

THEMIS SIGNAL ANALYSIS STATISTICS RESEARCH PROGRAM

GREATLY INCREASED PRACTICAL USEFULNESS OF TWO-PERSON
GAME THEORY BY ADOPTION OF MEDIAN CRITERION

by

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It is desirable to have optimum solutions that are of a "forcing" nature. That is, an optimum choice of strategies controls the game outcome according to some meaningful criterion (such as expected payoff). The minimax method of solution for expected-value game theory has this property (for example, see ref.1). Also, the two approaches that have been developed for median game theory are of this nature. However, both approaches for median game theory have huge advantages over the minimax method with respect to practical application.

The purpose of this paper is first to outline the two approaches to median game theory. Then, comparisons are made between these approaches and minimax game theory with respect to generality of application and effort needed for application (at the end of each outline). Finally, the two approaches are compared with each other.

The first approach, which is based on rankings within payoff matrices, has advantages over the second approach in the effort needed for application but strong disadvantages in generality of application. The second approach, which is based on ranking of the outcomes, has general applicability. The minimax method and the first approach are for situations where the players act competitively toward each other. The second approach is usable for almost any kind of situation (including that of competitive players). A special case occurs (for competitive players) that is a combination of the two approaches and has advantages of both. A strong advantage of the two median approaches is that the payoffs can be of a general nature. In fact, some payoffs might not even be numbers (for example, could represent categories). The allowable payoffs

to himself. A strategy that allows a player to be protective and win-

imize the payoff to the other player without consideration of the payoff

of the payoff to the other player. A vindictive player attempts to mini-

a protective player tries to maximize his payoff without consideration

dictively, are useful for situations with competitive players. That is,

competitively. The concepts of a player behaving protectively, or win-

The first approach is intended for use only when the players act

payoffs.

on the rankings, however, these results are applicable for any kinds of

payoffs have "values". Subject to the condition that the players agree

rices. For simplicity the description given here is phrased as if the

creasing desirability) are in agreement for the players in the two mat-

hold for payoffs of a general nature when the rankings (according to in-

in agreement for the two players. However, the results obviously also

bility) occurs. Then the rankings within the matrices are necessarily

be numbers for which a natural ranking (according to increasing desira-

for the first approach, the payoffs within a matrix were considered to

the basis for the first approach. In development of the current results

A ranking of the payoffs within each of the two matrices provides

FIRST APPROACH

The final section contains a comparison between the two approaches.

game theory. This is followed by a section devoted to the second approach.

The next section is concerned with the first approach for median

in the same unit for game theory with an expected-value basis.

must satisfy the arithmetical operations and ordinarily are expressed

dictive simultaneously is optimum for him when the players behave

competitively.

Let the players be designated as I and II. The following properties

always hold: A largest payoff P_I (P_{II}) occurs in the matrix for player

I (II) such that, acting protectively, he can assure himself at least

this payoff with probability at least $1/2$. Also, a smallest payoff

P_I (P_{II}) occurs in the matrix for player I (II) such that vindictive

player II (I) can assure, with probability at least $1/2$, that player

I (II) receives at most this payoff. The relations $P_I \leq P_{II}$ and

$P_{II} \leq P_I$ hold, with equality possible. Methods for evaluation of P_I ,

P_{II} , P_I , P_{II} are given in refs. 2 and 3, where the viewpoint of ref. 3

is ordinarily preferable. These references also contain methods for

determination of protective median optimum strategies and of vindictive

median optimum strategies.

Payoff matrices occur such that a player can be protective and vin-

dictive simultaneously (according to the median criterion). When this

can happen for a given player, the game is said to be one player median

competitive (OPMC) for him. A game is median competitive if and only

if it is OPMC for both players. A subclass of the median competitive

games is identified in ref. 2. The complete class is identified in ref. 3.

Identification of whether a game is OPMC for a player is not diffi-

cult. Consider the possible outcomes. Those outcomes such that the pay-

off to player I (II) is at least P_I (P_{II}) and also the payoff to player

II (I) is at most P_I (P_{II}) constitute set I (II). The game is OPMC

for player I (II) if and only if player I (II) can assure, with probability at least $1/2$, that an outcome in set I (II) occurs. A procedure for determination of whether a game is OPMC for a player, and for determining a median optimum strategy for him, is given in ref. 3. Now, let us compare the results for the first approach with those for the minimax method. In expected-value game theory, the players can be protective and vindictive simultaneously which corresponds to existence of minimax solutions, when the payoff matrices satisfy a zero-sum condition (sum of payoffs is zero for every strategy combination) or one of some mild modifications of this condition. These "zero-sum" games are a very small subclass of the median competitive games. Also, the OPMC concept, for just one player, does not seem to have an analogue in expected-value game theory. Thus, the first approach has very strong advantages over expected-value game theory with respect to generality of application.

The first approach also has very strong advantages over expected-value game theory with respect to application effort. For realistic games, a very large number of strategy combinations can occur. As an example, suppose that each player has 100 strategies. Then, there are 100,000 strategy combinations. An accurate payoff value must be determined for virtually every strategy combination in expected-value game theory. This can result in a huge amount of effort. For the first approach, it is sufficient to know the order of the payoffs within each matrix and to have accurate values for $P_I, P_{II}, P'_I, P'_{II}$. Moreover, knowledge of the order within the matrix for player I (II) identifies

Consider all the possible outcomes for a game. Suppose that, separately for each player, these outcomes can be ranked according to increasing desirability to that player (with equal desirability possible). This should virtually always be possible on a paired comparison basis. Each player could use almost any possible way of ranking the outcomes. If (p_I, p_{II}) is a general outcome, however, virtually any ordering method should be such that, for player I (II), desirability is a nondecreasing function of p_I (p_{II}) for fixed p_{II} (p_I), when payoffs can be ordered according to increasing desirability (separate orderings can occur for each player). Once an ordering according to increasing desirability has been determined for player I (II), he can identify a smallest set of outcomes S_I (S_{II}) such that the other outcomes are less desirable and such

SECOND APPROACH

A moderately thorough discussion of application advantages of the first approach over the expected-value method, for discrete two-person games, is given in ref. 4. These advantages are augmented by the more general types of payoffs that can be considered, as described at the beginning of this section. Hence, the first approach has very strong advantages with respect to application effort. of the smallest payoffs, is not needed for use of the first approach. ordering among a set of the largest payoffs of a matrix, and among a set the matrix locations for p_I (p_{II}) and p_I (p_{II}). Also, knowledge of

that he can assure an outcome of S_I (S_{II}) with probability at least $1/2$. A procedure for determining S_I and S_{II} is outlined in the Appendix. A procedure for determining corresponding optimum median strategies is also outlined there. These results are "the best obtainable", according to the median criterion used. The idea of ranking outcomes in a general fashion is based on a method used in ref. 5.

Now consider a special case of the second approach that has some aspects of the first approach, including competitive behavior of the players. In addition, the payoffs are ranked for each matrix and the players agree on the rankings. The methods for ranking of the outcomes are such that all outcomes of set I (II) have maximum desirability to player I (II). Also, the relations among desirability that are stated in terms of P_I and P_{II} , at the beginning of this section, are satisfied. The material using this type of ranking is taken from ref. 6.

Finally, let us compare results for the second approach with those for expected-value game theory in the general case. First, the players do not necessarily behave in a competitive manner for the second approach. Second, the second approach is applicable for virtually all discrete two-person games. Thus, the second approach is hugely preferable to expected-value game theory with respect to generality of application. Next, consider the effort needed for applying a method. The second approach and the expected-value method are about the same with respect to necessity for accurately evaluating payoffs. That is, virtually all payoffs need to be accurately evaluated in both cases. The ranking of outcomes can require a huge amount of effort if a paired comparison

method is used, due to a huge number of possible pairs, which is $N(N-1)/2$ if N is the number of outcomes. However, ranking is easy when a suitable function of P_I and P_{II} is available for this purpose.

The special case of the second approach has strong advantages over expected-value game theory with respect to effort needed for application. Within set I , and set II , it is sufficient to know relative order among payoffs and accurate values for at most two payoffs (which are identified by the orderings). Often, set I (II) contains the predominant number of the outcomes in S_I (S_{II}), so that a relatively few additional outcomes are needed to obtain S_I (S_{II}). However, this special case has less advantages with regard to generality of application. That is, the behavior of the players must be competitive and the players must be in agreement with respect to the orderings within payoff matrices.

COMPARISON OF APPROACHES

Now, the two approaches are compared with regard to generality of application and effort for application. The properties considered have already been described in the comparisons of the approaches with the expected-value method and, for brevity, will not be stated in much detail. The greatest advantage of the first approach over the second approach is with respect to effort for application. The necessity to accurately evaluate payoffs is almost entirely replaced by the requirement that they be ordered (separately in each payoff matrix). At most two payoffs need to be accurately evaluated for each matrix and their matrix positions are identified by the orderings. As discussed in ref. 4, great amounts of time and effort can be needed to accurately evaluate very large num-

bers of payoffs, and huge numbers of payoffs can easily occur in practice. Also, there can be conceptual difficulties in even approximate determination of some payoffs, and the first approach can have strong advantages when this is the case (ref. 4). Ordinarily, virtually all of the payoffs need to be accurately evaluated for the second approach although exceptions occur (one, the special case of the preceding section, is discussed later).

The ordering of payoffs within matrices is virtually always very

much easier than the ordering of outcomes. This is especially the case if paired comparisons, which is a general method, needs to be used for ordering the outcomes. Paired comparisons seldom needs to be used for ordering payoffs, even when some of the payoffs are not numbers. Also, if there is difficulty in ordering payoffs, the difficulty in ordering outcomes is usually increased correspondingly. One case, however, where ordering outcomes is not difficult is that where, for a player, relative desirability is expressible as an explicit function of (p_I, p_{II}) , where (p_I, p_{II}) is a general outcome. Unfortunately, the development of a

suitable function of (p_I, p_{II}) can itself be very difficult.

The two approaches have roughly the same properties with regard to

allowable kinds of payoffs. However, for very general types of payoffs, ordering of outcomes can be much more difficult than ordering of payoffs.

The second approach has an exceeding strong advantage over the first

approach with respect to generality of application. First, the situation

need not be that where the players act as if they were competitors. In

fact, one player could prefer that the other player receives larger pay-

offs. Second, the second approach can be used by a player whenever he is able to rank the outcomes according to their desirability to him.

It would seem that this ordering is possible for virtually all situations, so that the second approach is (virtually) always applicable.

On the other hand, the players act competitively for the first approach. Also, the players are required to agree on the rankings for the payoffs in the two matrices. Most important, however, is that an optimum solution occurs for a player only if the game is OPMC for him. The set of games that are OPMC for at least one player is a very small subclass of the class of all discrete two-person games.

The special case of the second approach has properties of both approaches, which results in advantages and disadvantages. The disadvantages, compared to the ordinary second approach, is that the players

are required to behave as competitors and that they must agree on the orderings of the payoffs in the two matrices. That is, the generality

of application is reduced somewhat. However, there are strong advantages with respect to the necessity to evaluate payoffs and with respect to the effort needed to order outcomes (as discussed in the preceding

section).

Finally, it should be noted that only subsets of payoffs and subsets of outcomes actually need to be determined. Only the set of payoffs at least as desirable to player I (II) as P_I (P_{II}) and the set of payoffs at most as desirable to player I (II) as P_I (P_{II}) need to be determined for the first approach. Only the outcomes of sets S_I and S_{II} need to be determined for the second approach. Also, situations occur where

cooperation is definitely preferable to optimum solutions for median game theory. Cases of this nature are considered in ref. 5.

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13. ABSTRACT

Considered is discrete two-person game theory where the players choose their strategies independently. Two approaches using a median criterion have been developed for obtaining optimum solution. In the first approach, the payoffs are ranked according to desirability separately for each matrix (with agreement between the players on these rankings). For the second approach, the possible outcomes of the game (pairs of payoffs, one to each player) are ordered according to desirability separately by each player. Only situations of players behaving competitively are considered for the first approach. The class of games with optimum solutions for the first approach is huge compared to (and includes) the class with minimax solutions for expected-value game theory, but is a very small subclass of all games. A strong application advantage of the first approach is that ranking of the payoffs for each player, plus accurate evaluation of most two payoffs for each player, is sufficient for application. Rather general situations can be considered for the second approach, which is usable for virtually all games. Ordinarily, however, virtually all payoffs need to be accurately evaluated for the second approach, and determination of the ranking for outcomes can require huge effort. An exception occurs for a case where the two approaches are combined. For both approaches, the payoffs can be of a very general nature (some payoffs may not even be numbers). In this paper, the two approaches are outlined and their application properties are discussed.