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ON THE APPLICATION OF THE METHOD OF DAS IN EVALUATING A MULTIVARIATE NORMAL INTEGRAL

by

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On the application of the method of Das in evaluating a multivariate normal integral

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Summary

Das (1956) presented a method of evaluating Prob $[x_1 > a_1, x_2 > a_2, \ldots x_n > a_n]$ for a multivariate normal distribution which reduced the dimension of integration by the use of the univariate cummulative normal. This note investigates properties of the variance-covariance matrix that indicate how small this dimension of integration might be and presents a method of determining the coefficients for the expression of Das.

1. Introduction

Das (1956) presents a method of evaluating the integral

$$G = \int_{a_1}^{\infty} \cdots \int_{a_n}^{\infty} f(x_1, x_2, \ldots, x_n) dx_1 \, dx_2 \ldots dx_n$$

(where $f(x_1, x_2, \ldots, x_n)$ is the joint multivariate normal density function with zero means and nonsingular variance-covariance matrix V) through the combining of n + k independent normal variables with zero means and unit variances. Later Marsaglia (1963) shows that this is a special case of a convolution formula. The complexity of implementing the method is highly dependent upon the size of k and Marsaglia (1963) notes that k equal to n minus the multiplicity of the smallest latent root of V can always be achieved. The author in an unpublished report has given some geometric conditions on the latent vectors of V that allow a change of scale to increase the multiplicity of a latent root.

This note presents a method of setting up an expression for G when small values of k are possible.

2. A slight modification on the method of Das.

As a method of evaluating G, Das (1956) considers two row vectors $\underline{y}' = (y_1, y_2, \ldots, y_n)$ and $\underline{z}' = (z_1, z_2, \ldots z_k)$ all of whose elements are normally and independently distributed with zero means and unit variances. The approach is to determine a constant c and an n x k real matrix B such that

$$V = c^2 I_n + BB'; \tag{1}$$

for then $\underline{x}' = (x_1, x_2, \ldots, x_n)$ can be expressed as

$$\underline{x} = c\underline{y} - B\underline{z}.$$

Then $G = \Pr\left[\, x_1 \geq a_1,\ x_2 \geq a_2, \ldots x_n \geq a_n \right]$

$$= \Pr\left[\, y_1 \geq (a_1 + \sum_{j=1}^{k} b_{1j} z_j)/c,\ y_2 \geq (a_2 + \sum_{j=1}^{k} b_{2j} z_j)/c, \ldots y_n \geq (a_n + \sum_{j=1}^{k} b_{nj} z_j)/c \right]$$

$$= (2\pi)^{-k/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} P\left[(a_i + \sum_{j=1}^{k} b_{ij} z_j)/c\right] \exp\left(-\underline{z}'\underline{z}/2\right) \prod_{j=1}^{k} dz_{ij} \tag{2}$$

where $P[a] = \Pr\ (y_1 \geq a)$.

The slight change in 1)

$$V = C^2 + BB', \tag{3}$$

where C is a diagonal matrix with positive diagonal elements c_i results in

$$\underline{x} = C\underline{y} - B\underline{z}$$

and an expression 2) of the form

$$G = (2\pi)^{-k/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} P\left[(a_i + \sum_{j=1}^{k} b_{ij} z_j)/c_i\right] \exp\left(-z'z/2\right) \prod_{j=1}^{k} dz_j. \quad 4)$$

3. Conditions for the existence of $V = C^2 + BB'$.

Sufficient conditions will be noted here for expressing an n x n symmetric positive matrix, $V = \{v_{ij}\}$, in the form

$$V = C^2 + BC_1 B';$$

where

C^2 is a diagonal matrix with positive diagonal elements,

B is an n x k real matrix of rank $k \leq (n+1)/2$, and

C_1 is an k x k diagonal matrix with diagonal elements either plus or minus one.

The term submatrix of V will refer to a matrix obtained by deleting rows and (or) columns of the matrix V.

Let V_i denote any (k+1) x (k+1) submatrix of V containing v_{ii} but no other diagonal element of V and V_{ii} denote the cofactor of v_{ii} in V_i. Conditions for expressing V in the form 1) are:

i) For each i at least one V_{ii} is non-zero.

ii) For each V_{ii} equal to zero all k x k submatricies of V (not including a diagonal element) formed from the rows (or the columns) of V used in the matrix of the V_{ii} are singular.

iii) For each i, $v_{ii} - |V_i|/V_{ii} = v^*_{ii} < v_{ii}$ for all V_i with $V_{ii} \neq 0$.

If these conditions hold let $c_i^2 = v_{ii} - v^*_{ii}$ and $BC_1B' = V^*$, where V^* is formed by replacing the v_{ii} in V with v^*_{ii}. It need only be shown that V^* is of rank k.

Consider any V_i^* with $V_{ii} \neq 0$, from condition iii) $|V_i^*| = 0$ and V_i^* is of rank k. Augment V_i^* with one more sub row of V* not containing a diagonal element of V*. Since all (k+1) x (k+1) submatrix of V* (containing exactly one diagonal element) are of rank $\leq$ k, then this augmented matrix is of rank k. From this and condition ii) all submatrix of V* containing at most one diagonal element of V* cannot be of rank greater than k.

Now if n is even consider H, an ½ n x ½ n submatrix of V* of rank k that contains no diagonal element of V* (if n is odd, let H be ½ (n+1) x ½ (n-1) containing no diagonal element of V* and the following argument still holds). Augment H with another column of V*; this augmented matrix will have rank k since at most one diagonal element of V* is included. Thus the added column is a linear combination of the colums of H. This augmentation can be done with each column of V* not in H and the same procedure with each row of V* not in H. This and the symmetry of V* suffice to show that V* is of rank k.

It readily follows that conditions ii) and iii) are also necessary conditions and the adjustment when i) does not hold will be mentioned later.

In order for the method of Das to be of use in the evaluation of G one further condition must hold.

iv) The k non-zero latent roots of V* must be positive (this makes C_1 an identity matrix).

If these four conditions hold the columns of B are the latent vectors (standardized to length one) of V* multiplied by the square root of their respective latent root. The b_{ij} in 4) are the ij elements of B and $c_i = \sqrt{v_{ii} - v_{ii}^*}$.

4. An example determining B when k is small

The determination of B is not difficult for small values of k which are perhaps the only ones of practical use. Any method of finding the latent roots and vectors of V* will suffice and those used in the following example are simply taking advantage of the symmetry and small rank of V*. Condition iv) above is verified (or disproven) as the latent roots are determined.

As an example of determining B consider the following variance-covariance matrix,

$$V = \begin{bmatrix} 10 & 1 & 3 & 7 & 6 \\ 1 & 11 & -5 & 0 & -3 \\ 3 & -5 & 10 & 5 & 6 \\ 7 & 0 & 5 & 12 & 9 \\ 6 & -3 & 6 & 9 & 10 \end{bmatrix}.$$

Each 3x3 submatrix of V not containing a diagnal element is singular and the following hold for 3x3 V_i:

$v_{ii} - |V_i|/V_{ii} = v^*_{ii} < v_{ii}$ $\qquad$ $i = 3;$ $\quad 10 - 5 = 5 < 10$

$i = 1;$ $\quad 10 - 5 = 5 < 10$ $\qquad$ $i = 4;$ $\quad 12 - 2 = 10 < 12$

$i = 2;$ $\quad 11 - 1 = 10 < 11$ $\qquad$ $i = 5;$ $\quad 10 - 1 = 9 < 10.$

Thus $k = 2$ is possible and V can be written as

$$V = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 1 & 3 & 7 & 6 \\ 1 & 10 & -5 & 0 & -3 \\ 3 & -5 & 5 & 5 & 6 \\ 7 & 0 & 5 & 10 & 9 \\ 6 & -3 & 6 & 9 & 9 \end{bmatrix}$$

The sum of the latent roots of V* equal the trace, $\lambda_1 + \lambda_2 = 39$, and the product equals the sum of the second order principle minors, $\lambda_1 \lambda_2 = 324$. Thus the two latent roots are $\lambda_1 = 12$, $\lambda_2 = 27$; satisfying condition iv).

Since the columns of V* are linear combinations of its latent vectors any two independent columns may be used to find the latent vectors of V*; say the first ($\underline{\gamma}_1$) and second ($\underline{\gamma}_2$). Then

$$V^*(a\underline{\gamma}_1 + b\underline{\gamma}_2) = \lambda(a\underline{\gamma}_1 + b\underline{\gamma}_2).$$

From the first element of this vector equation

$$120\ a - 18b = \lambda(5a + b).$$

For $\lambda = 12$; $a = 1$, $b = 2$ suffice and for $\lambda = 27$; $a = 3$, $b = -1$. The standardized latent vectors are then $(1,3,-1,1,0)/\sqrt{12}$ and $(2,-1,2,3,3)/\sqrt{27}$ and

$$B' = \begin{bmatrix} 1 & 3 & -1 & 1 & 0 \\ 2 & -1 & 2 & 3 & 3 \end{bmatrix}.$$

Determining V* and then B in situations where condition i) does not hold can be accomplished in a stepwise manner parallel to the above. First adjust the portion of V related to the V_{ii} for which i) holds and subtract this V* from V; then adjust the remainder. The resulting complete B will necessarily contain zeros which then reduce the magnitude of evaluating G.

5. Bounds for G

The emphasis of this discussion has been towards setting up the expression 4) for G when the variance-covariance matrix, V, is such that a small value of k is possible. In situations where the first three conditions of Section 3 are "almost" satisfied, it may be possible to find bounds on G by changing some of the off diagonal elements of V slightly and applying the following minor extension of a theorem by Slepian (1962).

<u>Theorem:</u> If $\underline{x}$ (nxl) is multivariately normally distributed with mean $\underline{\mu}$ and non-singular variance-covariance matrix V; then $G = \text{Prob}\ (x_1 > a_1, x_2 > a_2, \ldots, x_n > a_n)$ increases with an increase in any off diagonal element of V.

<u>Proof:</u> Slepian (1962) proved this for the particular case of all a_i equal to zero utilizing the property

$$\partial f/\partial \rho_{ij} = \partial^2 f/\partial x_i \partial x_j$$

where $f(x_1, x_2, \ldots, x_n)$ is the joint density function.

In exactly the same manner and, without lack of generality, treating the 1,2 case

$$G = \int_{a_1}^{\infty}\int_{a_2}^{\infty}\cdots\int_{a_n}^{\infty} f(x_1, x_2, \ldots, x_n)\, dx_1 dx_2 \ldots dx_n$$

and $$\partial G/\partial \rho_{12} = \int_{a_1}^{\infty}\int_{a_2}^{\infty}\cdots\int_{a_n}^{\infty} \partial^2 f/\partial x_1 \partial x_2 \; dx_1 dx_2 \ldots dx_n$$

$$= \int_{a_3}^{\infty}\int_{a_4}^{\infty}\cdots\int_{a_n}^{\infty} f(a_1, a_2, x_3, \ldots x_n)\, dx_3 dx_4 \ldots dx_n$$

> 0 since f is a positive valued function.

6. Conclusion

The method of Das (1956) for evaluating a multivariate normal probability is of practical use for only a restricted class of variance-covariance matrices. This paper has attempted to increase this class somewhat and also provide a method of recognition and application for the more tracticable cases of small k. With sufficient ingenuity and good fortune bounds may also be found on a probability through the method of Das and repeated application of the theorem in Section 5.

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