends on the circumstances. Often, however, the null value for the ratio will be unity (for example, when unit ratio represents equal reliability for two systems).

The case where $\prod_{i=1}^{N} p_i$ is investigated is considered first. The basic

2

statistic for this case is

$$T_P = \log_e\left[\prod_{i=1}^{N} p_i'\right] + \sum_{i=1}^{N} (1-p_i')/2n_i p_i'.$$

When all $n_i p_i' \geq 1$, all $p_i \geq 2/3$, all $n_i p_i \geq 10$, and $n_i(1-p_i) \geq 5$ for at least two values of i, the distribution of $T_P$ is, approximately, normal with mean $\log_e\left[\prod_{i=1}^{N} p_i\right]$ and variance

$$\sum_{i=1}^{N} (1-p_i)/n_i p_i, \tag{1}$$

which is estimated by

$$s^2 = \sum_{i=1}^{N} (1-p_i')/n_i p_i'.$$

Thus, approximately,

$$\left\{T_P - \log_e\left[\prod_{i=1}^{N} p_i\right]\right\} s^{-1} \tag{2}$$

has a standardized normal distribution (zero mean and unit variance).

One-sided and two-sided confidence intervals are easily obtained by use of (2). For example,

$$P\left\{\exp[T_P - sK_{\alpha/2}] \leq \prod_{i=1}^{N} p_i \leq \exp[T_P + sK_{\alpha/2}]\right\}$$

$$\doteq 1 - \alpha,$$

where $K_{\alpha/2}$ is the standardized normal deviate exceeded with probability $\alpha/2$, defines an equal-tail confidence interval for $\prod_{i=1}^{N} p_i$ whose confidence coefficient is approximately $1 - \alpha$.

Now, consider the case where a ratio is investigated. The basic
statistic for this case is

$$
T_R = \log_e \left[ \prod_{i=1}^{N} p_i' \right] - \log_e \left[ \prod_{i=m+1}^{N} p_i' \right]
$$

$$
+ \sum_{i=1}^{m} (1-p_i')/2n_i p_i' - \sum_{i=m+1}^{N} (1-p_i')/2n_i p_i'.
$$

When all $n_i p_i' \geq 1$, all $p_i \geq 2/3$, all $n_i p_i \geq 10$, and $n_i(1-p_i) \geq 5$ for at
least two values of i, the distribution of $T_R$ is, approximately, normal
with mean

$$
\log_e \left[ \left( \prod_{i=1}^{m} p_i \right) \Big/ \left( \prod_{i=m+1}^{N} p_i \right) \right]
$$

and the variance expression (1). Since (1) is estimated by $s^2$,

$$
\left\{ T_R - \log_e \left[ \left( \prod_{i=1}^{m} p_i \right) \Big/ \left( \prod_{i=m+1}^{N} p_i \right) \right] \right\} s^{-1}
$$

has a standardized normal distribution, approximately, and provides the
basis for obtaining one-sided and two-sided confidence intervals for the ratio.

In general, let the value of a confidence coefficient be represented
as $1 - \alpha$ (then $\alpha$ is the significance level of the corresponding test).
The accuracy of a confidence coefficient tends to decrease as $\alpha$ decreases.
That is, the normal approximation tends to become less accurate as values
farther and farther into the tails are considered. Use of $\alpha \geq .001$ should
be satisfactory if all the $n_i$ are at least moderately large (say, all
$n_i \geq 50$). In any case, when the other conditions hold, use of $\alpha \geq .01$
should be satisfactory. Equal-tail results tend to have the most accurate-
ly determined probabilities (for example, see ref. 1) and, in general, use

4

of $\alpha \geq .005$ should be satisfactory.

The final section contains verification of the stated results, including a discussion of the conditions on the $n_i$, $p_i'$, and $p_i$.

## VERIFICATION

Consider $\log_e p_i'$, where $p_i' > 0$. This can be expressed as

$$\log_e(p_i' - p_i + p_i) = \log_e\{p_i[1 + (p_i' - p_i)/p_i]\}$$

$$= \log_e p_i + \log_e[1 + (p_i' - p_i)/p_i].$$

Thus,

$$t_i = \log_e p_i' + (1 - p_i')/2n_i p_i'$$

$$= \log_e p_i + (p_i' - p_i)/p_i + (1 - p_i')/2n_i p_i' - (p_i' - p_i)^2/2p_i^2$$

$$+ (p_i' - p_i)^3/3p_i^3 - (p_i' - p_i)^4/4p_i^4 + \ldots$$

defines $t_i$ and

$$E(t_i) = \log_e p_i + O(n_i^{-3/2}).$$

Also, the variance of $t_i$ is

$$(1 - p_i)/n_i p_i + O(n_i^{-2}),$$

and the third moment about the mean, when divided by the 3/2 power of the variance, equals

$$(1 - 2p_i)/[n_i p_i(1 - p_i)]^{1/2} + O(n_i^{-3/2}).$$

Moreover, the fourth moment about the mean, when divided by the square of the variance, equals $3 + O(n_i^{-1})$.

5

These moment values, in combination with the form of the expansion, seem to imply that the distribution of $t_i$ is approximately normal when $n_i p_i' \geq 1$, $p_i \geq 1/2$, $n_i p_i \geq 10$, and $n_i(1 - p_i) \geq 5$. This is evidently the case (for example, see page 87 of ref. 2) when all the terms in the expression for $t_i - \log_e p_i$ except $(p_i' - p_i)/p_i$ are unimportant, which occurs when $n_i$ is at least moderately large (say, $n_i \geq 15$). Of course, $n_i(1 - p_i) \geq 5$ implies that $n_i \geq 15$. The requirement $n_i p_i' \geq 1$ should not have much conditional effect on the distribution of $t_i$, since the probability of $n_i p_i' \geq 1$ is very nearly unity when $n_i p_i \geq 10$.

At least rough normality of $t_i$ should occur when $n_i p_i' \geq 1$, $p_i \geq 2/3$, $n_i p_i \geq 10$, and $n_i(1 - p_i) \geq 2$. Addition of such $t_i$ to at least two others that satisfy $n_i p_i(1 - p_i) \geq 5$ should yield a statistic whose distribution is nearly normal. In fact, the variance of a $t_i$ for which $n_i p_i(1 - p_i) = 5$ is definitely greater than the variance of a $t_i$ for which $n_i p_i(1 - p_i) = 2$, and can be much greater. Again, the requirement $n p_i' \geq 1$ should have an unimportant conditional effect.

The remaining possibility is $t_i$ for which $n_i p_i' \geq 1$, $p_i \geq 2/3$, $n_i p_i \geq 10$, and $n_i(1 - p_i) < 2$. Here, the variance of $t_i$ decreases as $n_i(1 - p_i)$ decreases and is very definitely smaller than that for a $t_i$ where $n_i(1 - p_i) = 5$. Also, $t_i$ tends to have a unimodal distribution. Addition of a $t_i$ with $n_i(1 - p_i) < 2$ to at least two $t_i$ with $n_i(1 - p_i) \geq 5$ should yield a statistic that is very nearly normal. The effect of $n_i p_i' \geq 1$ is unimportant.

Since $T_P$ is the summation of the $t_i$, and $T_R$ is the summation of

6

# REFERENCES

1. John E. Walsh and Grace J. Kelleher, "Robust variables control charts based on sample means," Proceedings of the International Quality Control Conference-Tokyo, 1969, pp. 507-509.

2. Paul G. Hoel, Introduction to Mathematical Statistics (2nd Ed.), John Wiley and Sons, New York, 1954.

## DOCUMENT CONTROL DATA - R & D

*Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)*

| 1. ORIGINATING ACTIVITY (Corporate author) | 2a. REPORT SECURITY CLASSIFICATION |
|---|---|
| | UNCLASSIFIED |
| SOUTHERN METHODIST UNIVERSITY | 2b. GROUP |
| | UNCLASSIFIED |

3. REPORT TITLE

"Approximate intervals and tests for products and ratios of binomial probabilities"

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report

5. AUTHOR(S) (First name, middle initial, last name)

John E. Walsh

| 6. REPORT DATE | 7a. TOTAL NO. OF PAGES | 7b. NO. OF REFS |
|---|---|---|
| December 14, 1970 | 8 | 2 |

| 8a. CONTRACT OR GRANT NO. | 9a. ORIGINATOR'S REPORT NUMBER(S) |
|---|---|
| N00014-68-A-0515 | |
| b. PROJECT NO. | |
| NR 042-260 | 89 |
| c. | 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) |
| d. | |

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited. Reproduction in whole or in part is permitted for any purpose of the United States Government.

| 11. SUPPLEMENTARY NOTES | 12. SPONSORING MILITARY ACTIVITY |
|---|---|
| | Office of Naval Research |

13. ABSTRACT

The data are the observed numbers of successes for sets of independent trials from binomial distributions (success and failure are the possible outcomes for each binomial event). Desired are confidence intervals and tests for the product of the success probabilities for these binomial distributions. Also desired are intervals and tests for the product of a specified set of these probabilities divided by the product of the remaining probabilities. The situation of interest is that where, for all distributions, the success probabilities are of at least moderate size (say, at least 2/3) and the numbers of trials are at least moderately large. Approximate intervals and tests are developed by use of logarithms, expansions, and normal approximations. These results have application in investigating and comparing system reliabilities.